Bjorken-like Sum Rules and the Lorentz Group

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Workshop

Decay $B \to D^{**}$ and related issues

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Bjorken-like Sum Rules and the Lorentz Group

Well-known that the transitions $H_b \to H_c \ell \nu$ like

Meson transitions	$\overline{B}_d \to D \ell \nu$	$\overline{B}_d \to D^* \ell \nu$
Baryon transition	$\Lambda_b \to \Lambda_c \ell \nu$	

are related to the exclusive determination of $|V_{cb}|$

Many form factors but Heavy Quark Symmetry $SU(2N_f)$
$\to$ form factors given by a single function $\xi(w)$ (IW function)

Tension between inclusive and exclusive determinations of $|V_{cb}|$

But my purpose is only to expose new interesting theoretical results on the properties of the Heavy Quark Effective Theory of QCD

Heavy Quark Symmetry

Elastic meson transitions $\overline{B}_d \to D\ell\nu$ $\overline{B}_d \to D^*\ell\nu$

Light cloud $\frac{1}{2}^-$ combines with heavy quark spin $s_Q = \frac{1}{2}$

$\to$ $J^P = 0^-(D)$ and $1^-(D^*)$ ground states

By spin-flavor Heavy Quark Symmetry $SU(2N_f)$ (N_f heavy flavors)

six form factors (f_0, f_+ for $\overline{B} \to D$), (V, A_0, A_1, A_2 for $\overline{B} \to D^*$)
reduce to a single Isgur-Wise function $\boxed{\xi(w)}$ $\left(w = \frac{m_B^2+m_D^2-q^2}{2m_B m_D}\right)$
for the light cloud $(L = 0, s_q = \frac{1}{2})$ transition $\frac{1}{2}^- \to \frac{1}{2}^-$

Excited meson transitions $\overline{B}_d \to D^{**}\ell\nu$ ($L = 1$, D^{**} of $P = +$)

$L = 1, s_q = \frac{1}{2}$: light cloud transitions $\frac{1}{2}^- \to \frac{1}{2}^+$ and $\frac{1}{2}^- \to \frac{3}{2}^+$

two IW functions $\boxed{\tau_{1/2}(w), \tau_{3/2}(w)}$ $D^{**} : 0^+_{1/2}, 1^+_{1/2}, 1^+_{3/2}, 2^+_{3/2}$

Bjorken and Uraltsev Sum Rules

Bjorken SR $\rho^2 = \frac{1}{4} + \sum_n \left[|\tau_{1/2}^{(n)}(1)|^2 + 2|\tau_{3/2}^{(n)}(1)|^2 \right] \quad \rightarrow \quad \rho^2 \geqslant \frac{1}{4}$

Uraltsev SR $\sum_n \left[|\tau_{3/2}^{(n)}(1)|^2 - |\tau_{1/2}^{(n)}(1)|^2 \right] = \frac{1}{4}$

Bjorken (1990-1991) + Uraltsev (2001) $\rightarrow \quad \rho^2 \geqslant \frac{3}{4}$

Bound obtained in Bakamjian-Thomas quark models
(Le Yaouanc et al. 1996)

- covariant for $m_Q \rightarrow \infty$
- explicit Isgur-Wise scaling
- satisfying Bjorken and Uraltsev SR

Isgur-Wise functions and Sum Rules in HQET

(Bjorken; Isgur and Wise; Uraltsev; Le Yaouanc et al.)

Consider the non-forward amplitude

$$\overline{B}(v_i) \to D^{(n)}(v') \to \overline{B}(v_f) \quad (w_i = v_i \cdot v', w_f = v_f \cdot v', w_{if} = v_i \cdot v_f)$$

SR obtained from the OPE

$$L_{Hadrons}(w_i, w_f, w_{if}) = R_{OPE}(w_i, w_f, w_{if})$$

$L_{Hadrons}$: sum over $D^{(n)}$ states $\qquad R_{OPE}$: OPE counterpart

$$\sum_{D^{(n)}} < \overline{B}_f(v_f)|\Gamma_f|D^{(n)}(v') >< \overline{D}^{(n)}(v')|\Gamma_i|B_i(v_i) > \xi^{(n)}(w_i)\xi^{(n)}(w_f)$$

$$+ \textit{ Other excited states and IW functions} = -2\xi(w_{if}) \ < \overline{B}_f(v_f)|\Gamma_f P'_+ \Gamma_i|B_i(v_i) >$$

$$P'_+ = \frac{1 + \not{v}'}{2}$$: positive energy projector on the intermediate c

Light cloud angular momentum j and bound state spin J

$\overline{B}$: pseudoscalar ground state $(j^P, J^P) = \left(\frac{1}{2}^-, 0^-\right)$

$D^{(n)}$: tower $(j^P, J^P), J = j \pm \frac{1}{2}, j = L \pm \frac{1}{2}, P = (-1)^{L+1}$ (Falk, 1992)

Heavy quark currents : $\quad \overline{h}_{v'} \Gamma_i h_{v_i} \qquad \overline{h}_{v_f} \Gamma_f h_{v'}$

Domain of the variables (w_i, w_f, w_{if}) :

$$w_i \geq 1 \qquad\qquad w_f \geq 1$$

$$w_i w_f - \sqrt{(w_i^2 - 1)(w_f^2 - 1)} \leq w_{if} \leq w_i w_f + \sqrt{(w_i^2 - 1)(w_f^2 - 1)}$$

For $w_i = w_f = w$, the domain becomes :

$$w \geq 1 \qquad\qquad 1 \leq w_{if} \leq 2w^2 - 1$$

$$\Gamma_i = \not v_i \qquad \Gamma_f = \not v_f \qquad \rightarrow \qquad \textit{Vector SR}$$

$$(w+1)^2 \sum_{L\geq 0} \frac{L+1}{2L+1} \ S_L(w, w_{if}) \sum_n \left[\tau_{L+1/2}^{(L)(n)}(w)\right]^2$$
$$+ \sum_{L\geq 1} S_L(w, w_{if}) \sum_n \left[\tau_{L-1/2}^{(L)(n)}(w)\right]^2 = (1 + 2w + w_{if})\, \xi(w_{if})$$

$$\Gamma_i = \not v_i \gamma_5 \qquad \Gamma_f = \not v_f \gamma_5 \qquad \rightarrow \qquad \textit{Axial SR}$$

$$\sum_{L\geq 0} S_{L+1}(w, w_{if}) \sum_n \left[\tau_{L+1/2}^{(L)(n)}(w)\right]^2$$
$$+ (w-1)^2 \sum_{L\geq 1} \frac{L}{2L-1} S_{L-1}(w, w_{if}) \sum_n \left[\tau_{L-1/2}^{(L)(n)}(w)\right]^2$$
$$= -(1 - 2w + w_{if})\, \xi(w_{if})$$

