Radiation-reaction correction to scattering binary dynamics at the Next-to-Leading Post-Newtonian Order

Sara Rufrano Aliberti

Based on **Phys. Rev. D 112, 104005**
with Donato Bini and Andrea Geralico

Journée des groupes de travail "Formes d'onde" et "Tests de la relativité générale et théories alternatives" du GdR Ondes Gravitationnelles

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- Gravitational interaction between two bodies:
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- Radiation-reaction effects play important role in dynamics of binary system;
- Presence of radiation-reaction force: modification of the conservative equation of motion (EoM);

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- Aim: determine the radiation-reaction correction to the quasi-Keplerian orbital parameters at 3.5PN order.

Radiation-reaction correction

- Relative acceleration $\mathbf{a} = \mathbf{a}_1 - \mathbf{a}_2$ in the center-of-mass (CM) frame:

$$\eta = \frac{1}{c}, M = m_1 + m_2$$

$$\mathbf{a} = \mathbf{a}_{\text{cons}} + \mathbf{A}^{\text{rr}},$$

with

$$\mathbf{A}^{\text{rr}} = \eta^5 \mathbf{A}^{\text{rr}}_{2.5PN} + \eta^7 \mathbf{A}^{\text{rr}}_{3.5PN} + \eta^9 \mathbf{A}^{\text{rr}}_{4.5PN} + \mathcal{O}(\eta^{10}).$$

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$$\eta = \frac{1}{c}, M = m_1 + m_2,$$

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- CM radiation reaction relative force and acceleration:

$$\mathbf{F}^{\mathrm{rr}} = \mu \mathbf{A}^{\mathrm{rr}},$$

$$\rightarrow \mathbf{A}^{\mathrm{rr}} = -\frac{8}{5}\nu \left(\frac{GM}{r^2}\right)\left(\frac{GM}{c^2 r}\right)\left[-\left(A_{2.5\mathrm{PN}} + A_{3.5\mathrm{PN}} + \ldots\right)\frac{\dot{r}}{c}\mathbf{n} + \left(B_{2.5\mathrm{PN}} + B_{3.5\mathrm{PN}} + \ldots\right)\frac{\mathbf{v}}{c}\right]$$

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- Next-to-Leading order $\mathcal{O}(\eta^{7/2}) \rightarrow A_{3.5\mathrm{PN}}$ and $B_{3.5\mathrm{PN}}$ functions of $\mathcal{O}(\eta^4) \rightarrow$ 12 parameters.

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A_i and B_i $\rightarrow$ energy and angular momentum balance $\rightarrow$ $\begin{cases} \dfrac{dE}{dt} \propto -\Phi_E \\ \dfrac{dJ}{dt} \propto -\Phi_J \end{cases}$

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- Energy and angular momentum can be ambiguous when they are not conserved:
  - (6) + 6 parameters at LO;
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[G. A. Schott, Philos. Mag. 29,49 (1915)]

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$$\dot{r}\,\mathrm{F}_r^{\mathrm{rr}} + \dot{\phi}\,\mathrm{F}_\phi^{\mathrm{rr}} + \frac{dE_{Schott}}{dt} = -\Phi_E,$$

$$\mathrm{F}_\phi^{\mathrm{rr}} + \frac{dJ_{Schott}}{dt} = -\Phi_J\,.$$

- LO: 10 constraints over 12 parameters → 2 residual gauge parameters;
- NLO: 18 constraints over 24 parameters → 6 residual gauge parameters.

Variation of constants method at the 3.5PN order

- Radiation reaction corrections to dynamics → Lagrange method of variation of constants;

- Relative acceleration in CM frame $\mathbf{a}$:

[Damour, Gopakumar, Iyer, gr-qc/0404128]
[Bini, Damour, Geralico, 2210.07165]

$$\mathbf{a} = \mathbf{a}_{\text{cons}} + \mathbf{A}^{\text{rr}},$$

  - $\mathbf{a}_{\text{cons}}$ → unperturbed integrable part;
  - $\mathbf{A}^{\text{rr}}$ → perturbation → from the unperturbed solution, we promote $\{c_i\} \to \{c_i(t)\}$ and replace the solution in the perturbative problem.

- Motion confined in a plane → polar coordinates;

- Functional form of the unperturbed equations of motion:

$$\begin{cases} r = S(l, c_1, c_2), \\ \dot{r} = \bar{n}(c_1, c_2)\dfrac{\partial S(l, c_1, c_2))}{\partial l}, \\ \phi = c_\phi + \bar{W}(l, c_1, c_2), \\ \dot{\phi} = \bar{n}(c_1, c_2)\dfrac{\partial \bar{W}(l, c_1, c_2)}{\partial l}, \end{cases}$$

Average anomaly ↓

with $l = \bar{n}(c_1, c_2)(t - t_0) + c_l$

↑ Mean motion

c_1, c_2, c_l, c_ϕ
Constants

Variation of constants method at the 3.5PN order

- From quasi-Keplerian parametrization of the hyperbolic motion:

