Dynamics of Spinning, Eccentric Binary Black Holes at Second Post-Newtonian Order

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GdR Ondes Gravitationnelles
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Outline

1. **Motivations & Post-Newtonian Framework**
2. **The Spin Sector: A Hybrid Analytical Solution**
   - Orbit-Averaging and Exact Solutions
   - The Hybrid Approxach
3. **The Orbital Sector: 2PN Quasi-Keplerian Parametrization**
   - Radial & Azimuthal Dynamics
   - Solving the QKP Ansatz

Based on *[TC, Tanay & Bernard, 2603.20031]*

Motivation: Modeling Eccentric, Precessing Binaries

Ground-Based (LVK)

$10\,\text{Hz} < f < 5\,\text{kHz}$
Target: Late inspiral & merger of BBH and BNS

Eccentricity $e \to 0$

Space-Based (LISA)

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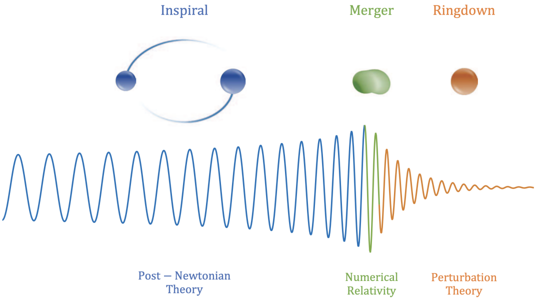
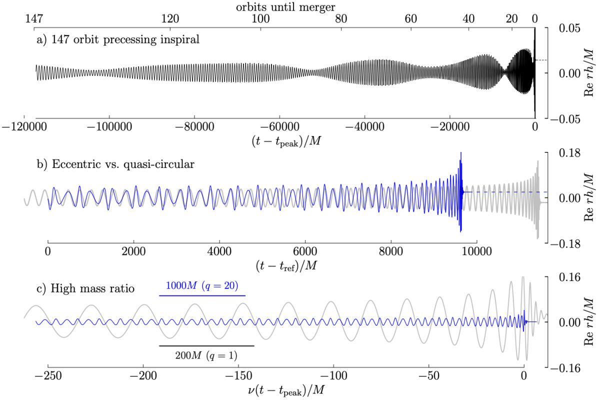
- **Goal:** Develop fast and accurate GW templates for advanced detectors (LISA, ET)



[Antelis & Moreno, 2016]

Effect of Spins and Eccentricity on Waveforms

- The inspiral phase is strongly affected by orbital eccentricity and precessing spins



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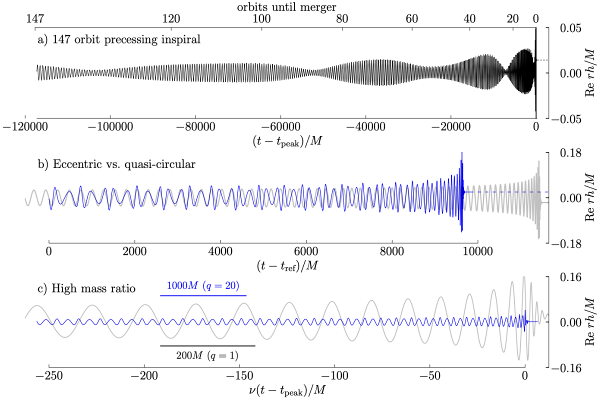

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- To compute accurate waveforms, we require a precise description of BBH dynamics.
- Aim: Compute an analytical solution as function of time of the dynamics of **spinning, eccentric inspiralling BBHs at 2PN within GR**.

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- **Spin Scaling:** Black hole spins carry an inherent $1/c$ factor: $\mathbf{S}_a = \chi_a \frac{Gm_a^2}{c}$ with $\|\chi_a\| \leq 1$.
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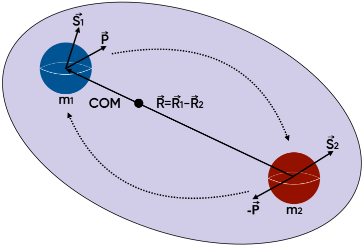
- The dynamics for $H_{\mathrm{p.p.}} = H_{\mathrm{N}} + H_{\mathrm{1PN}} + H_{\mathrm{2PN}}$ has been solved by *[Damour, Wex & Schäfer, 1988, 1993]*
- The dynamics for $H_{\mathrm{1.5PN}} = H_{\mathrm{N}} + H_{\mathrm{1PN}} + H_{\mathrm{SO}}$ has been solved by *[Cho & Lee, 2019]*

