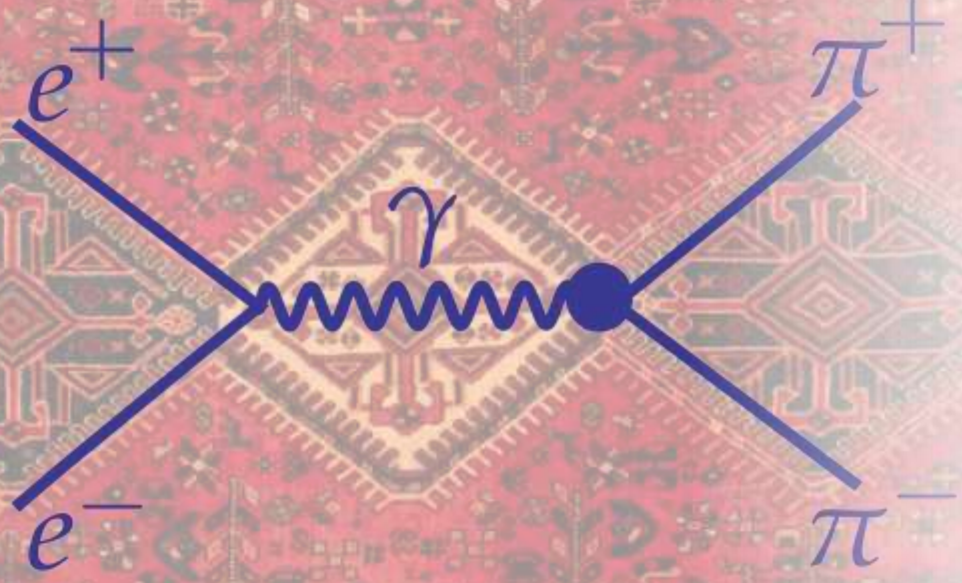


Analytical Studies of the Pion Vector Form Factor

GDR-InF Annual Workshop 2025

INTENSITY frontier GDR-InF

Fatemeh Nouri

université PARIS-SACLAY

IJCLab Irène Joliot-Curie Laboratoire de Physique des 2 Infinis

Motivation

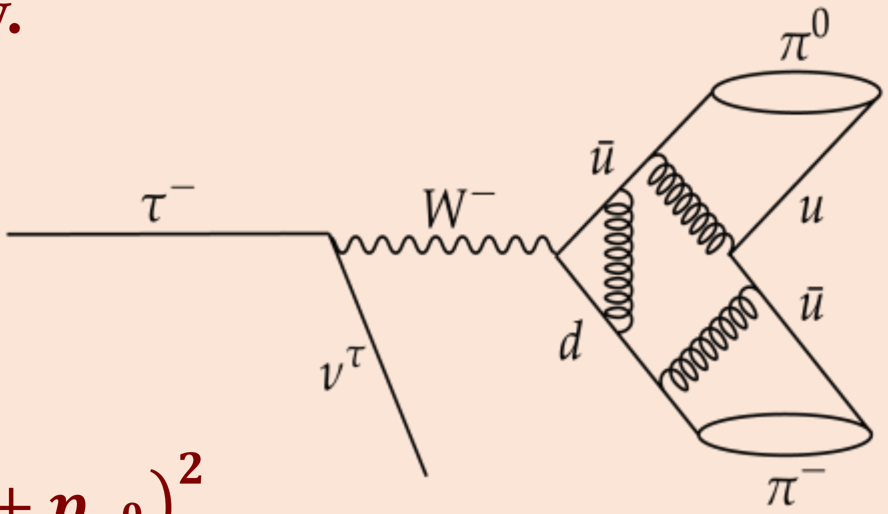
Motivation

- **Non-perturbative QCD interactions are often a major source of theory uncertainty.**

Motivation

- Non-perturbative QCD interactions are often a major source of theory uncertainty.

- In some processes, this uncertainties are encoded in form factors.

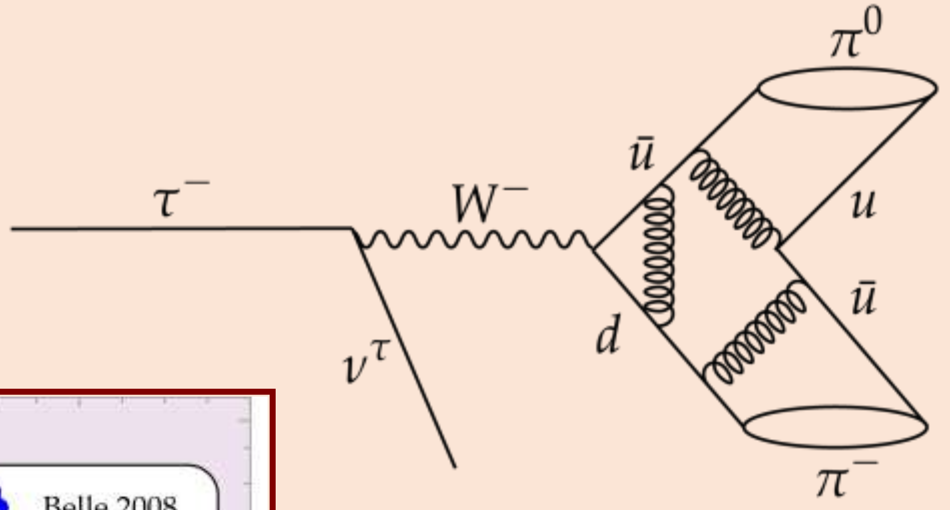


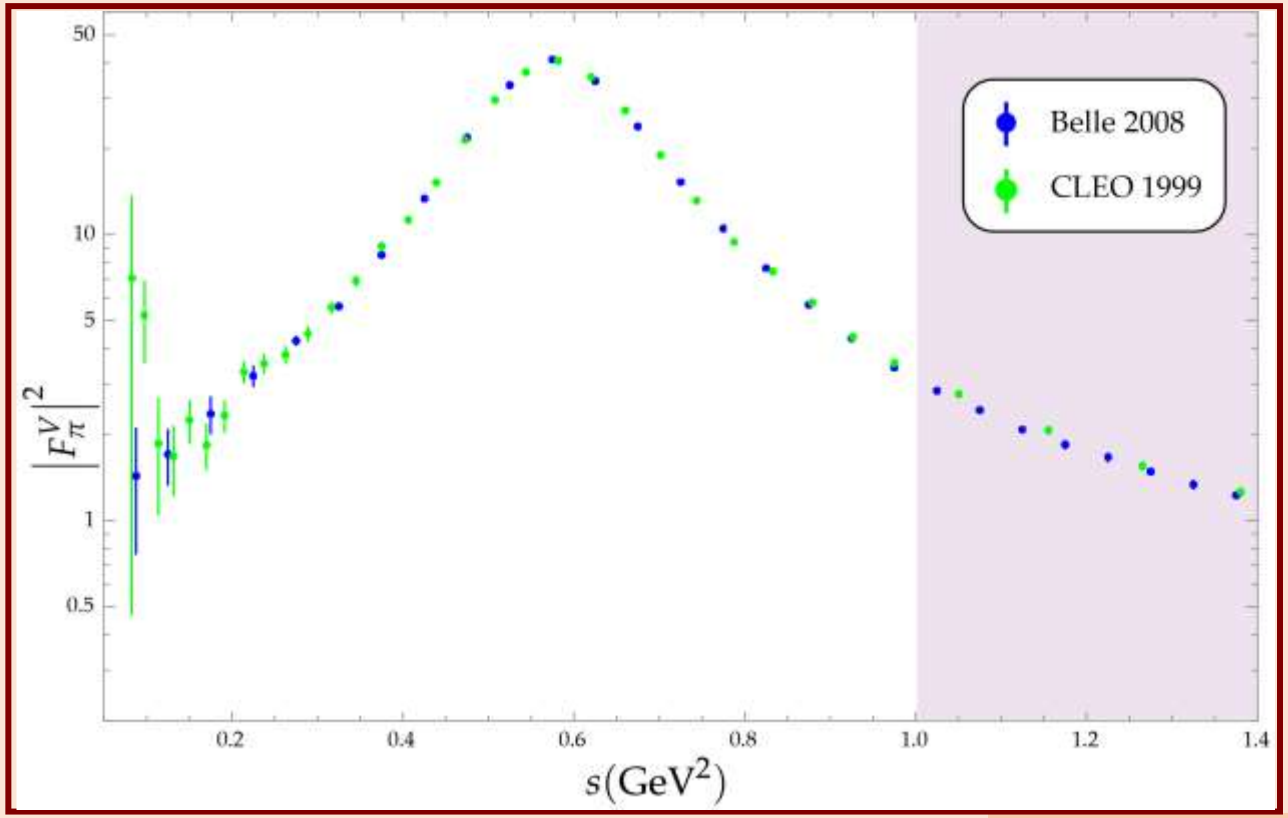
$$\left\langle \pi^0\pi^- \middle| J_W^{V,\mu} \middle| 0 \right\rangle = \left(p_{\pi^-} - p_{\pi^0}\right)^\mu F_\pi^V(s); \quad s = \left(p_{\pi^-} + p_{\pi^0}\right)^2$$

Pion vector form factor

Motivation

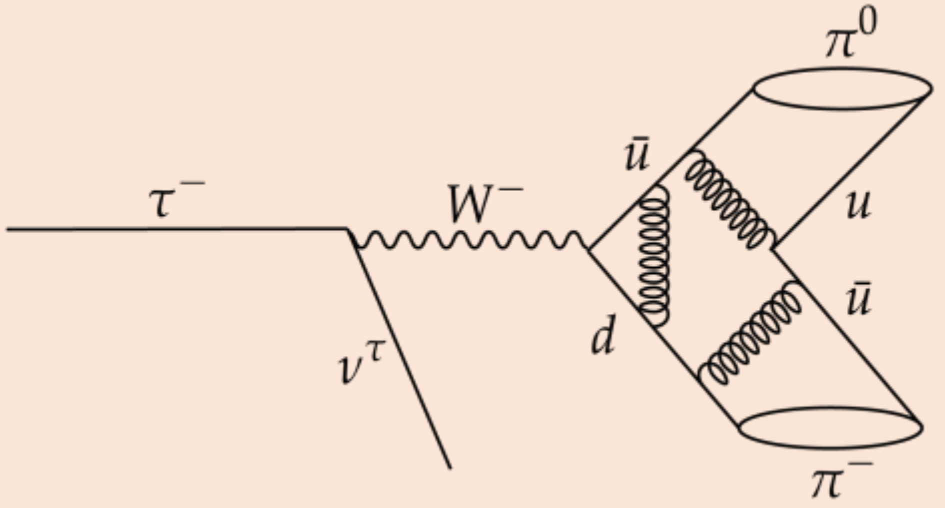
$$\left\langle \pi^0\pi^- \middle| J_W^{V,\mu} \middle| 0 \right\rangle = (p_{\pi^-} - p_{\pi^0})^\mu F_\pi^V(s); \quad s = (p_{\pi^-} + p_{\pi^0})^2$$

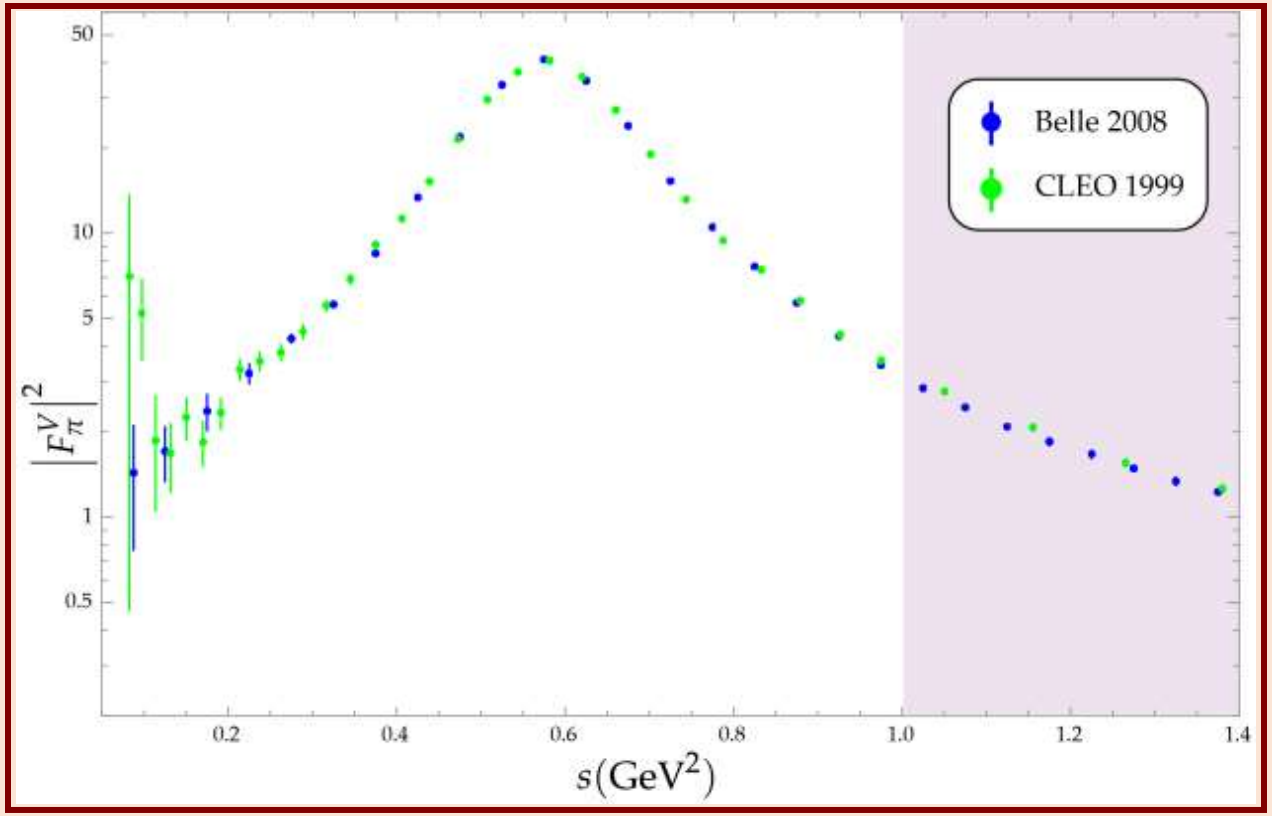




Motivation

$$\left\langle \pi^0\pi^- \middle| J_W^{V,\mu} \middle| 0 \right\rangle = (p_{\pi^-} - p_{\pi^0})^\mu F_\pi^V(s); \quad s = (p_{\pi^-} + p_{\pi^0})^2$$

