Constants of motion for elliptic orbits: 4PN and 1SF

[Trestini 2025b, arXiv:2511.10735] [Trestini, Nasipak & Pound 2026, arXiv:2601.05223]

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Wednesday, May 26, 2026

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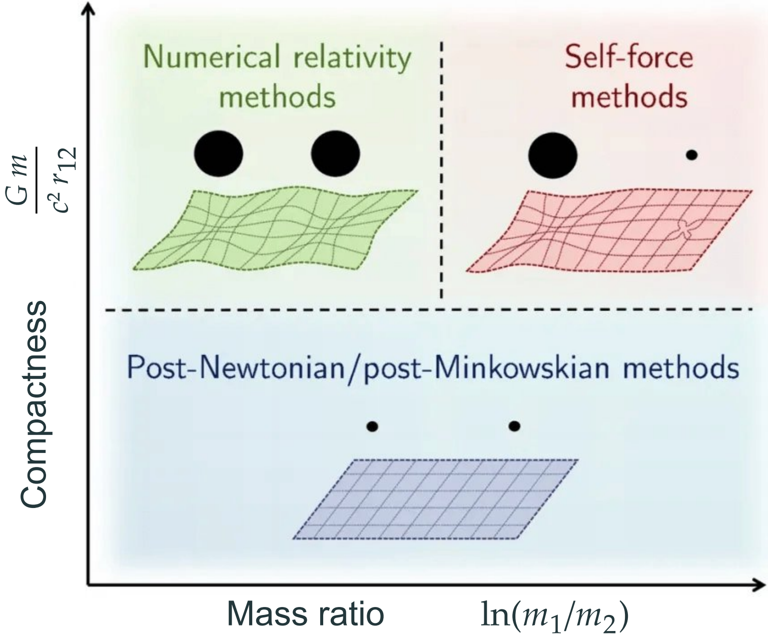
Modeling throughout parameter space



What is the post-Newtonian (PN) approximation

Perturbative study of motion and gravitational waves

Small parameter : relative velocity of the binary $v \ll c$

Motion of compact binary: **Newton**'s law of motion
+ corrections in $(v/c)^2$
$\Longrightarrow$ hence the name, post-**Newtonian** (PN)

Gravitational waves : Einstein's quadrupole formula
+ corrections in $(v/c)^2$

Corrections of order $(v/c)^{2n}$ to the leading order are said to be nPN
e.g.: a 4.5PN correction is $(v/c)^9$

What is gravitational self force (GSF or SF)

Point of view: small particle in Schwarschild spacetime.

Leading order: test particle follows a geodesic in Schwarzschild metric $g^{(0)}_{\mu\nu}$ with mass m_1

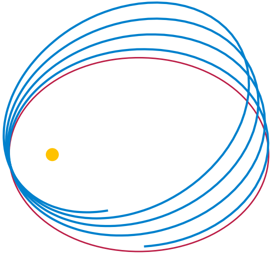
How to correct this trajectory due to finite mass m_2 of the particle ?
Perturb the metric in $\epsilon = m_2/m_1 \ll 1$:

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + \epsilon h^{(1)}_{\mu\nu}$$

This metric is singular at the location of the small particle ! Remove the Detweiler-Whiting singular field. Work with effective (regularized) metric

$$\tilde{g}_{\mu\nu} = g^{(0)}_{\mu\nu} + \epsilon \tilde{h}^{(1)}_{\mu\nu}$$

Quasi-elliptic motion



No spin ⇒ motion almost elliptic & in-plane; described in a gauge-invariant manner by the (orbital) fundamental frequencies

- the radial frequency $\Omega_r = n$ [frequency of periastron passages)].
- the azimuthal frequency $\Omega_\phi = \omega$ [frequency of ϕ variable]

The periastron advance is measured by $K = 1 + k = \Omega_\phi / \Omega_r = \omega / n$.

I will neglect dissipation (conservative motion)

Evolution equations for nonspinning eccentric orbits

The gravitational waveform is given by

$$h = h_+ - \mathrm{i}h_\times = \frac{1}{r}\sum_{\ell mn} h_{\ell mn}(t) Y_{\ell m}^{-2}(\theta,\phi)\mathrm{e}^{\mathrm{i}[m\varphi_\phi(t)+n\varphi_r(t)]}$$

The azimuthal phase φ_ϕ and radial phase φ_r evolve as

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\mathcal{F}_E(E,J) \qquad \frac{\mathrm{d}J}{\mathrm{d}t} = -\mathcal{F}_J(E,J)$$

$$\frac{\mathrm{d}\varphi_\phi}{\mathrm{d}t} = \Omega_\phi(E,J) \qquad \frac{\mathrm{d}\varphi_r}{\mathrm{d}t} = \Omega_r(E,J)$$

The map $(\Omega_r, \Omega_\phi) \leftrightarrow (E, J)$ important ingredient to predict the phase !