Notation & Reduced Variables

We work in the center-of-mass frame:

$$\mathbf{R} = \mathbf{R}_1 - \mathbf{R}_2 \,, \quad \& \quad \mathbf{P} = \mathbf{P}_1 = -\mathbf{P}_2$$



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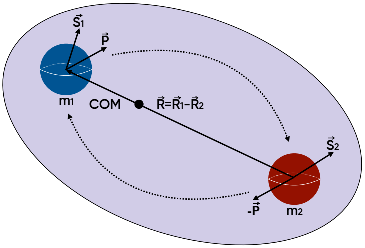


Variables are scaled by the total mass $m = m_1 + m_2$ and the reduced mass $\mu = m_1 m_2/m$.

- Time: $t = t^{\mathrm{phys}}/(Gm)$
- Hamiltonian: $h = H/\mu$

Orbital Sector

- Separation: $\mathbf{r} = \mathbf{R}/(Gm)$ and $\mathbf{n} = \mathbf{r}/r$
- Momentum: $\mathbf{p} = \mathbf{P}/\mu$

Angular Momenta Sector

- Orbital: $\mathbf{l} = \mathbf{r} \times \mathbf{p}$
- Individual spins: $\mathbf{s}_a = \mathbf{S}_a/(\mu G m)$
- Total angular momenum : $\mathbf{j} = \mathbf{l} + \mathbf{s}_1 + \mathbf{s}_2$

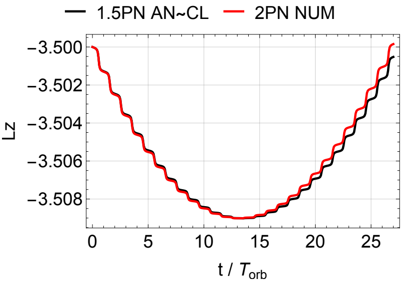
Spins Combinations

- $\mathbf{s}_{\mathrm{eff}} = \delta_1 \mathbf{s}_1 + \delta_2 \mathbf{s}_2$ [1.5PN SO]
- $\mathbf{s}_0 = \nu_1 \mathbf{s}_1 + \nu_2 \mathbf{s}_2$ [2PN SS]

Mass weights: δ_a *and* $\nu_a = m_b/m$.

The Full 2PN Spin Evolution

$$\frac{d\mathbf{s}_a}{dt} = \frac{1}{c^2 r^3}\left(\delta_a \mathbf{l} + \nu_a \left[3(\mathbf{n}\cdot\mathbf{s}_0)\mathbf{n} - \mathbf{s}_0\right]\right) \times \mathbf{s}_a \,,$$
$$\frac{d\mathbf{l}}{dt} = -\frac{1}{c^2 r^3}\left(\mathbf{l}\times\mathbf{s}_{\rm eff} + 3\left[(\mathbf{n}\cdot\mathbf{s}_0)(\mathbf{n}\times\mathbf{s}_0)\right]\right) .$$



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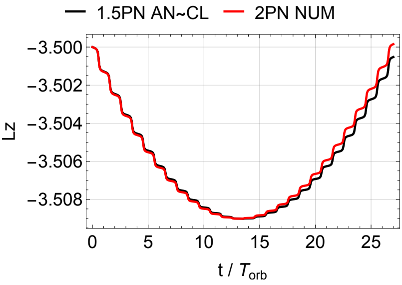
Timescale Separation

$$T_{\text{prec}} \sim \frac{c^2 r^3}{l} \sim \mathcal{O}(c^2)T_{\text{orb}}$$



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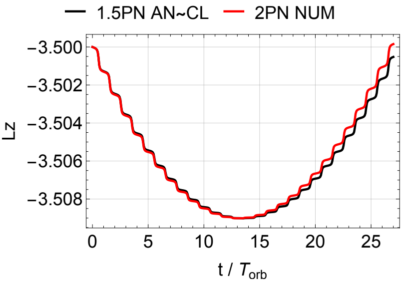

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Orbit-Averaged (OA) EOMs

[Racine, 2001]

$$\left\langle \frac{d\mathbf{s}_a}{dt}\right\rangle_{T_{\text{orb}}} = \frac{1}{c^2 d^3}\left(\delta_a \mathbf{l} + \frac{\nu_a}{2}\left[\mathbf{s}_0 - 3\lambda\mathbf{l}\right]\right)\times\mathbf{s}_a,$$