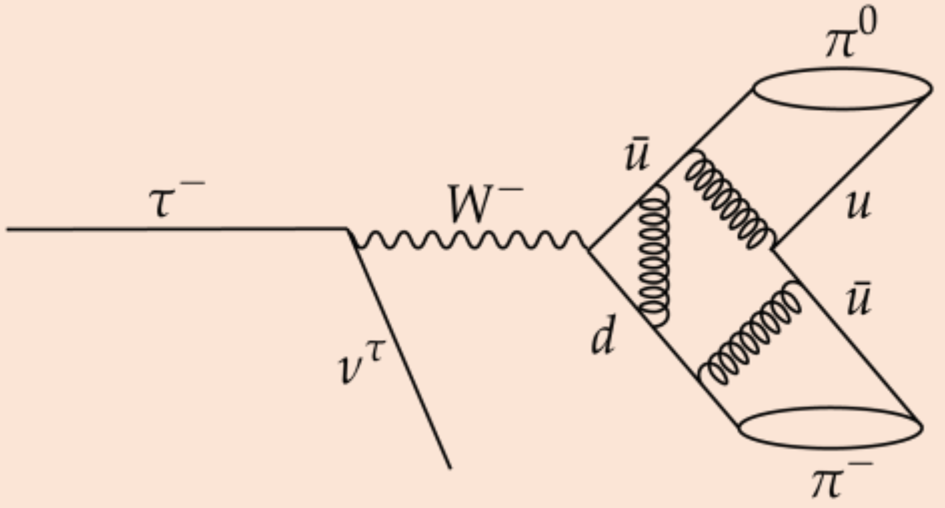


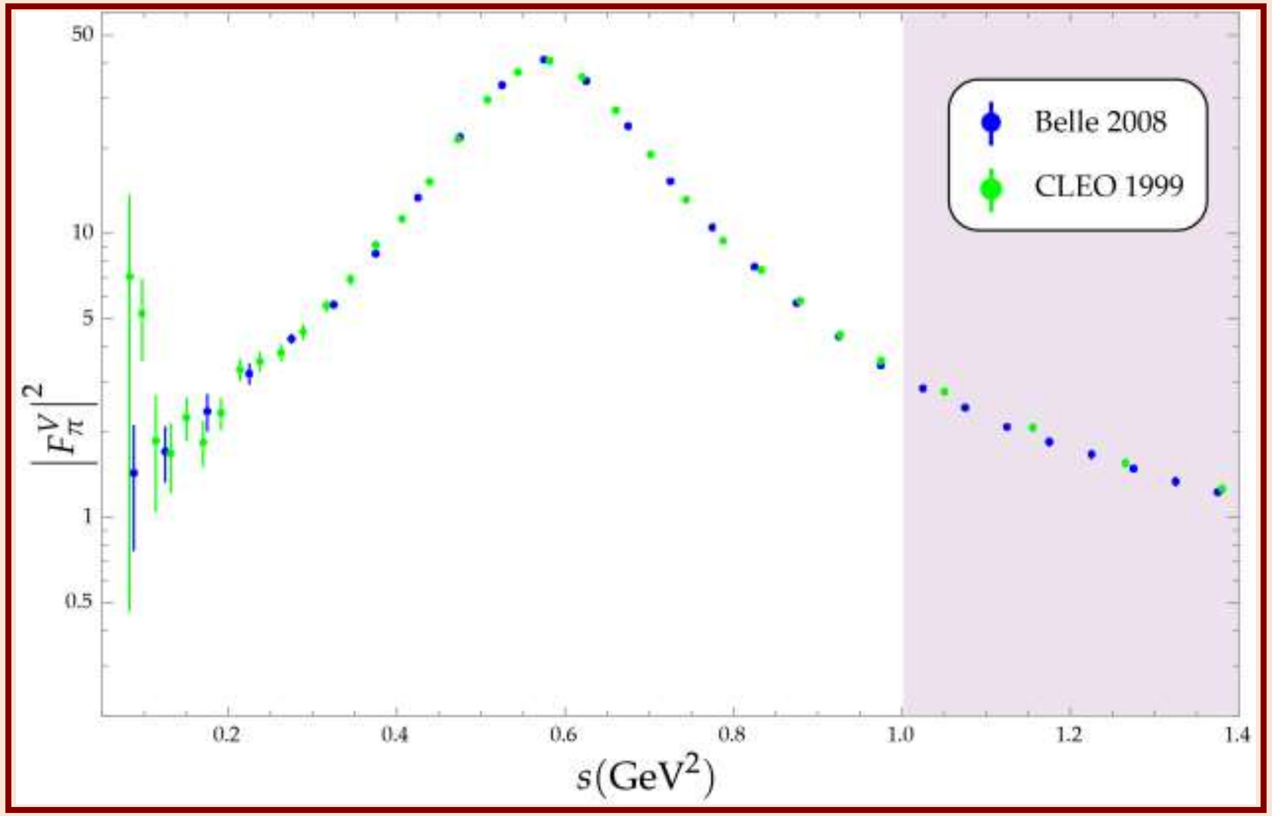


- **The peak corresponds to the isovector ρ (770) resonance.**

Motivation

$$\left\langle \pi^0\pi^- \middle| J_W^{V,\mu} \middle| 0 \right\rangle = (p_{\pi^-} - p_{\pi^0})^\mu F_\pi^V(s); \quad s = (p_{\pi^-} + p_{\pi^0})^2$$





- **The peak corresponds to the isovector ρ (770) resonance.**
- **How to describe the peak ?**

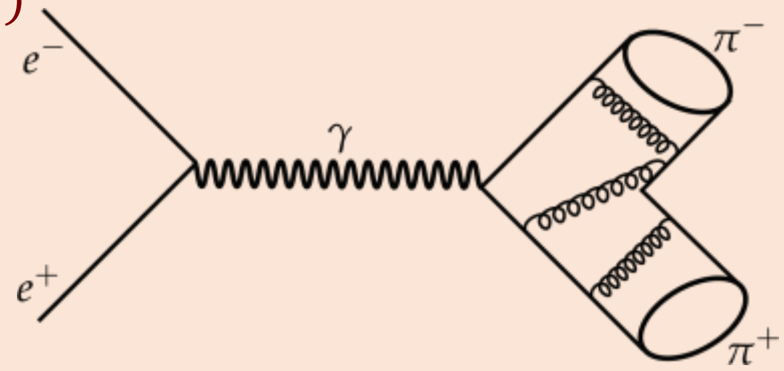
Motivation

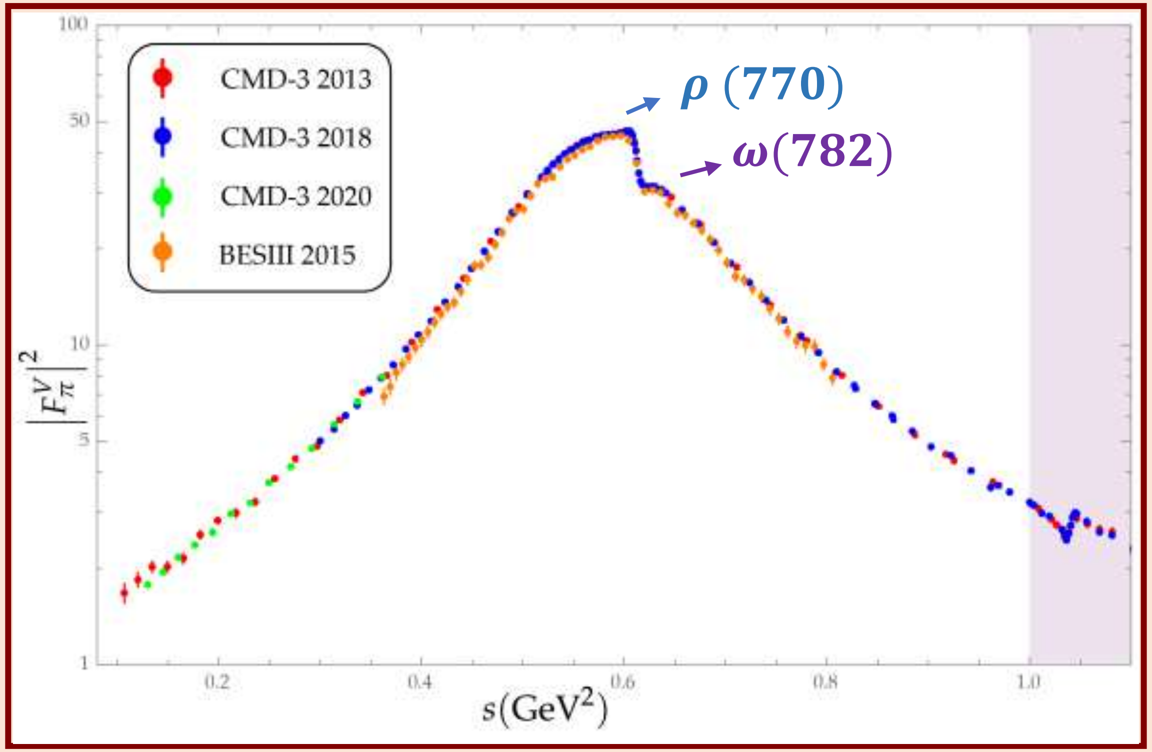
- **Breit-Wigner ?**

Motivation

$$\langle \pi^+\pi^- | J^{\mu}_{EM} | 0 \rangle = (p_{\pi^+} - p_{\pi^-})^{\mu} F^V_{\pi}(s)$$

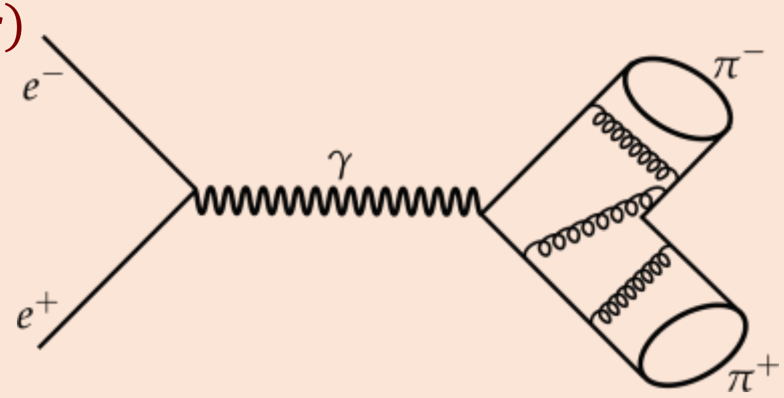
- **Breit-Wigner ? Can not describe the interference, can not produce thresholds, etc.**





Motivation

$$\langle \pi^+\pi^- | J^\mu_{EM} | 0 \rangle = (p_{\pi^+} - p_{\pi^-})^\mu F^V_\pi(s)$$



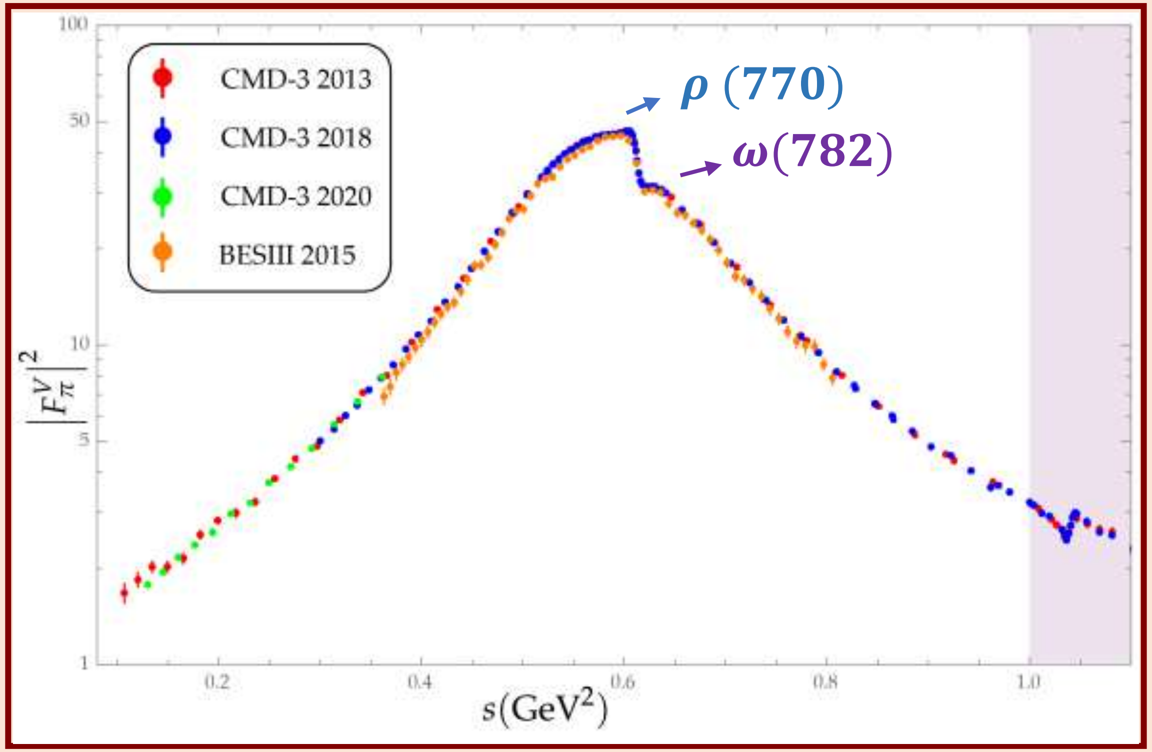
- Breit-Wigner ? Can not describe the interference, can not produce thresholds, etc.