[Cho, Gopakumar, Haney, Lee, 1807.02380]
[Bini, Damour, Geralico, 2007.11239]

$$\begin{cases} r = S(l, c_1, c_2), \\ \phi = c_\phi + \bar{W}(l, c_1, c_2), \end{cases} \rightarrow \begin{cases} S = \bar{a}_r(e_r \cosh u - 1), \\ \bar{W} = KV + c^{-4}(f_{4\phi} \sin 2V + g_{4\phi} \sin 3V) + \mathcal{O}(c^{-6}), \end{cases}$$

with $V = 2K \arctan \left[\sqrt{\frac{e_\phi + 1}{e_\phi - 1}} \tanh \frac{u}{2} \right]$ and $u = u(l, c_1, c_2)$ through $l = e_t \sinh u - u + c^{-4} f_{4t} V + c^{-4} g_{4t} \sin V + \mathcal{O}(c^{-6})$;

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- Perturbed solution: $\{c_1, c_2, c_l, c_\phi\} \rightarrow \{c_1(t), c_2(t), c_l(t), c_\phi(t)\}$ satisfying $l = \int_{t_0}^{t} \bar{n}(c_1(t'), c_2(t'))dt' + c_l(t)$ and

$$\begin{cases} \frac{dc_1}{dt} = \frac{\partial c_1}{\partial \mathbf{v}} \cdot \mathbf{A}^{\text{rr}}, \\ \frac{dc_2}{dt} = \frac{\partial c_2}{\partial \mathbf{v}} \cdot \mathbf{A}^{\text{rr}}, \\ \frac{dc_l}{dt} = -\left(\frac{\partial S}{\partial l}\right)^{-1} \left[\frac{\partial S}{\partial c_1}\frac{dc_1}{dt} + \frac{\partial S}{\partial c_2}\frac{dc_2}{dt}\right], \\ \frac{dc_\phi}{dt} = -\frac{\partial \bar{W}}{\partial l}\frac{dc_l}{dt} - \frac{\partial \bar{W}}{\partial c_1}\frac{dc_1}{dt} - \frac{\partial \bar{W}}{\partial c_2}\frac{dc_2}{dt}. \end{cases}$$

Variation of constants method at the 3.5PN order

[Cho, Gopakumar, Haney, Lee, 1807.02380]
[Bini, Damour, Geralico, 2007.11239]
[Arun, Blanchet, Iyer, Qusailah, 2007.11239]

- $c_1 = \bar{a}_r, c_2 = e_r$:

$$\bar{a}_r = \frac{1}{2\bar{E}}\left[1 - \frac{\bar{E}}{2}(-7+\nu)\eta^2 + \mathcal{O}(\eta^4)\right],$$

$$e_r = \sqrt{1+2\bar{E}j^2}\left[1 + \frac{\bar{E}}{2}\frac{5\bar{E}j^2(\nu-3)+2(\nu-6)}{1+2\bar{E}j^2}\eta^2 + \mathcal{O}(\eta^4)\right],$$

with

$$\bar{E} = \frac{1}{2}v^2 - \frac{GM}{r} + \eta^2\left[\frac{3}{8}(1-3\nu)v^4 + \frac{1}{2}(3+\nu)v^2\frac{GM}{r} + \frac{1}{2}\nu\frac{GM}{r}\dot{r}^2 + \frac{1}{2}\frac{G^2M^2}{r^2}\right] + \mathcal{O}(\eta^4),$$

$$j = (xv_y - yv_x)\left[1 + \eta^2\left(\frac{1}{2}(1-3\nu)v^2 + (3+\nu)\frac{GM}{r}\right) + \mathcal{O}(\eta^4)\right].$$

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Variation of constants method at the 3.5PN order

- Leading order:

[Bini, Damour, Geralico, 2210.07165]

$$\frac{d\bar{a}_r}{dt} = -2\bar{a}_r^2 \mathbf{v}\cdot\mathbf{A}^{\rm rr},$$

$$\frac{de_r}{dt} = \frac{e_r^2-1}{e_r}\bar{a}_r \mathbf{v}\cdot\mathbf{A}^{\rm rr} + \frac{\sqrt{e_r^2-1}}{e_r\sqrt{a}_r}[\mathbf{x}\times\mathbf{A}^{\rm rr}]_z\,.$$