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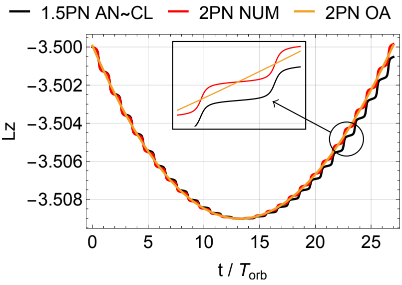
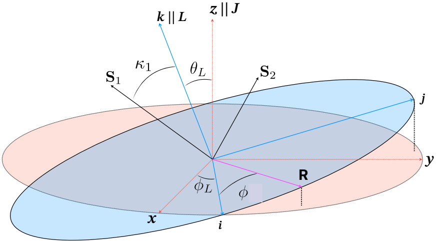
where $d^{-3} = \langle 1/r^3\rangle$ and $\lambda \equiv \mathbf{l}\cdot\mathbf{s}_0/l^2 = \text{const.}$



Orbit-averaging the 2PN phase space:

$$\{\mathbf{R}(t), \mathbf{P}(t), \mathbf{S}_1(t), \mathbf{S}_2(t)\} \longrightarrow \{\mathbf{L}(t), \mathbf{S}_1(t), \mathbf{S}_2(t)\}$$

9 dynamical variables



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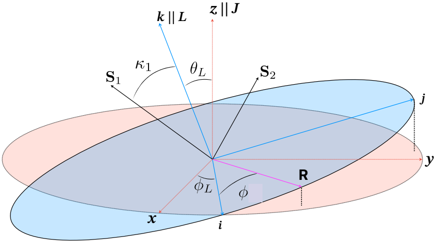

The OA system possesses **7 independent constants** $(s_1, s_2, l, j^2, j_z$ + 2 more relating angles between the momenta).

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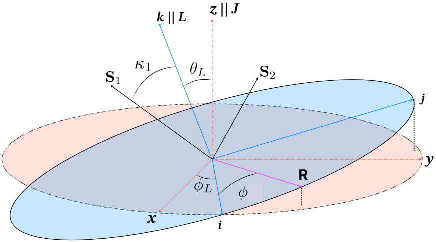

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The Two Slow Frequencies

These 2 degrees of freedom dictate the fundamental spin frequencies:

- **Nutation** (ω_{nut}): Internal shape evolution of the triad, governed by $\cos\kappa_1$.
- **Precession** (ω_{prec}): Azimuthal rotation of the entire triad around **J**, governed by ϕ_L.

Exact Analytical Solution: Nutation Boundaries

OA Equation of Motion

$$\left\langle \frac{d\cos\kappa_1}{dt} \right\rangle_{T_{\rm orb}} = \frac{\delta_2 - \nu/2}{c^2\, d^3\, l\, s_1}\,(1-\lambda)\,\mathbf{l}\cdot(\mathbf{s}_1 \times \mathbf{s}_2).$$

This can be rewritten in a separable form:

$$\frac{d\cos\kappa_1}{\sqrt{A(\cos\kappa_1 - x_-)(\cos\kappa_1 - x_+)(\cos\kappa_1 - x_3)}} = \frac{\pm dt}{c^2 d^3}\,,$$

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The Cubic Roots (x_-, x_+, x_3):

- $x_3 \sim \mathcal{O}(c) \longrightarrow$ **Unphysical root** (lies outside $[-1, 1]$).
- $x_-, x_+ \in [-1, 1] \longrightarrow$ **Physical boundaries**. The triad nutates within $[x_-, x_+]$ with an amplitude scaling as:

$$x_+ - x_- \simeq \frac{2\, m_1}{m_1 - m_2} \frac{s_2}{l} |\sin\kappa_1 \sin\kappa_2| \sim \mathcal{O}(c^{-1})$$

Analytical OA Solution

- $\cos\kappa_1$ is bounded between x_- and x_+, the integral yields a Jacobi elliptic sine function:

$$\cos\kappa_1(t) = x_- + (x_+ - x_-)\,\mathrm{sn}^2(\Upsilon(t), \beta)$$

$$\Upsilon(t) = \frac{\sqrt{A(x_3 - x_-)}}{2}\left(\alpha(0) + \frac{t}{c^2 d^3}\right), \qquad \beta = \sqrt{\frac{x_+ - x_-}{x_3 - x_-}}$$

Note: $\mathrm{sn}(u,k) = \sin(\mathrm{am}(u,k)) = \sin\phi$, where $u = \int_0^{\phi} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$ is the incomplete elliptic integral of the first kind.