- It is common to use Omnès representation:

[see e.g. Colangelo, Hoferichter, Stoffer, 1810.00007, '19]

$$F^V_\pi(s) \propto \Omega^1_1(s) = \exp\left\{\frac{s}{\pi}\int_{4m_\pi^2}^{\infty} ds' \frac{\delta^1_1}{s'(s'-s)}\right\}$$

Phase of the scattering amplitude, an external input



Motivation

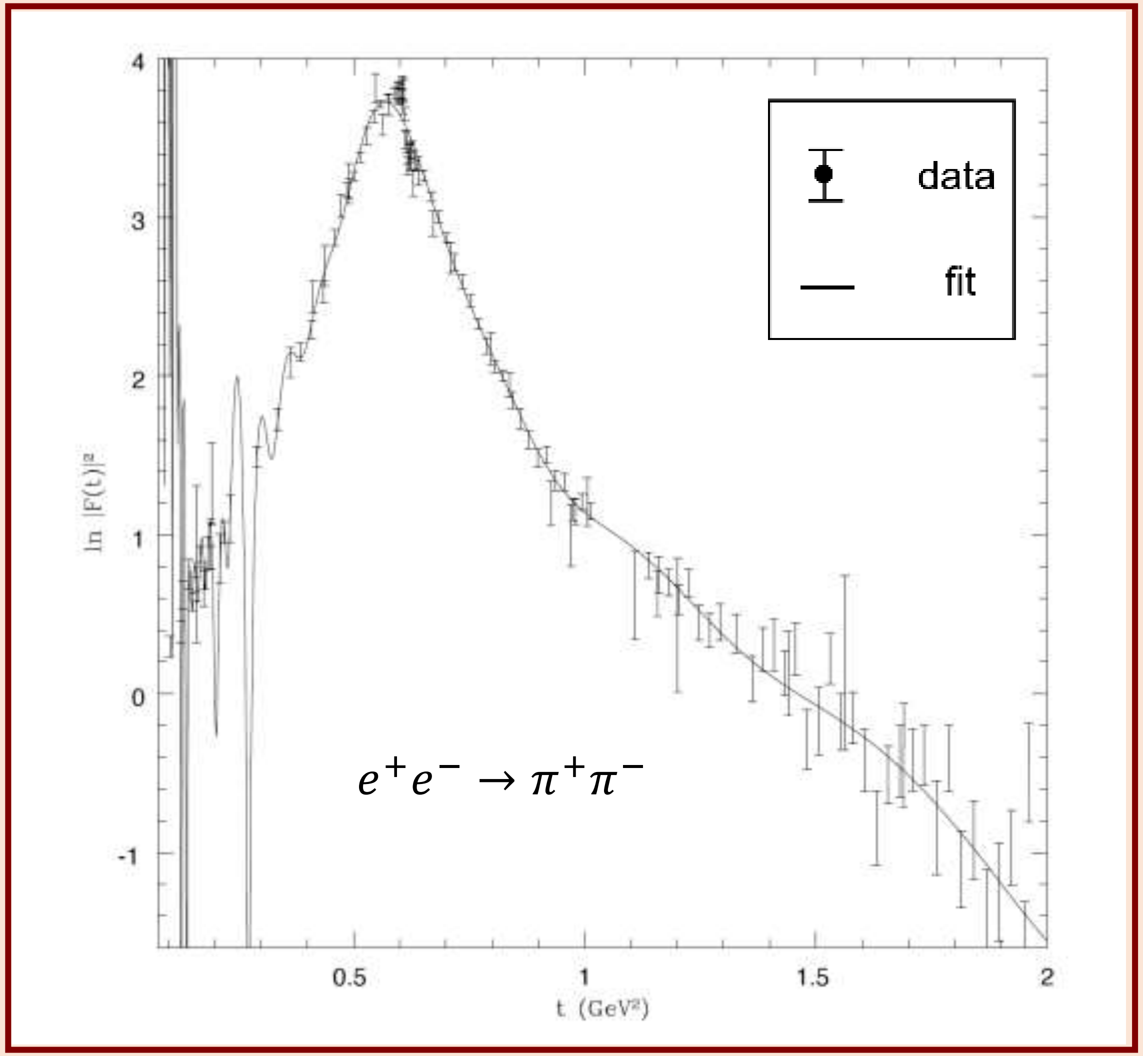
- **Another approach: a Taylor expansion in a conformal variable.**
  [Grinstein, Boyd, Lebed, 90's]

Motivation

- Another approach: a Taylor expansion in a conformal variable. [Grinstein, Boyd, Lebed, 90's]

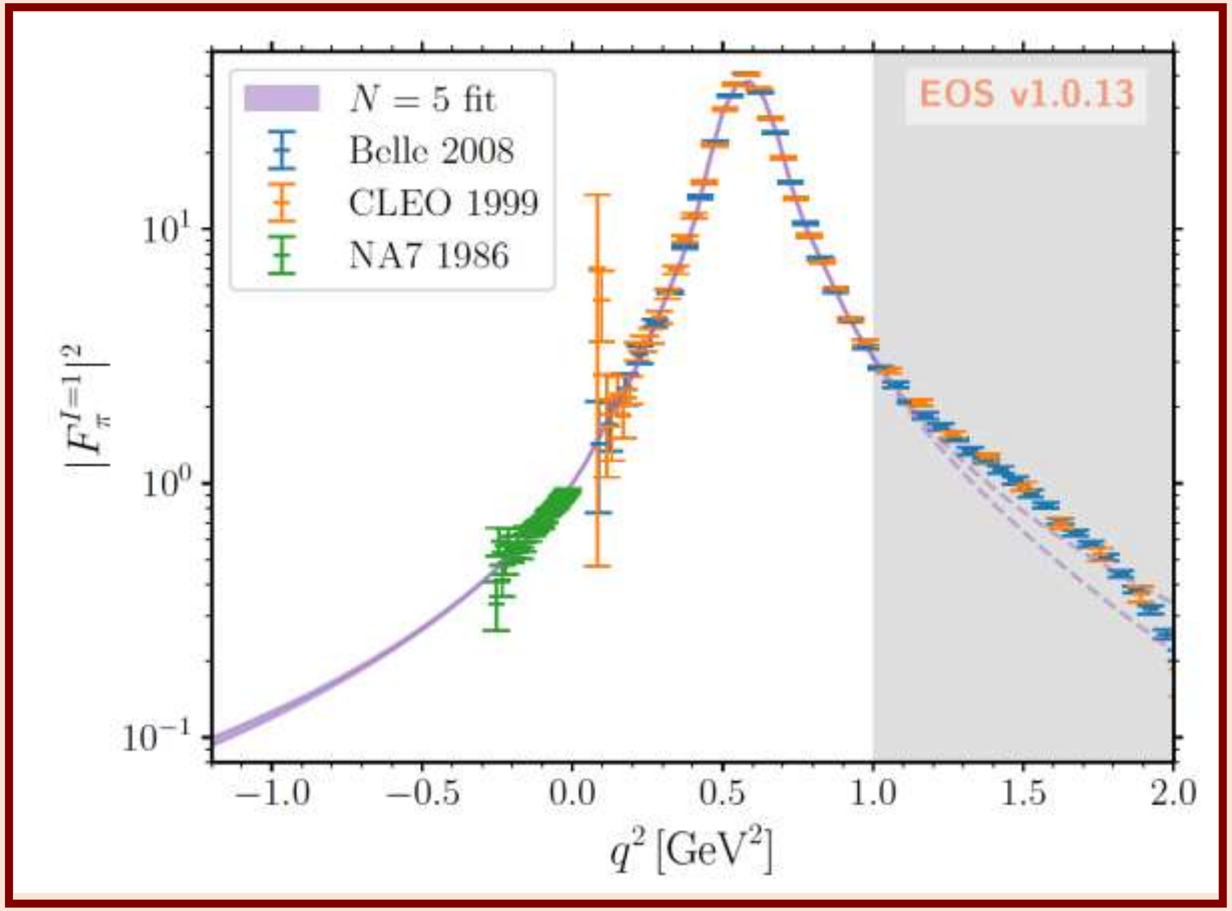
- Convergent Taylor series but unphysical oscillations in the fitted FF even after keeping 60 terms (Runge's phenomenon).

- We need a better description of form factors.



[Buck, Lebed, hep-ph/9802369, '98]

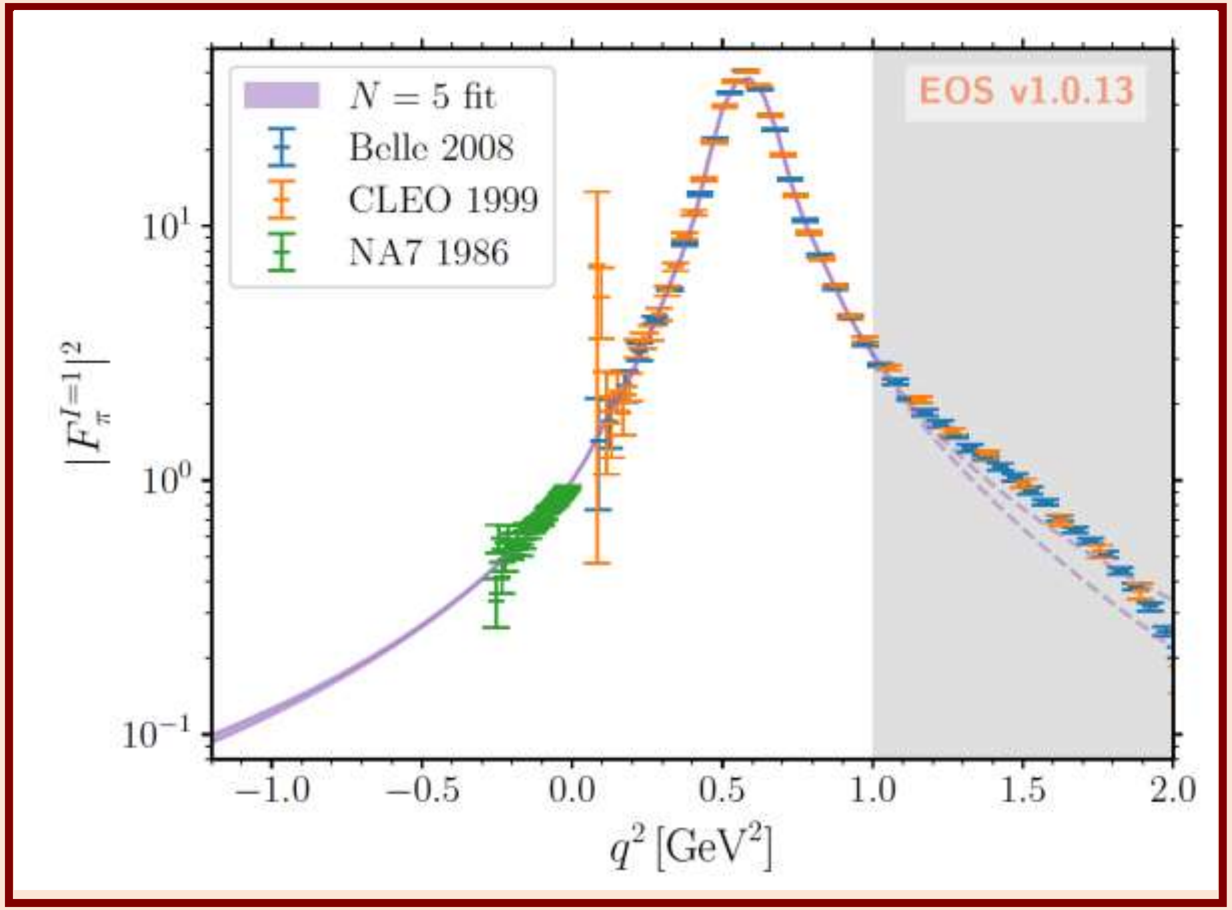
Improved Parametrization [Kirk, Kubis, Reboud, van Dyk, '25]



[Kirk, Kubis, Reboud, van Dyk, 2410.13764, '25]

- Successful to describe $\tau^- \to \pi^-\pi^0\upsilon_\tau$ data with only 5 terms of expansion.

Improved Parametrization [Kirk, Kubis, Reboud, van Dyk, '25]

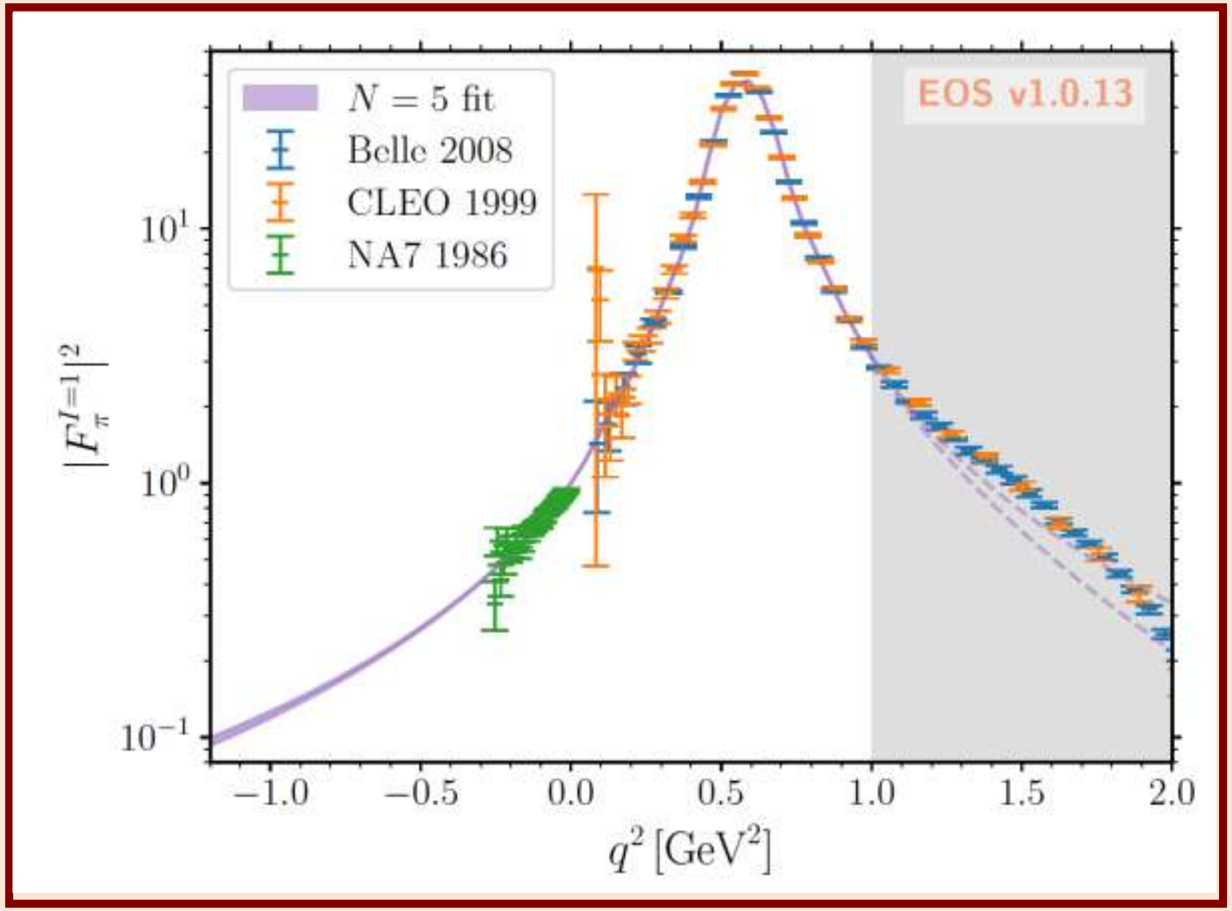


[Kirk, Kubis, Reboud, van Dyk, 2410.13764, '25]