Here we will focus on the conservative sector of this map; small dissipative corrections (Schott terms) need to be added separately

For circular orbits reduces to

$$\frac{\mathrm{d}E_{\mathrm{circ}}}{\mathrm{d}t} = -\mathcal{F}_E(E)\,, \qquad \frac{\mathrm{d}\varphi_\phi}{\mathrm{d}t} = \Omega_\phi(E)$$

Equations of motion
at fourth post-Newtonian order

Action formulation

The conservative 4PN EOM derive from the 4PN action

$$S[\boldsymbol{y_1}, \boldsymbol{y_2}] = \int \mathrm{d}t\, L_{\mathsf{loc}}\Big(\boldsymbol{y}_A(t), \boldsymbol{v}_A(t), \boldsymbol{a}_A(t), ...\Big) + S_{\mathsf{tail}}[\boldsymbol{y_1}, \boldsymbol{y_2}]$$

where

$$S_{\mathsf{tail}}[\boldsymbol{y_1}, \boldsymbol{y_2}] = \frac{G^2 \mathrm{M}}{5c^8} \mathsf{Pf}_{\frac{2r_{12}(t)}{c}} \int_{-\infty}^{+\infty} \mathrm{d}t \int_{-\infty}^{+\infty} \frac{\mathrm{d}t'}{|t-t'|} \mathrm{I}_{ij}^{(3)}(t) \mathrm{I}_{ij}^{(3)}(t')$$

where $\mathrm{I}_{ij}^{(3)} = 2m_1(3v_1^{\langle i}a_1^{j\rangle} + y_1^{\langle i}b_1^{j\rangle}) + (1 \leftrightarrow 2) + \mathcal{O}(1/c^2)$

One should not order-reduce the accelerations and jerks at this point!

The variation of the tail action is straightforward to perform thanks to the symmetry under $t \leftrightarrow t'$

Hamiltonian

The conservative Hamiltonian is defined from the action

$$S = \int \mathrm{d}t \, \left\{ \boldsymbol{p}(t) \cdot \boldsymbol{x}(t) - H\Big(\boldsymbol{x}(t), \boldsymbol{p}(t); t' \mapsto \boldsymbol{x}(t')\Big) \right\}$$

Split into local and tail part :

$$H = H^{\mathsf{loc}}\Big(\boldsymbol{x}(t), \boldsymbol{p}(t)\Big) + H^{\mathsf{tail}}\Big(\boldsymbol{x}(t), \boldsymbol{p}(t); t' \mapsto \boldsymbol{x}(t')\Big)$$

where

$$H^{\mathsf{tail}}(t) = -\frac{G^2 \mathrm{M}}{5c^8} \mathsf{Pf}_{\frac{2r_{12}(t)}{c}} \int_{-\infty}^{+\infty} \frac{\mathrm{d}t'}{|t-t'|} \mathrm{I}_{ij}^{(3)}(t) \mathrm{I}_{ij}^{(3)}(t')$$

The generalized Hamilton equations read

$$\dot{\boldsymbol{x}} = \frac{\partial H^{\mathsf{local}}}{\partial \boldsymbol{p}} + 2\frac{\delta H^{\mathsf{tail}}}{\delta \boldsymbol{p}(t)} \qquad \dot{\boldsymbol{p}} = -\frac{\partial H^{\mathsf{local}}}{\partial \boldsymbol{x}} - 2\frac{\delta H^{\mathsf{tail}}}{\delta \boldsymbol{x}(t)}$$

where the notation $\delta/\delta\boldsymbol{x}(t)$ and $\delta/\delta\boldsymbol{p}(t)$ means that one does not vary with respect to dynamical variable that depend on t'.

Conservative Hamiltonian and energy

Due to hereditary tails, the Hamiltonian is not conserved :

$$\mathrm{d}H/\mathrm{d}t\Big|_{\text{on-shell}} \neq 0$$

However, one can *construct* a conserved energy, which is constant *under the conservative EOM*.

$$\frac{\mathrm{d}E_{\text{cons}}[\boldsymbol{x}(t),\boldsymbol{v}(t)]}{\mathrm{d}t}\Bigg|_{\boldsymbol{a}_{\text{cons}}} = 0$$

where here $\mathrm{d}\boldsymbol{v}/\mathrm{d}t$ is replaced by $\boldsymbol{a}_{\text{cons}}$.

This can e.g. be achieved by:

- localizing Hamiltonian using well-chosen contact transformations
- evaluating the localized Hamiltonian (in terms of new phase-space variables) on shell — it is conserved
- going back to the original variables

Conservative 4PN constants of motion for elliptic orbits

Action-angle formulation

Consider the Hamiltonian $H(r, \phi, p_r, p_\phi)$

Hamilton's equations read

$$\dot{r} = \frac{\partial H}{\partial p_r} \qquad \dot{\phi} = \frac{\partial H}{\partial p_\phi} \qquad \dot{p}_r = -\frac{\partial H}{\partial r} \qquad \dot{p}_\phi = -\frac{\partial H}{\partial \phi}$$

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Define $i_\phi = p_\phi/(Gm^2\nu)$. We want to construct a contact transformation such that the new variable $i_r = f(r, p_r, p_\phi)$ makes the Hamiltonian depend only on momentum variables (called 'action variables'): $H(i_r, i_\phi)$.

