Gravitational radiation reaction for compact binary systems at 4.5 post-Newtonian order

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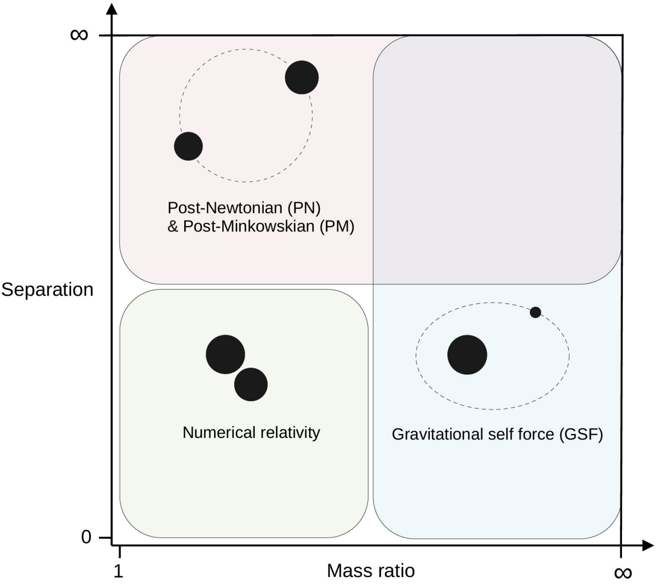
Based on Blanchet, Faye, Seraille & Trestini arxiv 2601.06743

The two body problem in General Relativity



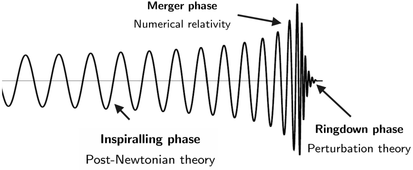
PN : $\frac{v}{c} \ll 1$

PM : $G \ll 1$

GSF : $\frac{m}{M} \ll 1$

The radiation reaction in post-Newtonian

We are interested in post-Newtonian expansion
PN EOMs = Newton's law + $\frac{v}{c}$ corrections



The radiation reaction is associated with the **loss of energy** of a gravitational system and the emission of **gravitational waves**.
It is possible to decompose of the equations of motion in a **conservative** and a **dissipative** part.

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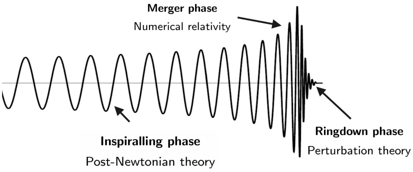

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Several expressions have been obtained in different coordinate systems at the leading PN order [Burke & Thorne 1970], [Damour 1982]

$$a_1^i\Big|_{\text{BT}} = -\frac{2G}{5c^5}\, y_1^j \partial_t^5 \mathrm{I}^i_j \quad \text{where} \quad \mathrm{I}_{ij} \quad \text{is the quadrupole}$$

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The radiation reaction is fundamentally **coordinate dependent**.

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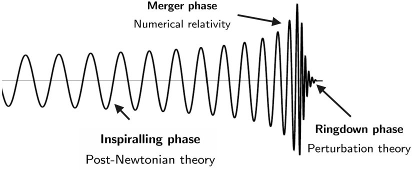

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State of the art | BT coordinates : 4.5PN | harmonic coordinates : 3.5PN |

Coordinate transformation, Burke -Thorne to harmonic

We look for a **coordinate shift** from BT to harmonic φ^α such that $x'^\alpha = x^\alpha + \varphi^\alpha(x)$. We search for a solution under the form

$$h^{\alpha\beta}_{\text{harm}}(x) = h^{\alpha\beta}_{\text{BT}}(x) + \partial\varphi^{\alpha\beta} + \Omega^{\alpha\beta}[\varphi, h_{\text{BT}}]$$

We decompose φ^α in **Post-Minkowskian** $\varphi^\alpha = G\varphi^\alpha_{(1)} + G^2\varphi^\alpha_{(2)} + \mathcal{O}(G^3)$
The recurrence relation that can be solved order by order

$$h^{\alpha\beta}_{(1)} = h^{\alpha\beta}_{\text{BT}\,(1)} + \partial\varphi^{\alpha\beta}_{(1)} \qquad \text{(1PM relation)}$$

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Non linear transformation (only 2PM is relevant for 4.5PN) : The harmonicity condition implies $\Box\varphi^\alpha_{(2)} = -\partial_\beta\Omega^{\alpha\beta}_{(2)}$ which can be solved as

$$\boxed{\varphi^\alpha_{(2)} = \varphi^\alpha_{(2)\,\text{part}} + \varphi^\alpha_{(2)\,\text{hom}}}$$

- homogeneous solution : $\varphi^\alpha_{(2)\,\text{hom}} = 0$ because of its **PN parity**
- particular solution : $\varphi^\alpha_{(2)\,\text{part}}$ explicitly given as a function of the **multipolar moments**

Computation of the multipolar moments

With the previous steps we have obtained the expression of the harmonic acceleration as a function of **6** sets of multipole moments.

$$\boxed{a^i_{1\,\mathrm{harm}}(\mathrm{I}, \mathrm{J}, \mathrm{W}, \mathrm{X}, \mathrm{Y}, \mathrm{Z})}$$

The **source moments** I and J are known up to a sufficient PN order.

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The **gauge moments** required for the explicit computation of the acceleration are

- **Newtonian** order: $W_{ij}, X_{ij}, Y_{ijk}, Z_{ij}$
- 1 **PN**: W_i, X, X_i, Y_{ij}, Z_i
- 2 **PN**: W, Y_i

For instance: $$\mathrm{Y}_L \approx \underset{B=0}{\mathrm{FP}} \int \mathrm{d}^d\mathbf{x}\, \tilde{r}^B \left\{ C_1(\ell, d) x \Sigma + C_2(\ell, d) x \dot{\Sigma} + \frac{1}{c^2} C_4(\ell, d) x \ddot{\Sigma} \right\}$$

where Σ depends of the **matter** and **energy** distribution.

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Our primary goal is to compute these integrals. For this we need

- Careful computation in the sense of the distribution theory
- A regularization which is necessary as we model compact bodies by point particles leading to divergences. For this we use **Dimensional regularization** computed with the **Hadamard finite part**.

RR acceleration in the harmonic coordinate system

Consistency with the literature up to 4PN

$$a^i_{\mathrm{RR1\ 2.5PN}} = \frac{G^2 m_1 m_2 {v_{12}}^i}{c^5 r_{12}^4}\left[\left(\frac{8}{5}m_1 - \frac{32}{5}m_2\right)G - \frac{4}{5}r_{12}v_{12}^2\right] + \frac{G^2 m_1 m_2 n_{12}^i}{c^5 r_{12}^4}\left[G\left(-\frac{24}{5}m_1(n_{12}v_{12}) + \frac{208}{15}m_2(n_{12}v_{12})\right) + \frac{12}{5}r_{12}(n_{12}v_{12})v_{12}^2\right]$$

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- The **pole is a coordinate effect**, it vanishes in the center of mass frame. It also explains why it has been so difficult to obtain directly the harmonic acceleration.
- The harmonic acceleration is **significantly simpler** than the Burke Thorne acceleration ($\approx$ an order of magnitude).
- There is also a **non-local contribution in the center of mass frame** at 4.5PN due to the recoil of the center of mass.

Flux balance laws in a general frame

We have **proved the flux-balance equations** associated with the different conserved quantities

- Energy E
- Angular momentum J^i
- Linear momentum P^i
- Center of mass position G^i

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\mathcal{F}_E \qquad \frac{\mathrm{d}J^i}{\mathrm{d}t} = -\mathcal{F}^i_{\boldsymbol{J}}$$
$$\frac{\mathrm{d}P^i}{\mathrm{d}t} = -\mathcal{F}^i_{\boldsymbol{P}} \qquad \frac{\mathrm{d}G^i}{\mathrm{d}t} = P^i - \mathcal{F}^i_{\boldsymbol{G}}$$

The fluxes at infinity being observable they have to be **pole-free** even if the conserved quantities include poles.

the flux balance equations are used in the literature at high PN order but are, in general, **not proven from first principles**. This computation is a proof at **2PN relative order**.

Consistency checks

- **Manifest Lorentz invariance in harmonic coordinates**

$$\partial'_\nu h'^{\mu\nu} = \Lambda^\alpha_\nu \Lambda^\mu_\beta \Lambda^\nu_\gamma \partial_\alpha h^{\beta\gamma} = \Lambda^\mu_\beta \partial_\alpha h^{\beta\alpha} = 0$$

We compute the difference between the boosted and unboosted acceleration (taken at the same time) at 4.5 PN order. After replacement and using the dimensional identities we find that the acceleration is **manifestly Lorentz invariant** $\boxed{\delta_\Lambda \boldsymbol{a}_1 = 0}$

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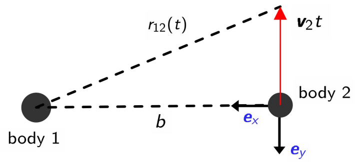
- **Consistency with the 2PM equations of motion**

$$\frac{\mathrm{d}u_a{}^\alpha(\tau_a)}{d\tau_a} = G[...] + G^2[...] + \mathcal{O}(G^3)$$ [Bini, Damour & Geralico 2024]



Developing in PN they obtain :

$A_x^{4.5\text{PN}\,2\text{PM}}$ & $A_y^{4.5\text{PN}\,2\text{PM}}$

Projecting our result truncated at 2PM in the $\boldsymbol{e}_x$, $\boldsymbol{e}_y$ basis **we found complete agreement with the truncated 4.5PN expression**.