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The periodicity of the sn^2 function dictates the timescale of the triad's internal shape evolution, directly yielding the **nutation frequency**:

$$\omega_{\mathrm{nut}} = \frac{\pi\sqrt{A(x_3 - x_-)}}{2c^2 d^3 K(\beta)}$$

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Slow Variables Dynamics: OA Solution for $\cos\kappa_1$ and ϕ_L

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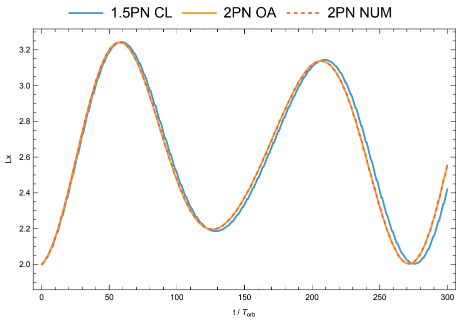
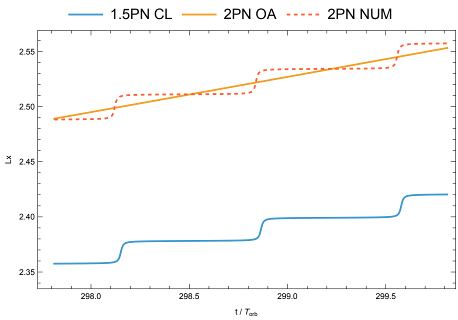
- **Global Precession:** Similar integration with elliptic functions yields an exact analytical solution for $\phi_L(t)$, from which we get the precession frequency ω_{prec}.

The Orbit-Averaged Limitation: Missing Fast Oscillations





- The OA model captures the **secular evolution** driven by the 2PN spin-spin couplings.
- However, the mathematical averaging artificially **filters out all fast orbital-timescale oscillations** present in the exact dynamics.

Slow Variables Dynamics: The Hybrid Model Framework

Motivation:

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Mnemonic technique:
Take the orbit-averaged precession equations and replace d^3 with r^3.

Slow Variables Dynamics: The Hybrid Solution

The 2PN Hybrid Equation of Motion : $x \equiv \cos \kappa_1^{(\mathrm{H})}$

$$\frac{dx}{\sqrt{A(x-x_-)(x-x_+)(x-x_3)}} = \frac{\pm dt}{c^2 r^3}\,.$$

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- Integration requires injecting the **1PN Quasi-Keplerian** $r(u)$.

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$$\Upsilon^{(\mathrm{H})}(u) = \frac{\sqrt{A(x_3 - x_-)}}{2}\left[\alpha(0) + \frac{1}{2c^2\, n a_r^3}\left(\frac{(2 + e_r^2 - 3e_r e_t)v_r}{(1-e_r^2)^{5/2}} + \frac{(e_r - e_t)\sin u}{(1-e_r^2)(1-e_r\cos u)^2} + \frac{(3e_r - e_t - 2e_r^2 e_t)\sin u}{(1-e_r^2)^2(1-e_r\cos u)}\right)\right]$$