Then, half of the Hamilton equations read

$$\frac{\mathrm{d}i_r}{\mathrm{d}t} = 0 \qquad \frac{\mathrm{d}i_\phi}{\mathrm{d}t} = 0$$

Action-angle formulation

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Then, half of the Hamilton equations read

$$\frac{\mathrm{d}i_r}{\mathrm{d}t} = 0 \qquad \frac{\mathrm{d}i_\phi}{\mathrm{d}t} = 0$$

But this choice is not the best because of the degeneracy of radial and azimuthal frequency at Newtonian order

Action-angle formulation

Introduce $i_{r\phi} = i_r + i_\phi$

These are the so-called Delaunay variables [(i_2, i_3) in [Goldstein 1980]]

Then, half of the Hamilton equations read

$$\frac{\mathrm{d}i_{r\phi}}{\mathrm{d}t} = 0 \qquad \frac{\mathrm{d}i_\phi}{\mathrm{d}t} = 0$$

The angle variables are *defined* by the other half of Hamilton's equations

$$\frac{\mathrm{d}\ell}{\mathrm{d}t} = \frac{\partial H}{\partial i_{r\phi}} = n = \Omega_r \qquad \frac{\mathrm{d}g}{\mathrm{d}t} = \frac{\partial H}{\partial i_\phi} = \omega - n = \Omega_\phi - \Omega_r$$

ℓ is the "mean anomaly"' (related to "mean motion" n)

g is the angle of the periastron (constant in Newtonian case)

Note that only the first equation is informative at Newtonian order

Local sector — working out the radial action

For separable systems, the radial action is defined as $i_r = \frac{1}{2\pi} \oint \mathrm{d}r \, \frac{p_r}{Gm^2\nu}$.

Step 1 : Invert $\dot{r} = \frac{\partial H}{\partial p_r}(p_r, p_\phi, r)$ and $\dot{\phi} = \frac{\partial H}{\partial p_\phi}(p_r, p_\phi, r)$ and find

$$p_r = m\nu\dot{r}\left\{1 + \frac{1}{c^2}\left[\frac{\dot{r}^2 + r^2\dot{\phi}^2}{2}(1-3\nu) + \frac{Gm}{r}(3+\nu)\right] + \mathcal{O}\left(\frac{1}{c^4}\right)\right\}$$

$$p_\phi = m\nu r^2\dot{\phi}\left\{1 + \frac{1}{c^2}\left[\frac{\dot{r}^2 + r^2\dot{\phi}^2}{2}(1-3\nu) + \frac{Gm}{r}(3+\nu)\right] + \mathcal{O}\left(\frac{1}{c^4}\right)\right\}$$

Step 2 : invert $E = H(r, \dot{r}, \dot{\phi})$ and $J = p_\phi$ to obtain

$$\dot{r}^2 = A + 2B/r + C/r^2 + \cdots \qquad \dot{\phi} = F/r^2 + \cdots$$

where A,B,C, F, $\cdots$ are functions of (E, J).

Step 3 : Using all this, express $p_r^2 = \mathcal{I}(1/r) = \mathcal{A} + \mathcal{B}/r + \mathcal{C}/r^2 + \cdots$ where $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ are functions of (E, J).

Step 4 : Perform contour integral: $i_r(E, J) = \frac{1}{2\pi Gm^2\nu} \oint \mathrm{d}r \, \sqrt{\mathcal{I}(1/r)}$

Local sector — action-angle formulation of the Hamiltonian

Thus we have expressed $i_r^{\text{loc}}(E, J)$

Recall $i_\phi^{\text{loc}} = J/(Gm^2\nu)$

Introduce $i_{r\phi}^{\text{loc}} = i_r^{\text{loc}} + i_\phi^{\text{loc}}$

Identifying $H = E$, invert these relations :

$$H^{\text{loc}}(i_{r\phi}, i_\phi) = -\frac{m\nu}{2i_{r\phi}^2}\left\{1 + \frac{1}{c^2}\left[\frac{6}{i_\phi i_{r\phi}} + \frac{1}{i_{r\phi}^2}\left(-\frac{15}{4} + \frac{\nu}{4}\right)\right] + \mathcal{O}\left(\frac{1}{c^4}\right)\right\}$$

I compute H^{loc} at 4PN (extends 3PN result of [Damour, Jaranowski & Schäfer, arXiv:gr-qc/9912092.])

Next goal: add H^{tail} as a small 4PN perturbation to obtain

$$H(i_{r\phi}, i_\phi) = H^{\text{loc}}(i_{r\phi}, i_\phi) + H^{\text{tail}}(i_{r\phi}, i_\phi)$$

Tail sector — split between logarithmic and hereditary pieces

The tail Hamiltonian reads [1702.06839] $H^{\text{tail}} = H^{\log} + H^{\text{hered}}$ where

$$H^{\log} = -\frac{2G^2m}{5c^8}\mathrm{I}_{ij}^{(3)}\mathrm{I}_{ij}^{(3)}\ln\left(\frac{r(t)}{\eta}\right) \qquad \text{where } \eta = \frac{\exp(-\gamma_E)}{4}\sqrt{\frac{c^2a^3}{Gm}}$$

$$H^{\text{hered}} = \frac{G^2m}{5c^8}\mathrm{I}_{ij}^{(3)}\int_0^\infty \mathrm{d}\tau \ln\left(\frac{c\tau}{2\eta}\right)\left[\mathrm{I}_{ij}^{(4)}(t-\tau) - \mathrm{I}_{ij}^{(4)}(t+\tau)\right]$$

All operations hereafter are licit because they can be framed as canonical transformations — phase-space variables implicitly redefined at each step.