- **Successful to describe $\tau^- \to \pi^-\pi^0\upsilon_\tau$ data with only 5 terms of expansion.**
- **The peak is due to the isospin-preserving decay $\rho^- \to \pi^-\pi^0$.**

$$|I = 1, I_3 = -1\rangle$$

Improved Parametrization [Kirk, Kubis, Reboud, van Dyk, '25]



[Kirk, Kubis, Reboud, van Dyk, 2410.13764, '25]

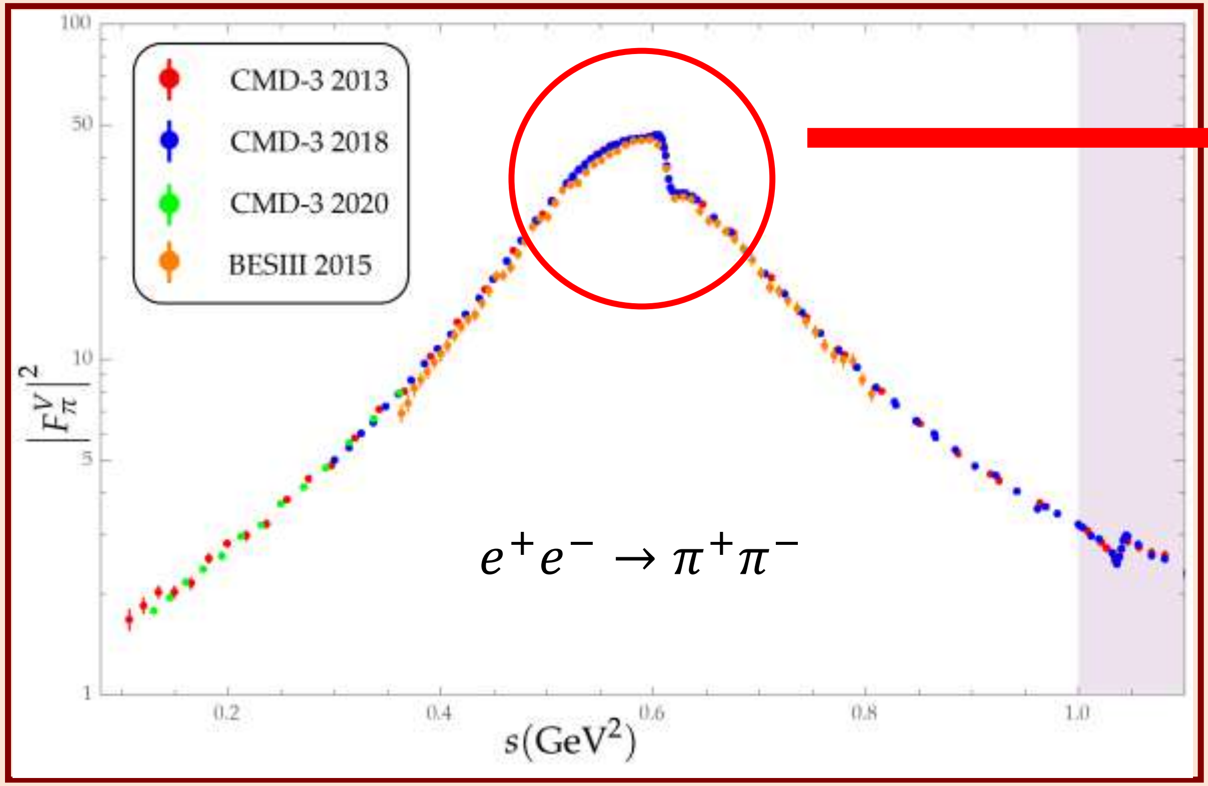
- **Successful to describe $\tau^- \to \pi^-\pi^0 \upsilon_\tau$ data with only 5 terms of expansion.**
- **The peak is due to the isospin-preserving decay $\rho^- \to \pi^-\pi^0$.**

$|I = 1, I_3 = -1\rangle$

Very good parametrization for isospin-1 projection of the pion vector form factor $F_\pi^{I=1}$ ☺

The aim of this project ...

How to generalize this parametrization to account for the $\rho - \omega$ mixing ?



The dominant peak is due to isospin preserving $\rho^0 \to \pi^+\pi^-$ decay but the small distortion is due to isospin violating $\omega^0 \to \pi^+\pi^-$ decay.

$I = 0 \neq I = 1$

We need to parametrize $F_\pi^{I=0}$

Form Factor Properties [Caprini, Functional Analysis and Optimization Methods, '00]

- Consider a semileptonic decay: $P_1 \to P_2 \, \bar{l} \, l'$

vector ff scalar ff momentum transfer

$$\langle P_2(p_2)|J^\mu|P_1(p_1)\rangle = f^+(q^2)\left[(p_1+p_2)^\mu - \frac{M_1^2-M_2^2}{q^2}\,q^\mu\right] + f^0(q^2)\,\frac{M_1^2-M_2^2}{q^2}\,q^\mu$$

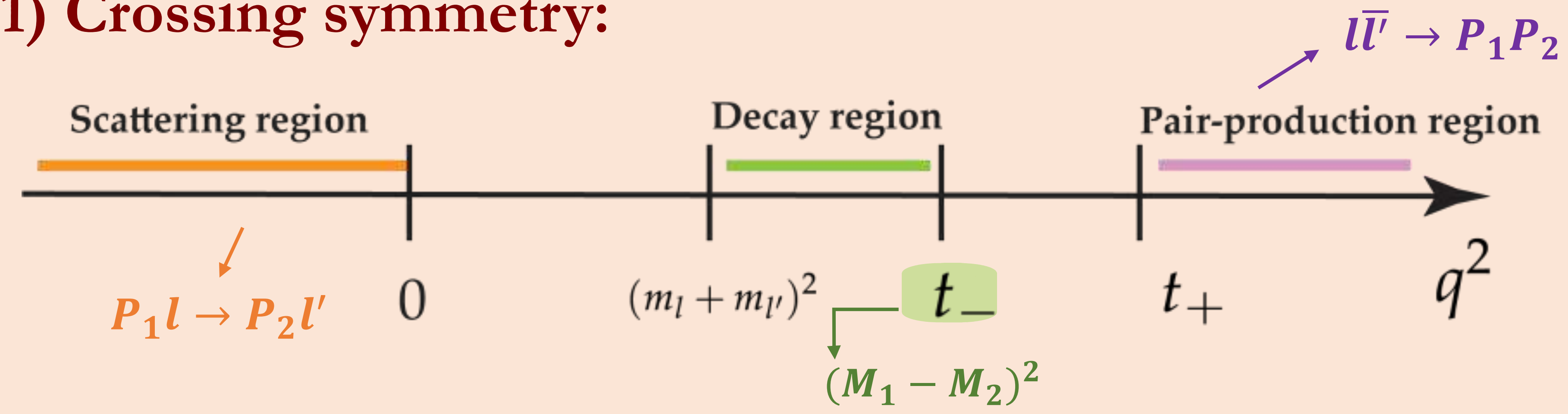
Form Factor Properties [Caprini, Functional Analysis and Optimization Methods, ’00]

- Consider a semileptonic decay: $P_1 \to P_2 \, \bar{l} \, l'$

vector ff — scalar ff — momentum transfer

$$\langle P_2(p_2)|J^\mu|P_1(p_1)\rangle = f^+(q^2)\left[(p_1+p_2)^\mu - \frac{M_1^2-M_2^2}{q^2}q^\mu\right] + f^0(q^2)\frac{M_1^2-M_2^2}{q^2}q^\mu$$

1) Crossing symmetry:



2) Branch cut starts at pair-production threshold $t_+ = (M_1 + M_2)^2$.

Form Factor Properties [Caprini, Functional Analysis and Optimization Methods, '00]

- Consider a semileptonic decay: $P_1 \to P_2 \bar{l}\, l'$

vector ff — scalar ff — momentum transfer

$$\langle P_2(p_2)|J^\mu|P_1(p_1)\rangle = f^+(q^2)\left[(p_1+p_2)^\mu - \frac{M_1^2-M_2^2}{q^2}q^\mu\right] + f^0(q^2)\frac{M_1^2-M_2^2}{q^2}q^\mu$$

3) Unitarity and optical theorem requires that :

Phase factor

$$\frac{1}{\pi\,\chi_T(q^2)}\int_{t_+}^{\infty} dt\,\frac{\omega(t)|f_+(t)|^2}{(t-q^2)^3} \leq 1$$

Dispersive bound

Known function, from theory or lattice computations

[Bharucha, Feldmann, Wick, 1004.3249, '11]

Resonances

Resonances

- **Pair-production threshold opens another Riemann sheet.**

- **Resonance: Pole on the 2^{nd} (unphysical) sheet**

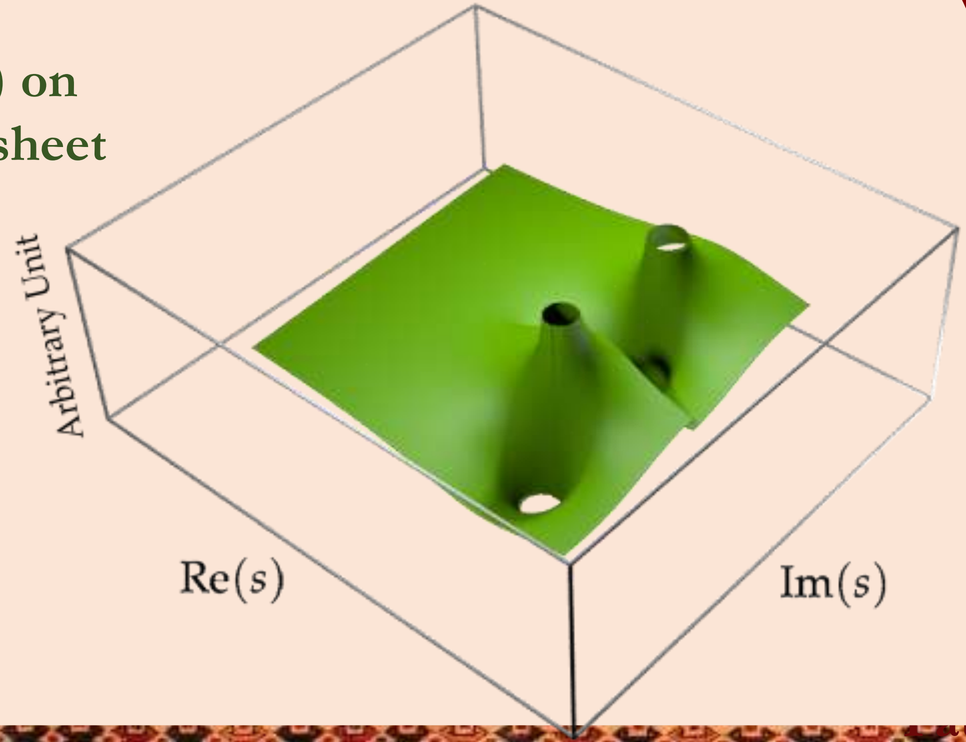
Resonances

- **Pair-production threshold opens another Riemann sheet.**

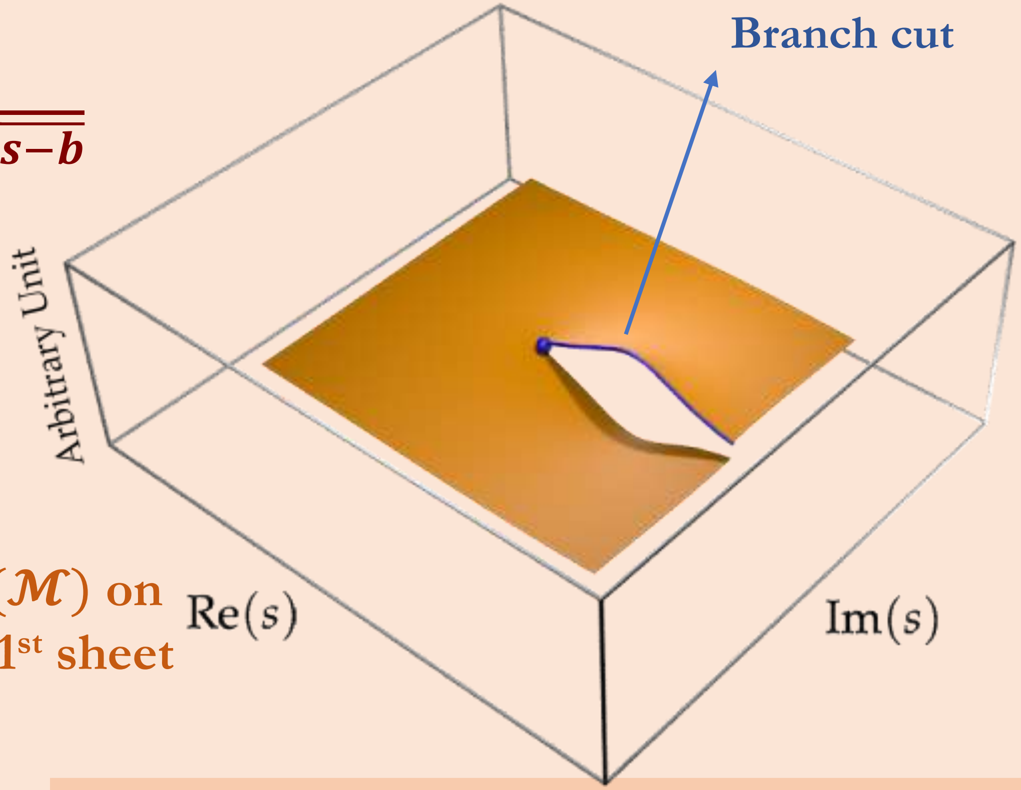
- **Resonance: Pole on the 2nd (unphysical) sheet, leaves an imprint on the 1st (physical) sheet.**

Example: consider an amplitude $\mathcal{M} \sim \frac{1}{\sqrt{s-a}\sqrt{s-b}}$

Im ($\mathcal{M}$) on the 2nd sheet



Im ($\mathcal{M}$) on the 1st sheet



A Conformal Map

- In q^2 plane two Riemann sheets are connected non-trivially.

A Conformal Map

- In q^2 plane two Riemann sheets are connected non-trivially.