- Next-to-Leading order:

$$\bar{a}_r(\mathbf{x},\mathbf{v}) = \bar{a}_r^{(0)}(\mathbf{x},\mathbf{v}) + \eta^2\bar{a}_r^{(2)}(\mathbf{x},\mathbf{v}),$$
$$e_r(\mathbf{x},\mathbf{v}) = e_r^{(0)}(\mathbf{x},\mathbf{v}) + \eta^2 e_r^{(2)}(\mathbf{x},\mathbf{v}),$$
$$\mathbf{A}^{\rm rr}(\mathbf{x},\mathbf{v}) = \eta^5\mathbf{A}^{(0){\rm rr}}(\mathbf{x},\mathbf{v}) + \eta^7\mathbf{A}^{(2){\rm rr}}(\mathbf{x},\mathbf{v}),$$

where $\mathbf{A}^{(0){\rm rr}} \equiv \mathbf{A}^{\rm rr}_{2.5PN}$, $\mathbf{A}^{(2){\rm rr}} \equiv \mathbf{A}^{\rm rr}_{3.5PN}$. Therefore:

$$\begin{aligned}\bar{a}_r(\mathbf{x},\mathbf{v}) &= \bar{a}_r^{(0)}(\mathbf{x},\mathbf{v}) + \eta^2\bar{a}_r^{(2)}(\mathbf{x},\mathbf{v})\\ \mathbf{A}^{\rm rr}(\mathbf{x},\mathbf{v}) &= \eta^5\mathbf{A}^{(0){\rm rr}}(\mathbf{x},\mathbf{v}) + \eta^7\mathbf{A}^{(2){\rm rr}}(\mathbf{x},\mathbf{v})\end{aligned} \rightarrow \frac{d\bar{a}_r}{dt} = \frac{\partial\bar{a}_r}{\partial\mathbf{v}}\cdot\mathbf{A}_{\rm rr} \rightarrow \begin{aligned}\frac{d\bar{a}_r}{dt}^{(2.5)} &= \frac{\partial\bar{a}_r^{(0)}}{\partial {\rm v}^i}{\rm A}_i^{(0){\rm rr}},\\ \frac{d\bar{a}_r}{dt}^{(3.5)} &= \frac{\partial\bar{a}_r^{(0)}}{\partial {\rm v}^i}{\rm A}_i^{(2){\rm rr}} + \frac{\partial\bar{a}_r^{(2)}}{\partial {\rm v}^i}{\rm A}_i^{(0){\rm rr}},\end{aligned}$$

Variation of constants method at the 3.5PN order

- Evolution of the radiative-reaction part $\delta^{\rm rr} c_\alpha(t)$ from:

$$\begin{cases} \dfrac{dc_1}{dt} = \dfrac{\partial c_1}{\partial \mathbf{v}} \cdot \mathbf{A}^{\rm rr} \\ \dfrac{dc_2}{dt} = \dfrac{\partial c_2}{\partial \mathbf{v}} \cdot \mathbf{A}^{\rm rr} \\ \dfrac{dc_l}{dt} = -\left(\dfrac{\partial S}{\partial l}\right)^{-1} \left[\dfrac{\partial S}{\partial c_1}\dfrac{dc_1}{dt} + \dfrac{\partial S}{\partial c_2}\dfrac{dc_2}{dt}\right] \\ \dfrac{dc_\phi}{dt} = -\dfrac{\partial \bar{W}}{\partial l}\dfrac{dc_l}{dt} - \dfrac{\partial \bar{W}}{\partial c_1}\dfrac{dc_1}{dt} - \dfrac{\partial \bar{W}}{\partial c_2}\dfrac{dc_2}{dt} \end{cases}$$

$\longrightarrow$

$$\delta^{\rm rr} c_\alpha(u) = \int_{-\infty}^{u} \frac{dc_\alpha}{dt}\frac{dt}{du^*}du^*$$

$\downarrow$

$$\delta^{\rm rr} X = \mathrm{D}X = \frac{\partial X}{\partial c_1}\delta^{\rm rr} c_1 + \frac{\partial X}{\partial c_2}\delta^{\rm rr} c_2 + \frac{\partial X}{\partial l}\delta^{\rm rr} l$$

$\downarrow$

$$\begin{cases} \delta^{\rm rr} r = \mathrm{D}S \\ \delta^{\rm rr} \phi = \delta^{\rm rr} c_\phi + \mathrm{D}\bar{W} \end{cases}$$

Results

- Radiation-reaction contribution to the relative scattering angle:

[Bini, Damour, Geralico, 2107.08896]

$$[\delta^{rr}\phi]^{(0)} = \frac{\nu}{\bar{a}_r^{5/2}}\left[\frac{2(121e_r^2+304)}{15(e_r^2-1)^3}\arccos\left(-\frac{1}{e_r}\right)+\frac{2(72e_r^4+1069e_r^2+134)}{45e_r^2(e_r^2-1)^{5/2}}\right],$$

$$[\delta^{rr}\phi]^{(2)} = \frac{\nu}{\bar{a}_r^{7/2}}\left[\frac{48(7e_r^2+8)}{5(e_r^2-1)^{7/2}}\arccos^2\left(-\frac{1}{e_r}\right)\right.$$
$$+\frac{[(1316\nu+2783)e_r^4-(36400\nu+90420)e_r^2-51296\nu-58376]}{420(e_r^2-1)^4}\arccos\left(-\frac{1}{e_r}\right)$$
$$\left.+\frac{[(10080\nu+51840)e_r^6-(175420\nu+1396049)e_r^4-(1049720\nu+1003562)e_r^2-80640\nu+157576]}{6300e_r^2(e_r^2-1)^{7/2}}\right],$$

$$\rightarrow \chi_{\rm rr}^{2.5\rm PN+3.5PN} = [\delta^{\rm rr}\phi]^{(0)} + \eta^2[\delta^{\rm rr}\phi]^{(2)}$$

What's next?