IW functions $\tau_{L\pm 1/2}^{(L)(n)}(w) :\ \frac{1}{2}^- \rightarrow \left(L \pm \frac{1}{2}\right)^P, P = (-1)^{L+1}$

$S_L(w, w_{if})$ is a Legendre polynomial :

$$S_L(w, w_{if}) = \sum_{0 \leq k \leq L/2} C_{L,k} \left(w^2 - 1\right)^{2k} \left(w^2 - w_{if}\right)^{L-2k}$$

$$C_{L,k} = (-1)^k \frac{(L!)^2}{(2L)!} \, \frac{(2L-2k)!}{k!(L-k)!(L-2k)!}$$

Differentiating the Sum Rules $\left[\frac{d^{p+q}(L_{Hadrons} - R_{OPE})}{dw_{if}^p dw^q}\right]_{w_{if}=w=1} = 0$

(going to the corner of the domain $w \to 1$, $w_{if} \to 1$)

one finds constraints on the derivatives $\xi^{(n)}(1)$, in particular

$$\boxed{\rho^2 = -\xi'(1) \geq \frac{3}{4}} \qquad \boxed{\xi''(1) \geq \frac{1}{5}\left[4\rho^2 + 3(\rho^2)^2\right]}$$

Non-trivial inequalities

Non-forward amplitude (Uraltsev) $\overline{B}(v_i) \to D^{(n)}(v') \to \overline{B}(v_f)$

The Legendre polynomial $S_L(w_i, w_f, w_{if})$

$$S_L(w_i, w_f, w_{if}) = v_{f\nu_1} ... v_{f\nu_L} T^{v_{f\nu_1} \cdots v_{f\nu_L}, v_{i\mu_1} \cdots v_{i\mu_L}} v_{i\mu_1} ... v_{i\mu_L}$$

Projector on polarization tensor of integer spin L

$T^{v_{f\nu_1} \cdots v_{f\nu_L}, v_{i\mu_1} \cdots v_{i\mu_L}} = \sum_\lambda \epsilon'^{(\lambda)*\nu_1 \dots \nu_L} \epsilon'^{(\lambda)\mu_1 \dots \mu_L}$ (depends on v')

Polarization tensor $\epsilon'^{(\lambda)\mu_1 \dots \mu_L}$ is symmetric, traceless and transverse

$g_{\mu_i \mu_j} \epsilon'^{(\lambda)\mu_1 \dots \mu_L} = v'_{\mu_i} \epsilon'^{(\lambda)\mu_1 \dots \mu_L} = 0$ Examples of projector :

$$L = 1 \qquad T^{\mu\nu} = -g^{\mu\nu} + v'^\mu v'^\nu$$

$$\begin{aligned} L = 2 \qquad T^{\mu\nu,\rho\sigma} = \tfrac{1}{6} [& -2 g^{\mu\nu} g^{\rho\sigma} + 3 \left(g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho} \right) \\ & + 2 \left(g^{\mu\nu} v'^\rho v'^\sigma + g^{\rho\sigma} v'^\mu v'^\nu \right) + 4 v'^\mu v'^\nu v'^\rho v'^\sigma \\ & - 3 \left(g^{\mu\rho} v'^\nu v'^\sigma + g^{\nu\sigma} v'^\mu v'^\rho + g^{\nu\rho} v'^\mu v'^\sigma + g^{\mu\sigma} v'^\nu v'^\rho \right)] \end{aligned}$$

$$S_L(w_i, w_f, w_{if}) = \sum_{0 \leq k \leq L/2} C_{L,k} (w_i^2 - 1)^k (w_f^2 - 1)^k (w_i w_f - w_{if})^{L-2k}$$

$$C_{L,k} = (-1)^k \frac{(L!)^2}{(2L)!} \, \frac{(2L-2k)!}{k!(L-k)!(L-2k)!}$$

Derivation of sum rules and inequalities

Differentiating the Sum Rule $\quad L_{Hadrons}(w, w_{if}) = R_{OPE}(w, w_{if})$

$$\left(\frac{d^{p+q} L_{Hadrons}}{dw_{if}^p dw^q} \right)_{w_{if}=w=1} = \left(\frac{d^{p+q} R_{OPE}}{dw_{if}^p dw^q} \right)_{w_{if}=w=1}$$

Choosing the currents

$$\xi^{(L)}(1) = \frac{1}{4}(-1)^L L! \sum_n \left[\frac{L+1}{2L+1} 4[\tau^{(L)(n)}_{L+1/2}(1)]^2 + [\tau^{(L-1)(n)}_{L-1/2}(1)]^2 + [\tau^{(L)(n)}_{L-1/2}(1) \right]^2$$

$L = 1 \ \rightarrow$ Bjorken SR $\qquad \rho^2 = \frac{1}{4} + \sum_n \left[|\tau^{(n)}_{1/2}(1)|^2 + 2|\tau^{(n)}_{3/2}(1)|^2 \right]$

$$\sum_n \left[\frac{L}{2L+1} [\tau^{(L)(n)}_{L+1/2}(1)]^2 - \frac{1}{4} [\tau^{(L)(n)}_{L-1/2}(1)]^2 \right] = \sum_n \frac{1}{4} [\tau^{(L-1)(n)}_{L-1/2}(1)]^2$$

$L = 1 \ \rightarrow$ Uraltsev SR $\qquad \sum_n \left[|\tau^{(n)}_{3/2}(1)|^2 - |\tau^{(n)}_{1/2}(1)|^2 \right] = \frac{1}{4}$

Inequalities for derivatives

Slope $\rho^2 = -\xi'(1) = \frac{3}{4}[1 + [\tau_{1/2}^{1)(n)}(1)]^2] \ \rightarrow \ \rho^2 \geqslant \frac{3}{4}$

Curvature $\sigma^2 = \xi''(1) = \frac{5}{4} \sum_n \left[[\tau_{3/2}^{(1)(n)}(1)]^2 + [\tau_{3/2}^{(2)(n)}(1)]^2 \right]$