Summary and Prospects

We have obtained the acceleration of compact bodies in the two-body problem of General Relativity **without spins** for **general orbits** in a **general frame** at the 4.5PN order in the (uniquely defined) **harmonic coordinate** system.

This new expression is **simpler** than the Burke-Thorne acceleration due to its manifest **Lorentz-invariant formulation** and is valid in the **whole space**.

As a cross check the **balance equations** are **satisfied** and this result is **compatible with the 2PM expression**.

This result can be useful for **comparisons with other approximation methods**, such as the Gravitational self force or the post-Minkowskian approach and could enter in **EOB** models. With this result we can also compute the **black holes trajectories** at 4.5PN order.

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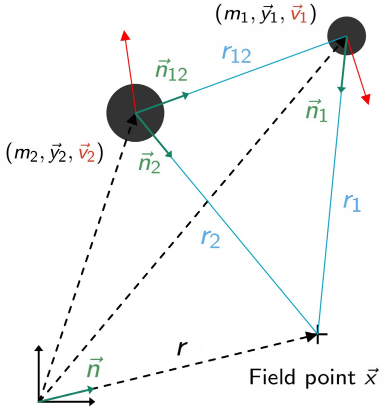
Thank you for your attention !

Additional slides

Post-Newtonian dynamics



Decomposition of the equations of motion in a conservative and a dissipative part

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The Burke -Thorne coordinates [Burke & Thorne 1970], [Blanchet 1993]

$$h^{\alpha\beta}_{\mathrm{BT}\,(1)}(\mathrm{M}_L, \mathrm{S}_L) = h^{\alpha\beta}_{\mathrm{can}\,(1)}(\mathrm{I}_L, \mathrm{J}_L) - \partial\xi^{\alpha\beta}_{(1)}$$

$$\mathrm{I}_L = \mathrm{M}_L + \mathcal{O}(1/c^5) \text{ and } \mathrm{J}_L = \mathrm{S}_L + \mathcal{O}(1/c^5)$$

$$\xi^0_{(1)} = -\frac{2}{c}\sum_{\ell\geqslant 2}\frac{(-)^\ell}{\ell!}\frac{2\ell+1}{\ell(\ell-1)}\partial_L\left\{\frac{\mathrm{I}_L^{(-1)}(t-r/c) - \mathrm{I}_L^{(-1)}(t+r/c)}{2r}\right\}$$

$$\begin{aligned}\xi^i_{(1)} = &\; 2\sum_{\ell\geqslant 2}\frac{(-)^\ell}{\ell!}\frac{(2\ell+1)(2\ell+3)}{\ell(\ell-1)}\partial_{iL}\left\{\frac{\mathrm{I}_L^{(-2)}(t-r/c) - \mathrm{I}_L^{(-2)}(t+r/c)}{2r}\right\}\\ &- \frac{4}{c^2}\sum_{\ell\geqslant 2}\frac{(-)^\ell}{\ell!}\frac{2\ell+1}{\ell-1}\partial_{L-1}\left\{\frac{\mathrm{I}_{iL-1}(t-r/c) - \mathrm{I}_{iL-1}(t+r/c)}{2r}\right\}\\ &- \frac{4}{c^2}\sum_{\ell\geqslant 2}\frac{(-)^\ell \ell}{(\ell+1)!}\frac{2\ell+1}{\ell-1}\epsilon_{iab}\partial_{aL-1}\left\{\frac{\mathrm{J}_{bL-1}^{(-1)}(t-r/c) - \mathrm{J}_{bL-1}^{(-1)}(t+r/c)}{2r}\right\}\end{aligned}$$

The equations of motion for compact binaries are known up to 4.5PN order in the Burke-Thorne gauge [Blanchet, Faye & Trestini 2024].

$$\boxed{a^i_{1\,\mathrm{BT}} = -\frac{2Gy_1^a}{5c^5}\partial_t^5 M^i_a + \frac{1}{c^7}[\ldots] + \frac{1}{c^9}[\ldots]}$$

Unicity of the harmonic coordinate system

Harmonic condition : $\partial_\alpha h^{\alpha\beta} = 0$

Shortcut notation : $\partial\varphi^{\alpha\beta} \equiv \partial^\alpha\varphi^\beta + \partial^\beta\varphi^\alpha - \eta^{\alpha\beta}\partial_\gamma\varphi^\gamma$

The harmonic coordinates are uniquely defined under 2 conditions

- The metric is generated by a **regular and isolated source**
- No incoming radiation at past null infinity $\mathcal{J}^-$: $r \to +\infty$ with $t + r/c = \text{const}$

If we consider two harmonic metrics that differ by a gauge transformation

$$h'^{\alpha\beta} = h^{\alpha\beta} + \partial\varphi^{\alpha\beta}$$

then $\partial_\beta\partial\varphi^{\alpha\beta} = \Box\varphi^\alpha = 0$. Therefore, for a regular source at any field point $(\mathbf{x}, t)$ (outside or inside the source) the **Fresnel-Kirchhoff formula** writes

$$\varphi^\alpha(\mathbf{x}, t) = \int \frac{\mathrm{d}\Omega'}{4\pi} \left[\left(\frac{\partial}{\partial r} + \frac{\partial}{c\partial t}\right)(r\varphi^\alpha)\right]\left(\mathbf{x}', t - \frac{|\mathbf{x} - \mathbf{x}'|}{c}\right)$$

In the absence of incoming radiations :

$$\lim_{\substack{r\to+\infty \\ t+r/c=\text{const}}} \left(\frac{\partial}{\partial r} + \frac{\partial}{c\partial t}\right)(r\varphi^\alpha) = 0$$

Finally,

$$\boxed{\varphi^\alpha = 0} \quad \text{and} \quad \boxed{h'^{\alpha\beta} = h^{\alpha\beta}} \quad \text{(up to the Poincare invariance)}$$

Post-Minkowskian metric in harmonic gauge in the exterior zone [Blanchet & Damour 1986]

PN convention for the metric $\quad h^{\alpha\beta} = \sqrt{-g}\, g^{\alpha\beta} - \eta^{\alpha\beta}$

$$\mathcal{M}(h^{\alpha\beta}) \equiv h^{\alpha\beta}_{\text{MPM}} = \sum_{n=1}^{+\infty} G^n\, h^{\alpha\beta}_{(n)}\,, \qquad h^{\alpha\beta}_{(1)} = h^{\alpha\beta}_{\text{can}\,(1)} + \partial^\alpha \zeta^\beta_{(1)} + \partial^\beta \zeta^\alpha_{(1)} - \eta^{\alpha\beta}\partial_\gamma \zeta^\gamma_{(1)}$$

The form of the metric is entirely determined in **vacuum** by the **harmonicity condition** $\partial_\alpha h^{\alpha\beta} = 0$ and the **absence of incoming radiation**.

The canonical part is described by 2 sets of multipolar moments :

$$h_{\text{can}\,(1)} \approx \sum_\ell \partial_L \left(\frac{1}{r}\mathrm{I}_L\right) + \sum_\ell \left(\frac{1}{r}\mathrm{J}_L\right)$$

The gauge freedom is parameterized by 4 sets of additional multipole moments

$$\zeta^0_{(1)} \approx \sum_\ell \partial_L \left(\frac{1}{r}\mathrm{W}_L\right)\,,$$

$$\zeta^i_{(1)} \approx \sum_\ell \partial_L \left(\frac{1}{r}\mathrm{W}_L\right) + \sum_\ell \left(\frac{1}{r}\mathrm{X}_L\right) + \sum_\ell \partial_L \left(\frac{1}{r}\mathrm{Y}_L\right) + \sum_\ell \left(\frac{1}{r}\mathrm{Z}_L\right)$$

Matching with the PN expansion - MPM-PN

$$\mathrm{I}_L(u) = \underset{B=0}{\mathrm{FP}} \int \mathrm{d}^3\mathbf{x}\, \widetilde{r}^B \int_{-1}^{1} \mathrm{d}z \Big\{ \delta_\ell(z)\hat{x}_L \Sigma - \frac{4(2\ell+1)}{c^2(\ell+1)(2\ell+3)} \delta_{\ell+1}(z)\hat{x}_{iL}\Sigma_i^{(1)} + \frac{2(2\ell+1)}{c^4(\ell+1)(\ell+2)(2\ell+5)} \delta_{\ell+2}(z)\hat{x}_{ijL}\Sigma_{ij}^{(2)} \Big\}$$

$$\Sigma = \frac{\overline{\tau}^{00} + \overline{\tau}^{ii}}{c^2} \qquad \Sigma_i = \frac{\overline{\tau}^{0i}}{c} \qquad \Sigma_{ij} = \overline{\tau}^{ij}$$

The post-Newtonian field in the near zone of a post-Newtonian source with no incoming radiation writes

$$\overline{h}^{\alpha\beta} = \frac{16\pi G}{c^4} \Box^{-1}_{\mathrm{ret}} \left[\overline{\tau}^{\alpha\beta} \right] - \frac{4G}{c^4} \sum_{\ell=0}^{+\infty} \frac{(-)^\ell}{\ell!} \hat{\partial}_L \left\{ \frac{\mathcal{R}_L^{\alpha\beta}(t-r/c) - \mathcal{R}_L^{\alpha\beta}(t+r/c)}{2r} \right\}$$

$$\text{Matching condition :} \quad \overline{\mathcal{M}(h^{\alpha\beta})} \equiv \mathcal{M}(\overline{h}^{\alpha\beta})$$

The matching procedure fully determines $\mathcal{R}_L^{\alpha\beta}$ and the moments I_L,J_L,W_L,X_L,Y_L, Z_L.