- A conformal mapping allows us to "unfold" the branch cut :

$$z\left(q^2; t_+, t_0\right) = \frac{\sqrt{t_+ - q^2} - \sqrt{t_+ - t_0}}{\sqrt{t_+ - q^2} + \sqrt{t_+ - t_0}}$$

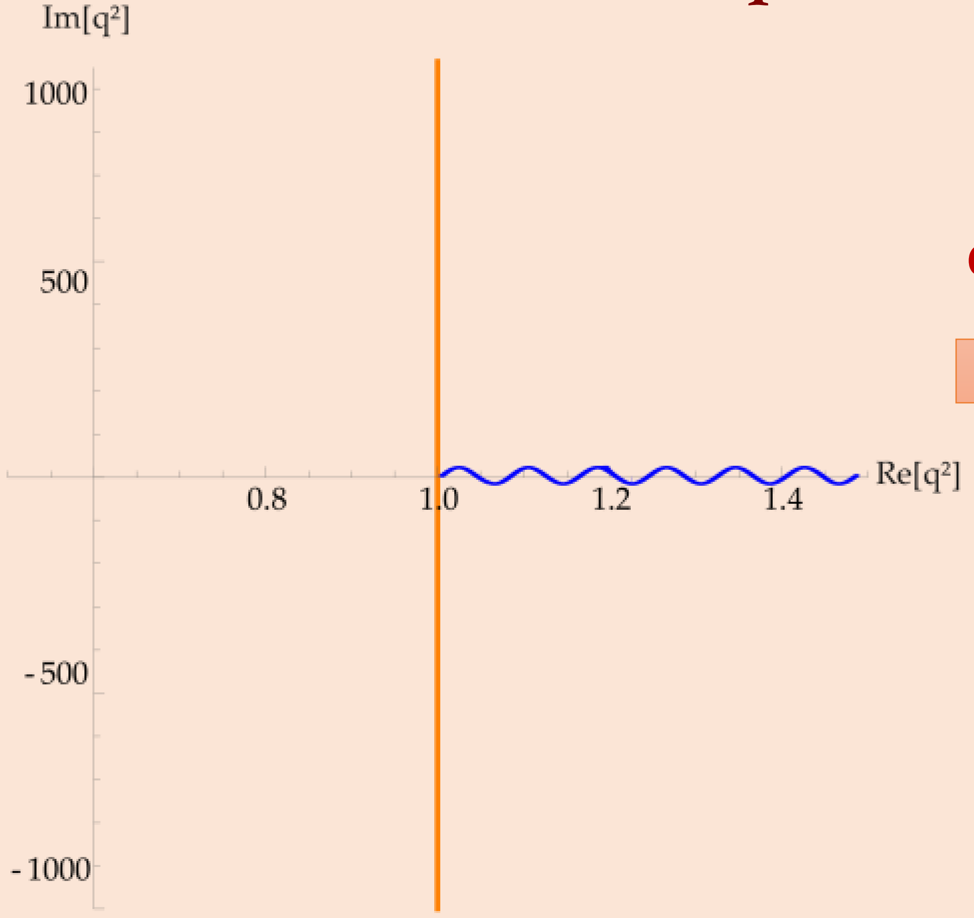
Free parameter

[Grinstein, Boyd, Lebed, 90's]

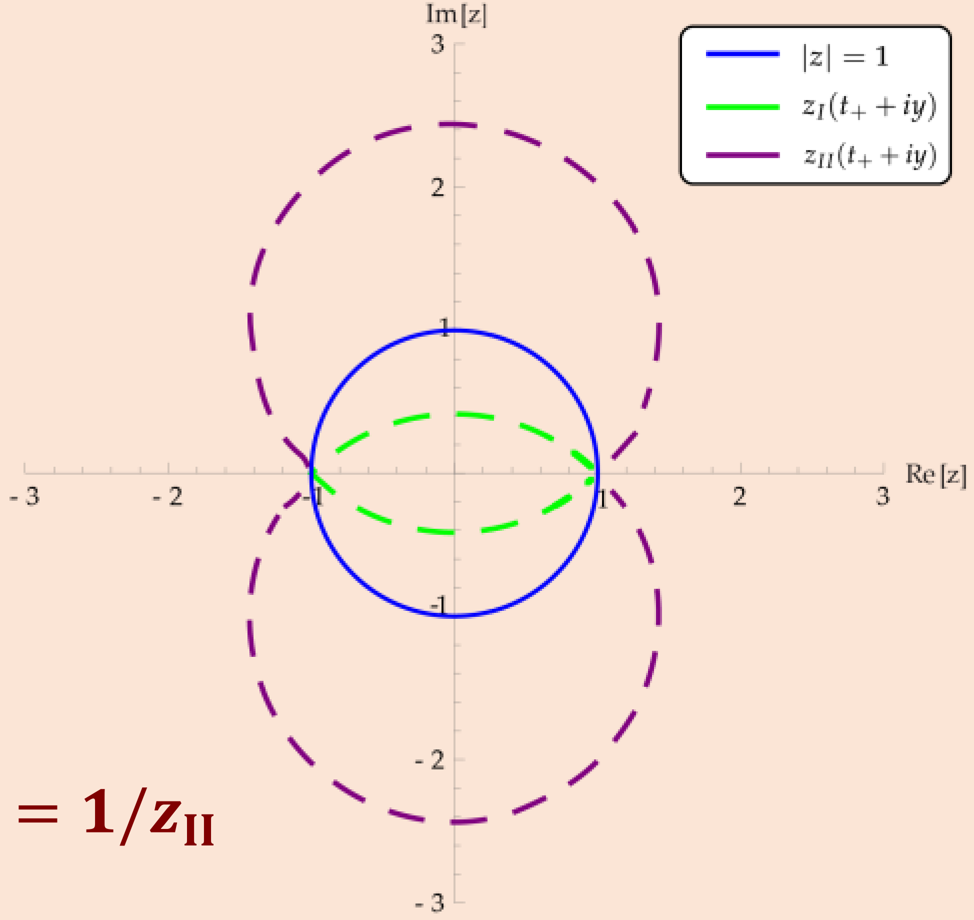
A Conformal Map

$$z\left(q^2; t_+, t_0\right) = \frac{\sqrt{t_+ - q^2} - \sqrt{t_+ - t_0}}{\sqrt{t_+ - q^2} + \sqrt{t_+ - t_0}}$$

- **Branch cut** is mapped on the unit circle and two sheets are separated:



Conformal map

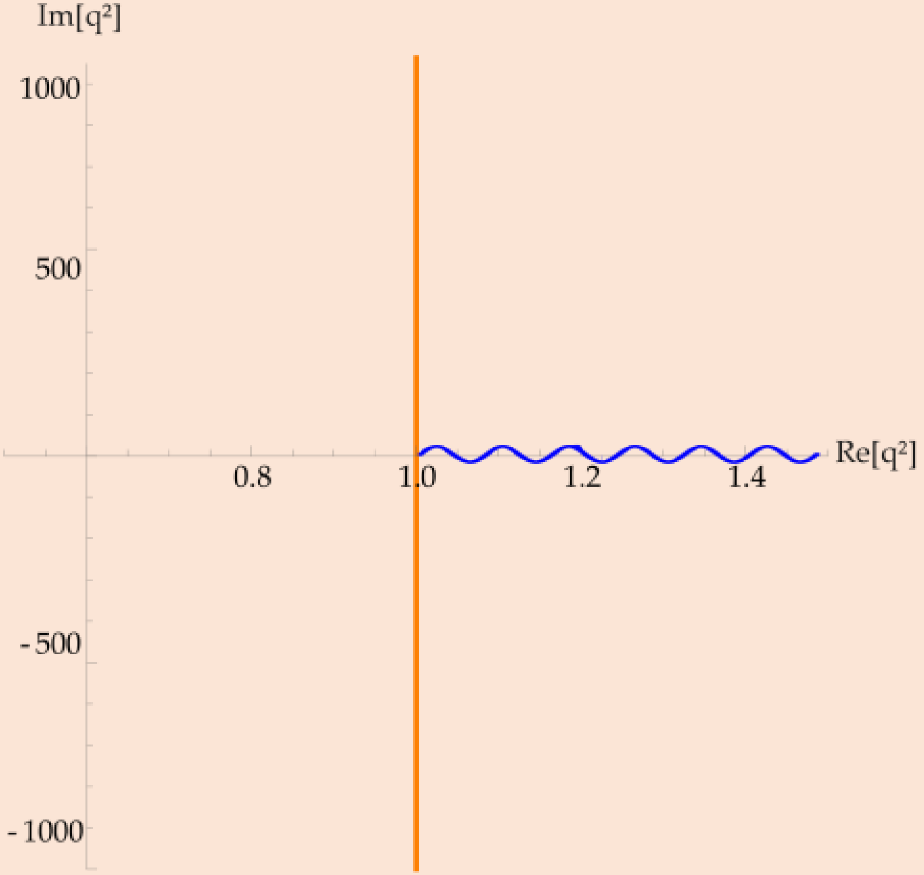


$$z_{\mathrm{I}} = 1/z_{\mathrm{II}}$$

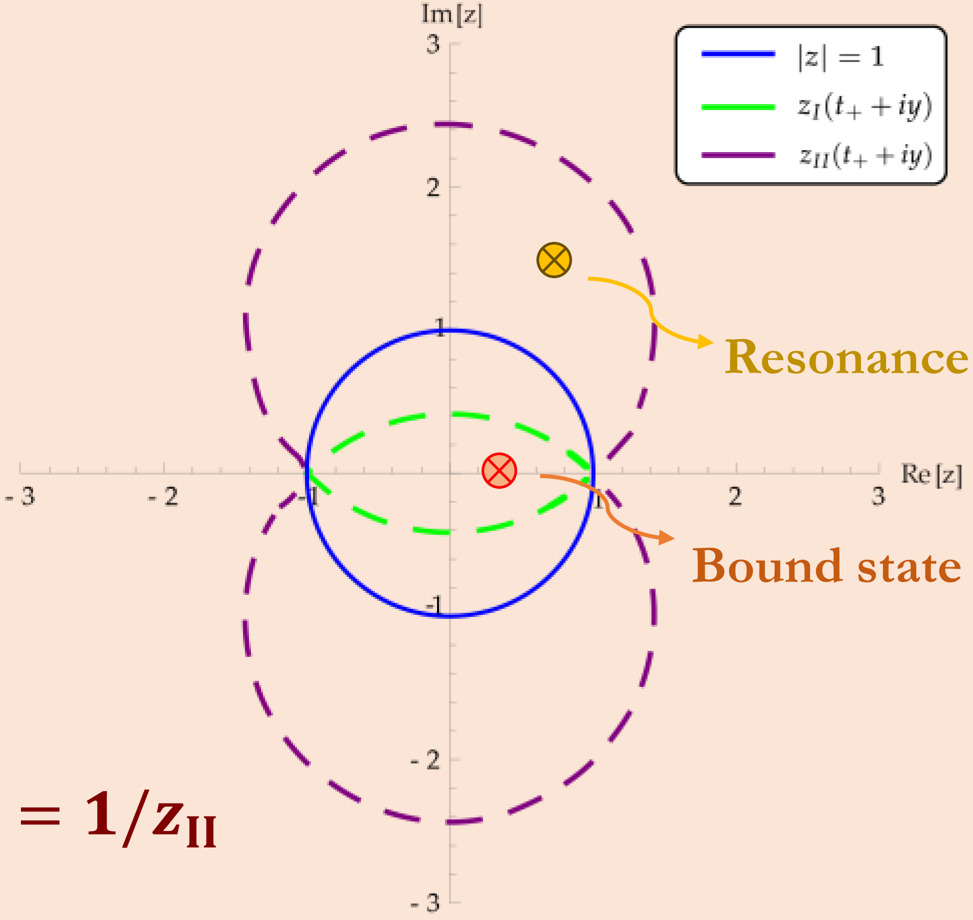
A Conformal Map

$$z\left(q^2; t_+, t_0\right) = \frac{\sqrt{t_+ - q^2} - \sqrt{t_+ - t_0}}{\sqrt{t_+ - q^2} + \sqrt{t_+ - t_0}}$$

- **Branch cut** is mapped on the unit circle and two sheets are separated:



Conformal map



$$z_{\mathrm{I}} = 1/z_{\mathrm{II}}$$

An Expansion for f_+

- Dispersive bound in terms of z :

$$\frac{1}{2\pi i}\oint_{|z|=1}\frac{dz}{z}\,|\phi(z)f_+(z)|^2 \leq 1$$

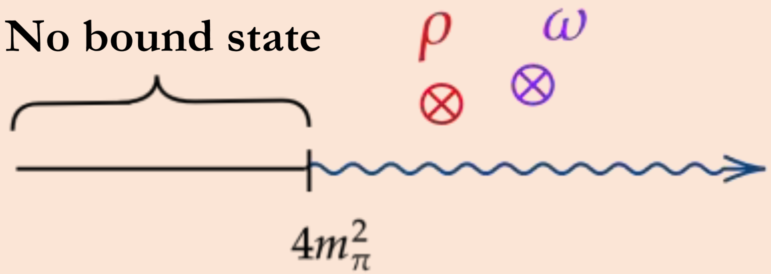
- Outer function $\phi(z)$ is a known function which is analytic and non-vanishing inside the unit disk by construction.