Quasi-Keplerian parametrization

The quadrupole moment reads

$$\mathrm{I}_{ij} = m\nu r^2 \begin{pmatrix} \cos^2\phi - \frac{1}{3} & \cos\phi\sin\phi & 0 \\ \cos\phi\sin\phi & \sin^2\phi - \frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix}_{ij}$$

To express in terms of Newtonian action-angle variables, use Keplerian parametrization:

$$\ell = u - e\sin u \qquad \text{[defines eccentric anomaly } u \text{ from mean anomaly } \ell\text{]}$$

$$r = a(1 - e\cos u)$$

$$\phi = g + u + 2\arctan\left[\frac{\beta(e)\sin u}{1 - \beta(e)\cos u}\right] \qquad [g \text{ constant so choose } g = 0]$$

where $a = Gm\, i_{r\phi}$, $e = \sqrt{1 - (i_\phi / i_{r\phi})^2}$ and $\beta(e) = e/(1 + \sqrt{1 - e^2})$

Logarithmic piece of the Hamiltonian

Using this find

$$\mathrm{I}_{ij}^{(3)}\mathrm{I}_{ij}^{(3)} = \frac{G^3 m^5 \nu^2}{3a^5}\left[\frac{88(1-e^2)}{(1-e\cos u)^2} + \frac{16}{(1-e\cos u)^5} - \frac{8}{(1-e\cos u)^4}\right]$$

Recall $r = a(1-e\cos u)$

Immediate to evaluate

$$H^{\log} = -\frac{2G^2 m}{5c^8}\mathrm{I}_{ij}^{(3)}\mathrm{I}_{ij}^{(3)}\ln\left(\frac{r(t)}{b_0}\right)$$

After orbit averaging:

$$\langle H^{\log}\rangle = \frac{G^5 m^6 \nu^2}{a^5 c^8 (1-e^2)^{7/2}}\Bigg\{-\frac{1204}{45} - \frac{632}{15}e^2 - \frac{17}{6}e^4 + \frac{1204+1346e^2}{45}\sqrt{1-e^2} + \frac{96+292e^2+37e^4}{15}\left[2\ln\left(\frac{1-e^2}{1+\sqrt{1-e^2}}\right) + \ln\left(\frac{Gm}{c^2 a}\right) + 6\ln 2 + 2\gamma_{\mathrm{E}}\right]\Bigg\}$$

Hereditary piece of the Hamiltonian

We have everything to evaluate the integrand in

$$H^{\mathrm{hered}} = \frac{G^2 m}{5c^8} \mathrm{I}_{ij}^{(3)} \int_0^\infty \mathrm{d}\tau \, \ln\left(\frac{c\tau}{2b_0}\right) \left[\mathrm{I}_{ij}^{(4)}(t-\tau) - \mathrm{I}_{ij}^{(4)}(t+\tau)\right]$$

but cannot perform the integral in closed form. **But** the circular case can be performed in closed form. **Solution**: Fourier series (=epicycles).

$$\mathrm{I}_{ij}(\ell, i_{r\phi}, i_\phi) = m\nu a^2 \sum_{p\in\mathbb{Z}} {}_p\widehat{\mathrm{I}}_{ij}(i_{r\phi}, i_\phi)\, \mathrm{e}^{\mathrm{i}p\ell} \,,$$

For example:

$${}_p\widehat{\mathrm{I}}_{xx} = -\frac{2}{3}\frac{3-e^2}{e^2}\frac{J_p(pe)}{p^2} + \frac{2(1-e^2)}{e}\frac{J_p'(pe)}{p}$$

We then find

$$H^{\mathrm{hered}} = -\frac{2G^2 m}{5c^8} n^6 \left(\mathcal{I}_2\right)^2 \sum_{(p,q)\in\mathbb{Z}^2} p^3 q^3 \ln(|p|) \; {}_p\widehat{\mathrm{I}}_{ij} \; {}_q\widehat{\mathrm{I}}_{ij} \mathrm{e}^{\mathrm{i}(p+q)\ell}$$

Hereditary piece of the Hamiltonian

We finally want to orbit average

$$H^{\text{hered}} = -\frac{2G^2 m}{5c^8} n^6 m^2 \nu^2 a^4 \sum_{(p,q)\in\mathbb{Z}^2} p^3 q^3 \ln(|p|) \; {}_p\widehat{\mathrm{I}}_{ij} \; {}_q\widehat{\mathrm{I}}_{ij} \mathrm{e}^{\mathrm{i}(p+q)\ell}$$

We find

$$\boxed{\langle H^{\text{hered}} \rangle = \frac{64 G^5 m^6 \nu^2}{5 a^5 c^8} \Lambda_0(e)\,.}$$

expressed in terms of the *enhancement function*

$$\boxed{\Lambda_0(e) = \frac{1}{16} \sum_{p=1}^{\infty} p^6 \ln\left(\frac{p}{2}\right) \; {}_p\widehat{\mathrm{I}}_{ij} \; {}_{-p}\widehat{\mathrm{I}}_{ij}\,,}$$

This is a pure function of $e \in [0,1]$ which we will study more

Action-angle Hamiltonian

We have succeeded in constructing

$$H(i_{r\phi}, i_\phi) = H^{\mathsf{loc}}(i_{r\phi}, i_\phi) + \langle H^{\mathsf{log}} \rangle (i_{r\phi}, i_\phi) + \langle H^{\mathsf{hered}} \rangle (i_{r\phi}, i_\phi)$$

This is exact for all eccentricity [requires a non-standard function $\Lambda_0(e)$]