- Radiative-reaction corrections to quasi-Keplerian parameters at 4.5PN order;

  [Blanchet, Faye, Seraille, Trestini, 2601.06743]

- Application to losses of:

  [Arun, Blanchet, Iyer, Qusailah, 0711.0302]

  - Energy;
  - Angular momentum;
  - Linear momentum.

What's next?

- Radiative-reaction corrections to quasi-Keplerian parameters at 4.5PN order;

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  - Energy;
  - Angular momentum;
  - Linear momentum.

Thank you for the attention!

Results

- $\delta^{\rm rr} c_\alpha = \delta^{\rm rr} c_\alpha^{(0)} + \eta^2 \delta^{\rm rr} c_\alpha^{(2)}$:

$$AT(u) \equiv \arctan\left[\sqrt{\frac{e_r+1}{e_r-1}}\tanh\left(\frac{u}{2}\right)\right] + \arctan\left[\sqrt{\frac{e_r+1}{e_r-1}}\right],\ \chi \equiv e_r\cosh(u) - 1$$

$$\delta^{\rm rr}\bar{a}_r^{(2)} = \frac{\nu}{\bar{a}_r^{5/2}}\Bigg\{\Bigg[-\frac{32(e_r^2-1)^2}{\chi^7} + \frac{(\frac{56}{3}\nu+\frac{274}{15})(e_r^2-1)}{\chi^6} + \frac{((-\frac{24}{5}\nu+\frac{12}{5})e_r^2 - 24\nu - \frac{9166}{75})}{\chi^5} + \frac{((-\frac{256}{15}\nu-\frac{157}{2})e_r^2+\frac{1223}{25}+\frac{104}{15}\nu)}{\chi^4(e_r^2-1)}$$

$$+\frac{((-\frac{16}{5}\nu-\frac{272}{35})e_r^4+(\frac{112}{45}\nu-\frac{28277}{1050})e_r^2-\frac{18017}{525}-\frac{344}{15}\nu)}{\chi^3(e_r^2-1)^2} + \frac{(-\frac{3823}{420}e_r^4+(-\frac{23161}{175}-\frac{1124}{45}\nu)e_r^2-\frac{16327}{525}-\frac{512}{15}\nu)}{\chi^2(e_r^2-1)^3}$$

$$+\frac{((-\frac{235733}{2100}-\frac{352}{45}\nu)e_r^4+(-\frac{191726}{525}-\frac{4796}{45}\nu)e_r^2-\frac{21107}{525}-\frac{944}{15}\nu)}{\chi(e_r^2-1)^4}\Bigg] e_r\sinh u + \frac{((-\frac{235733}{2100}-\frac{352}{45}\nu)e_r^4+(-\frac{191726}{525}-\frac{4796}{45}\nu)e_r^2-\frac{21107}{525}-\frac{944}{15}\nu)}{(e_r^2-1)^4}$$

$$+\frac{(-\frac{3823}{210}e_r^6+(-\frac{328}{5}\nu-471)e_r^4+(-\frac{3472}{15}\nu-\frac{18476}{35})e_r^2-\frac{1912}{105}-\frac{288}{5}\nu)}{(e_r^2-1)^{9/2})}AT(u)\Bigg]$$

Results

- $\delta^{\rm rr} c_\alpha = \delta^{\rm rr} c_\alpha^{(0)} + \eta^2 \delta^{\rm rr} c_\alpha^{(2)}$:

$$AT(u) \equiv \arctan\left[\sqrt{\frac{e_r+1}{e_r-1}}\tanh\left(\frac{u}{2}\right)\right] + \arctan\left[\sqrt{\frac{e_r+1}{e_r-1}}\right], \ \chi \equiv e_r\cosh(u) - 1$$

$$\delta^{\rm rr} e_r^{(2)} = \frac{\nu}{\bar{a}_r^{7/2}}\Bigg\{\Bigg[\frac{16(e_r^2-1)^3}{\chi^7} + \frac{\left(-\frac{137}{15}-\frac{28}{3}\nu\right)(e_r^2-1)^2}{\chi^6} + \frac{(\left(\frac{114}{5}+\frac{12}{5}\nu\right)e_r^2+\frac{2783}{75}+12\nu)(e_r^2-1)}{\chi^5}$$