An Expansion for f_+

- **Dispersive bound in terms of z :**

$$\frac{1}{2\pi i}\oint_{|z|=1}\frac{dz}{z}\,|\phi(z)f_+(z)|^2 \leq 1$$

- **Outer function $\phi(z)$ is a known function which is analytic and non-vanishing inside the unit disk by construction.**



$F_\pi^V(z)\phi(z)$ is analytic inside the unit disk

$$F_\pi^V(z) = \frac{1}{\phi(z)}\sum_{n=0}^{\infty} a_n z^n$$

[Grinstein, Boyd, Lebed, 90's]

Constraints on f_+ [Buck, Lebed, hep-ph/9802369, '98]

- Angular momentum conservation require that :

$$\mathrm{Im}\, f_+(t) \sim (t - t_+)^{\frac{3}{2}} \quad \text{as} \quad t \to t_+ \,,$$

equivalently in terms of z it will be :

$$\frac{df_+}{dz}\Big|_{z=-1} = 0$$ ➡ P-wave constraint

- Perturbative calculations at high energy require :

$$f_+(q^2) \sim \frac{1}{q^2} \quad \text{as} \quad q^2 \to \infty.$$

- Charge conservation implies that :

$$F_\pi^V(q^2 = 0) = 1$$ ➡ Normalization constraint

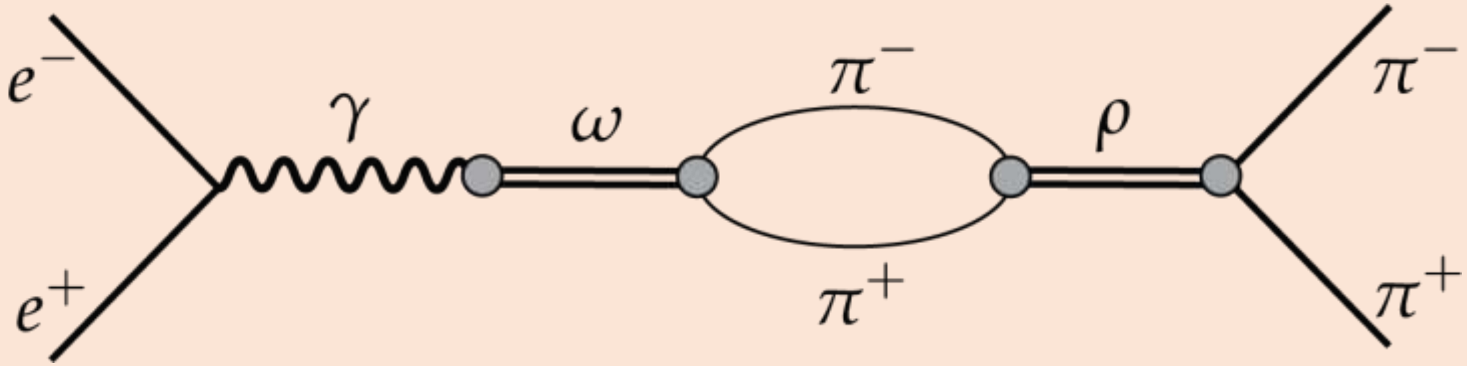
Our Ansatz for $F_{\pi}^{I=0}$

- The simplest ansatz is an additive form, i.e., $F_{\pi}^{V}(z) = F_{\pi}^{I=1}(z) + F_{\pi}^{I=0}(z)$

Our Ansatz for $F_\pi^{I=0}$

- The simplest ansatz is an additive form, i.e., $F_\pi^V(z) = F_\pi^{I=1}(z) + F_\pi^{I=0}(z)$

- NO! Due to final state interactions, e.g. :



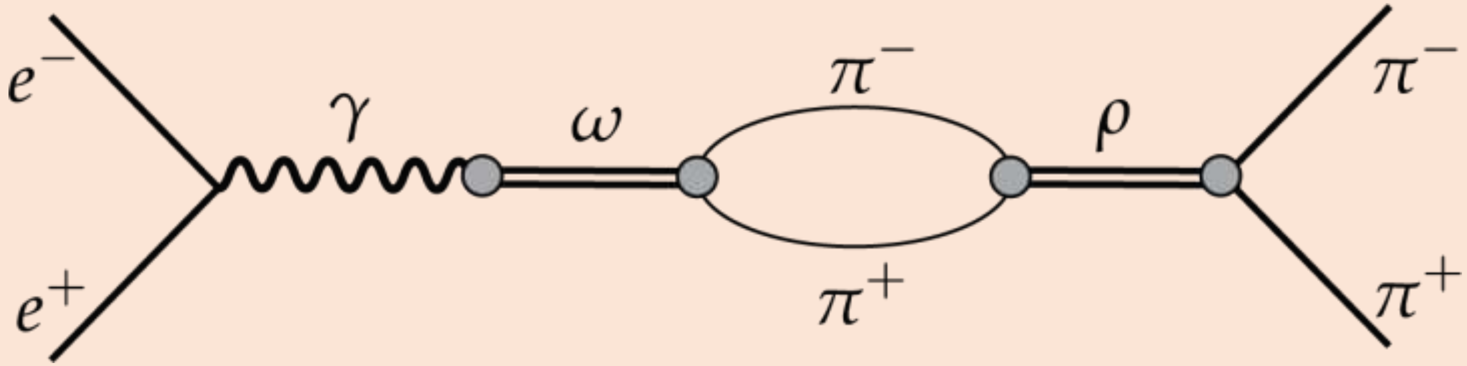
- Multiplicative ansatz is favorable, we take :

$$F_\pi^V(z) = F_\pi^{I=1}(z)[1 + F_\pi^{I=0}(z)]$$

Our Ansatz for $F_\pi^{I=0}$

- The simplest ansatz is an additive form, i.e., $F_\pi^V(z) = F_\pi^{I=1}(z) + F_\pi^{I=0}(z)$

- NO! Due to final state interactions, e.g. :



- Multiplicative ansatz is favorable, we take :

$$F_\pi^V(z) = F_\pi^{I=1}(z)[1 + F_\pi^{I=0}(z)]$$

In isospin limit $F_\pi^V(z) \cong F_\pi^{I=1}(z)$

Parametrization of $F_\pi^{I=1}$

$$F_\pi^{I=1}(z) = \frac{W(z)}{\phi(z)} \frac{\sum_{n=0}^{N} b_n z^n}{(z - z_\rho)(z - \bar{z}_\rho)}$$

Analytic inside the unit disk

[Kirk, Kubis, Reboud, van Dyk, 2410.13764, '25]

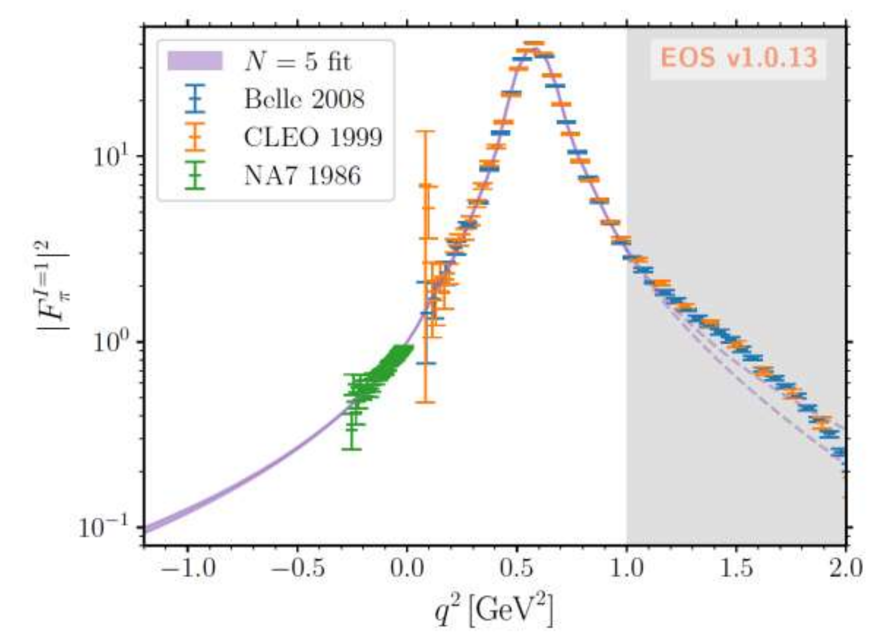
- $W(z) = (1+z)^2(1-z)^{5/2}$ ensures the large q^2 behavior and cancels the kinematical singularities of the outer function $\phi(z)$ “on” the unit circle.

- $z_\rho = \frac{1}{z((M_\rho - i\Gamma_\rho)^2)}$ is the position of ρ resonance on the second sheet, M_ρ and Γ_ρ are its mass and width.

Parametrization of $F_\pi^{I=1}$

$$F_\pi^{I=1}(z) = \frac{W(z)}{\phi(z)} \frac{\sum_{n=0}^N b_n z^n}{\underbrace{(z - z_\rho)(z - \bar{z}_\rho)}}$$

Factorizing the ρ pole avoid the Runge problem that [Buck, Lebed, hep-ph/9802369, '98] faced



[Kirk, Kubis, Reboud, van Dyk, 2410.13764, '25]

truncation N	χ^2	d.o.f.	p value [%]	$\langle r_\pi^2 \rangle$ [fm^2]	M_ρ [MeV]	Γ_ρ [MeV]	bound saturation
1	≈ 3300	92	$< 10^{-10}$	—	—	—	0.46
2	≈ 1500	91	$< 10^{-10}$	—	—	—	0.44
3	117.4	90	2.8	0.474 ± 0.0022	760.4 ± 0.4	143.0 ± 0.6	0.46
4	98.17	89	23.8	0.457 ± 0.0045	760.2 ± 0.4	145.9 ± 0.9	0.46
5	97.9	88	22.1	0.460 ± 0.0061	760.0 ± 0.6	146.1 ± 0.9	0.46

Dispersive bound check

[Kirk, Kubis, Reboud, van Dyk, 2410.13764, '25]

Adding $F_\pi^{I=0}$ to the picture

$$F_\pi^{I=0}(z) = \frac{\sum_{m=0}^{N'} c_m z^m}{(z - z_\omega)(z - \bar{z}_\omega)}$$

Fatemeh N

Adding $F_\pi^{I=0}$ to the picture

$$F_\pi^V(z) = \frac{W(z)}{\phi(z)} \frac{\sum_{n=0}^{N} b_n z^n}{(z - z_\rho)(z - \bar{z}_\rho)} \left[1 + \frac{\sum_{m=0}^{N'} c_m z^m}{(z - z_\omega)(z - \bar{z}_\omega)} \right]$$

Adding $F_\pi^{I=0}$ to the picture

$$F_\pi^V(z) = \frac{W(z)}{\phi(z)} \frac{\sum_{n=0}^{N} b_n z^n}{(z - z_\rho)(z - \bar{z}_\rho)} \left[1 + \frac{\sum_{m=0}^{N'} c_m z^m}{(z - z_\omega)(z - \bar{z}_\omega)}\right]$$

- $N = 4$ was the best result for $F_\pi^{I=1}$.
- $N' = 2$; Small ω effect does not need many terms to be described.

Adding $F_\pi^{I=0}$ to the picture

$$F_\pi^V(z) = \frac{W(z)}{\phi(z)} \frac{\sum_{n=0}^{N} b_n z^n}{(z - z_\rho)(z - \bar{z}_\rho)} \left[1 + \frac{\sum_{m=0}^{N'} c_m z^m}{(z - z_\omega)(z - \bar{z}_\omega)} \right]$$

- $N = 4$ was the best result for $F_\pi^{I=1}$.
- $N' = 2$; Small ω effect does not need many terms to be described.
- P-wave and normalization constraints need to be implemented.