Fundamental frequencies expressed in terms of action-angle variables:

$$n = \Omega_r = \frac{\partial H}{\partial i_{r\phi}} \qquad \omega - n = \Omega_\phi - \Omega_r = \frac{\partial H}{\partial i_\phi}$$

We know $i_{r\phi} = f(E, J)$ and $i_\phi = J/(Gm^2\nu)$ so we can obtain at 4PN

$$n(E, J) \qquad \omega(E, J)$$

We can invert these relations and obtain at 4PN

$$E(n, \omega) \qquad J(n, \omega)$$

This was the result we were aiming for

Enhancement function

We want to study

$$\Lambda_0(e) = \frac{1}{16}\sum_{p=1}^{\infty} p^6 \ln\left(\frac{p}{2}\right)\ {}_p\widehat{\mathrm{I}}_{ij}\ {}_{-p}\widehat{\mathrm{I}}_{ij}$$

The easiest limit is to Taylor-expand as $e \to 0$

Dominated by small values of p

$$\begin{aligned}\Lambda_0(e) \underset{e\to 0}{\sim} e^2 &\left[-\frac{277}{24}\ln 2 + \frac{729}{64}\ln 3\right] + e^4\left[\frac{11353}{96}\ln 2 - \frac{13851}{256}\ln 3\right] \\ &+ e^6\left[-\frac{21997}{32}\ln 2 + \frac{419661}{4096}\ln 3 + \frac{9765625}{36864}\ln 5\right] \\ &+ e^8\left[\frac{5056751}{2304}\ln 2 + \frac{26915409}{32768}\ln 3 - \frac{419921875}{294912}\ln 5\right] + \mathcal{O}(e^{10})\end{aligned}$$

Enhancement function

We want to study

$$\Lambda_0(e) = \frac{1}{16}\sum_{p=1}^{\infty} p^6 \ln\left(\frac{p}{2}\right)\ {}_p\widehat{\mathrm{I}}_{ij}\ {}_{-p}\widehat{\mathrm{I}}_{ij}$$

Possible (but more difficult) to study $e \to 1$

Dominated by large values of p (i.e. $p \to \infty$)

Following [Loutrel & Yunes 1607.05409] and [Forseth, Evans Hopper, arXiv:1512.03051], one must

- obtain uniform asymptotic expansion as $p \to \infty$ of

$$p^6 \ln(p/2)\ {}_p\widehat{\mathrm{I}}_{ij}\ {}_{-p}\widehat{\mathrm{I}}_{ij}$$

  which involves Airy functions
- $\sum_{p=1}^{\infty} \Rightarrow \int_1^{\infty} \mathrm{d}p$
- Taylor-expand as $\epsilon = 1 - e^2 \to 0$

Enhancement function

We want to study

$$\Lambda_0(e) = \frac{1}{16}\sum_{p=1}^{\infty} p^6 \ln\left(\frac{p}{2}\right)\ {}_p\widehat{\mathrm{I}}_{ij}\ {}_{-p}\widehat{\mathrm{I}}_{ij}$$

Possible (but more difficult) to study $e \to 1$

$$\begin{aligned}\Lambda_0(e) \underset{e\to 1}{\sim} \quad & \frac{1}{(1-e^2)^{7/2}}\left[\frac{65}{3} - \frac{425}{96}\gamma_\mathsf{E} - \frac{425}{24}\ln 2 - \frac{425}{192}\ln 3 - \frac{425}{64}\ln(1-e^2)\right] \\ & + \frac{1}{(1-e^2)^{5/2}}\left[-\frac{3301}{160} + \frac{61}{16}\gamma_\mathsf{E} + \frac{61}{4}\ln 2 + \frac{61}{32}\ln 3 + \frac{183}{32}\ln\left(1-e^2\right)\right] \\ & + \frac{1}{(1-e^2)^{3/2}}\left[\frac{31707}{11200} - \frac{37}{96}\gamma_\mathsf{E} - \frac{37}{24}\ln 2 - \frac{37}{192}\ln 3 - \frac{37}{64}\ln\left(1-e^2\right)\right] \\ & + \mathcal{O}\left(\frac{1}{\sqrt{1-e^2}}\right)\end{aligned}$$

Enhancement function

We want to study

$$\Lambda_0(e) = \frac{1}{16}\sum_{p=1}^{\infty} p^6 \ln\left(\frac{p}{2}\right)\ {}_p\widehat{\mathrm{I}}_{ij}\ {}_{-p}\widehat{\mathrm{I}}_{ij}$$

Possible (but more difficult) to study $e \to 1$

$$\begin{aligned}\Lambda_0(e) \underset{e\to1}{\sim}\quad & -\frac{3}{2} f(e)\ln\big(1-e^2\big)\\ & + \frac{1}{(1-e^2)^{7/2}}\left[\frac{65}{3} - \frac{425}{96}\gamma_{\mathsf{E}} - \frac{425}{24}\ln 2 - \frac{425}{192}\ln 3\right]\\ & + \frac{1}{(1-e^2)^{5/2}}\left[-\frac{3301}{160} + \frac{61}{16}\gamma_{\mathsf{E}} + \frac{61}{4}\ln 2 + \frac{61}{32}\ln 3\right]\\ & + \frac{1}{(1-e^2)^{3/2}}\left[\frac{31707}{11200} - \frac{37}{96}\gamma_{\mathsf{E}} - \frac{37}{24}\ln 2 - \frac{37}{192}\ln 3\right] + \mathcal{O}\left(\frac{1}{\sqrt{1-e^2}}\right)\end{aligned}$$

where we recognize the Peters enhancement function $f(e) = \frac{1+\frac{73}{24}e^2+\frac{37}{96}e^4}{(1-e^2)^{7/2}}$