$$+\frac{(\left(\frac{2333}{60}+\frac{68}{15}\nu\right)e_r^2-\frac{1807}{75}+\frac{8}{15}\nu)}{\chi^4} + \frac{(\left(\frac{94}{35}+\frac{28}{5}\nu\right)e_r^4+\left(\frac{223361}{6300}-\frac{296}{45}\nu\right)e_r^2-\frac{5717}{1575}+\frac{64}{5}\nu)}{\chi^3(e_r^2-1)}$$

$$+\frac{(\frac{8737}{840}e_r^4+(\frac{1042}{45}\nu+\frac{500513}{6300})e_r^2+\frac{32}{5}\nu-\frac{5627}{1575})}{\chi^2(e_r^2-1)^2} + \frac{(-\frac{144}{35}e_r^6+(\frac{488}{45}\nu+\frac{1456529}{12600})e_r^4+(\frac{3214}{45}\nu+\frac{135823}{900})e_r^2+\frac{32}{5}\nu-\frac{5627}{1575})}{\chi(e_r^2-1)^3}\Bigg)\sinh(u)$$

$$+\frac{(-\frac{144}{35}e_r^6+(\frac{488}{45}\nu+\frac{1456529}{12600})e_r^4+(\frac{3214}{45}\nu+\frac{135823}{900})e_r^2+\frac{32}{5}\nu-\frac{5627}{1575})}{e_r(e_r^2-1)^3} + \frac{(\frac{5281}{420}e_r^4+(\frac{2585}{7}+68\nu)e_r^2+\frac{14258}{105}+\frac{328}{3}\nu)e_r}{(e_r^2-1)^{7/2}}AT(u)\Bigg\}$$

Results

- $\delta^{\rm rr} c_\alpha = \delta^{\rm rr} c_\alpha^{(0)} + \eta^2 \delta^{\rm rr} c_\alpha^{(2)}$:

$\chi \equiv e_r \cosh(u) - 1$

$$\delta^{\rm rr} c_l^{(2)} = \frac{\nu\left(-25200 e_r^8 + 100800 e_r^6 - 151200 e_r^4 + 100800 e_r^2 - 25200\right)}{1575\,\chi^7\,\bar{a}_r^{7/2}\,e_r^2}$$
$$+\frac{\nu\left[105(140\nu+377)e_r^6 - 315(140\nu+377)e_r^4 + 315(140\nu+377)e_r^2 - 105(140\nu+377)\right]}{1575\,\chi^6\,\bar{a}_r^{7/2}\,e_r^2}$$
$$+\frac{\nu\left[-630(6\nu-47)e_r^6 - 168(155\nu+786)e_r^4 + 42(1510\nu+4173)e_r^2 - 84(400\nu+867)\right]}{1575\,\chi^5\,\bar{a}_r^{7/2}\,e_r^2}$$
$$+\frac{\nu\left[-630(122\nu-37)e_r^4 + 840(113\nu+87)e_r^2 - 210(86\nu+459)\right]}{1575\,\chi^4\,\bar{a}_r^{7/2}\,e_r^2}$$
$$+\frac{\nu\left[-60(147\nu-269)e_r^4 + 10(13594\nu+13369)e_r^2 - 10(1932\nu+3223)\right]}{1575\,\chi^3\,\bar{a}_r^{7/2}\,e_r^2}$$
$$+\frac{\nu\left[90(1386\nu-113)e_r^2 - 90(112\nu-1)\right]}{1575\,\chi^2\,\bar{a}_r^{7/2}\,e_r^2} + \frac{32\nu\,(28\nu-39)}{35\,\chi\,\bar{a}_r^{7/2}}$$

Results

$\chi \equiv e_r \cosh(u) - 1$

- $\delta^{\rm rr} c_\alpha = \delta^{\rm rr} c_\alpha^{(0)} + \eta^2 \delta^{\rm rr} c_\alpha^{(2)}$:

$$\delta^{\rm rr} c_\phi^{(2)} = \frac{\nu}{1575\,\bar{a}_r^{7/2}} \Bigg\{ -\frac{25200\,(e_r^2-1)^{7/2}}{\chi^7\, e_r^2} + \frac{105\,(140\nu+377)\,\left(e_r^2-1\right)^{5/2}}{\chi^6\, e_r^2} - \frac{42\,\left(e_r^2-1\right)^{3/2}\left[15(6\nu-55)e_r^2+800\nu+1734\right]}{\chi^5\, e_r^2}$$

$$-\frac{105\sqrt{e_r^2-1}\,\left[(117\nu+158)e_r^2-4(43\nu+282)\right]}{\chi^4\, e_r^2} + \frac{10\left[6(42\nu-53)e_r^4+(1750\nu+6817)e_r^2-1932\nu-4399\right]}{\chi^3\, e_r^2\sqrt{e_r^2-1}}$$