$\geqslant \frac{5}{4} \sum_n [\tau_{3/2}^{(1)(n)}(1)]^2 = \frac{5}{4}\rho^2 \geqslant \frac{15}{16}$

L-th derivative $(-1)^L \xi^{(L)}(1) \geqslant \frac{2L+1}{4} (-1)^{L-1} \xi^{(L-1)}(1) \geqslant \frac{(2L+1)!!}{2^{2L}}$

$\frac{4}{3}\rho^2 + (\rho^2)^2 - \frac{5}{3}\sigma^2 + \sum_{n \neq 0} [\xi'^{(n)}(1)]^2 = 0$ ($\frac{1}{2}^-$ excited states)

$\rightarrow$ $\sigma^2 \geqslant \frac{1}{5} \left[4\rho^2 + 3(\rho^2)^2 \right]$ new improved bound

term $\frac{3}{5}(\rho^2)^2$ dominant in non-relativistic limit for the light quark

The so-called BPS limit of HQET

$\mu_\pi^2 = \mu_G^2 \quad \rightarrow \quad -\xi'(1) = \rho^2 = \frac{3}{4}$ (Uraltsev, 2001)

Using the Sum Rules and by induction $\rightarrow (-1)^L \xi^{(L)}(1) = \frac{(2L+1)!!}{2^{2L}}$

Therefore BPS implies the explicit form $\boxed{\xi(w) = \left(\frac{2}{w+1}\right)^{3/2}}$

Defined limit of HQET $\rightarrow$ explicit form for the elastic IW function

This limit has a simple group theoretical interpretation

Isgur-Wise functions and the Lorentz group

Matrix element of a current between heavy hadrons **factorizes** into a trivial **heavy quark current matrix element**
and a **light cloud overlap** (that contains the long distance physics)

$$< H'(v')|J^{Q'Q}(q)|H(v) > =$$

$$< Q'(v'), \pm\frac{1}{2}|J^{Q'Q}(q)|Q(v), \pm\frac{1}{2} >< v', j', M'|v, j, M >$$

The light cloud follows the heavy quark with the same four-velocity

Isgur-Wise functions : light cloud overlaps $\xi(v.v') = < v'|v >$

Factorization valid only in absence of **hard radiative corrections**

Light cloud Hilbert space

Sensible hypothesis : light cloud states form a Hilbert space on which acts a unitary representation of the Lorentz group

$\Lambda \to U(\Lambda)$ $\qquad$ $U(\Lambda)|v, j, \epsilon > = |\Lambda v, j, \Lambda\epsilon >$

$$\boxed{|v, j, \epsilon > = \sum_M (\Lambda^{-1}\epsilon)_M \; U(\Lambda)|v_0, j, M >}$$

$\Lambda v_0 = v$ $\qquad$ $v_0 = (1, 0, 0, 0)$ $\qquad$ $\Lambda^{-1}\epsilon$: polarization vector at rest

Defines in Hilbert space $\mathcal{H}$ of unitary representation of $SL(2, C)$ the states $|v, j, \epsilon >$ whose scalar products define the IW functions in terms of $|v_0, j, M >$ $\quad$ ($SU(2)$ multiplets in $SU(2) \subset SL(2, C)$)

Illustration with the simpler case of baryons with $\mathbf{j = 0}$

Baryons $\Lambda_b(v)$, $\Lambda_c(v)$ ($S_{qq} = 0, L = 0$ in quark model language)

The Isgur-Wise function writes

$$\xi(v.v') = < U(B_{v'})\phi_0 | U(B_v)\phi_0 >$$

$|\phi_0 >$ represents the light cloud at rest and B_v, $B_{v'}$ are boosts

$$\xi(w) = < \phi_0 | U(\Lambda)\phi_0 > \qquad \Lambda v_0 = v \qquad v^0 = w$$

Λ is for instance the boost along Oz

$$\Lambda_\tau = \begin{pmatrix} e^{\tau/2} & 0 \\ 0 & e^{-\tau/2} \end{pmatrix} \qquad w = ch(\tau)$$

Method completely general, for any j and any transition $j \to j'$

Decomposition into irreducible representations

The unitary representation $U(\Lambda)$ is in general reducible

Decompose it into irreducible representations $U_\chi(\Lambda)$

Hilbert space $\mathcal{H}$ made of functions $\psi : \chi \in X \to \psi_\chi \in \mathcal{H}_\chi$

Scalar product in $\mathcal{H}$

$$< \psi' | \psi > = \int_X < \psi'_\chi | \psi_\chi > d\mu(\chi)$$

$\chi \in X$: irreducible unitary representation
$d\mu(\chi)$: a positive measure

$$(U(\Lambda)\,\psi)_\chi = U_\chi(\Lambda)\psi_\chi \qquad \psi_\chi \in \mathcal{H}_\chi$$

$\mathcal{H}_\chi$: Hilbert space of χ on which acts $U_\chi(\Lambda)$

Integral formula for the Isgur-Wise function

Notation $\xi_\chi(w) = < \phi_{0,\chi} | U_\chi(\Lambda) \phi_{0,\chi} >$

irreducible Isgur-Wise function corresponding to irreducible χ

Isgur-Wise function $\boxed{\xi(w) = \int_{X_0} \xi_\chi(w)\ d\nu(\chi)}$

positive normalized measure $d\nu(\chi)$ $\int_{X_0} d\nu(\chi) = 1$

$X_0 \subset X$ irreducible representations of $SL(2, C)$
containing a non-zero $SU(2)$ scalar subspace ($j = 0$ case)

Irreducible IW function $\xi_\chi(w)$ when ν is a δ function

Irreducible unitary representations of the Lorentz group

Naïmark (1962)

Principal series $\chi = (n, \rho)$

$n \in Z$ and $\rho \in R$ $\quad (n = 0, \rho \geq 0; n > 0, \rho \in R)$

Hilbert space $\mathcal{H}_{n,\rho}$

$< \phi' | \phi > = \int \overline{\phi'(z)}\ \phi(z)\ d^2 z \qquad d^2 z = d(Rez) d(Imz)$

Unitary operator $U_{n,\rho}(\Lambda)$

$$(U_{n,\rho}(\Lambda)\phi)(z) = \left(\frac{\alpha - \gamma z}{|\alpha - \gamma z|}\right)^n |\alpha - \gamma z|^{2i\rho - 2}\ \phi\left(\frac{\delta z - \beta}{\alpha - \gamma z}\right)$$

$$\Lambda = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \qquad \alpha\delta - \beta\gamma = 1 \qquad (\alpha, \beta, \gamma, \delta) \in C$$