It also guarantees the harmonicity condition $\partial_\alpha \overline{h}^{\alpha\beta} = 0$ in the interior zone.

Change of coordinate, from Burke -Thorne to Harmonic

[Blanchet, Faye & Larrouturou 2022] , [Trestini, Larrouturou & Blanchet 2023]

$$h^{\alpha\beta}_{\text{harm}}(x') = \frac{1}{|J|}\frac{\partial x'^{\alpha}}{\partial x^{\gamma}}\frac{\partial x'^{\beta}}{\partial x^{\lambda}}(h^{\gamma\lambda}_{\text{BT}}(x) + \eta^{\gamma\lambda}) - \eta^{\alpha\beta} \quad \text{where} \quad J \equiv det(\partial x'/\partial x)$$

We look for a coordinate shift φ^α such that $x'^{\alpha} = x^\alpha + \varphi^\alpha(x)$.

$$h^{\alpha\beta}_{\text{harm}}(x') = h^{\alpha\beta}_{\text{harm}}(x+\varphi(x)) = \sum_{n\geqslant 2} \varphi^{\lambda_1}...\varphi^{\lambda_n}\partial_{\lambda_1}...\partial_{\lambda_n} h^{\alpha\beta}_{\text{harm}}(x)/n!$$

We search for a solution under the form

$$h^{\alpha\beta}_{\text{harm}}(x) = h^{\alpha\beta}_{\text{BT}}(x) + \partial\varphi^{\alpha\beta} + \Omega^{\alpha\beta}[\varphi, h_{\text{BT}}]$$

Using the identity $\partial_\beta\partial\varphi^{\alpha\beta} = \Box\varphi^\alpha$ we obtain $\Box\varphi^\alpha + \partial_\beta\Omega^{\alpha\beta} = 0$
We can decompose the coordinate transformation and the metric in PM

$$\begin{aligned}
\varphi^\alpha &= G\varphi^\alpha_{(1)} + G^2\varphi^\alpha_{(2)} + \mathcal{O}(G^3) \\
h^{\alpha\beta}_{(1)} &= h^{\alpha\beta}_{\text{BT}\,(1)} + \partial\varphi^{\alpha\beta}_{(1)} \\
h^{\alpha\beta}_{(2)} &= h^{\alpha\beta}_{\text{BT}\,(2)} + \partial\varphi^{\alpha\beta}_{(2)} + \Omega^{\alpha\beta}_{(2)}[\varphi_{(1)}h_{(1)}] \\
&\cdots
\end{aligned}$$

Linear coordinate transformation

At linear order in G we can undo the transformation

$$h^{\alpha\beta}_{\text{BT}}(\mathrm{M}_L, \mathrm{S}_L) \underset{\nwarrow}{\rightarrow} h^{\alpha\beta}_{\text{can}}(\mathrm{I}_L, \mathrm{J}_L) \overset{\partial\zeta^{\alpha\beta} \searrow}{\rightarrow} h^{\alpha\beta}_{\text{harm}}(\mathrm{I}_L, \mathrm{J}_L, \mathrm{W}_L, \mathrm{X}_L, \mathrm{Y}_L, \mathrm{Z}_L)$$

By construction of BT, $h^{\alpha\beta}_{\text{BT}\,(1)}(\mathrm{M}_L, \mathrm{S}_L) = h^{\alpha\beta}_{\text{can}\,(1)}(\mathrm{I}_L, \mathrm{J}_L) - \partial\xi^{\alpha\beta}_{(1)}$

Where

$$\xi^0_{(1)} \approx \sum_\ell \partial_L \left\{ \frac{\mathrm{I}_L(t - r/c) - \mathrm{I}_L(t + r/c)}{2r} \right\}$$

$$\xi^i_{(1)} \approx \sum_\ell \partial_L \left\{ \frac{\mathrm{I}_L(t - r/c) - \mathrm{I}_L(t + r/c)}{2r} \right\} + \sum_\ell \partial_L \left\{ \frac{\mathrm{J}_L(t - r/c) - \mathrm{J}_L(t + r/c)}{2r} \right\}$$

Non linear corrections to the gauge transformation

We decompose $\Delta^{\alpha}_{(2)}\left(\frac{r}{r_0}\right)^B \to \hat{n}_L \sum_k \left(\frac{r}{r_0}\right)^B \frac{F^{\alpha}_L(t)}{r^k}$

- homogeneous solution

$$\varphi^{\alpha}_{(2)\,\mathrm{hom}} = \hat{\partial}_L \left\{ \frac{G(t-r/c) - G(t+r/c)}{r} \right\}$$

We can show that

$$G(t) = \underset{B\to 0}{FP}\, G_B(t) \quad \text{where} \quad G_B(t) = \frac{C(B,k,l)}{c^{k-l-3}} \int_{-\infty}^{u} d\tau F^{(k-l-2)}(\tau) \left(\frac{u-\tau}{r_0}\right)^B$$

We obtain that no term has the PN parity to contribute $\to$ $\boxed{\varphi^{\alpha}_{(2)\,\mathrm{hom}} = 0}$

- particular solution $\boxed{\varphi^{\alpha}_{(2)\,\mathrm{part}} = \sum_{i=0}^{+\infty} \left(\frac{\partial}{c\partial_t}\right)^{2i} \Delta^{-1-i}\left[\Delta^{\alpha}_{(2)}\right]}$

The non linear gauge transformation writes

$$\varphi^0_{(2)} = -\frac{8G}{c^8}(m_1 r_1 + m_2 r_2)\partial_t^3 \mathrm{W}$$

$$\varphi^i_{(2)} = -\frac{G r_1 m_1}{c^9}\left(8\partial_t^3 \mathrm{Y}^i - 4 v_1^a \partial_t^4 {\mathrm{I}'}^i_{\ a} - 2x^a \partial_t^5 {\mathrm{I}'}^i_{\ a} + n_1^a r_1 \partial_t^5 {\mathrm{I}'}^i_{\ a}\right) + (1 \leftrightarrow 2)$$

Total shift and harmonic acceleration as a function of the multipolar moments

The total shift is given by the sum $\boxed{\varphi^\alpha = \zeta^\alpha + \xi^\alpha + \varphi^\alpha_{(2)}}$

$$\vec{y'}(t') = \vec{y}(t) + \varphi^i(\vec{y}(t), t) \qquad t' = t + \varphi^0(\vec{y}(t), t)$$

The shift $\vec{\psi}_p$ associated to the point p is defined such that

$$\vec{y'}_p(t) = \vec{y}_p(t) + \vec{\psi}_p(t) \quad \text{which implies} \quad \vec{a'}_p(t) = \vec{a}_p(t) + \ddot{\vec{\psi}}_p(t)$$

$$\text{where} \quad \boxed{\psi^i_p = \varphi^i(\vec{y_p}) - \frac{v^i_p}{c}\varphi^0(\vec{y_p}) + \mathcal{O}(\psi^2)}$$

$$\delta_\psi a^i_{1,2} = a^i_{1,2\,\mathrm{BT}}[\vec{y_1}, \vec{y_2}, \vec{v_1}, \vec{v_2}] - a^i_{1,2\,\mathrm{harm}}[\vec{y_1}, \vec{y_2}, \vec{v_1}, \vec{v_2}] \quad \text{is the shifted acceleration}$$

$$= \ddot{\psi}^i_1 - \psi^j_1 \frac{\partial a^i_{1,2\,\mathrm{BT}}}{\partial y^j_1} - \psi^j_2 \frac{\partial a^i_{1,2\,\mathrm{BT}}}{\partial y^j_2} - \dot{\psi}^j_1 \frac{\partial a^i_{1,2\,\mathrm{BT}}}{\partial v^j_1} - \dot{\psi}^j_2 \frac{\partial a^i_{1,2\,\mathrm{BT}}}{\partial v^j_2}$$

We obtain
$$a^i_{1\,\mathrm{harm}} = \frac{G}{c^5}\left(4v^i_1\partial^3_t\mathrm{W} - 4\partial^3_t\mathrm{Y}^i + 2v^a_1\partial^4_t\mathrm{I}^i{}_a + \frac{3y^a_1}{5}\partial^5_t\mathrm{I}^i{}_a\right) - \frac{G^2 m_2 n^i_{12}}{c^5 (r_{12})^2}\left(3n^a_{12}n^b_{12}\partial^3_t\mathrm{I}_{ab} + 8\partial^2_t\mathrm{W}\right) + \frac{1}{c^7}[\ldots] + \frac{1}{c^9}[\ldots]$$

The Hadamard partie finie regularization

A regularization is necessary as we model compact bodies by point particles leading to divergences. We are interested in functions $F^{(d)}(\mathbf{x})$, which are smooth except at the source points $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$ around which it admits a power-like singular expansion. When $r_1 \equiv |\mathbf{x} - \boldsymbol{y}_1| \to 0$ and for any $N \in \mathbb{N}$, we have

$$F^{(d)}(\mathbf{x}) = \sum_{p_0 \leqslant p \leqslant N} \sum_{q_0 \leqslant q \leqslant q_1} r_1^{p+q\varepsilon} \ell_0^{-q\varepsilon} \underset{1}{f}{}^{(\varepsilon)}_{p,q}((\mathbf{x} - \boldsymbol{y}_1)/r_1) + o(r_1^N)$$

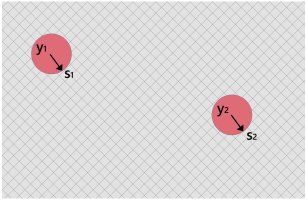
where ℓ_0 is a scale associated to dimensional regularization and $\epsilon = d - 3$.
$\to$ The 3D function is obtained taking $\epsilon = 0$.