The 2PN Hybrid Equation of Motion : $x \equiv \cos\kappa_1^{(\mathrm{H})}$

$$\frac{dx}{\sqrt{A(x-x_-)(x-x_+)(x-x_3)}} = \frac{\pm dt}{c^2 r^3}\,.$$

- Roots x_i match the OA case (both algebraically and numerically).
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- The true anomaly v_r drives the **secular growth**: $v_r = u + 2\arctan\left(\frac{\beta_r\sin u}{1-\beta_r\cos u}\right)$
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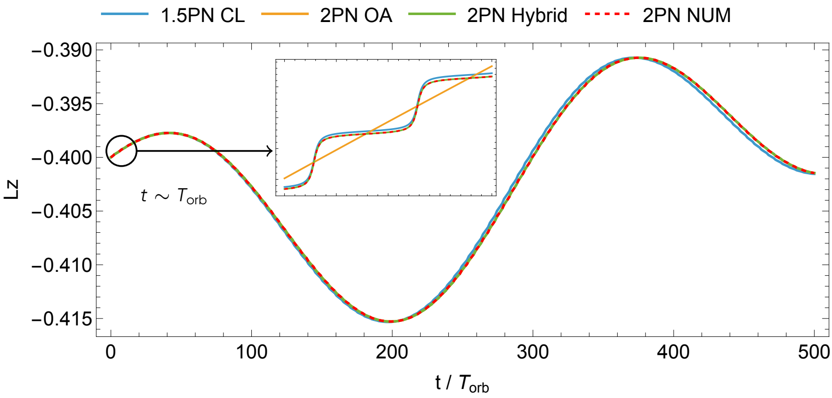
Global Precession: $\phi_L^{(\mathrm{H})}$ is evaluated by substituting $\Upsilon \to \Upsilon^{(\mathrm{H})}$ into its OA solution.

Initial Inputs: $q = 0.2,$ $e = 0.60$ $x_{\rm PN} = 1/(rc^2) \sim 0.5\%$

$\chi_1 = 0.19,$ $\chi_2 = 0.78$ $\kappa_1 = 21°,$ $\kappa_2 = 58°,$ $\gamma = 79°$



Slow Variables Dynamics: Hybrid Solution

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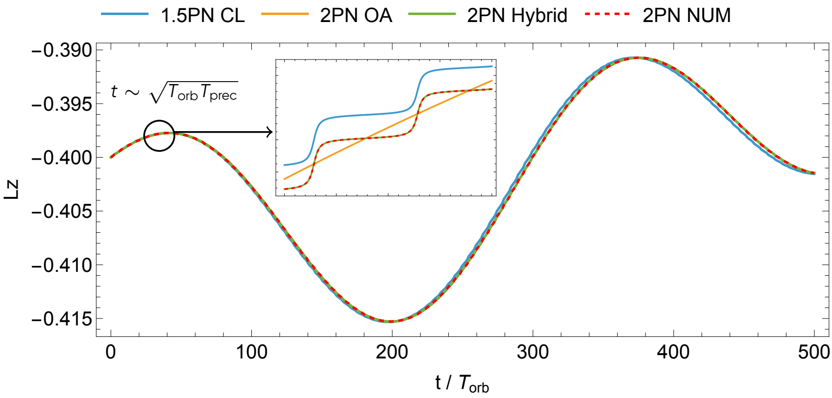

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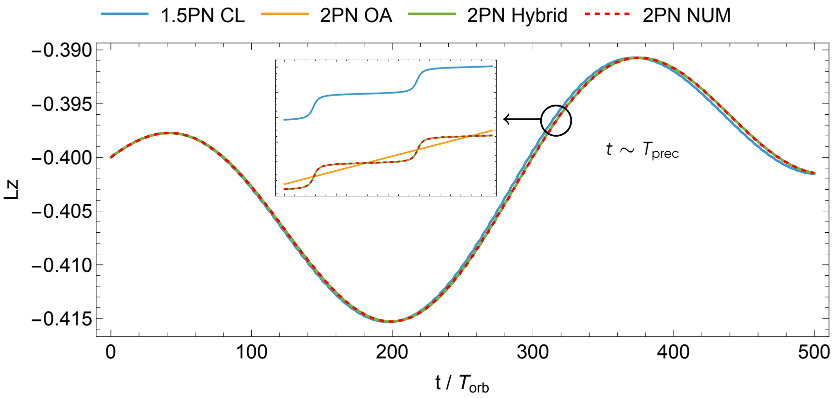
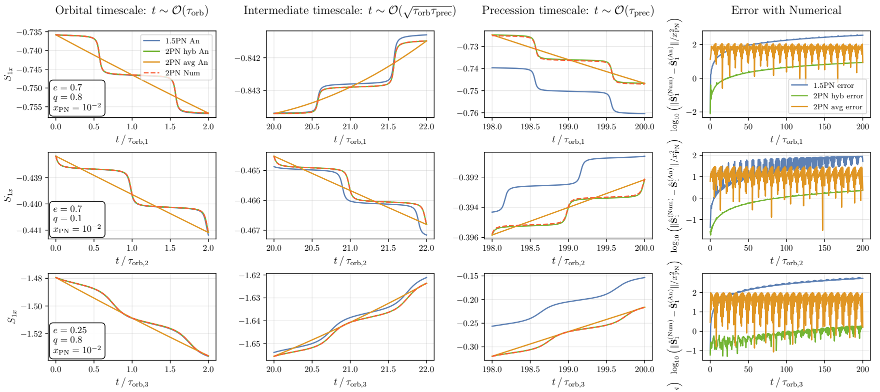