If n odd $\quad \frac{n}{2} \leq j \quad \rightarrow \quad \boxed{j = \frac{1}{2} \rightarrow n = 1}$ for the meson case

Irreducible IW functions in the meson case $\mathbf{j}^P = \frac{1}{2}^-$

Need $\xi_\chi(w) = < \phi^\chi_{\frac{1}{2},M} | U_\chi(\Lambda_\tau) \phi^\chi_{\frac{1}{2},M} >$ $(\Lambda_\tau$: boost, $w = ch(\tau))$

$\phi^\chi_{\frac{1}{2},M}$ orthonormal basis of $\mathcal{H}_\chi$ adapted to rotation group $SU(2)$

Compute transformed elements $U_\chi(\Lambda_\tau)\phi^\chi_{\frac{1}{2},M}$ (spin complications)

For $j = \frac{1}{2}$ only the principal series of representations contributes

Using scalar products for principal class of representations (ρ real)

$$\boxed{\xi_\rho(w) = \frac{1}{\cosh(\tau)+1} \frac{1}{\sinh(\tau)} \frac{4}{4\rho^2+1} \left[\sinh\left(\frac{\tau}{2}\right)\cos(\rho\tau) + 2\rho\cosh\left(\frac{\tau}{2}\right)\sin(\rho\tau)\right]}$$

Integral formula for the Isgur-Wise function $\xi(w)$

$\boxed{\xi(w) = \int \xi_\rho(w)\ d\nu(\rho)}$ d$\nu(\rho)$ positive measure $\int d\nu(\rho) = 1$

Constraints on the derivatives of the Isgur-Wise function

Derivative $\xi^{(k)}(1)$: *expectation value* of a polynomial of degree k

$$\xi^{(k)}(1) = (-1)^k \ \frac{1}{2^{2k}(2k+1)!!} < \prod_{i=1}^{k}[(2i+1)^2 + 4\rho^2] >$$

In terms of moments of a positive variable $\mu_n = < x^n > \ (x = \rho^2)$

$$\xi(1) = \mu_0 = 1$$

$$-\xi'(1) = \frac{3}{4} + \frac{1}{3}\mu_1$$

$$\xi''(1) = \frac{1}{240} \ (225 + 136\mu_1 + 16\mu_2)$$

...

Moments μ_k in terms of derivatives $\xi(1)$, $\xi'(1)$, ... $\xi^{(k)}(1)$

$$\mu_0 = \xi(1) = 1$$

$$\mu_1 = \frac{9}{4} - 3 \ \xi'(1)$$

$$\mu_2 = \frac{3}{16} \, [27 + 136\xi'(1) + 80\xi''(1)]$$

...

Constraints on moments of a variable with positive values

$det\left[(\mu_{i+j})_{0\leq i,j\leq n}\right] \geq 0$ $det\left[(\mu_{i+j+1})_{0\leq i,j\leq n}\right] \geq 0$

Lower moments

$\mu_1 \geq 0$

$\mu_2 \geq \mu_1^2$

...

That imply for the derivatives of the Isgur-Wise function

$\rho^2 \geq 0$

$\xi''(1) \geq \frac{1}{5}\left[4\rho^2 + 3(\rho^2)^2\right]$

...

Same results as with the Sum Rule approach

Consistency test for any Ansatz of the Isgur-Wise function

Integral representation of the Isgur-Wise function $(w = \cosh(\tau))$

$$\xi(w) = \int \frac{1}{\cosh(\tau)+1} \frac{1}{\sinh(\tau)} \frac{4}{4\rho^2+1} [\sinh\left(\frac{\tau}{2}\right)\cos(\rho\tau) + 2\rho\cosh\left(\frac{\tau}{2}\right)\sin(\rho\tau)]\ d\nu(\rho)$$

$d\nu(\rho)$ is a positive measure satisfying $\int d\nu(\rho) = 1$

Can invert by Fourier transform

$$\widehat{\xi}(\tau) \equiv (\cosh(\tau)+1)\sinh(\tau)\xi(ch(\tau))$$

$$(\mathcal{F}\widehat{\xi})(\sigma) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\tau\sigma}(\cosh(\tau)+1)\sinh(\tau)\ \xi(ch(\tau))\ d\tau$$

$\rightarrow$ check if an Ansatz for $\xi(w)$ satisfies it with *positive measures*

Phenomenological one-parameter examples

- Linear form $\xi(w) = 1 - c(w-1)$

Does not satisfy the integral representation for any value of c

- Exponential form $\xi(w) = exp[-c(w-1)]$

Does not satisfy the integral representation for any value of c

- "Dipole" form $\xi(w) = \left(\frac{2}{1+w}\right)^{2c}$

Satisfies the integral representation if the slope $c \geq \frac{3}{4}$

- The BPS form $\xi(w) = \left(\frac{2}{1+w}\right)^{3/2}$ $(c = \frac{3}{4})$

is an irreducible Isgur-Wise function (representation with $\rho = 0$)

Two other new rigorous results on Isgur-Wise functions

• The Bjorken-like Sum Rules imply that
the Isgur-Wise function is a function of positive type :

$$\int \frac{d^3\vec{v}}{v^0}\frac{d^3\vec{v}\,'}{v\,'^0}\ \psi(v')^*\ \xi(v.v')\ \psi(v) \geq 0 \qquad \text{for any } \psi(v)$$

• There is a complete equivalence between the Sum Rule approach and the Lorentz group approach :

- The Lorentz group approach implies that $\xi(w)$ is of positive type

- The Sum Rule approach implies the Lorentz group approach

Conclusions

• Considering the non-forward amplitude in the heavy quark limit, Bjorken-like Sum Rules give strong bounds on the derivatives of the Isgur-Wise function

• Decomposing into irreducible representations a unitary representation of the Lorentz group $\rightarrow$ one gets an integral formula for the Isgur-Wise function with positive measure

• Derivatives of the IW function given in terms of moments of a positive variable $\rightarrow$ inequalities between the derivatives the same as obtained from Bjorken-like Sum Rules

• Consistency test for any Ansatz of the IW function

• Applications to phenomenological examples

• Sum Rules $\rightarrow$ IW function is a function of positive type

• Equivalence between Sum Rule and Lorentz group approaches

Back up slides

New rigorous results on Isgur-Wise functions : motivations

At LHC, many more urgent subjects than $b \to c\ell\nu$ transitions :