Resumming enhancement function

We want to study

$$\Lambda_0(e) = \frac{1}{16}\sum_{p=1}^{\infty} p^6 \ln\left(\frac{p}{2}\right)\ {}_p\widehat{\mathrm{I}}_{ij}\ {}_{-p}\widehat{\mathrm{I}}_{ij}$$

I propose the following resummation, which

- agrees with the $e \to 0$ to any desired order
- captures the leading divergence $(1-e^2)^{-7/2}$ as $e \to 1$
- exactly resums the part $\propto \ln\big(1-e^2\big)$

$$\Lambda_0(e) = -\frac{3}{2(1-e^2)^{7/2}}\left[\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right)\ln\big(1-e^2\big)+e^2\lambda_0(e)\right]$$

where the coefficients α_k of

$$\lambda_0(e) = \sum_{k=0}^{k_{\mathsf{max}}} \alpha_k e^{2k}$$

are determined by the $e \to 0$ expansion of $\Lambda_0(e)$

Resumming enhancement function

We want to study

$$\Lambda_0(e) = \frac{1}{16}\sum_{p=1}^{\infty} p^6 \ln\left(\frac{p}{2}\right)\ {}_p\widehat{\mathrm{I}}_{ij}\ {}_{-p}\widehat{\mathrm{I}}_{ij}$$

I also study the following 'naive' resummation, which

- agrees with the $e \to 0$ to any desired order
- captures the leading divergence $(1-e^2)^{7/2}$ as $e \to 1$
- ~~exactly resums the part $\propto \ln(1-e^2)$~~

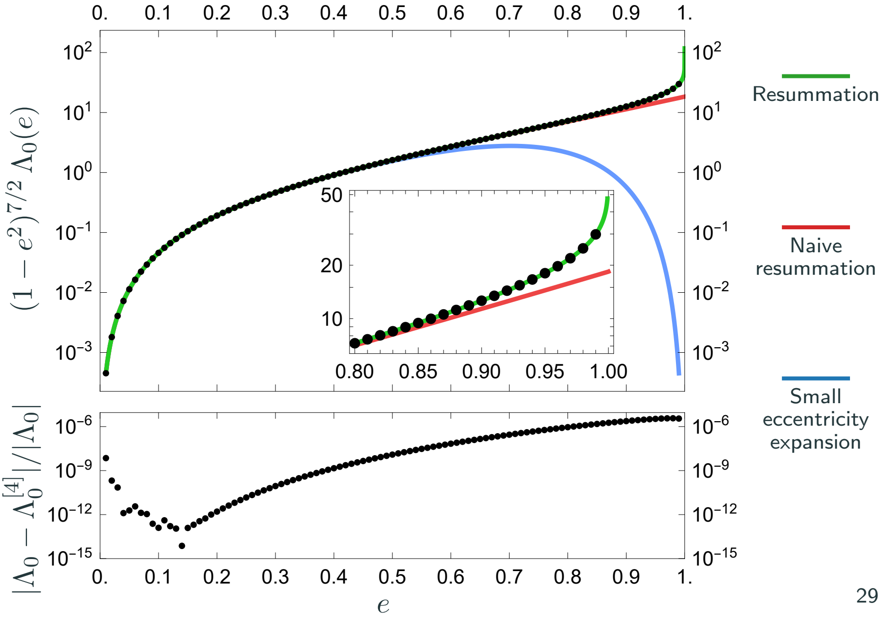
$$\Lambda_0(e) = -\frac{3e^2}{2(1-e^2)^{7/2}}\lambda_0^{\text{naive}}(e)$$

where the coefficients α_k^{naive} of

$$\lambda_0^{\text{naive}}(e) = \sum_{k=0}^{k_{\text{max}}} \alpha_k^{\text{naive}} e^{2k}$$

are determined by the $e \to 0$ expansion of $\Lambda_0(e)$

Comparison of resummations



Constants of motion at 1SF order

Map at geodesic order

At geodesic order one defines the Darwin parameters (p, e) as

$$p = \frac{2r_{\text{min}}r_{\text{max}}}{r_{\text{max}} + r_{\text{min}}} \qquad e = \frac{r_{\text{max}} - r_{\text{min}}}{r_{\text{max}} + r_{\text{min}}}$$

$$\hat{E} = \sqrt{\frac{(p-2+2e)(p-2-2e)}{p(p-3-e^2)}} \qquad \hat{L} = \frac{p}{\sqrt{p-3-e^2}}$$

$\hat{\Omega}_r(p, e)$ and $\hat{\Omega}_\phi(p, e)$ are expressed explicitly in terms of elliptic functions

The map between $(\hat{E}, \hat{L})$ and $(\hat{\Omega}_r, \hat{\Omega}_\phi)$ is known analytically at geodesic order

N.B.: GSF normalizes by m_1 ; PN normalized by m_{tot}

Darwin parameter at postgeodesic order

At postgeodesic order, r_{min} and r_{max} are gauge-dependent

We have different choices for defining (p, e).