The Hadamard partie finie (Pf) of the integral in 3 dimensions is an **analytic continuation** of two complex parameters α and β and two regularization scales s_1 and s_2.

$$H \equiv \underset{s_1, s_2}{\mathrm{Pf}} \int \mathrm{d}^3\mathbf{x}\, F(\mathbf{x}) = \underset{\substack{\alpha \to 0 \\ \beta \to 0}}{\mathrm{FP}} \int \mathrm{d}^3\mathbf{x} \left(\frac{r_1}{s_1}\right)^{\alpha} \left(\frac{r_2}{s_2}\right)^{\beta} F$$

$$= \lim_{s \to 0} \left\{ \int_{\mathbb{R}^3 \setminus B_1(s) \setminus B_2(s)} \mathrm{d}^3\mathbf{x}\, F + 4\pi \sum_{p+3 \leqslant -1} \frac{s^{p+3}}{p+3} \left(\frac{F}{r_1^p}\right)_1 \right.$$

$$\left. + 4\pi \ln\left(\frac{s}{s_1}\right) \left(r_1^3 F\right)_1 + 1 \leftrightarrow 2 \right\}$$



Dimensional Regularization

The Hadamard partie finie is a convenient way to regularize but is in principle not sufficient as it leads to ambiguous terms at high PN order $\rightarrow$ which is the case in our computation.

We can obtain the full Dimensional regularization result using the Hadamard partie finie

$$H^{(d)} = \int \mathrm{d}^d \mathbf{x}\, F^{(d)}(\mathbf{x})$$

In practice $\lim_{d\to 3} H^{(d)} = \mathcal{D}H + H$ with

$$\mathcal{D}H = \frac{1}{\varepsilon} \sum_{q_0 \leqslant q \leqslant q_1} \left[\frac{1}{q+1} + \varepsilon \ln\left(\frac{s_1}{\ell_0}\right) \right] \langle \underset{1}{f}{}^{(\varepsilon)}_{-3,q} \rangle + \mathcal{O}(\varepsilon) + (1 \leftrightarrow 2)$$

and the angular integral has to be taken in d dimension.

$$\langle \underset{1}{f}{}^{(\varepsilon)}_{p,q} \rangle \equiv \int \mathrm{d}\Omega_1^{(d-1)}\, \underset{1}{f}{}^{(\varepsilon)}_{p,q}(\mathbf{n}_1)$$

Distributional derivatives

The function we use may be singular in $\boldsymbol{y_1}$ and $\boldsymbol{y_2}$. To ensure that the derivatives are correct in the sense of the distribution,

$$\text{ie, for any test function } \phi, \quad \big\langle \partial_i F^{(d)}, \phi \big\rangle = -\big\langle F^{(d)}, \partial_i \phi \big\rangle$$

It is **necessary to correct** the derivatives.

- Space partial derivatives

$$D_i\big[F^{(d)}\big] = \sum_{\ell=0}^{+\infty} \frac{(-)^\ell}{\ell!}\, \partial_L \delta^{(d)}(\mathbf{x} - \boldsymbol{y}_1)\, \langle n_1^{iL} \underset{1}{f}{}^{(\varepsilon)}_{-\ell-2,-1} \rangle + \big(1 \leftrightarrow 2\big)$$

- Time partial derivatives
  We need to define a derivative with respect to $\boldsymbol{y_1}$ and $\boldsymbol{y_2}$.

$$\underset{1}{D_i}\big[F^{(d)}\big] = -\sum_{\ell=0}^{+\infty} \frac{(-)^\ell}{\ell!}\, \partial_L \delta^{(d)}(\mathbf{x} - \boldsymbol{y}_1)\, \langle n_1^{iL} \underset{1}{f}{}^{(\varepsilon)}_{-\ell-2,-1} \rangle \quad \text{and} \quad \big(1 \leftrightarrow 2\big)$$

$$\text{and} \quad D_t\big[F^{(d)}\big] = v_1^i \underset{1}{D_i}\big[F^{(d)}\big] + v_2^i \underset{2}{D_i}\big[F^{(d)}\big]\,.$$

In practice these formulas are used to compute second derivatives

$$D_{\alpha\beta}\big[F^{(d)}\big] = D_\alpha\big[\partial_\beta F^{(d)}\big] + \partial_\alpha D_\beta\big[F^{(d)}\big]$$

In most cases we can keep the 3D formalism and stick to the computation of $D_i\big(\frac{n_a^L}{r_a^m}\big)$.

Necessity of the dimensional regularization - pole contribution in Y^i at 2PN

The **dimensional regularization** is necessary for the computation of Y^i at 2PN.

$$\mathrm{Y}_i = \mathrm{Y}_i^{3D} + \mathrm{Y}_i^{\text{pole}}$$

$$\mathrm{Y}_i^{\text{pole}} = \frac{G^3 m_1^3 m_2}{c^4} \frac{n_{12}^i}{r_{12}^2} \left[-\frac{1}{\varepsilon} + 3 \ln \left(\frac{\sqrt{\bar{q}}\, r_{12}}{\ell_0} \right) \right] + (1 \leftrightarrow 2)$$

where

- $\epsilon = d - 3$ is the small parameter in dimensional regularization
- ℓ_0 is the length scale associated to dimensional regularization
- $\bar{q} = 4\pi \mathrm{e}^{\gamma_\mathrm{E}}$ is a numerical factor

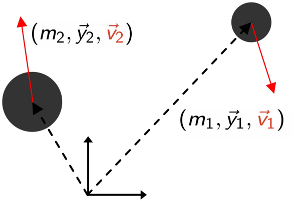
This term contributes to the total acceleration of the first body such that

$$a^{\text{pole}}_{\text{RR1}\,i} = \frac{4G}{c^5} \left(-1 + \varepsilon \left[1 + \frac{\gamma_\mathrm{E}}{2} \right] \right) \frac{\mathrm{d}^3 \mathrm{Y}_i^{\text{pole}}}{\mathrm{d}t^3} + \frac{2G\varepsilon}{c^5} \frac{\mathrm{d}^4}{\mathrm{d}t^4} \int_0^{+\infty} \mathrm{d}\tau \ln \left(\frac{c\tau\sqrt{\pi}}{\ell_0} \right) \left[\mathrm{Y}_i^{\text{pole}}(t-\tau) - \mathrm{Y}_i^{\text{pole}}(t+\tau) \right]$$

Dimensional identities - Motivations

In practice, a direct verification of the balance equations **seems incorrect as the equality is not recovered**. This can be explained by our representation of the binary parameters. For instance, a typical term in the 4.5PN acceleration of the first body before simplification looks like

$$\frac{180 G^2 m_1 m_2 n_{12}^i}{c^9 r_{12}^3}(n_{12} v_1)^2 (y_1 v_2)(v_1 v_2) v_2^2 \in a^i_{\mathrm{RR1\ 4.5PN}}$$



For convenience we use $(\boldsymbol{y_1}, \boldsymbol{n_{12}}, \boldsymbol{v_1}, \boldsymbol{v_2})$ where $n_{12}^i = (y_1^i - y_2^i)/r_{12}$.

We have 4 vectors in a 3 dimensional space so there is an obvious relation between them

$$\boxed{n_{12}^{[i} y_1^j v_1^k v_2^{l]} = 0}$$

This relation is of course **trivial** if we **develop in components** but this can be **difficult in practice** as the expressions are too long.

Dimensional identities - Practical implementation

In the **absence of cross-products** we transform our expressions in a **polynomial**

$$X_1 = \frac{Gm_1}{r_{12}c^2} \quad X_2 = \frac{Gm_2}{r_{12}c^2} \quad X_3 = \frac{(n_{12}v_1)}{c} \quad X_4 = \frac{(n_{12}v_2)}{c} \quad X_5 = \frac{v_1^2}{c^2} \quad X_6 = \frac{(v_1 v_2)}{c^2}$$

$$X_7 = \frac{v_2^2}{c^2} \quad X_8 = \frac{(n_{12}y_1)}{r_{12}} \quad X_9 = \frac{y_1^2}{r_{12}^2} \quad X_{10} = \frac{(y_1 v_1)}{r_{12}c} \quad X_{11} = \frac{(y_1 v_2)}{r_{12}c}$$

$$X_{12}^{(\mathrm{p})} = n_{12}^i \quad X_{12}^{(\mathrm{p})} = \frac{v_1^i}{c} \quad X_{14}^{(\mathrm{p})} = \frac{v_2^i}{c} \quad X_{15}^{(\mathrm{p})} = \frac{y_1^i}{r_{12}}$$

and we have **5** relations between these terms

$$P_S(X_1, \cdots, X_{11}) \equiv \frac{24}{r_{12}^2 c^4} n_{12}^{[i} y_1^j v_1^k v_2^{l]} n_{12}^i y_1^j v_1^k v_2^l = 0$$

$$P_1^{(\mathrm{p})}(X_1, \cdots, X_{11}, X_{12}^{(\mathrm{p})}, \cdots, X_{15}^{(\mathrm{p})}) \equiv \frac{24}{r_{12} c^4} n_{12}^{[i} y_1^j v_1^k v_2^{l]} n_{12}^j v_1^k v_2^l = 0$$

$$P_2^{(\mathrm{p})}(X_1, \cdots, X_{11}, X_{12}^{(\mathrm{p})}, \cdots, X_{15}^{(\mathrm{p})}) \equiv \frac{24}{r_{12}^2 c^4} n_{12}^{[i} y_1^j v_1^k v_2^{l]} v_1^j v_2^k y_1^l = 0$$

$$P_3^{(\mathrm{p})}(X_1, \cdots, X_{11}, X_{12}^{(\mathrm{p})}, \cdots, X_{15}^{(\mathrm{p})}) \equiv \frac{24}{r_{12}^2 c^3} n_{12}^{[i} y_1^j v_1^k v_2^{l]} v_2^j y_1^k n_{12}^l = 0$$

$$P_4^{(\mathrm{p})}(X_1, \cdots, X_{11}, X_{12}^{(\mathrm{p})}, \cdots, X_{15}^{(\mathrm{p})}) \equiv \frac{24}{r_{12}^2 c^3} n_{12}^{[i} y_1^j v_1^k v_2^{l]} y_1^j n_{12}^k v_1^l = 0$$