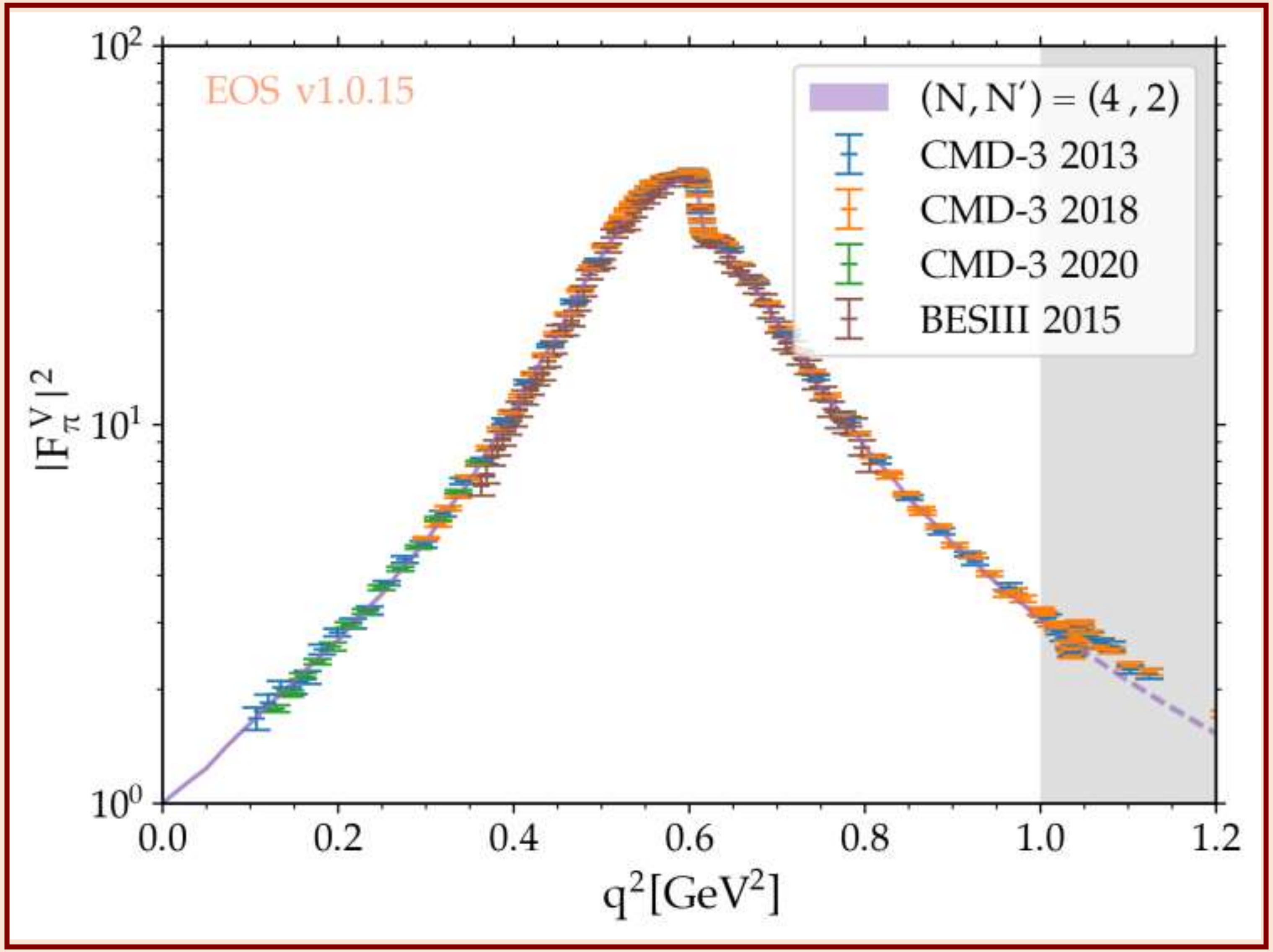
Fit Results

(N, N')	χ^2	d.o.f.	P-value [%]
$(4, 2)$	413.8	318	0.0234

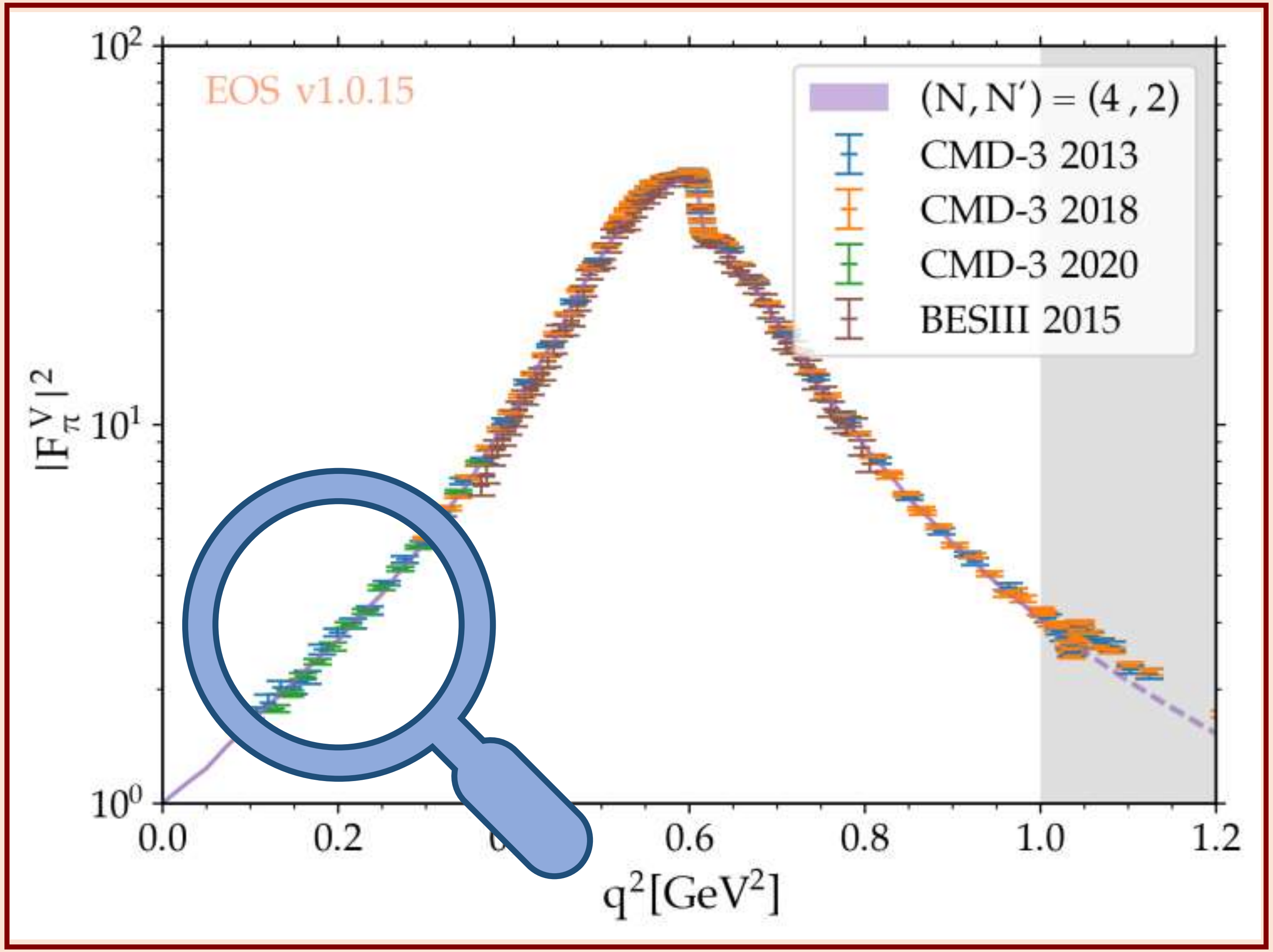
Fit Results

Dataset	χ^2	d.o.f	Local p-value [%]
BESIII 2015	61.05	60	43.81
Belle 2008	23.60	19	21.19
CLEO 1999	43.33	29	4.42
CMD-3 2013	76.67	66	17.36
CMD-3 2018	93.74	93	45.89
CMD-3 2020	69.36	13	0.00
NA7 1986	45.80	45	43.87
Total	413.79	318	0.0234

Fit Results

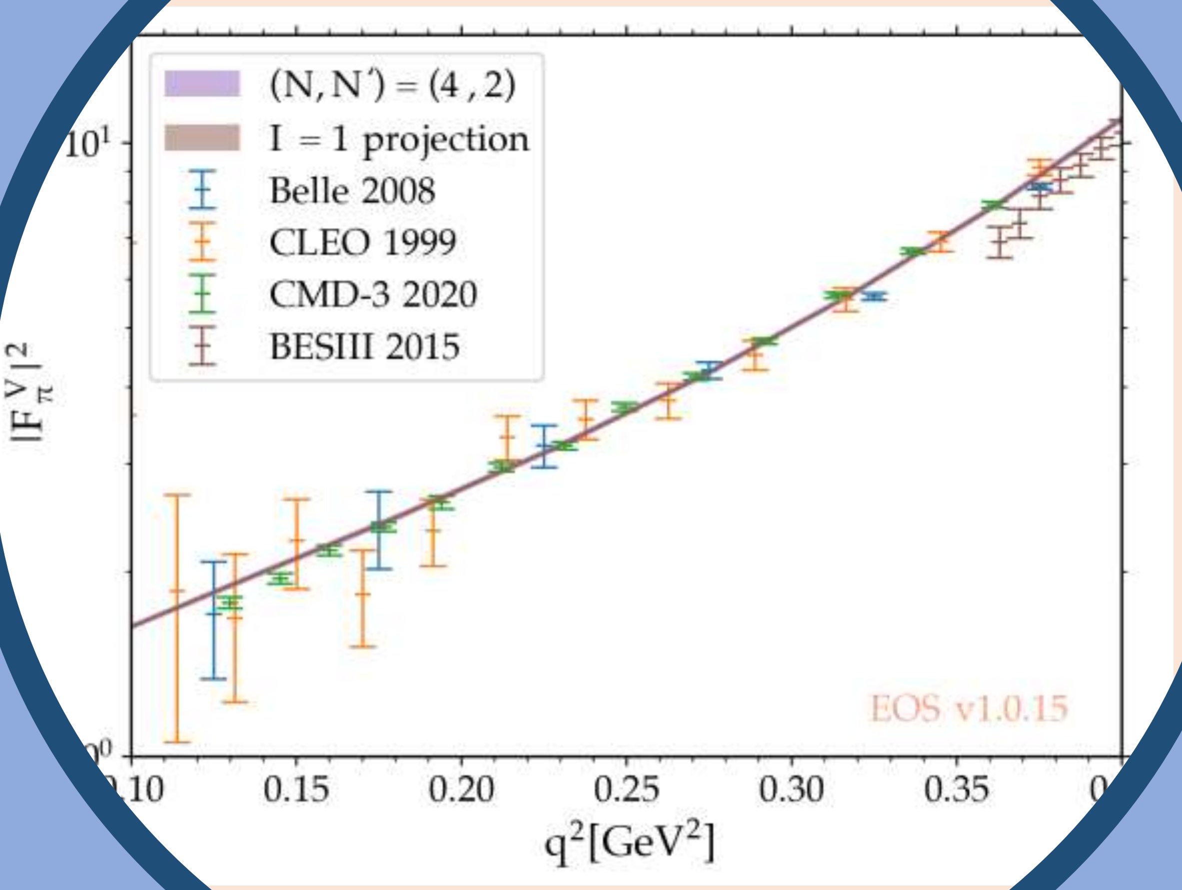

EOS v1.0.15
$(N, N') = (4, 2)$
CMD-3 2013
CMD-3 2018
CMD-3 2020
BESIII 2015
$|F_\pi^V|^2$
10^0
10^1
10^2
0.0
0.2
0.4
0.6
0.8
1.0
1.2
$q^2[\mathrm{GeV}^2]$

Fit Results



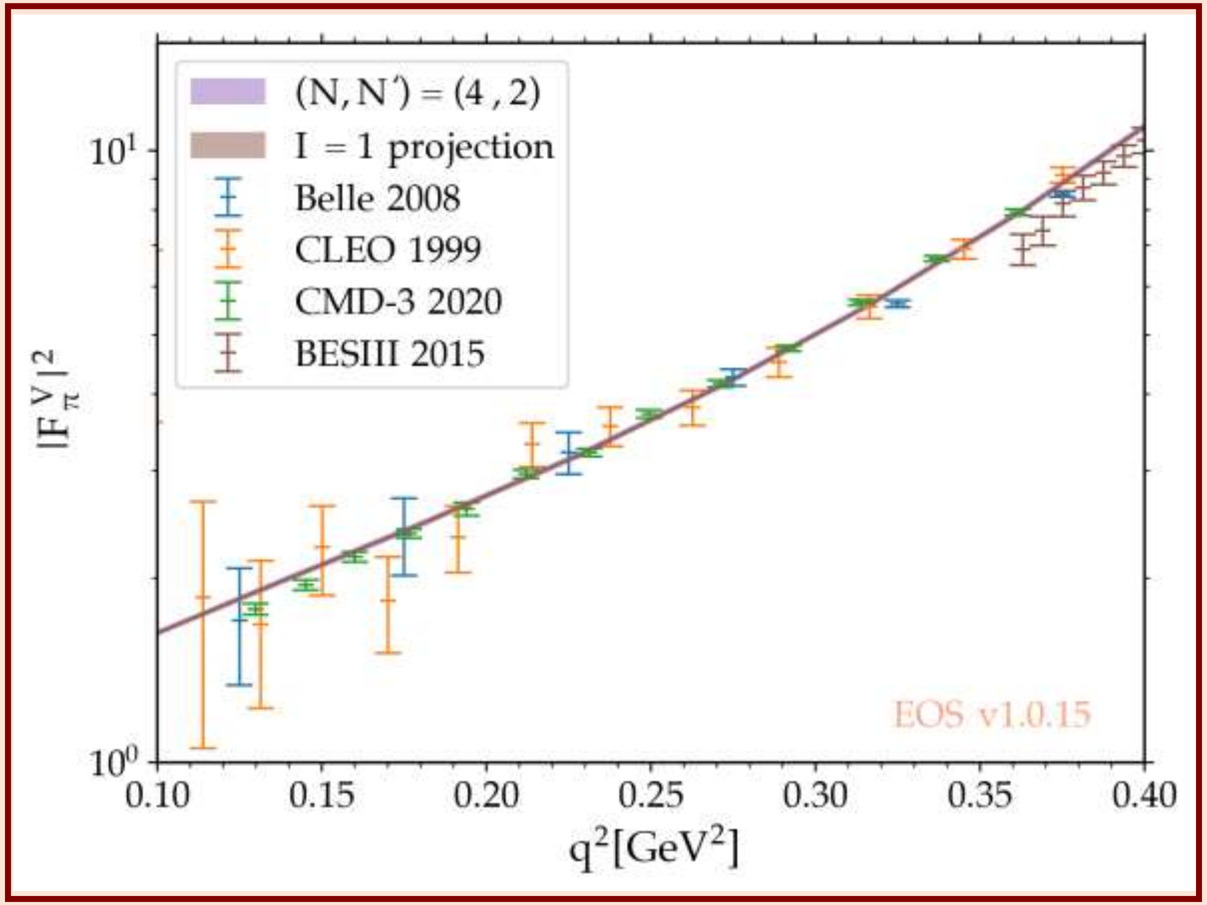
10^2
10^1
10^0
$|F_\pi^V|^2$
EOS v1.0.15
$(N, N') = (4, 2)$
CMD-3 2013
CMD-3 2018
CMD-3 2020
BESIII 2015
0.0
0.2
0.6
0.8
1.0
1.2
$q^2[\mathrm{GeV}^2]$

Fit Results



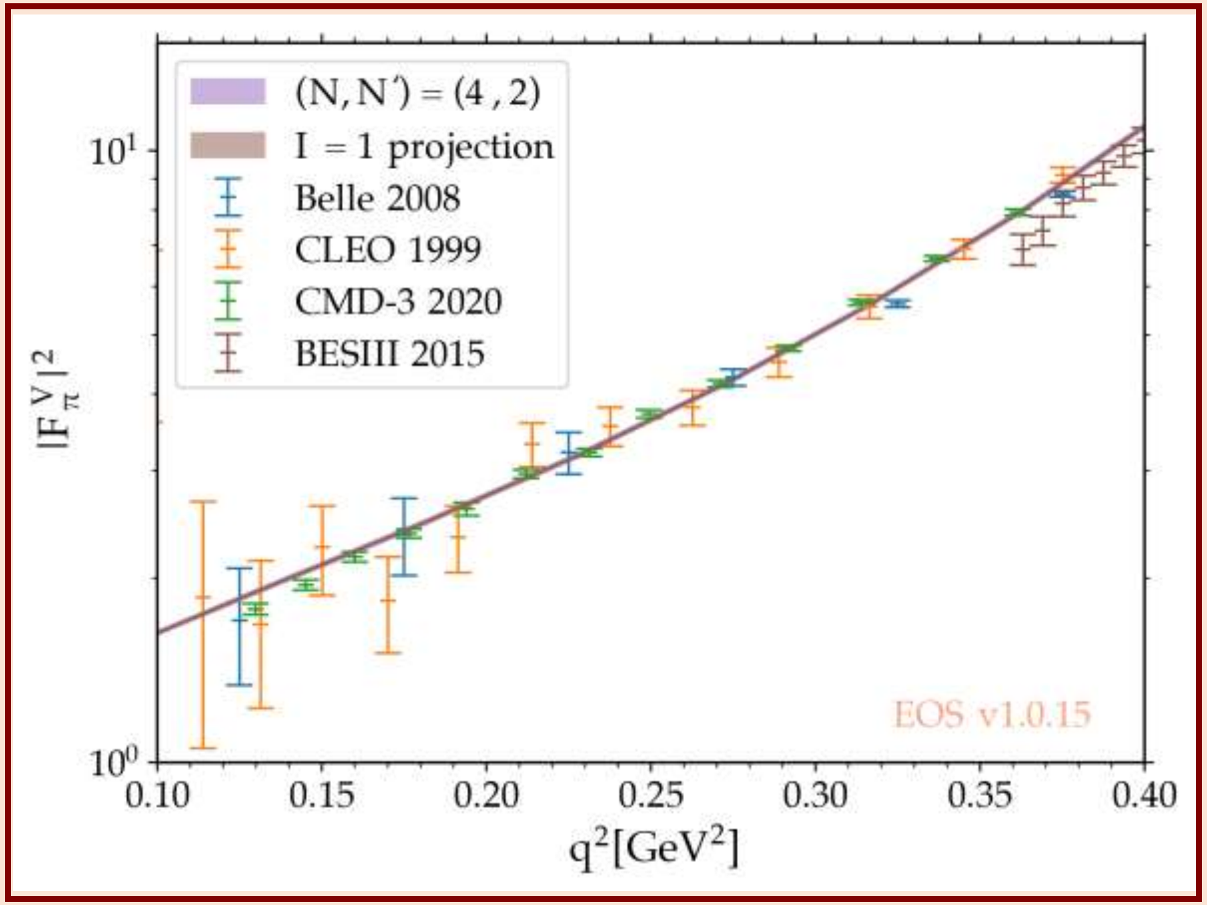
(N, N′) = (4 , 2)
I = 1 projection
Belle 2008
CLEO 1999
CMD-3 2020
BESIII 2015
EOS v1.0.15
$|F_\pi^V|^2$
q^2[GeV^2]
0.15
0.20
0.25
0.30
0.35