We choose a *fixed-frequency* expansion: we define (p, e) such that the usual relations $\hat{\Omega}_r(p, e)$ and $\hat{\Omega}_\phi(p, e)$ hold

We seek to compute the post-geodesic corrections to the constants of motion:

$$\hat{E} = \hat{E}_{(0)}(p, e) + \epsilon\, \hat{E}_{(1)}(p, e)$$
$$\hat{L} = \hat{L}_{(0)}(p, e) + \epsilon\, \hat{L}_{(1)}(p, e)$$

Redshift invariant

All the information needed for this is contained in one quantity: the Detweiler redshift invariant:

$$\langle z \rangle = \left\langle \frac{\mathrm{d}\tilde{\tau}}{\mathrm{d}t} \right\rangle = \frac{\mathcal{T}_r}{T_r}$$

- $\tilde{\tau}$ is the proper time of the small particle in the effective metric.
- $T_r = 2\pi/\Omega_r$ is the radial period in coordinate time
- $\mathcal{T}_r$ is the radial period in proper time

We similarly expand the redshift in mass ratio (fixed frequency):

$$\langle z \rangle(p,e) = \langle z_{(0)} \rangle(p,e) + \epsilon \, \langle z_{(1)} \rangle(p,e)$$

The geodesic $\langle z_{(0)} \rangle(p,e)$ known analytically in terms of elliptic functions

The 1SF redshift is obtained numerically from the regularized metric:

$$\langle z_{(1)} \rangle = -\frac{1}{2} \langle z_{(0)} \rangle \tilde{h}^{(1)}_{\mu\nu} u^\mu u^\nu \qquad \text{where} \quad u^\mu = \mathrm{d}x^\mu_p / \mathrm{d}\tilde{\tau}$$

Expressing everything in terms of redshift

The constants of motion can be obtained from the redshift as

$$\hat{E}_{(1)}(p,e) = \frac{1}{2}\left[\langle z_{(1)}\rangle(p,e) - \Omega_\phi(p,e)\frac{\partial\langle z_{(1)}\rangle}{\partial\Omega_\phi} - \Omega_r(p,e)\frac{\partial\langle z_{(1)}\rangle}{\partial\Omega_r}\right]$$

$$\hat{L}_{(1)}(p,e) = -\frac{1}{2}\Omega_\phi(p,e)\frac{\partial\langle z_{(1)}\rangle}{\partial\Omega_\phi}$$

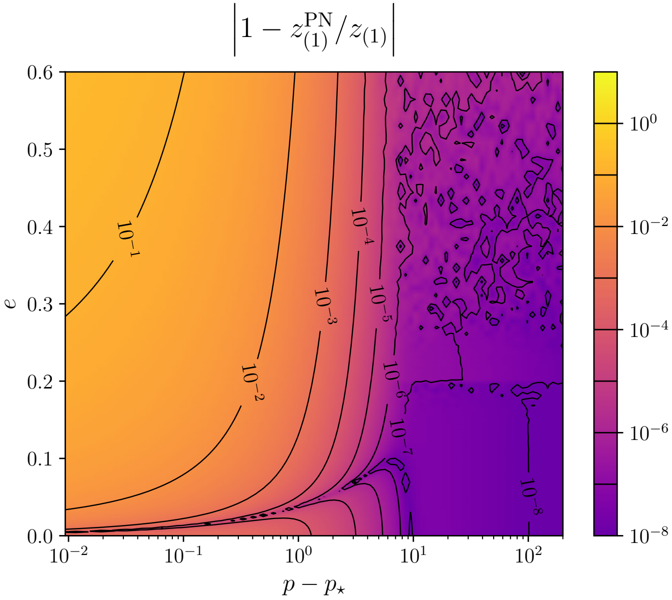
The redshift $\langle z_{(1)}\rangle(p,e)$ is known

- numerically [e.g. Zach Nasipak's `pybhpt` code]
- as a post-Newtonian expansion to 10PN [Munna and Evans (2022)]

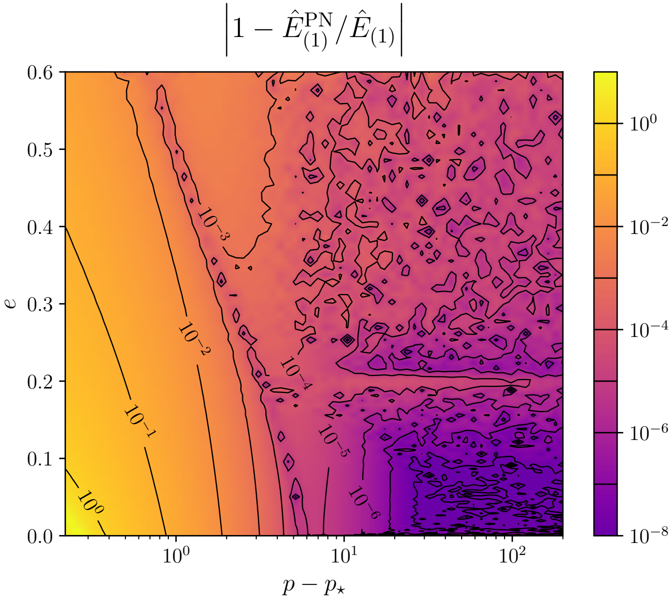
However, some difficulties of implementation: numerical derivatives, large cancellations, cancellation of the leading PN order, ...