Additional polynomials for dimensional identities with cross-products

$$X_{12}^{(a)} = \frac{(\boldsymbol{n}_{12} \times \boldsymbol{v}_1)^i}{c} \qquad X_{13}^{(a)} = \frac{(\boldsymbol{n}_{12} \times \boldsymbol{v}_2)^i}{c} \qquad X_{14}^{(a)} = \frac{(\boldsymbol{v}_1 \times \boldsymbol{v}_2)^i}{c^2}$$

$$X_{15}^{(a)} = \frac{(\boldsymbol{n}_{12} \times \boldsymbol{y}_1)^i}{r_{12}} \qquad X_{16}^{(a)} = \frac{(\boldsymbol{y}_1 \times \boldsymbol{v}_1)^i}{r_{12} c} \qquad X_{17}^{(a)} = \frac{(\boldsymbol{y}_1 \times \boldsymbol{v}_2)^i}{r_{12} c}$$

$\varepsilon_{ijk} n_{12}^{[j} y_1^k v_1^l v_2^{m]} U_1^l U_2^m = 0$ implies the vanishing of **six new polynomials**

$$P_s^{(a)}(X_1, \cdots, X_{11}, X_{12}^{(a)}, \cdots, X_{15}^{(a)}) = 0$$

Construction of the passage to the center of mass

CM quantities : $x^i = y_1^i - y_2^i$, $v^i = v_1^i - v_2^i$, $m = m_1 + m_2$, $\nu = \frac{m_1 m_2}{m^2}$ and $\Delta = \frac{m_1 - m_2}{m}$

We define the integrated fluxes of linear momentum and CM position

$$\Pi^i \equiv \int_{-\infty}^{t} \mathrm{d}t' \, \mathcal{F}^i_{\boldsymbol{P}}(t') \,, \quad \Gamma^i \equiv \int_{-\infty}^{t} \mathrm{d}t' \int_{-\infty}^{t} \mathrm{d}t'' \, \mathcal{F}^i_{\boldsymbol{P}}(t'') + \int_{-\infty}^{t} \mathrm{d}t' \, \mathcal{F}^i_{\boldsymbol{G}}(t')$$

The center of mass position is defined by $G^i + \Gamma^i = 0$ where Γ^i is a 3.5PN quantity representing the radiation contribution. The formula for the passage to the center of mass frame takes the form

$$y_1^i = x^i \Big(X_2 + \nu \, \Delta \, \mathcal{P} \Big) + \nu \, \Delta \, \mathcal{Q} \, v^i + \mathcal{R}^i \,, \quad y_2^i = x^i \Big(-X_1 + \nu \, \Delta \, \mathcal{P} \Big) + \nu \, \Delta \, \mathcal{Q} \, v^i + \mathcal{R}^i$$

where $\mathcal{R}^i$ is gauge independent and writes

$$\mathcal{R}^i = -\frac{\Gamma^i}{m} + \frac{\nu}{mc^2} \left[\left(\frac{v^2}{2} - \frac{Gm}{r} \right) \Gamma^i + v^j \left(\Pi^j + \mathcal{F}^j_{\boldsymbol{G}} \right) x^i \right] + \mathcal{O}\left(\frac{1}{c^{11}} \right)$$

To solve this we decompose $\mathcal{P} = \mathcal{P}_{\text{cons}} + \mathcal{P}_{\text{RR}}$ and $\mathcal{Q} = \mathcal{Q}_{\text{cons}} + \mathcal{Q}_{\text{RR}}$ and obtain

$$\mathcal{Q}_{\text{RR}} = Gm \Bigg\{ \frac{1}{c^5} \left[\frac{4}{5} v^2 - \frac{8Gm}{5r} \right] + \frac{1}{c^7} \Bigg[\frac{G^2 m^2}{r^2} \left(-\frac{172}{105} - \frac{64}{35} \nu \right) + \frac{Gm}{r} \dot{r}^2 \left(\frac{44}{15} + \frac{68}{35} \nu \right)$$

$$+ \frac{Gm}{r} v^2 \left(\frac{6}{35} + \frac{132}{35} \nu \right) + v^4 \left(\frac{6}{7} - \frac{22}{7} \nu \right) \Bigg] \Bigg\} + \frac{1}{c^9} [\ldots] \,, \quad \mathcal{P}_{\text{RR}} = \frac{1}{c^7} [\ldots] + \frac{1}{c^9} [\ldots]$$

Center of mass radiation reaction acceleration and flux balance laws

$$a^i_{\rm RR}\Big|_{\rm harm} = a^i_{\rm RR\ 2.5PN} + a^i_{\rm RR\ 3.5PN} + a^i_{\rm RR\ 4PN} + a^i_{\rm RR\ 4.5PN\ mat} + a^i_{\rm RR\ 4.5PN\ rad}$$

$$a^i_{\rm RR\,2.5PN} = \frac{G^2m^2\nu}{c^5r^3}\left\{n^i\left(\frac{136}{15}\frac{Gm}{r}\dot r + \frac{24}{5}\dot r v^2\right) - v^i\left(\frac{24}{5}\frac{Gm}{r} + \frac{8}{5}v^2\right)\right\}$$

$$\begin{aligned} a^i_{\rm RR\,3.5PN} = \frac{G^2m^2\nu}{c^7r^3}\Bigg\{ & n^i\Bigg[\frac{G^2m^2}{r^2}\dot r\left(-\frac{3956}{35} - \frac{184}{5}\nu\right) + \frac{Gm}{r}\dot r^3\left(-\frac{294}{5} - \frac{376}{5}\nu\right) \\ & + \frac{Gm}{r}\dot r v^2\left(-\frac{692}{35} + \frac{724}{15}\nu\right) - 112\,\dot r^5 + \dot r^3 v^2(114+12\nu) + \dot r v^4\left(-\frac{366}{35} - 12\nu\right)\Bigg] \\ & + v^i\Bigg[\frac{G^2m^2}{r^2}\left(\frac{1060}{21} + \frac{104}{5}\nu\right) + \frac{Gm}{r}\dot r^2\left(\frac{82}{3} + \frac{848}{15}\nu\right) + \frac{Gm}{r}v^2\left(-\frac{164}{21} - \frac{148}{5}\nu\right) \\ & + 120\,\dot r^4 + \dot r^2 v^2\left(-\frac{678}{5} - \frac{12}{5}\nu\right) + v^4\left(\frac{626}{35} + \frac{12}{5}\nu\right)\Bigg]\Bigg\} \end{aligned}$$

$$a^i_{\rm RR\ 4.5PN\ mat} \approx 60 \text{ terms}$$

$$a^i_{\rm RR\ 4.5PN\ rad} = \frac{G\Delta}{r^2c^2}\left(2n^iv^j + n^jv^i\right)\left[\Pi^j + \mathcal{F}^j_{\boldsymbol{G}}\right]$$

The same way we can obtain the matter and radiation contribution for the Energy E and angular momentum J^i in the CM frame.