Slow Variables Dynamics: Hybrid Model Accuracy

Initial angular momenta configuration:

$$\chi_1 = 0.62, \quad \chi_2 = 0.42, \quad \kappa_1 = 68^\circ, \quad \kappa_2 = 10^\circ, \quad \gamma = 73^\circ$$

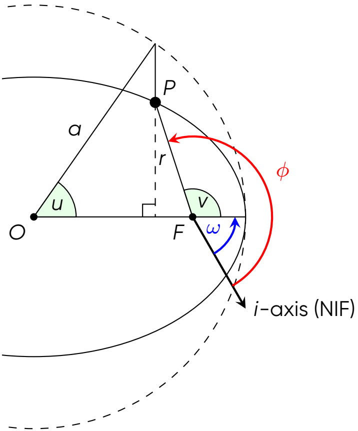


Model	1.5PN Oscillations	1.5PN Secular	2PN Oscillations	2PN Secular	Error with Numerical
Exact 2PN	✔	✔	✔	✔	0
1.5PN	✔	✔	✘	✘	$\mathcal{O}(t/c^4)$
2PN (OA)	✘	✔	✘	✔	$\mathcal{O}(1/c^3)$
2PN (H)	✔	✔	✘	✔	$\mathcal{O}(1/c^4)$

The separation vector **R** in the orbital plane is $\mathbf{R} = rGm(\cos\phi, \sin\phi, 0)$

Keplerian Dynamics:

$$r = a(1 - e\cos u)$$
$$n(t - t_0) = u - e\sin u$$
$$v = u + 2\arctan\left(\frac{\beta\sin u}{1 - \beta\cos u}\right)$$
$$\beta = \frac{e}{1 + \sqrt{1 - e^2}}$$



Parameters:

- r: Radial separation
- a: Semi-major axis
- e: Eccentricity
- n: Mean motion
- u: Eccentric anomaly
- v: True anomaly
- ω: Argument of periapsis
- ϕ: Azimuthal phase ($\omega + v$)

The spinning 2PN ADM Hamiltonian:

$$h = h_{\text{p.p.}} + \frac{1}{2c^2 r^3}\left[2\mathbf{l}\cdot\mathbf{s}_{\text{eff}} + 3(\mathbf{n}\cdot\mathbf{s}_0)^2 - s_0^2\right]$$

Orbital Sector: Spinning Relativistic Kepler Problem

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Radial Equation of Motion:

$$\left(\frac{dr}{dt}\right)^2 = A + \frac{B}{r} + \frac{C}{r^2} + \frac{D_1}{r^3} + \frac{D_2}{r^4} + \frac{D_3}{r^5}$$

- A, B, D_2, D_3 remain constant (matching the non-spinning case).
- **The Problem:** C and D_1 evolve over the orbital timescale:

$$C = -l^2(t) + \frac{1}{c^2}\left[5(\nu-2) + 2(1-3\nu)\mathcal{E}l^2\right] + \dots$$

$$D_1 = D_{1\text{orb}} - \frac{1}{c^2}\left[2(\mathbf{l}\cdot\mathbf{s}_{\text{eff}}) - s_0^2 + 3(\mathbf{n}(t)\cdot\mathbf{s}_0)^2\right]$$

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The time-dependence of l^2 and $(\mathbf{n}\cdot\mathbf{s}_0)^2$ prevents the standard integration to compute the closed-form QKP.

Hybrid azimuthal EOM:
ϕ is measured relative to the i-axis of the NIF.

$$\frac{d\phi}{dt} = \frac{F}{r^2} + \frac{l_{1\mathrm{orb}} + l_{1\mathrm{spin}}}{r^3} + \frac{l_2}{r^4} + \frac{l_3}{r^5} + \frac{1}{c^2 r^3} \sum_{i=1}^{2} \left(\frac{\beta_{iL}}{\alpha_{iL} + \cos \kappa_1^{(2,\mathrm{H})}} \right)$$

$$l_{1\mathrm{spin}} = -\frac{1}{2c^2} \Big[(2\delta_1 + 2\delta_2 - \nu)(1-\lambda)\bar{l} + \lambda\big(\nu\bar{l} - 3\nu_2 s_2 \Sigma_2\big) \Big]$$