$$+\frac{90\left[41+112\nu-(334+231\nu)\,e_r^2+(83+119\nu)\,e_r^4\right]}{\chi^2\, e_r^2(e_r^2-1)^{3/2}} - \frac{7560\left(2e_r^2+13\right)}{\chi\left(e_r^2-1\right)^{5/2}} - \frac{75600\, e_r \sinh(u)}{\chi^2\left(e_r^2-1\right)^2} \arctan\left[\sqrt{\frac{e_r+1}{e_r-1}}\tanh\left(\frac{u}{2}\right)\right]$$

$$-\frac{15120\, e_r\left(2e_r^2+13\right)\sinh(u)}{\chi\left(e_r^2-1\right)^3} \arctan\left[\sqrt{\frac{e_r+1}{e_r-1}}\tanh\left(\frac{u}{2}\right)\right] - \frac{15120\left(7e_r^2+8\right)}{\left(e_r^2-1\right)^{7/2}} \arctan\left[\sqrt{\frac{e_r+1}{e_r-1}}\tanh\left(\frac{u}{2}\right)\right]^2$$

$$+\frac{15120\left(7e_r^2+8\right)}{\left(e_r^2-1\right)^{7/2}} \arctan\left[\sqrt{\frac{e_r+1}{e_r-1}}\right]^2 + \frac{15120\left(2e_r^2+13\right)}{\left(e_r^2-1\right)^3} \arctan\left[\sqrt{\frac{e_r+1}{e_r-1}}\right] \Bigg\}$$

Results

$$AT(u) \equiv \arctan\left[\sqrt{\frac{e_r+1}{e_r-1}}\tanh\left(\frac{u}{2}\right)\right] + \arctan\left[\sqrt{\frac{e_r+1}{e_r-1}}\right]$$

- $\delta^{\rm rr} n = \dfrac{\partial n}{\partial \bar{a}_r}\delta^{\rm rr}\bar{a}_r(t)$:

$$\delta^{\rm rr}\bar{n}^{(0)} = \frac{\nu}{\bar{a}_r^4}\left\{-\frac{2(37e_r^4+292e_r^2+96)}{5(e_r^2-1)^{7/2}}AT(u) - \frac{(673e_r^2+602)}{15(e_r^2-1)^3} - e_r\sinh(u)\left(\frac{(673e_r^2+602)}{15(e_r^2-1)^3\chi} + \frac{(111e_r^2+314)}{15(e_r^2-1)^2\chi^2} + \frac{2(36e_r^2+49)}{15(e_r^2-1)\chi^3} + \frac{14}{\chi^4}\right)\right\}$$

$$\begin{aligned}\delta^{\rm rr}\bar{n}^{(2)} = \frac{\nu}{\bar{a}_r^5}\Bigg\{ & e_r\sinh(u)\Bigg[\frac{48(e_r^2-1)^2}{\chi^7} + \frac{\left(-28\nu-\frac{137}{5}\right)(e_r^2-1)}{\chi^6} + \frac{\left((\frac{36}{5}\nu-\frac{18}{5})e_r^2+36\nu+\frac{4583}{25}\right)}{\chi^5} + \frac{\left((\frac{51}{4}+\frac{559}{15}\nu)e_r^2-\frac{331}{15}\nu+\frac{1581}{50}\right)}{\chi^4(e_r^2-1)} \\ & + \frac{\left((-\frac{852}{35}+\frac{44}{5}\nu)e_r^4+(-\frac{103}{45}\nu+\frac{19177}{700})e_r^2+\frac{1303}{45}\nu+\frac{35167}{350}\right)}{\chi^3(e_r^2-1)^2} + \frac{\left((\frac{37}{6}\nu-\frac{11717}{280})e_r^4+(\frac{16979}{175}+\frac{4387}{90}\nu)e_r^2+\frac{1519}{45}\nu+\frac{71277}{350}\right)}{\chi^2(e_r^2-1)^3} \\ & + \frac{\left((-\frac{235367}{1400}+\frac{4421}{90}\nu)e_r^4+(\frac{14033}{90}\nu+\frac{204151}{350})e_r^2+\frac{126457}{350}+\frac{2743}{45}\nu\right)}{\chi(e_r^2-1)^4}\Bigg] + \frac{\left((-\frac{235367}{1400}+\frac{4421}{90}\nu)e_r^4+(\frac{14033}{90}\nu+\frac{204151}{350})e_r^2+\frac{126457}{350}+\frac{2743}{45}\nu\right)}{(e_r^2-1)^4} \\ & + \frac{\left((-\frac{11717}{140}+\frac{37}{3}\nu)e_r^6+(\frac{917}{5}\nu-\frac{117}{2})e_r^4+(\frac{4228}{15}\nu+\frac{48294}{35})e_r^2+\frac{272}{5}\nu+\frac{11036}{35}\right)}{(e_r^2-1)^{9/2}}AT(u)\Bigg\}\end{aligned}$$