Perfect agreement with 4PN results (including $\Lambda_0(e)$)

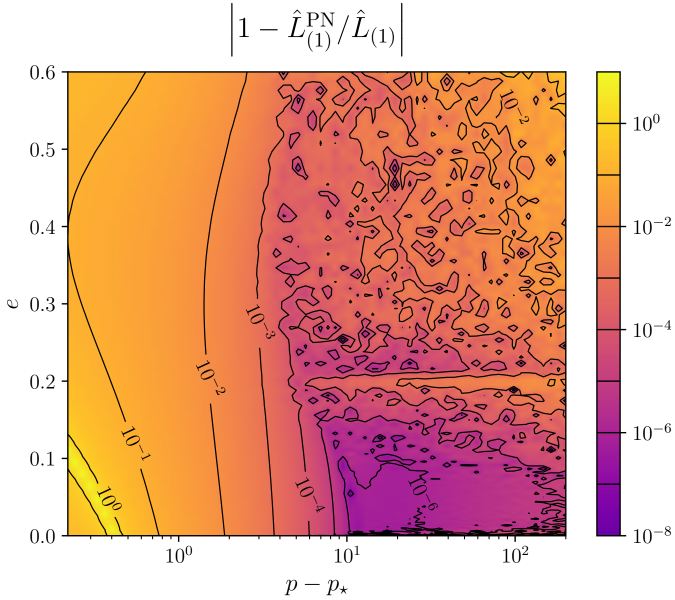
Redshift comparison: 10PN vs numerics



Energy comparison: 9PN vs numerics



Angular momentum comparison: 9PN vs numerics



Conclusion

Obtained conservative map $(E, J) \leftrightarrow (\Omega_\phi, \Omega_r)$ for nonspinning elliptic orbits

- at 4PN (all order in the mass ratio)
- at 1SF as a 10PN expansion
- at 1SF (numerically)

Checked that 4PN result agrees perfectly with 1SF-10PN in region of overlap

Determine region where 1SF-10PN is accurate enough and where (heavy) numerics are needed

Useful results for future post-adiabaitc EMRI models for eccentric orbits

Backup

Equations of motion at 4PN

Two point particles with positions and velocities $(\boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{v}_1, \boldsymbol{v}_2)$.

In center-of-mass frame: $\boldsymbol{x} = \boldsymbol{y}_1 - \boldsymbol{y}_2$ and $\boldsymbol{v} = \boldsymbol{v}_1 - \boldsymbol{v}_2$,
with $r = |\boldsymbol{x}|$ and $\boldsymbol{n} = \boldsymbol{x}/r$. The (relative) equations of motion (EOM) read [Damour, Jaranowski, Schäfer '14-'16][Bernard, Blanchet, Bohé, Faye, Marchand, Marsat '16-'18][Foffa, Porto, Rothstein, Sturani '19][Blümlein, Maier, Marquard, Schäfer '20]

Equations of motion at 4PN

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$$\boldsymbol{a} = \boldsymbol{a}_1 - \boldsymbol{a}_2 = \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \frac{\mathrm{d}^2\boldsymbol{x}}{\mathrm{d}t^2} = \boldsymbol{a}_{\text{cons}}(\boldsymbol{x}, \boldsymbol{v}) + \boldsymbol{a}_{\text{diss}}(\boldsymbol{x}, \boldsymbol{v})$$

where the conservative/dissipative EOM decompose as

$$\boldsymbol{a}_{\text{cons}} = -\frac{Gm}{r^2}\boldsymbol{n} + \boldsymbol{a}^{\text{1PN}}_{\text{cons}} + \boldsymbol{a}^{\text{2PN}}_{\text{cons}} + \boldsymbol{a}^{\text{3PN}}_{\text{cons}} + \boldsymbol{a}^{\text{4PN, inst}}_{\text{cons}} + \boldsymbol{a}^{\text{4PN, tail}}_{\text{cons}}$$

$$\boldsymbol{a}_{\text{diss}} = \boldsymbol{a}^{\text{2.5PN}}_{\text{diss}} + \boldsymbol{a}^{\text{3.5PN}}_{\text{diss}} + \boldsymbol{a}^{\text{4PN, tail}}_{\text{diss}} + \boldsymbol{a}^{\text{4.5PN}}_{\text{diss}}$$

where the "tail" pieces are hereditary functionals the whole trajectories $\{\boldsymbol{x}(t), \boldsymbol{v}(t)\}$. Thus, starting at 4PN, the EOM are not second-order differential equations anymore, but integro-differential equations.

Redshift

Using the first law of binary black hole dynamics, we can compute the invariant orbit-average redshift variable [1506.05648]

$$\langle z_1 \rangle = \frac{\partial \mathrm{M}}{\partial m_1}\bigg|_{J, I_r, m_2}$$

where we defined the ADM mass $\mathrm{M} = m + E/c^2$.

Our 4PN result perfectly agrees with PN-GSF literature at leading (geodesic) and subleading (1SF) orders in the mass-ratio [2203.13832].

Reduction to circular orbits

To compare these results with the known case of circular orbits, we set

$$i_r(E, J) = 0$$

This yields a relation $E^{\text{circ}}(J)$ in perfect agreement with [1401.4548].

Using this relation, one can obtain $E^{\text{circ}}(\omega)$ and $K^{\text{circ}}(\omega)$ at 4PN accuracy, all in perfect agreement with the literature [1610.07934]

Finally, the redshift $z^{\text{circ}}(\omega)$ can thus be computed at 4PN, also in agreement with the literature [1702.06839]