- Search of the Higgs boson
- Search of New Physics (Supersymmetry ?)
- Precise study of CP violation in B mesons, as in $B_s - \overline{B}_s$
- Look for photon polarization in rare decays $b \to s\gamma$

However, there are some motivations :

- It is never too late to get new rigorous results on this subject
- $BR(\Lambda_b \to \Lambda_c \ell\nu) \simeq 5\%$ (Tevatron), $\frac{d\Gamma}{dw}$ can be studied at LHC-b
- Exclusive (HQET) $\overline{B} \to D(D^*)\ell\nu \Rightarrow |V_{cb}| = (38.7 \pm 1.1) \times 10^{-3}$
  Inclusive (OPE) $\overline{B} \to X_c\ell\nu \Rightarrow |V_{cb}| = (41.5 \pm 0.7) \times 10^{-3}$
  Consistent within errors, but the situation is not satisfactory

Exclusive determination of $|V_{cb}|$

$$\frac{d\Gamma(\overline{B}\to D^*\ell\nu)}{dw} = \frac{G_F^2}{48\pi^3}(m_B - m_{D^*})^2 m_{D^*}^3 K(w,r)|V_{cb}|^2|\mathcal{F}^*(1)|^2|\xi(w)|^2$$

$$r = \frac{m_{D^*}}{m_B}, w = \frac{m_B^2+m_D^2-q^2}{2m_B m_D}, w = 1 \to q^2_{max} = (m_B - m_{D^*})^2$$

$$\mathcal{F}^*(1) = \eta_A\left(1 + \delta_{1/m^2} + ...\right) = 0.924 \pm 0.012 \pm 0.019 \qquad \text{(lattice QCD)}$$

$$\xi(1) = 1, \qquad \xi'(1) = -\rho^2$$

$$|V_{cb}| = (38.7 \pm 0.7 \pm 0.9) \times 10^{-3} \qquad \text{(HFAG 2007)}$$

Great dispersion of data in the $(|V_{cb}|, \rho^2)$ plane

Inclusive determination $\quad |V_{cb}| = (41.7 \pm 0.4 \pm 0.6) \times 10^{-3}$

[Buchmüller and Flächer (2005-2007), from Bigi et al., Bauer et al.]

$m_b = 4.59\ GeV$, $m_c = 1.14\ GeV$, $\mu_G^2 = 0.35\ GeV^2$, $\mu_\pi^2 = 0.40\ GeV^2$

Different hadronic uncertainties in inclusive vs. exclusive methods

Operator Product Expansion

$T = i \int d^4x e^{-iq.x} < \overline{B} | T[J(x) J^+(0)] | \overline{B} >$ $\qquad J = \overline{c} \Gamma b$

$T \sim \sum_X \frac{|<X|J(0)|\overline{B}>|^2}{m_B - q^0 - E_X} \delta(\mathbf{p}_X + \mathbf{q}) - \sum_{X'} \frac{|<X' B \overline{B}|J^+(0)|\overline{B}>|^2}{m_B + q^0 - (E_{X'} + 2m_B)} \delta(\mathbf{p}_{X'} - \mathbf{q})$

Direct channel virtuality $\qquad \mathcal{V} = m_B - q^0 - E_X$

Choose q^0 such that $\qquad \Lambda_{QCD} \ll \mathcal{V} \ll m_B$

Crossed channel denominator $\qquad \mathcal{V} + 2m_D \gg \mathcal{V}$

Leading contribution to the OPE

$T = i \int d^4x e^{-iq.x} < \overline{B} | \overline{b}(x) \Gamma^+ S_c^{free}(x,0) \Gamma b(0) | \overline{B} > + \ O(1/m_c^2)$

Varying independently $\mathcal{V}, m_b, m_c$ and equating residues

$\sum_{X_c} | < X_c | J(0) | \overline{B} > |^2 = < \overline{B} | \overline{b} \ \overline{\Gamma} \frac{\not{v}'_c + 1}{2 v'^0_c} \Gamma b(0) | \overline{B} >$

$\frac{\not{v}'_c + 1}{2 v'^0_c}$: positive energy residue of c quark propagator

Details of the calculations of the sum rules and bounds

- The excited states of arbitrary spin (Falk 1992)
- Calculation of the polynomial $S_L(w_i, w_f, w_{if})$ (Le Yaouanc et al. 2002)
- Simple derivation of Bjorken and Uraltsev SR (Le Yaouanc et al. 2002)
- Generalizations for higher derivatives (Le Yaouanc et al. 2002)
- Proof of improved bound on the curvature (Le Yaouanc et al. 2003)
- The Isgur-Wise function in the BPS limit (Jugeau et al. 2006)
- Radiative corrections (Dorsten 2003)
- Phenomenology (Dorsten 2003)

4×4 matrices for states of arbitrary spin

L : orbital angular momentum of light clound of half-integer spin j

$k = j - \frac{1}{2}$

- $j = L + \frac{1}{2}, J = j + \frac{1}{2}$ $\qquad \mathcal{M}^{\mu_1,\cdots\mu_k}(v) = P_+ \epsilon^{\mu_1,\cdots\mu_{k+1}} \gamma_{\mu_{k+1}}$

- $j = L + \frac{1}{2}, J = j - \frac{1}{2}$ $\qquad \mathcal{M}^{\mu_1,\cdots\mu_k}(v) = -\sqrt{\frac{2k+1}{k+1}} P_+ \gamma_5 \epsilon^{\nu_1,\cdots\nu_k}$

$$\times \left[g^{\mu_1}_{\nu_1} ... g^{\mu_k}_{\nu_k} - \frac{1}{k+1} \left[\gamma_{\nu_1} (\gamma^{\mu_1} - v^{\mu_1}) g^{\mu_2}_{\nu_2} ... g^{\mu_k}_{\nu_k} + g^{\mu_1}_{\nu_1} ... g^{\mu_{k-1}}_{\nu_{k-1}} \gamma_{\nu_k} (\gamma^{\mu_k} - v^{\mu_k}) \right] \right]$$

- $j = L - \frac{1}{2}, J = j + \frac{1}{2}$ $\qquad \mathcal{M}^{\mu_1,\cdots\mu_k}(v) = P_+ \epsilon^{\mu_1,\cdots\mu_{k+1}} \gamma_5 \gamma_{\mu_{k+1}}$