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$$F = I(t) + \frac{1}{c^2}(3\nu - 1)\, h \bar{I} + \frac{1}{2c^4}(2 - 9\nu + 6\nu^2)\, h^2 \bar{I}$$

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$$F = l(t) + \frac{1}{c^2}(3\nu - 1)\, h\, \bar{l} + \frac{1}{2c^4}(2 - 9\nu + 6\nu^2)\, h^2\, \bar{l}$$

We can express $l(\phi)$ in terms of its orbital average $\bar{l}$ + oscillatory corrections :

$$l^2(\phi) = \bar{l}^{\,2} - \frac{s_{0p}^2}{2c^2 \bar{l}^{\,2}} \left[3e \cos(\phi - 2\phi_{sp}) + 3 \cos(2\phi - 2\phi_{sp}) + e \cos(3\phi - 2\phi_{sp}) \right]$$

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We now have both $(dr/dt)^2$ and $d\phi/dt$ analytically expressed in terms of r, ϕ, $\mathcal{E}$, $\bar{l}$ and spin quantities.

Radial and Azimuthal EOMs

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The Quasi-Keplerian Ansatz

$$n(t-t_0) = u - e_t \sin u + \frac{1}{c^4}\left[f_t \sin v_\phi + g_t(v_\phi - u) + h_t \sin(v_\phi - 2\phi_{sp})\right],$$

$$r = a_r(1 - e_r \cos u) + \frac{1}{c^4}\left[h_{r1}\cos(v_\phi - 2\phi_{sp}) + h_{r2}\cos(2v_\phi - 2\phi_{sp})\right],$$

$$\phi(t) - \phi(t_0) = (1+k')v_\phi + \frac{1}{c^4}\left[f_{\phi 2}\sin 2v_\phi + f_{\phi 3}\sin 3v_\phi\right]$$

$$+ \frac{1}{c^4}\left[h_{\phi 1}\sin(v_\phi - 2\phi_{sp}) + h_{\phi 2}\sin(2v_\phi - 2\phi_{sp}) + h_{\phi 3}\sin(3v_\phi - 2\phi_{sp})\right] + \Pi_{\mathrm{osc}}(u),$$

$$\Pi_{\mathrm{osc}}(u) = \sum_{i=1}^{2}\frac{\beta_{iL}}{\alpha_{iL} + x_-}\left[\frac{2\,\Pi\left(n_i, \mathrm{am}(\Upsilon(u),\beta),\beta\right)}{\sqrt{A(x_3 - x_-)}} - \frac{\Pi(n_i,\beta)}{nc^2 d^3 K(\beta)}v_\phi\right]$$

Ansatz close to *[Klein & Jetzer, 2010]*

Solving the QKP Ansatz

Methodology

- **Parametrize:** Add harmonic functions and 2PN spin-spin free parameters to all QK parameters, e.g.:

$$a_r = a_r^{(N)} + \frac{1}{c^2} a_r^{(1)} + \frac{1}{c^3} a_r^{(\mathrm{SO})} + \frac{1}{c^4} a_r^{(2)} + \frac{1}{c^4} a_r^{(\mathrm{SS})}$$

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- **Match Coefficients:** Solve all equations to find the 2PN spin-spin corrections.

$$a_r = -\frac{1}{2\mathcal{E}} \left[1 + \frac{1}{2c^2} \mathcal{E} \Big((7-\nu) - \frac{1}{\bar{l}^2} \left(4\mathbf{l} \cdot \mathbf{s}_{\text{eff}} - 2s_0^2 + 3s_{0p}^2 \right) \Big) + \dots \right]$$

$$h_r = -\frac{s_{0p}^2}{4\,\bar{l}^2}, \quad h_{\phi 1} = -\frac{1}{2}\sqrt{1 + 2\,h\,\bar{l}^2}\,\frac{s_{0p}^2}{\bar{l}^4}, \quad h_{\phi 2} = -\frac{s_{0p}^2}{8\,\bar{l}^4}$$

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Observation: 2PN spin-spin interactions imply a slow secular evolution of the QK parameters themselves through s_0^2 and s_{0p}^2.