4.5PN contribution to the matter acceleration in the center of mass frame

$$
\begin{aligned}
a^i_{\text{RR 4.5PN mat}} = \frac{G^2 m^2 \nu}{c^9 r^3} \Bigg\{ n^i \Bigg[& \frac{G^3 m^3}{r^3} \dot{r} \left(\frac{336922}{945} + \frac{20644}{35}\nu - \frac{3632}{105}\nu^2 \right) + \frac{G^2 m^2}{r^2} \dot{r}^3 \left(-\frac{83177}{945} + \frac{41524}{135}\nu - \frac{30076}{105}\nu^2 \right) \\
& + \frac{G^2 m^2}{r^2} \dot{r} v^2 \left(-\frac{129769}{315} - \frac{58468}{315}\nu + \frac{4636}{105}\nu^2 \right) + \frac{Gm}{r} \dot{r}^5 \left(\frac{261883}{105} + \frac{5316}{5}\nu - \frac{1776}{5}\nu^2 \right) \\
& + \frac{Gm}{r} \dot{r}^3 v^2 \left(-\frac{22133}{6} - \frac{47643}{35}\nu + \frac{16028}{35}\nu^2 \right) + \frac{Gm}{r} \dot{r} v^4 \left(\frac{14307}{14} + \frac{46337}{105}\nu - \frac{2659}{21}\nu^2 \right) \\
& + \dot{r}^7 (-180 - 504\nu) + \dot{r}^5 v^2 \left(\frac{329}{2} + 1106\nu + 21\nu^2 \right) + \dot{r}^3 v^4 \left(\frac{88}{7} - \frac{4362}{7}\nu - 66\nu^2 \right) \\
& + \dot{r} v^6 \left(-\frac{1643}{210} + \frac{1248}{35}\nu + 45\nu^2 \right) \Bigg] \\
& + v^i \Bigg[\frac{G^3 m^3}{r^3} \left(-\frac{499286}{2835} - \frac{1376}{5}\nu + \frac{272}{35}\nu^2 \right) + \frac{G^2 m^2}{r^2} \dot{r}^2 \left(\frac{85991}{315} - \frac{97228}{315}\nu + \frac{4604}{21}\nu^2 \right) \\
& + \frac{G^2 m^2}{r^2} v^2 \left(\frac{47459}{315} + \frac{66632}{315}\nu - \frac{1796}{35}\nu^2 \right) + \frac{Gm}{r} \dot{r}^4 \left(-\frac{86323}{105} - \frac{10646}{35}\nu + \frac{1192}{7}\nu^2 \right) \\
& + \frac{Gm}{r} \dot{r}^2 v^2 \left(\frac{224351}{210} + \frac{7177}{35}\nu - \frac{2816}{15}\nu^2 \right) + \frac{Gm}{r} v^4 \left(-\frac{10747}{70} - \frac{539}{15}\nu + \frac{1769}{35}\nu^2 \right) \\
& + \dot{r}^6 (350 + 420\nu) + \dot{r}^4 v^2 \left(-\frac{1007}{2} - 984\nu - 3\nu^2 \right) + \dot{r}^2 v^4 \left(\frac{5778}{35} + \frac{4350}{7}\nu + \frac{54}{5}\nu^2 \right) \\
& + v^6 \left(-\frac{4873}{630} - \frac{298}{5}\nu - \frac{39}{5}\nu^2 \right) \Bigg] \Bigg\}
\end{aligned}
$$

Iyer-Will-Gopakumar parameters in the harmonic gauge

$$a^i_{\mathrm{RR}} = a^{i\,\mathrm{GII}}_{\mathrm{RR}} + \delta a^i_{\mathrm{RR}} + \mathcal{O}\left(\frac{1}{c^{11}}\right)$$

$$\delta a^i_{\mathrm{RR}} = \frac{\Delta}{c^2}\left\{\frac{G}{r^2}\left(2n^i v^j + n^j v^i\right)\left[\Pi^j + \mathcal{F}^j_{\boldsymbol{G}}\right] - \frac{v^i v^j}{m}\left[\mathcal{F}^j_{\boldsymbol{P}} + \dot{\mathcal{F}}^j_{\boldsymbol{G}}\right]\right\}$$

$$a^{i\,\mathrm{GII}}_{\mathrm{RR}} = -\frac{8}{5}\frac{G^2 m^2 \nu}{c^3 r^3}\left[-\left(A_{2.5\mathrm{PN}} + A_{3.5\mathrm{PN}} + A_{4.5\mathrm{PN}}\right)\dot{r} n^i + \left(B_{2.5\mathrm{PN}} + B_{3.5\mathrm{PN}} + B_{4.5\mathrm{PN}}\right)v^i\right]$$

$A_{n\mathrm{PN}}$ and $B_{n\mathrm{PN}}$ depend on 21 **arbitrary gauge parameters** at 4.5PN translating the freedom left by the choice of coordinates. We found for the harmonic coordinates

$$\alpha_3 = 0\,,\quad \beta_2 = -1$$

$$\xi_1 = \frac{271}{28} + 6\nu\,,\ \xi_2 = -\frac{77}{4} - \frac{3}{2}\nu\,,\ \xi_3 = \frac{79}{14} - \frac{92}{7}\nu\,,\ \xi_4 = 10\,,\ \xi_5 = \frac{5}{42} + \frac{242}{21}\nu\,,\ \rho_5 = -\frac{439}{28} + \frac{18}{7}\nu$$

$$\psi_1 = \frac{10139}{1008} - \frac{1977}{28}\nu - \frac{296}{7}\nu^2\,,\ \psi_2 = -\frac{53}{56} + \frac{10233}{56}\nu + \frac{261}{14}\nu^2\,,\ \psi_3 = -\frac{26561}{504} - \frac{3188}{63}\nu + \frac{886}{9}\nu^2$$

$$\psi_4 = -\frac{295}{16} - \frac{585}{4}\nu - \frac{15}{8}\nu^2\,,\ \psi_5 = -\frac{129}{2} - \frac{239}{63}\nu - \frac{908}{21}\nu^2\,,\ \psi_6 = \frac{170087}{756} + \frac{98299}{756}\nu - \frac{2942}{27}\nu^2$$

$$\psi_7 = \frac{25}{2} + 35\nu\,,\ \psi_8 = -\frac{74315}{504} - \frac{24935}{252}\nu + \frac{2179}{63}\nu^2\,,\ \psi_9 = \frac{12119}{126} + \frac{1241}{252}\nu + \frac{8696}{189}\nu^2$$

$$\chi_6 = \frac{107}{48} + \frac{88957}{504}\nu - \frac{2419}{504}\nu^2\ \chi_8 = -\frac{1655}{84} - \frac{4660}{63}\nu + \frac{485}{126}\nu^2\,,\ \chi_9 = \frac{53687}{1512} + \frac{7757}{378}\nu - \frac{589}{12}\nu^2$$

Manifest Lorentz invariance of the harmonic acceleration

Harmonic coordinates are Lorentz Invariant $\partial'_\nu h'^{\mu\nu} = \Lambda^\alpha_\nu \Lambda^\mu_\beta \Lambda^\nu_\gamma \partial_\alpha h^{\beta\gamma} = \Lambda^\mu_\beta \partial_\alpha h^{\beta\alpha} = 0$

To prove it we focus on a coordinate boost such that

$$t' = \gamma\Big(t - \frac{1}{c^2}(\boldsymbol{V}\cdot\mathbf{x})\Big)\,, \quad \mathbf{x}' = \mathbf{x} - \gamma\boldsymbol{V}\Big(t - \frac{\gamma}{c^2(\gamma+1)}(\boldsymbol{V}\cdot\mathbf{x})\Big)$$

At any PN order one can show that the position, velocity and acceleration transform such that

$$\boldsymbol{y}'_1 = \boldsymbol{y}_1 - \gamma\boldsymbol{V}\left(t - \frac{1}{c^2}\frac{\gamma}{\gamma+1}(\boldsymbol{V}\cdot\boldsymbol{x})\right) + \sum_{n=1}^{+\infty}\frac{(-)^n}{c^{2n}n!}\left(\frac{\partial}{\partial t}\right)^{n-1}\left[(\boldsymbol{V}\cdot\boldsymbol{r}_1)^n\left(\boldsymbol{v}_1 - \frac{\gamma}{\gamma+1}\boldsymbol{V}\right)\right]$$

$$\boldsymbol{v}'_1 = \frac{1}{\gamma}\boldsymbol{v}_1 - \boldsymbol{V} + \frac{1}{\gamma}\sum_{n=1}^{+\infty}\frac{(-)^n}{c^{2n}n!}\left(\frac{\partial}{\partial t}\right)^{n}\left[(\boldsymbol{V}\cdot\boldsymbol{r}_1)^n\left(\boldsymbol{v}_1 - \frac{\gamma}{\gamma+1}\boldsymbol{V}\right)\right]$$

$$\boldsymbol{a}'_1 = \frac{1}{\gamma^2}\left\{\boldsymbol{a}_1 + \sum_{n=1}^{+\infty}\frac{(-)^n}{c^{2n}n!}\left(\frac{\partial}{\partial t}\right)^{n+1}\left[(\boldsymbol{V}\cdot\boldsymbol{r}_1)^n\left(\boldsymbol{v}_1 - \frac{\gamma}{\gamma+1}\boldsymbol{V}\right)\right]\right\}$$

We compute the difference between the boosted and unboosted acceleration (taken at the same time)

$$\delta_\Lambda \boldsymbol{a}_1 \equiv \boldsymbol{a}'_1(t') - \boldsymbol{a}_1(t') = \boldsymbol{a}'_1\left[\boldsymbol{y}'_1, \boldsymbol{y}'_2, \boldsymbol{v}'_1, \boldsymbol{v}'_2\right] - \boldsymbol{a}_1\left[\boldsymbol{y}'_1, \boldsymbol{y}'_2, \boldsymbol{v}'_1, \boldsymbol{v}'_2\right]$$

After replacement and using the dimensional identities the result is indeed $\boxed{\delta_\Lambda \boldsymbol{a}_1 = 0}$ and the acceleration is **manifestly Lorentz invariant**.