- $j = L - \frac{1}{2}, J = j - \frac{1}{2}$ $\qquad \mathcal{M}^{\mu_1,\cdots\mu_k}(v) = \sqrt{\frac{2k+1}{k+1}} P_+ \epsilon^{\nu_1,\cdots\nu_k}$

$$\times \left[g^{\mu_1}_{\nu_1} ... g^{\mu_k}_{\nu_k} - \frac{1}{k+1} \left[\gamma_{\nu_1} (\gamma^{\mu_1} - v^{\mu_1}) g^{\mu_2}_{\nu_2} ... g^{\mu_k}_{\nu_k} + g^{\mu_1}_{\nu_1} ... g^{\mu_{k-1}}_{\nu_{k-1}} \gamma_{\nu_k} (\gamma^{\mu_k} - v^{\mu_k}) \right] \right]$$

Sketch of the demonstration

Reduce to a three-dimensional problem at rest

$$v' = (1, \mathbf{0}), v_i = (\sqrt{1+\mathbf{v}_i^2}, \mathbf{v}_i), v_f = (\sqrt{1+\mathbf{v}_f^2}, \mathbf{v}_f) \ \rightarrow \ T^{j_1,\ldots j_L, i_1 \ldots i_L}$$

Couple L angular momenta $\vec{1}$ into total $\vec{L}$

$$S_L(\mathbf{v}_i^2, \mathbf{v}_f^2, \mathbf{v}_i.\mathbf{v}_f) = \sum_{j_1 \ldots j_L} \sum_{k_1 \ldots k_L} v_f^{k_1} \ldots v_f^{k_L} T^{k_1,\ldots k_L, j_1 \ldots j_L} v_i^{j_1} \ldots v_i^{j_L}$$

$$= \frac{2^L (L!)^2}{(2L+1)!} 4\pi \sum_{M=-L}^{M=L} \mathcal{Y}_L^M(\mathbf{v}_f)^* \mathcal{Y}_L^M(\mathbf{v}_i) = \frac{2^L (L!)^2}{(2L)!} |\mathbf{v}_i|^L |\mathbf{v}_f|^L P_L(\hat{\mathbf{v}}_i.\hat{\mathbf{v}}_f)$$

$$S_L(\mathbf{v}_i^2, \mathbf{v}_f^2, \mathbf{v}_i.\mathbf{v}_f) = \sum_{0 \leqslant k \leqslant \frac{L}{2}} \frac{(L!)^2}{(2L)!} (-1)^k \frac{(2L-2k)!}{k!(L-k)!(L-2k)!} (\mathbf{v}_i^2)^k (\mathbf{v}_f^2)^k (\mathbf{v}_i.\mathbf{v}_f)^{L-2k}$$

Covariant $\rightarrow$ $\mathbf{v}_i^2 = w_i^2 - 1, \ \mathbf{v}_f^2 = w_f^2 - 1, \ \mathbf{v}_i.\mathbf{v}_f = w_i w_f - w_{if}$

Improved bound on the curvature

$$\left[\frac{d^{p+q}L^V_{Hadrons}}{dw^p_{if}dw^q}\right]_{w_{if}=w=1} = \left[\frac{d^{p+q}R^V_{OPE}}{dw^p_{if}dw^q}\right]_{w_{if}=w=1} = 0 \qquad \text{(p+q = 0,1,2)}$$

$$\left[\frac{d^{p+q}L^A_{Hadrons}}{dw^p_{if}dw^q}\right]_{w_{if}=w=1} = \left[\frac{d^{p+q}R^A_{OPE}}{dw^p_{if}dw^q}\right]_{w_{if}=w=1} = 0 \qquad \text{(p+q = 0,1,2,3)}$$

4 linearly independent equations for the curvature $\sigma^2 = \xi''(1)$

$$\rho^2 - \frac{5}{4}\sigma^2 + \sum_n [\tau^{(1)(n)}_{3/2}(1)]^2 = 0 \quad \rightarrow \quad \sigma^2 \geqslant \frac{5}{4}\rho^2 \qquad \text{(see above)}$$

Shape of the Isgur-Wise function in a limit of HQET

Matrix elements of dimension 5 operators in HQET

$\mu_\pi^2 = -\frac{1}{2m_B} < \overline{B}|\overline{h}_v(iD)^2 h_v|\overline{B} >$ kinetic operator

$\mu_G^2 = \frac{1}{2m_B} < \overline{B}|\frac{g_s}{2}\overline{h}_v \sigma_{\alpha\beta} G^{\alpha\beta} h_v|\overline{B} >$ chromomagnetic operator

Sum Rules in terms of $\frac{1}{2}^- \to \frac{1}{2}^+, \frac{3}{2}^+$ IW functions $\tau_j^{(n)}$ and level spacings $\Delta E_j^{(n)}$ (Bigi et al., 1995) :

$$\mu_\pi^2 = 6 \sum_n [\Delta E_{3/2}^{(n)}]^2 [\tau_{3/2}^{(n)}(1)]^2 + 3 \sum_n [\Delta E_{1/2}^{(n)}]^2 [\tau_{1/2}^{(n)}(1)]^2$$

$$\mu_G^2 = 6 \sum_n [\Delta E_{3/2}^{(n)}]^2 [\tau_{3/2}^{(n)}(1)]^2 - 6 \sum_n [\Delta E_{1/2}^{(n)}]^2 [\tau_{1/2}^{(n)}(1)]^2$$

Inequality $\mu_\pi^2 \geqslant \mu_G^2$ (expt. $\mu_\pi^2 \cong 0.40\ GeV^2$, $\mu_G^2 \cong 0.35\ GeV^2$)

The so-called BPS limit of HQET

$\mu_\pi^2 = \mu_G^2 \quad \rightarrow \quad \tau_{1/2}^{(n)}(1) = 0$ (Uraltsev, 2001)

BPS with two derivatives $\rightarrow \quad \tau_{3/2}^{(2)(n)}(1) = 0 \ \rightarrow \ \sigma^2 = \frac{15}{16}$

To generalize need to demonstrate $\tau_{L-1/2}^{(L)(n)}(1) = 0$

By induction : $\tau_{1/2}^{(1)(n)}(1) = \tau_{3/2}^{(2)(n)}(1) = 0$, assume $\tau_{L-3/2}^{(L-1)(n)}(1) = 0$

Vector and Axial SR $\rightarrow \ \tau_{L-1/2}^{(L)(n)}(1) = 0 \ \rightarrow (-1)^L \xi^{(L)}(1) = \frac{(2L+1)!!}{2^{2L}}$