Results

- $\delta^{\rm rr} K = \dfrac{\partial K}{\partial \bar{a}_r}\delta^{\rm rr}\bar{a}_r(t) + \dfrac{\partial K}{\partial e_r}\delta^{\rm rr} e_r(t)$:

$$AT(u) \equiv \arctan\left[\sqrt{\frac{e_r+1}{e_r-1}}\tanh\left(\frac{u}{2}\right)\right] + \arctan\left[\sqrt{\frac{e_r+1}{e_r-1}}\right]$$

$$\delta^{\rm rr} K^{(0)} = 0$$

$$\delta^{\rm rr} K^{(2)} = \frac{\nu}{\bar{a}_r^{7/2}}\left[\frac{48(7e_r^2+8)}{5(e_r^2-1)^{7/2}}AT(u) + \frac{24(2e_r^2+13)}{5(e_r^2-1)^3} + \frac{24 e_r \sinh(u)}{(e_r^2-1)^2\chi^2} + \frac{24 e_r(2e_r^2+13)\sinh(u)}{5\chi(e_r^2-1)^3}\right]$$

Results

$$AT(u) \equiv \arctan\left[\sqrt{\frac{e_r+1}{e_r-1}}\tanh\left(\frac{u}{2}\right)\right] + \arctan\left[\sqrt{\frac{e_r+1}{e_r-1}}\right]$$

- $\delta^{\rm rr} l = \int_{-\infty}^{t} \delta^{\rm rr}\bar{n}(t')dt' + \delta^{\rm rr}c_l(t) \equiv \delta^{\rm rr}\tilde{l}(t) + \delta^{\rm rr}c_l(t)$:

$$\delta^{\rm rr}\tilde{l}^{(0)} = \frac{\nu}{\bar{a}_r^{5/2}}\left\{\left[\ln(\chi) - \chi - 1 - e_r\sinh(u) + u + \ln(2) - \ln(e_r)\right]B + \frac{2(49+36e_r^2)}{15\chi(e_r^2-1)} + \frac{7}{\chi^2}\right.$$
$$\left. + \left[-e_r\sinh(u)AT(u) + \frac{i}{2}\left(\mathrm{Li}_2(z) - \mathrm{Li}_2(\bar{z})\right) - \frac{\sqrt{e_r^2-1}}{2}\right]C\right\}$$

with

$$z = e^{v}\frac{e_r}{(1+i\sqrt{e_r^2-1})} = e^{v - i\arctan(\sqrt{e_r^2-1})} = -\,e^{v+2i\arctan(\alpha)}$$ due to $\arctan(\sqrt{e_r^2-1}) = -\,2\arctan\left(\sqrt{\frac{e_r+1}{e_r-1}}\right) + \pi,$

$$B = \frac{602 + 673\,e_r^2}{15(e_r^2-1)^3},\quad C = \frac{2(96 + 292\,e_r^2 + 37\,e_r^4)}{5(e_r^2-1)^{7/2}}$$

Results

$$AT(u) \equiv \arctan\left[\sqrt{\frac{e_r+1}{e_r-1}}\tanh\left(\frac{u}{2}\right)\right] + \arctan\left[\sqrt{\frac{e_r+1}{e_r-1}}\right]$$

- $\delta^{\rm rr} l = \int_{-\infty}^{t} \delta^{\rm rr}\bar{n}(t')dt' + \delta^{\rm rr}c_l(t) \equiv \delta^{\rm rr}\tilde{l}(t) + \delta^{\rm rr}c_l(t)$:

$$\delta^{\rm rr}\tilde{l}^{(2)} = \frac{\nu}{\bar{a}_r^{7/2}}\Bigg\{-\frac{i}{2}\left(\mathrm{Li}_2(z)-\mathrm{Li}_2(\bar{z})\right)D + \left[e_r\sinh(u)AT(u) + \frac{\sqrt{e_r^2-1}}{2}\right]E + \left(\chi+1+e_r\sinh(u)\right)F - \left(u+\ln(\chi)+\ln(2)-\ln(e_r)\right)G$$

$$-\frac{e_r^4\left(\frac{68\nu}{5}-\frac{768}{35}\right)+e_r^2\left(\frac{949\nu}{90}-\frac{2809}{2100}\right)+\frac{2422\nu}{45}+\frac{4717}{350}}{(e_r^2-1)^2\chi} - \frac{e_r^2\left(\frac{877\nu}{30}+\frac{11}{40}\right)-\frac{197\nu}{15}-\frac{227}{300}}{(e_r^2-1)\chi^2} - \frac{e_r^2\left(\frac{12\nu}{5}-\frac{6}{5}\right)+19\nu+\frac{1061}{25}}{\chi^3}$$

$$+\frac{(e_r^2-1)(137+140\nu)}{20\chi^4} - \frac{48(e_r^2-1)^2}{5\chi^5}\Bigg\}$$

with

$$z = e^{v}\frac{e_r}{(1+i\sqrt{e_r^2-1})} = e^{v-i\arctan(\sqrt{e_r^2-1})} = -e^{v+2i\arctan(\alpha)}$$ due to $\arctan(\sqrt{e_r^2-1}) = -2\arctan\left(\sqrt{\frac{e_r+1}{e_r-1}}\right)+\pi$,

And D, E, F, G functions of ν, e_r (further details in the draft).