Fit Results



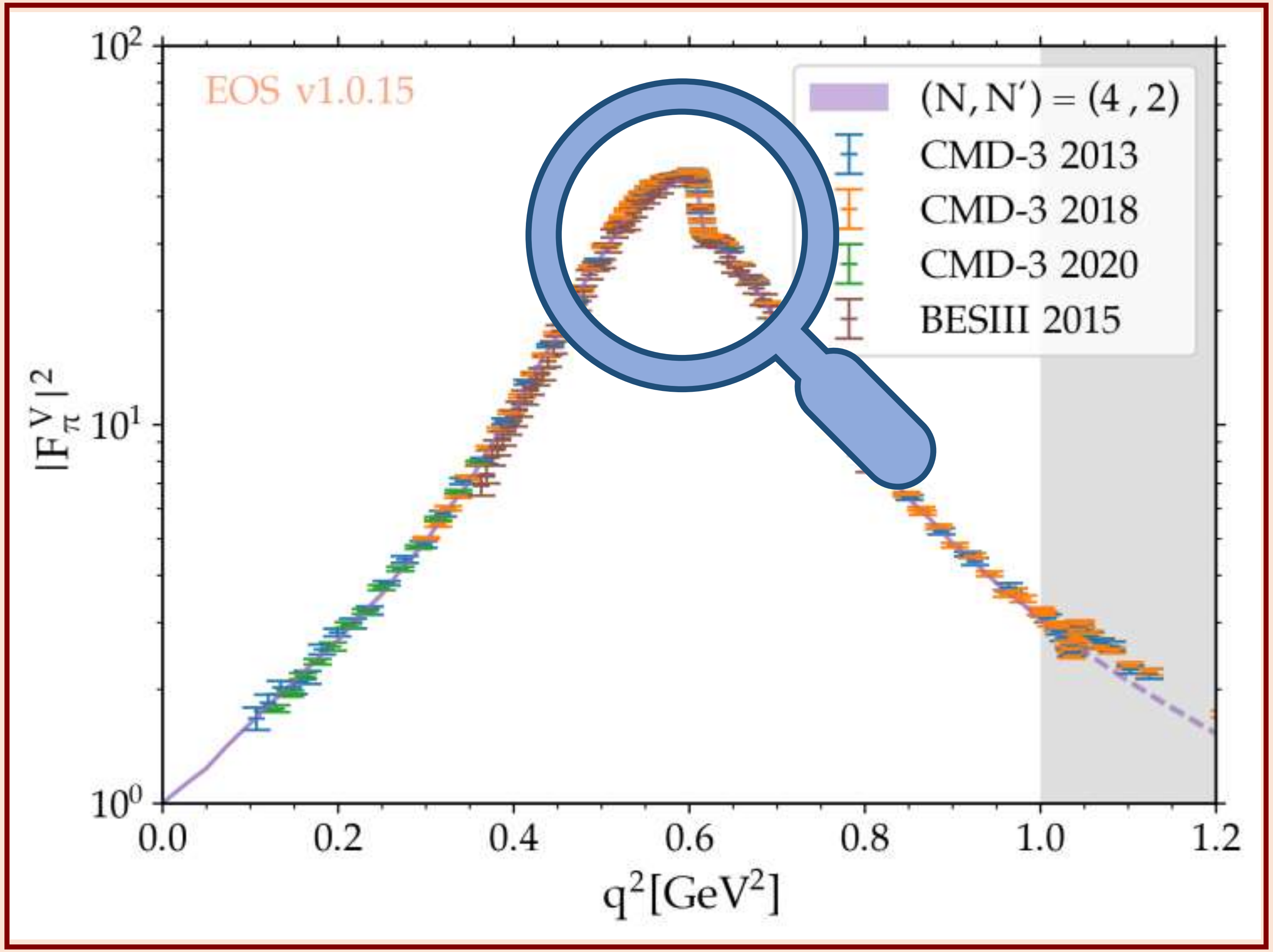
- **The best-fit curve is closer to the CMD-3 2020 date than what we expected !**

Fit Results

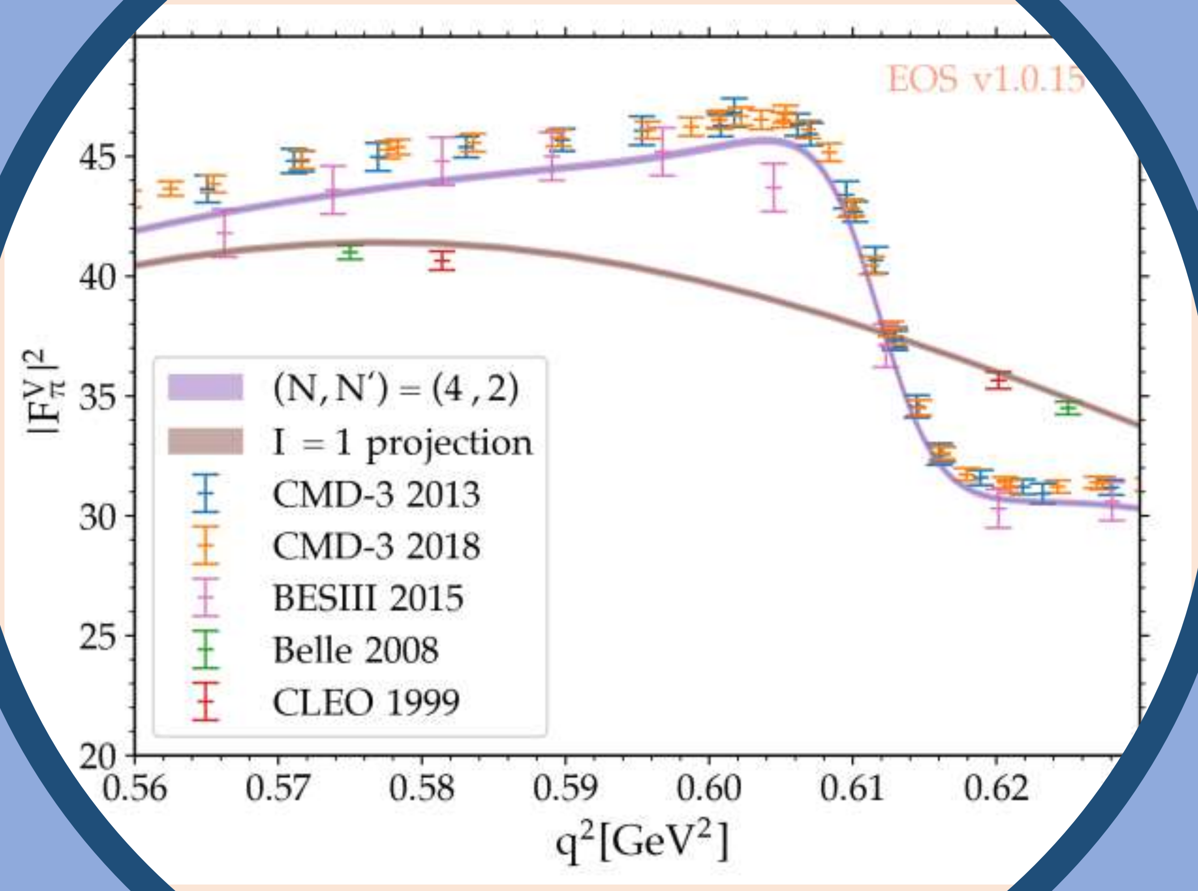


- The best-fit curve is closer to the CMD-3 2020 date than what we expected !
- Compare F_π^V with $F_\pi^{I=1}$: The two curves coincide perfectly in this region, confirming that, as expected, the ω contribution is negligible at low momentum transfer.

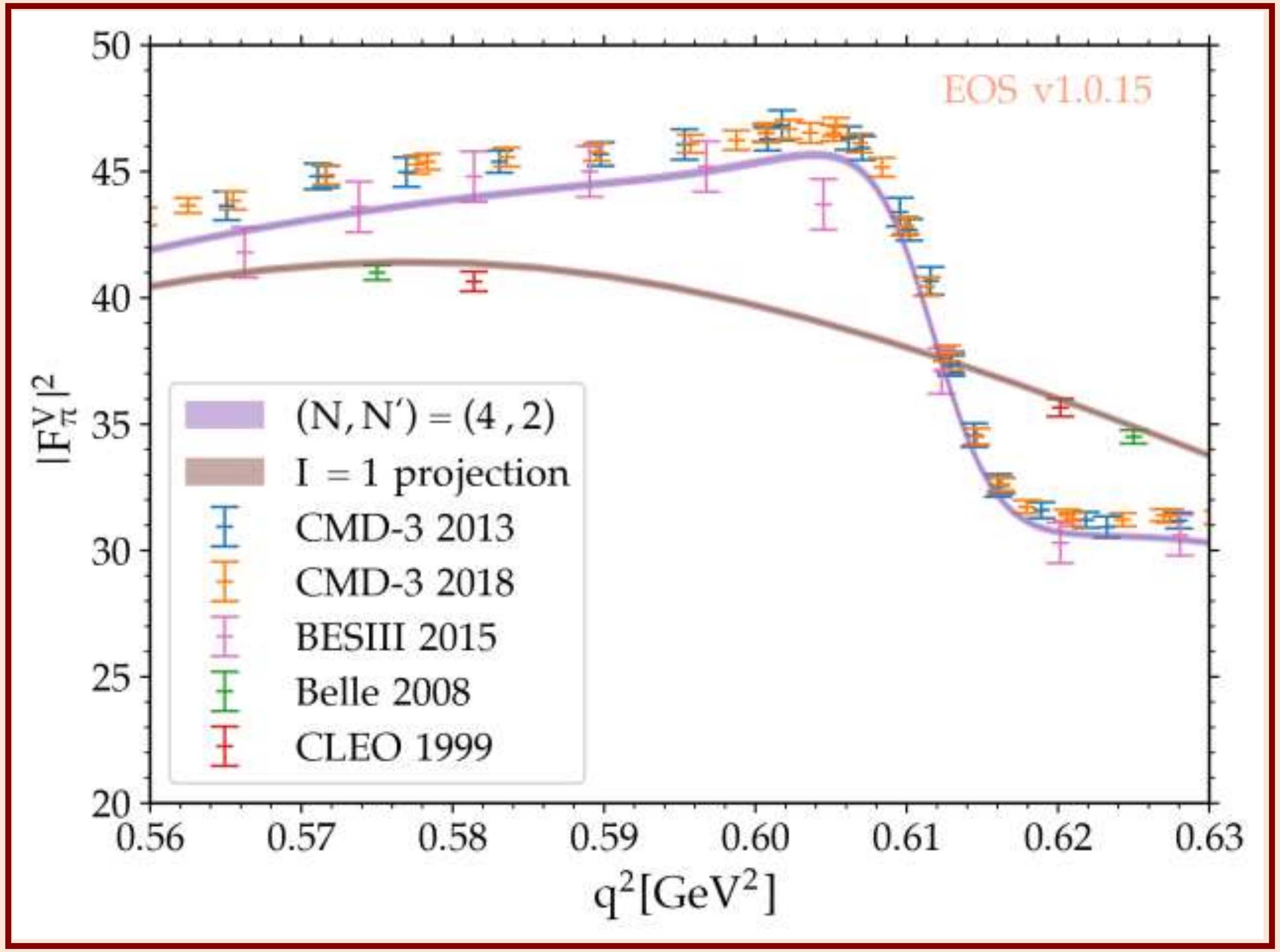
Fit Results


EOS v1.0.15
(N, N′) = (4 , 2)
CMD-3 2013
CMD-3 2018
CMD-3 2020
BESIII 2015
10^2
10^1
10^0
$|F_\pi^V|^2$
0.0
0.2
0.4
0.6
0.8
1.0
1.2
$q^2[\mathrm{GeV}^2]$

Fit Results


EOS v1.0.15
(N, N′) = (4 , 2)
I = 1 projection
CMD-3 2013
CMD-3 2018
BESIII 2015
Belle 2008
CLEO 1999
$|F_\pi^V|^2$
$q^2[\text{GeV}^2]$
20
25
30
35
40
45
0.56
0.57
0.58
0.59
0.60
0.61
0.62

Fit Results



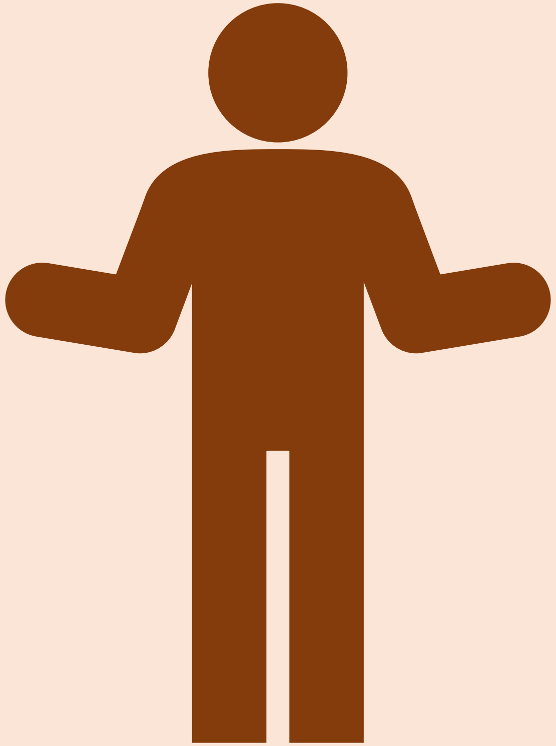
- Best-fit curve deviates from the experimental data in this region.
- Tension between different datasets.
- The tension originates from the parametrization, from inconsistencies in the experimental inputs, or both ?

Outlook

- The CMD-3 2020 dataset was extremely precise so maybe it is sensitive to the QED corrections or other isospin-breaking effects.

Outlook

- **Why does this matter at all?**



Outlook

- Why does this matter at all?

- Pion form factor is the playground for understanding more complicated form factors.

- It is the dominant contribution to the vacuum polarization and can help reducing theory uncertainty on muon g-2.

- It can help constraint rare decays like B decays.

Thank you for your attention