Therefore BPS implies the explicit form $\boxed{\xi(w) = \left(\frac{2}{w+1}\right)^{3/2}}$

Defined limit of HQET $\rightarrow$ explicit form for the elastic IW function

This limit has a simple group theoretical interpretation

Radiative corrections

Two types of radiative corrections : (1) within HQET
(2) Wilson coefficients to make the matching with QCD

Modified sum rule (Dorsten 2003)
μ-dependence in OPE side and cut-off Δ in hadronic sum

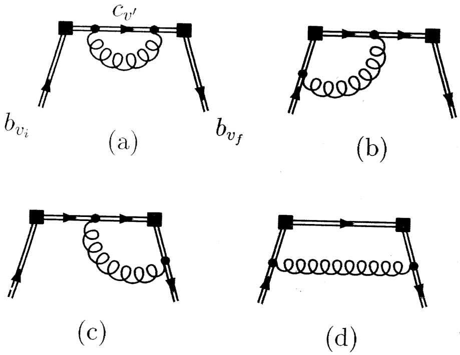
$$\sum_{X_c} W_\Delta(E_M - E_{X_c}) < \overline{B}_f|J_f(0)|X_c >< X_c|J_i(0)|\overline{B}_i >$$
$$= 2\xi(w_{if})\left[1 + \alpha_s(\mu)F(w_i, w_f, w_{if})\right] Tr\left[P_{f+}\not{v}_f(\gamma_5)P'_+\not{v}_i(\gamma_5)P_{i+}\right]$$

Universal function $F(w_i, w_f, w_{if}) \to F(1, w, w) = F(w, 1, w) = 0$

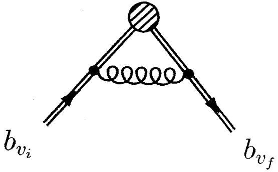
Modified bound due to radiative corrections within HQET

$$\sigma^2(\mu) > \frac{3}{5}\left[\rho^2(\mu)\right]^2 + \frac{4}{5}\rho^2(\mu)\left[1 + \frac{20\alpha_s(\mu)}{27\pi}\right] - \frac{148\alpha_s(\mu)}{675\pi} \qquad (\Delta = 2\mu)$$

Curvature of physical axial form factor $\sigma^2_{A_1} > 0.94 - 0.07_p - 0.2_{np}$



$c_{v'}$
b_{v_i}
(a)
b_{v_f}
(b)
(c)
(d)



b_{v_i}
b_{v_f}

The case of baryons $\Lambda_b(v_i) \to \Lambda_c^{(n)}(v') \to \Lambda_b(v_f)$

$\Lambda_b \,:\, (j^P, J^P) = \left(0^+, \frac{1}{2}^+\right)$

$\Lambda_c^{(n)} \,:\,$ tower $(j^P, J^P), J = j, j = L, P = (-1)^L$

Sum rule

$$\xi_\Lambda(w_{if}) = \sum_n \sum_{L\geq 0} \tau_L^{(n)}(w_i)^* \tau_L^{(n)}(w_f)$$
$$\sum_{0\leq k\leq L/2} C_{L,k}\,(w_i^2-1)^k(w_f^2-1)^k(w_i w_f - w_{if})^{L-2k}$$

IW functions $\tau_L(w) \,:\, 0^+ \to L^P, P = (-1)^L$

One finds the constraints on the derivatives :

$$\rho_\Lambda^2 = -\xi_\Lambda'(1) \geq 0 \qquad \xi_\Lambda''(1) \geq \frac{3}{5}[\rho_\Lambda^2 + (\rho_\Lambda^2)^2]$$

Supplementary series $\chi = (s, \rho)$

$\rho \in R$ $(0 < \rho < 1)$

Hilbert space $\mathcal{H}_{s,\rho}$

$$< \phi' | \phi > = \int \overline{\phi'(z_1)} \, |z_1 - z_2|^{2\rho - 2} \, \phi(z_2) \, d^2 z_1 d^2 z_2$$

(non-standard scalar product)

Unitary operator $U_{s,\rho}(\Lambda)$

$$(U_{s,\rho}(\Lambda)\phi)(z) = |\alpha - \gamma z|^{-2\rho - 2} \, \phi\left(\frac{\delta z - \beta}{\alpha - \gamma z}\right)$$

Trivial representation $\chi = t$

Hilbert space $\mathcal{H}_t = C$

$$< \phi' | \phi > = \overline{\phi'(z)}\phi(z)$$

Unitary operator $U_t(\Lambda) = 1$

Decomposition under the rotation group

Need restriction to $SU(2)$ of unitary representations χ of $SL(2,C)$

For a χ there is an orthonormal basis $\phi^{\chi}_{j,M}$ of $\mathcal{H}_\chi$ adapted to $SU(2)$

Particularizing to $j=0$: all types of representations contribute

$$\phi^{p,0,\rho}_{0,0}(z) = \frac{1}{\sqrt{\pi}}(1+|z|^2)^{i\rho-1} \qquad (\chi=(p,0,\rho),\ \rho\geq 0)$$

$$\phi^{s,\rho}_{0,0}(z) = \frac{\sqrt{\rho}}{\pi}(1+|z|^2)^{-\rho-1} \qquad (\chi=(s,\rho),\ 0<\rho<1)$$

$$\phi^{t}_{0,0}(z) = 1 \qquad (\chi=t)$$

For $j\neq 0$ enters also the matrix element

$$D^j_{M',M}(R) = < j, M'|U_j(R)|j,M > \qquad R\in SU(2)$$

Irreducible IW functions in the case $\mathbf{j = 0}$

Need $\xi_\chi(w) = < \phi^\chi_{0,0} | U_\chi(\Lambda_\tau) \phi^\chi_{0,0} >$ $\quad (\Lambda_\tau$: boost, $w = ch(\tau))$