Orbital Frequencies of the 2PN Dynamics

From the final QK solution, we determine the fundamental frequencies:

$$r = a_r(1 - e_r \cos u) + \frac{1}{c^4} + h_{r2} \cos(2v_\phi - 2\phi_{sp}),$$

$$n(t - t_0) = u - e_t \sin u + \frac{1}{c^4}\left[f_t \sin v_\phi + g_t(v_\phi - u)\right]$$

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- **Radial Mean Motion** n: Governs the radial period $T_{\text{orb}} = 2\pi/n$.

$$n = (-2\mathcal{E})^{3/2} \left[1 + \frac{(15 - \nu)\mathcal{E}}{4c^2} + \frac{1}{c^4} \left(\frac{3(2\nu - 5)(-2\mathcal{E})^{3/2}}{2\bar{l}} + \frac{(555 + 30\nu + 11\nu^2)\mathcal{E}^2}{32} \right)\right]$$

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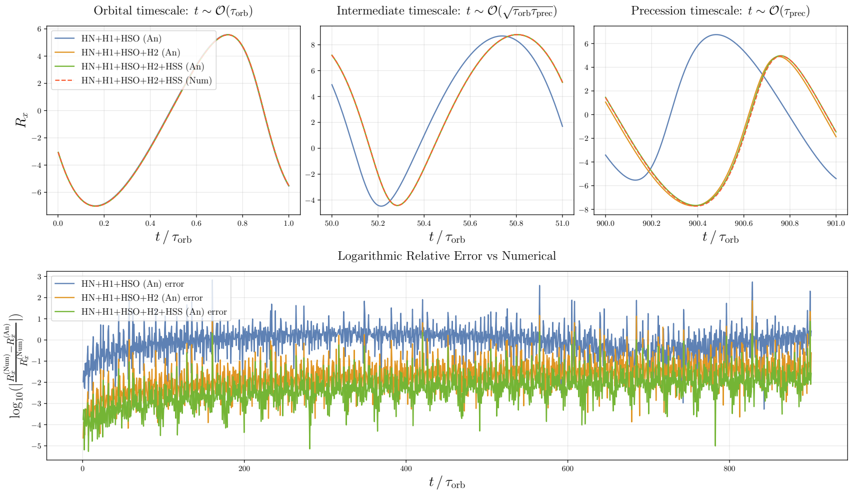
where $\delta_0 = \delta_1 + \delta_2 - \nu/2$.

Together with the spin-sector frequencies ω_{nut} and ω_{prec}, we complete the set of **4 non-zero fundamental frequencies**.

Initial Inputs: $q = 2/3$, $e = 0.40$ $x_{\text{PN}} = 1/(rc^2) \sim 1\%$

$\chi_1 = 0.31$, $\chi_2 = 0.62$ $\kappa_1 = 39°$, $\kappa_2 = 154°$, $\gamma = 127°$



The Spin-Aligned Limit

Geometric Reduction

- Vanishing Angles: $\theta_L, \kappa_1, \kappa_2, \gamma \to 0$ and $s_{0p} \to 0$.
- Constant Orbital Plane: **L** remains fixed; spins are orthogonal to the orbital plane (**R**, **P**).
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- The system recovers the **two degrees of freedom** of the non-spinning case (r and ϕ).
- **Nutation Amplitude Vanishes:** Even if the nutation frequency (ω_{nut}) is mathematically non-zero, its physical amplitude goes to 0.
- **Inertial Azimuthal Frequency:** The precession frequency simply adds to the non-inertial azimuthal frequency to yield the standard inertial azimuthal frequency ($\omega_\phi^{\text{inertial}} = \omega_\phi + \omega_{\text{prec}}$).
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Integrating the precessing equations of motion first, and then taking the spin-aligned limit, is a commutative procedure.

The Quasi-Circular Limit ($e \to 0$)

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- Does the orbit become circular? **No.**
- Due to the 2PN spin-spin interactions, neither the relative separation r nor the angular speed $\dot{\phi}$ can stay constant.
  $\longrightarrow$ The terms $h_r, h_{\phi 1}$, and $h_{\phi 2}$ induce small, fast oscillations on the orbital timescale.

$$r = a_r(1 - \cancel{e_r \cos u}) - \frac{s_{0p}^2}{4\,\bar{l}^2}\cos(2u - 2\phi_{sp})$$

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Constant r and $\dot{\phi}$ are only recovered if one performs an **orbit-average** to artificially smooth out these fast, spin-spin driven wiggles

Conservative Dynamics

- **Hybrid Spin Solution:** Constructed an analytical model for precessing spins that captures both spin–spin slow secular evolution and leading-order spin–orbit fast oscillations.
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Thank you!