PN potentials

$$\Box V = -4\pi G\,\sigma$$
$$\Box V_i = -4\pi G\,\sigma_i$$
$$\Box K = -4\pi G \sigma V$$
$$\Box \hat{W}_{ij} = -4\pi G \left(\sigma_{ij} - \delta_{ij}\,\frac{\sigma_{kk}}{d-2}\right) - \frac{1}{2}\left(\frac{d-1}{d-2}\right)\partial_i V \partial_j V$$
$$\Box \hat{R}_i = -\frac{4\pi G}{d-2}\left(\frac{5-d}{2}\,V\sigma_i - \frac{d-1}{2}\,V_i\,\sigma\right) - \frac{d-1}{d-2}\,\partial_k V \partial_i V_k - \frac{d(d-1)}{4(d-2)^2}\,\partial_t V \partial_i V$$
$$\Box \hat{X} = -4\pi G\left[\frac{V\sigma_{ii}}{d-2} + 2\left(\frac{d-3}{d-1}\right)\sigma_i V_i + \left(\frac{d-3}{d-2}\right)^2 \sigma\left(\frac{V^2}{2} + K\right)\right]$$
$$+ \hat{W}_{ij}\,\partial_{ij} V + 2V_i\,\partial_t \partial_i V + \frac{1}{2}\left(\frac{d-1}{d-2}\right) V \partial_t^2 V + \frac{d(d-1)}{4(d-2)^2}\,(\partial_t V)^2 - 2\partial_i V_j\,\partial_j V_i$$
$$\Box \hat{Z}_{ij} = -\frac{4\pi G}{d-2}\,V\left(\sigma_{ij} - \delta_{ij}\,\frac{\sigma_{kk}}{d-2}\right) - \frac{d-1}{d-2}\,\partial_t V_{(i}\,\partial_{j)} V + \partial_i V_k\,\partial_j V_k + \partial_k V_i\,\partial_k V_j - 2\partial_k V_{(i}\,\partial_{j)} V_k$$
$$- \frac{\delta_{ij}}{d-2}\,\partial_k V_m\,(\partial_k V_m - \partial_m V_k) - \frac{d(d-1)}{8(d-2)^3}\,\delta_{ij}\,(\partial_t V)^2 + \frac{(d-1)(d-3)}{2(d-2)^2}\,\partial_{(i} V \partial_{j)} K$$

The explicit 4.5PN contribution to the acceleration in a general frame - Part 1/5

$$
\begin{aligned}
a^i_{\text{RR1 4.5PN 3D}} = \frac{G^2 m_1 m_2 {v_{12}}^i}{c^9 r_{12}^6} \Bigg\{ & \Big(\frac{106972}{567} m_1^3 - \frac{5242}{135} m_1^2 m_2 - \frac{706996}{945} m_1 m_2^2 - \frac{1075618}{2835} m_2^3 \Big) G^3 \\
& + G^2 r_{12} \Big[m_2^2 \Big(-\frac{22384}{35} (n_{12} v_{12})^2 + \frac{16196}{105} (n_{12} v_{12})(n_{12} v_2) - \frac{16054}{105} (n_{12} v_2)^2 - \frac{12776}{105} (v_{12} v_2) \\
& + \frac{7412}{21} v_{12}^2 - \frac{3194}{105} v_2^2 \Big) + m_1 m_2 \Big(\frac{48749}{105} (n_{12} v_{12})^2 + \frac{21716}{105} (n_{12} v_{12})(n_{12} v_2) - \frac{5896}{35} (n_{12} v_2)^2 \\
& - \frac{12448}{105} (v_{12} v_2) + \frac{13273}{105} v_{12}^2 - \frac{3112}{105} v_2^2 \Big) + m_1^2 \Big(\frac{261323}{315} (n_{12} v_{12})^2 - \frac{1296}{35} (n_{12} v_{12})(n_{12} v_2) \\
& + \frac{2972}{105} (n_{12} v_2)^2 + \frac{368}{21} (v_{12} v_2) - \frac{55477}{315} v_{12}^2 + \frac{92}{21} v_2^2 \Big) \Big] + G r_{12}^2 \Big[m_1 \Big(\frac{4073}{35} (n_{12} v_{12})^4 \\
& + \frac{276}{5} (n_{12} v_{12})^3 (n_{12} v_2) - \frac{64}{5} (n_{12} v_{12})^2 (n_{12} v_2)^2 - \frac{48}{5} (n_{12} v_{12})(n_{12} v_2)^3 + \frac{24}{5} (n_{12} v_2)^4 \\
& + \frac{56}{15} (n_{12} v_{12})(n_{12} v_2)(v_{12} v_2) + \frac{48}{5} (n_{12} v_2)^2 (v_{12} v_2) - \frac{76}{35} (v_{12} v_2)^2 - \frac{18898}{105} (n_{12} v_{12})^2 v_{12}^2 \\
& - \frac{5048}{105} (n_{12} v_{12})(n_{12} v_2) v_{12}^2 + \frac{124}{35} (n_{12} v_2)^2 v_{12}^2 + \frac{479}{15} v_{12}^4 + \frac{26}{15} (n_{12} v_{12})^2 v_2^2 + \frac{8}{5} (n_{12} v_{12})(n_{12} v_2) v_2^2 \\
& - \frac{8}{5} (n_{12} v_2)^2 v_2^2 - \frac{8}{5} (v_{12} v_2) v_2^2 - \frac{66}{35} v_{12}^2 v_2^2 - \frac{1}{5} v_2^4 \Big) + m_2 \Big(-\frac{93166}{105} (n_{12} v_{12})^4 + \frac{582}{5} (n_{12} v_{12})^3 (n_{12} v_2) \\
& - \frac{454}{5} (n_{12} v_{12})^2 (n_{12} v_2)^2 + \frac{208}{5} (n_{12} v_{12})(n_{12} v_2)^3 - \frac{96}{5} (n_{12} v_2)^4 + \frac{372}{5} (n_{12} v_{12})(n_{12} v_2)(v_{12} v_2) \\
& - \frac{592}{15} (n_{12} v_2)^2 (v_{12} v_2) - \frac{1432}{105} (v_{12} v_2)^2 + \frac{36389}{30} (n_{12} v_{12})^2 v_{12}^2 - \frac{3736}{105} (n_{12} v_{12})(n_{12} v_2) v_{12}^2 \\
& + \frac{1562}{105} (n_{12} v_2)^2 v_{12}^2 - \frac{38933}{210} v_{12}^4 + \frac{227}{15} (n_{12} v_{12})^2 v_2^2 - \frac{104}{15} (n_{12} v_{12})(n_{12} v_2) v_2^2 + \frac{32}{5} (n_{12} v_2)^2 v_2^2
\end{aligned}
$$

The explicit 4.5PN contribution to the acceleration in a general frame - Part 2/5

$$+\frac{32}{5}(v_{12}v_2)v_2^2-\frac{76}{21}v_{12}^2v_2^2+\frac{4}{5}v_2^4\Big)\Big]+r_{12}^3\Big(308(n_{12}v_{12})^6+56(n_{12}v_{12})^5(n_{12}v_2)$$

$$-210(n_{12}v_{12})^4(n_{12}v_2)^2+120(n_{12}v_{12})^4(v_{12}v_2)+240(n_{12}v_{12})^3(n_{12}v_2)(v_{12}v_2)$$

$$-\frac{348}{5}(n_{12}v_{12})^2(v_{12}v_2)^2-\frac{12}{5}(n_{12}v_{12})(n_{12}v_2)(v_{12}v_2)^2+\frac{6}{5}(n_{12}v_2)^2(v_{12}v_2)^2-470(n_{12}v_{12})^4v_{12}^2$$

$$-60(n_{12}v_{12})^3(n_{12}v_2)v_{12}^2+174(n_{12}v_{12})^2(n_{12}v_2)^2v_{12}^2+6(n_{12}v_{12})(n_{12}v_2)^3v_{12}^2-\frac{3}{2}(n_{12}v_2)^4v_{12}^2$$

$$-\frac{696}{5}(n_{12}v_{12})^2(v_{12}v_2)v_{12}^2-\frac{708}{5}(n_{12}v_{12})(n_{12}v_2)(v_{12}v_2)v_{12}^2-\frac{12}{5}(n_{12}v_2)^2(v_{12}v_2)v_{12}^2$$

$$+\frac{668}{35}(v_{12}v_2)^2v_{12}^2+\frac{1230}{7}(n_{12}v_{12})^2v_{12}^4+\frac{246}{35}(n_{12}v_{12})(n_{12}v_2)v_{12}^4-\frac{501}{35}(n_{12}v_2)^2v_{12}^4$$

$$+\frac{668}{35}(v_{12}v_2)v_{12}^4-\frac{6473}{630}v_{12}^6+90(n_{12}v_{12})^4v_2^2-\frac{6}{5}(v_{12}v_2)^2v_2^2-\frac{522}{5}(n_{12}v_{12})^2v_{12}^2v_2^2$$

$$-\frac{18}{5}(n_{12}v_{12})(n_{12}v_2)v_{12}^2v_2^2+\frac{9}{5}(n_{12}v_2)^2v_{12}^2v_2^2+\frac{501}{35}v_{12}^4v_2^2-\frac{3}{10}v_{12}^2v_2^4\Big)\Big\}$$

$$+\frac{G^2m_1m_2n_{12}^i}{c^9r_{12}^6}\Big\{G^3\Big[\frac{1096496}{945}m_2^3(n_{12}v_{12})+m_1m_2^2\Big(\frac{634292}{315}(n_{12}v_{12})-\frac{4464}{35}(n_{12}v_2)\Big)$$

$$+m_1^2m_2\Big(-\frac{146686}{315}(n_{12}v_{12})-\frac{9592}{105}(n_{12}v_2)\Big)+m_1^3\Big(-\frac{744454}{945}(n_{12}v_{12})+\frac{1584}{35}(n_{12}v_2)\Big)\Big]$$

$$+G^2r_{12}\Big[m_2^2\Big(\frac{36428}{15}(n_{12}v_{12})^3+\frac{7894}{15}(n_{12}v_{12})(n_{12}v_2)^2+\frac{3172}{21}(n_{12}v_{12})(v_{12}v_2)$$

$$-\frac{3172}{21}(n_{12}v_2)(v_{12}v_2)-\frac{24034}{15}(n_{12}v_{12})v_{12}^2+\frac{32}{5}(n_{12}v_2)v_{12}^2+\frac{1586}{21}(n_{12}v_{12})v_2^2\Big)$$