Fourier space analysis

- Fourier transform of $f(t)$:

$$\hat{f}(\omega) = \int_{-\infty}^{+\infty} dt\, e^{i\omega t} f(t) \longrightarrow \hat{f}(\omega) = \int_{-\infty}^{+\infty} du\, \frac{dt}{du} e^{i\omega\, t(u)} f(t(u)).$$

- For $t \to \infty$, $f(t(v)) \to [f]$ (constant) $\sim$ memory.

- Easier to perform FT taking the large-e_r (eq. large-j) expansion of integrand, choosing:

$$T = \sinh(u) \to du = \frac{dT}{\sqrt{1+T^2}}$$

$$\to \hat{f}(\omega) = \int_{-\infty}^{+\infty} \frac{dT}{\sqrt{1+T^2}} \frac{dt}{du} e^{i\omega\, t(u)} f(t(u)) \Big|_{u=\mathrm{arcsinhT}} \quad \text{with} \quad \tilde{\omega} = \frac{\omega j}{p_\infty^2}$$

Fourier space analysis

- Fourier transforms:

$$\hat{\delta}^{\mathrm{rr}}\bar{a}_r(\tilde{\omega}) = \nu \left[\frac{32}{5}\tilde{\omega}K_0(\tilde{\omega})\frac{i}{p_\infty j} + \left(\frac{74}{15\tilde{\omega}} + \frac{118}{15}\tilde{\omega} + \frac{74}{15}\right)\frac{i\pi e^{-\tilde{\omega}}}{p_\infty^2 j^2} \right.$$

$$\left. + \left(\frac{196}{9}i\tilde{\omega}K_0(\tilde{\omega}) + \frac{376}{15}i\tilde{\omega}^2K_1(\tilde{\omega}) - \frac{74}{15}I_1^{\mathrm{at}}(\tilde{\omega}) + \frac{3136}{45}iK_1(\tilde{\omega}) - \frac{74}{15}K_0(\tilde{\omega})\pi\right)\frac{1}{p_\infty^3 j^3}\right]$$

$$\hat{\delta}^{\mathrm{rr}}e_r(\tilde{\omega}) = \nu \left[(-\frac{16}{5}K_1(\tilde{\omega}) - \frac{16}{5}\tilde{\omega}K_0(\tilde{\omega}))ip_\infty^2 \right.$$

$$+[\frac{16}{5}K_0(\tilde{\omega}) + (-\frac{121}{15\tilde{\omega}} - \frac{59}{15}\tilde{\omega} - \frac{97}{15})i\pi e^{-\tilde{\omega}}]\frac{p_\infty}{j}$$

$$\left. + \left((-\frac{778}{45}i\tilde{\omega} + \frac{121}{15}\pi)K_0(\tilde{\omega}) + \frac{121}{15}I_1^{\mathrm{at}}(\tilde{\omega}) + (-\frac{3152}{45} - \frac{188}{15}\tilde{\omega}^2)iK_1(\tilde{\omega})\right)\frac{1}{j^2}\right]$$

Fourier space analysis

- Fourier transforms:

$$\hat{\delta}^{\mathrm{rr}}\bar{n}(\tilde{\omega}) = \nu\left[-\frac{48}{5}i\tilde{\omega}K_0(\tilde{\omega})\frac{p_\infty^4}{j} + \left(-\frac{37}{5} - \frac{59}{5}\tilde{\omega} - \frac{37}{5\tilde{\omega}}\right)\frac{i\pi e^{-\tilde{\omega}}p_\infty^3}{j^2}\right.$$

$$\left. + \left(-\frac{98}{3}i\tilde{\omega}K_0(\tilde{\omega}) - \frac{188}{5}i\tilde{\omega}^2K_1(\tilde{\omega}) + \frac{37}{5}I_1^{\mathrm{at}}(\tilde{\omega}) - \frac{1568}{15}iK_1(\tilde{\omega}) + \frac{37}{5}K_0(\tilde{\omega})\pi\right)\frac{p_\infty^2}{j^3}\right]$$

$$\hat{\delta}^{\mathrm{rr}}K(\tilde{\omega}) = \nu\eta^2\left[\frac{96i}{5j^3}p_\infty K_1(\tilde{\omega}) + \frac{\frac{168}{5\tilde{\omega}}i\pi e^{-\tilde{\omega}} + 24i\pi e^{-\tilde{\omega}} - \frac{96}{5}K_0(\tilde{\omega})}{j^4}\right.$$

$$\left. + \frac{(-\frac{168}{5}\pi - \frac{336}{5}i\tilde{\omega})K_0(\tilde{\omega}) - \frac{168}{5}I_1^{\mathrm{at}}(\tilde{\omega}) + \frac{48}{5}iK_1(\tilde{\omega})}{p_\infty j^5}\right]$$