The explicit 4.5PN contribution to the acceleration in a general frame - Part 3/5

$$
\begin{aligned}
&+ m_1 m_2 \Big(-\frac{154853}{315}(n_{12}v_{12})^3 - \frac{6592}{21}(n_{12}v_{12})^2(n_{12}v_2) + \frac{7204}{15}(n_{12}v_{12})(n_{12}v_2)^2 - \frac{392}{15}(n_{12}v_2)^3 \\
&+ \frac{13576}{105}(n_{12}v_{12})(v_{12}v_2) - \frac{5048}{35}(n_{12}v_2)(v_{12}v_2) - \frac{1879}{7}(n_{12}v_{12})v_{12}^2 + \frac{848}{105}(n_{12}v_2)v_{12}^2 \\
&+ \frac{6788}{105}(n_{12}v_{12})v_2^2 - \frac{56}{15}(n_{12}v_2)v_2^2 \Big) + m_1^2 \Big(-\frac{2220173}{945}(n_{12}v_{12})^3 + \frac{940}{21}(n_{12}v_{12})^2(n_{12}v_2) \\
&- \frac{404}{3}(n_{12}v_{12})(n_{12}v_2)^2 + \frac{56}{5}(n_{12}v_2)^3 - \frac{3992}{105}(n_{12}v_{12})(v_{12}v_2) + \frac{4664}{105}(n_{12}v_2)(v_{12}v_2) \\
&+ \frac{342617}{315}(n_{12}v_{12})v_{12}^2 - \frac{6796}{105}(n_{12}v_2)v_{12}^2 - \frac{1996}{105}(n_{12}v_{12})v_2^2 + \frac{8}{5}(n_{12}v_2)v_2^2 \Big) \Big] + r_{12}^3 \Big(-216(n_{12}v_{12})^7 \\
&+ 252(n_{12}v_{12})^5(n_{12}v_2)^2 - 168(n_{12}v_{12})^5(v_{12}v_2) - 280(n_{12}v_{12})^4(n_{12}v_2)(v_{12}v_2) \\
&+ 60(n_{12}v_{12})^3(v_{12}v_2)^2 - 6(n_{12}v_{12})(n_{12}v_2)^2(v_{12}v_2)^2 + \frac{12}{5}(n_{12}v_{12})(v_{12}v_2)^3 + \frac{12}{5}(n_{12}v_2)(v_{12}v_2)^3 \\
&+ 280(n_{12}v_{12})^5 v_{12}^2 - 210(n_{12}v_{12})^3(n_{12}v_2)^2 v_{12}^2 + \frac{21}{2}(n_{12}v_{12})(n_{12}v_2)^4 v_{12}^2 + 180(n_{12}v_{12})^3(v_{12}v_2)v_{12}^2 \\
&+ 180(n_{12}v_{12})^2(n_{12}v_2)(v_{12}v_2)v_{12}^2 - 6(n_{12}v_{12})(n_{12}v_2)^2(v_{12}v_2)v_{12}^2 - 6(n_{12}v_2)^3(v_{12}v_2)v_{12}^2 \\
&- \frac{408}{35}(n_{12}v_{12})(v_{12}v_2)^2 v_{12}^2 + \frac{12}{5}(n_{12}v_2)(v_{12}v_2)^2 v_{12}^2 - 70(n_{12}v_{12})^3 v_{12}^4 + \frac{123}{7}(n_{12}v_{12})(n_{12}v_2)^2 v_{12}^4 \\
&- \frac{738}{35}(n_{12}v_{12})(v_{12}v_2)v_{12}^4 - \frac{246}{35}(n_{12}v_2)(v_{12}v_2)v_{12}^4 - \frac{187}{210}(n_{12}v_{12})v_{12}^6 - 84(n_{12}v_{12})^5 v_2^2 \\
&+ \frac{18}{5}(n_{12}v_{12})(v_{12}v_2)^2 v_2^2 + 90(n_{12}v_{12})^3 v_{12}^2 v_2^2 - 9(n_{12}v_{12})(n_{12}v_2)^2 v_{12}^2 v_2^2 + \frac{18}{5}(n_{12}v_{12})(v_{12}v_2)v_{12}^2 v_2^2
\end{aligned}
$$

The explicit 4.5PN contribution to the acceleration in a general frame - Part 4/5

$$
\begin{aligned}
&+\frac{18}{5}(n_{12}v_2)(v_{12}v_2)v_{12}^2v_2^2-\frac{369}{35}(n_{12}v_{12})v_{12}^4v_2^2+\frac{9}{10}(n_{12}v_{12})v_{12}^2v_2^4\Big)+Gr_{12}^2\Big[m_2\Big(\frac{61658}{21}(n_{12}v_{12})^5\\
&+\frac{2328}{5}(n_{12}v_{12})^3(n_{12}v_2)^2+\frac{416}{5}(n_{12}v_{12})(n_{12}v_2)^4-\frac{582}{5}(n_{12}v_{12})^3(v_{12}v_2)\\
&-\frac{1746}{5}(n_{12}v_{12})^2(n_{12}v_2)(v_{12}v_2)+\frac{208}{5}(n_{12}v_{12})(n_{12}v_2)^2(v_{12}v_2)-\frac{208}{5}(n_{12}v_2)^3(v_{12}v_2)\\
&+\frac{3568}{105}(n_{12}v_{12})(v_{12}v_2)^2-\frac{208}{15}(n_{12}v_2)(v_{12}v_2)^2-\frac{183863}{42}(n_{12}v_{12})^3v_{12}^2-\frac{3526}{35}(n_{12}v_{12})(n_{12}v_2)^2v_{12}^2\\
&+\frac{3568}{105}(n_{12}v_{12})(v_{12}v_2)v_{12}^2+\frac{3568}{105}(n_{12}v_2)(v_{12}v_2)v_{12}^2+\frac{261973}{210}(n_{12}v_{12})v_{12}^4-\frac{16}{5}(n_{12}v_2)v_{12}^4\\
&-\frac{291}{5}(n_{12}v_{12})^3v_2^2-\frac{104}{5}(n_{12}v_{12})(n_{12}v_2)^2v_2^2-\frac{104}{15}(n_{12}v_{12})(v_{12}v_2)v_2^2+\frac{104}{15}(n_{12}v_2)(v_{12}v_2)v_2^2\\
&+\frac{1784}{105}(n_{12}v_{12})v_{12}^2v_2^2-\frac{26}{15}(n_{12}v_{12})v_2^4\Big)+m_1\Big(-\frac{11353}{35}(n_{12}v_{12})^5-\frac{536}{5}(n_{12}v_{12})^4(n_{12}v_2)\\
&-\frac{996}{5}(n_{12}v_{12})^3(n_{12}v_2)^2-\frac{96}{5}(n_{12}v_{12})^2(n_{12}v_2)^3-\frac{144}{5}(n_{12}v_{12})(n_{12}v_2)^4+48(n_{12}v_{12})^3(v_{12}v_2)\\
&+144(n_{12}v_{12})^2(n_{12}v_2)(v_{12}v_2)-\frac{24}{5}(n_{12}v_{12})(n_{12}v_2)^2(v_{12}v_2)+\frac{72}{5}(n_{12}v_2)^3(v_{12}v_2)\\
&-\frac{4888}{105}(n_{12}v_{12})(v_{12}v_2)^2+\frac{32}{5}(n_{12}v_2)(v_{12}v_2)^2+\frac{15102}{35}(n_{12}v_{12})^3v_{12}^2+\frac{2292}{35}(n_{12}v_{12})^2(n_{12}v_2)v_{12}^2\\
&+\frac{5056}{35}(n_{12}v_{12})(n_{12}v_2)^2v_{12}^2-\frac{24}{5}(n_{12}v_2)^3v_{12}^2-\frac{4888}{105}(n_{12}v_{12})(v_{12}v_2)v_{12}^2-\frac{4888}{105}(n_{12}v_2)(v_{12}v_2)v_{12}^2\\
&-\frac{1829}{15}(n_{12}v_{12})v_{12}^4+\frac{40}{7}(n_{12}v_2)v_{12}^4+24(n_{12}v_{12})^3v_2^2+\frac{12}{5}(n_{12}v_{12})^2(n_{12}v_2)v_2^2
\end{aligned}
$$

The explicit 4.5PN contribution to the acceleration in a general frame - Part 5/5

$$+ \frac{36}{5}(n_{12}v_{12})(n_{12}v_2)^2 v_2^2 + \frac{12}{5}(n_{12}v_{12})(v_{12}v_2)v_2^2 - \frac{12}{5}(n_{12}v_2)(v_{12}v_2)v_2^2 - \frac{2444}{105}(n_{12}v_{12})v_{12}^2 v_2^2$$
$$+ \frac{4}{5}(n_{12}v_2)v_{12}^2 v_2^2 + \frac{3}{5}(n_{12}v_{12})v_2^4\Big)\Big]\Bigg\}$$