Transformed elements $U_\chi(\Lambda_\tau)\phi^\chi_{0,0}$

$$\left(U_{p,0,\rho}(\Lambda_\tau)\phi^{p,0,\rho}_{0,0}\right)(z) = \frac{1}{\sqrt{\pi}}(e^\tau + e^{-\tau}|z|^2)^{i\rho - 1}$$

$$\left(U_{s,\rho}(\Lambda_\tau)\phi^{s,\rho}_{0,0}\right)(z) = \frac{\sqrt{\rho}}{\sqrt{\pi}}(e^\tau + e^{-\tau}|z|^2)^{-\rho - 1}$$

$$U_t(\Lambda_\tau)\phi^t_{0,0} = 1$$

Using the scalar products for each class of representations

$$\xi_{p,0,\rho}(w) = \frac{sin(\rho\tau)}{\rho \ sh(\tau)} \qquad (\rho \geq 0)$$

$$\xi_{s,\rho}(w) = \frac{sh(\rho\tau)}{\rho \ sh(\tau)} \qquad (0 < \rho < 1)$$

$$\xi_t(w) = 1$$

Integral formula for the IW function in the case $j = 0$

$$\boxed{\xi(w) = \int_{[0,\infty[} \frac{sin(\rho\tau)}{\rho\ sh(\tau)}\ d\nu_p(\rho) + \int_{]0,1[} \frac{sh(\rho\tau)}{\rho\ sh(\tau)}\ d\nu_s(\rho) + \nu_t}$$

ν_p and ν_s are positive measures and ν_t a ≥ 0 real number

$$\boxed{\int_{[0,\infty[} d\nu_p(\rho) + \int_{]0,1[} d\nu_s(\rho) + \nu_t = 1}$$

One-parameter family $\qquad \xi_x(w) = \frac{sh(\tau\sqrt{1-x})}{sh(\tau)\sqrt{1-x}} = \frac{sin(\tau\sqrt{x-1})}{sh(\tau)\sqrt{x-1}}$

covers all irreducible representations $\rightarrow$ simplifies integral formula

$\xi(w) = \int_{[0,\infty[} \xi_x(w)\ d\nu(x) \quad$ (ν positive measure $\int_{[0,\infty[} d\nu(x) = 1$)

$\rightarrow$ a transparent deduction of constraints on the derivatives $\xi^{(n)}(1)$

Integral formula for the IW function in the case $j = 0$

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$\xi(w) = \int_{[0,\infty[} \xi_x(w) \; d\nu(x) \quad$ (ν positive measure $\int_{[0,\infty[} d\nu(x) = 1$)

$\rightarrow$ a transparent deduction of constraints on the derivatives $\xi^{(n)}(1)$

Example 3

From the integral representation

$$\xi(w) = \int_{[0,\infty[} \xi_x(w)\, d\nu(x) \quad (\nu \text{ positive measure } \int_{[0,\infty[} d\nu(x) = 1)$$

and $$\xi_x(w) = \frac{sh(\tau\sqrt{1-x})}{sh(\tau)\sqrt{1-x}} = \frac{sin(\tau\sqrt{x-1})}{sh(\tau)\sqrt{x-1}}$$

if the curvature saturates its lower bound $\xi''(1) = \frac{3}{5}\rho_\Lambda^2(1+\rho_\Lambda^2)$

$$\boxed{\xi(w) = \frac{sh\left(\tau\sqrt{1-3c}\right)}{sh(\tau)\sqrt{1-3c}} = \frac{sin\left(\tau\sqrt{3c-1}\right)}{sh(\tau)\sqrt{3c-1}}}$$

valid for any slope $c = \rho_\Lambda^2 \geqslant 0$

i.e. the lower bound predicted by HQET (Isgur et al.)

This is an irreducible Isgur-Wise function since only one irreducible representation contributes to the integral formula

One-parameter functions satisfying the Lorentz constraints

- Isgur-Wise function for baryons $j^P = 0^+$ $\qquad \Lambda_b \to \Lambda_c \ell \nu$

$$\xi_\Lambda(w) = \left(\frac{2}{w+1}\right)^{2\rho_\Lambda^2} \qquad \text{with} \qquad \rho_\Lambda^2 \geq \frac{1}{4}$$

Rigorous lower bound (Isgur et al. SR) : $\qquad \rho_\Lambda^2 \geq 0$

- Isgur-Wise function for mesons $j^P = \frac{1}{2}^-$ $\qquad \overline{B} \to D(D^*)\ell\nu$

One can apply the method to mesons (spin complications)

$$\xi(w) = \left(\frac{2}{w+1}\right)^{2\rho^2} \qquad \text{with} \qquad \rho^2 \geq \frac{3}{4}$$

Rigorous lower bound (Bjorken + Uraltsev SR) : $\qquad \rho^2 \geq \frac{3}{4}$

Clean group theoretical interpretation : only one irreducible representation contributes to the integral formula

BPS limit of HQET

$\mu_\pi^2 = \mu_G^2 \quad \rightarrow \quad \tau_{1/2}^{(n)}(1) = 0 \qquad$ (Uraltsev, 2001)

Limit of HQET $\quad (\vec{\sigma}.i\vec{D})h_v|\overline{B}>=0$ (small components in $\overline{B} \rightarrow 0$)

Covariant form $\quad \gamma_5 i\not{D} h_v|\overline{B}>=0 \quad$ (eq. of motion $\ (iD.v)h_v = 0$)

$\gamma_5 i\not{D}\gamma_5 i\not{D} = -\left[(iD)^2 + \frac{g_s}{2}\sigma_{\alpha\beta}G^{\alpha\beta}\right] \quad \rightarrow \quad \mu_\pi^2 = \mu_G^2$

Leading and subleading matrix elements $\quad \left(\frac{1}{2}^-, 0^-\right) \rightarrow \left(\frac{1}{2}^+, 0^+\right)$

$$< D(0^+)(v')|\overline{h}_{v'}^{(c)}\Gamma h_v^{(b)}|\overline{B}(v)> = 2\tau_{1/2}(w)\,Tr\left[P'_+\Gamma P_+(-\gamma_5)\right]$$

$$< D(0^+)(v')|\overline{h}_{v'}^{(c)}\Gamma i\overrightarrow{\mathcal{D}}_\lambda h_v^{(b)}|\overline{B}(v)> = Tr\left[S_\lambda^{(b)}P'_+\Gamma P_+(-\gamma_5)\right]$$

$$< D(0^+)(v')|\overline{h}_{v'}^{(c)} i\overleftarrow{\mathcal{D}}_\lambda\Gamma h_v^{(b)}|\overline{B}(v)> = Tr\left[S_\lambda^{(c)}P'_+\Gamma P_+(-\gamma_5)\right]$$

$$S_\lambda^{(Q)} = \zeta_1^{(Q)}v_\lambda + \zeta_2^{(Q)}v'_\lambda + \zeta_3^{(Q)}\gamma_\lambda$$

Shape of the Isgur-Wise function in the BPS limit of HQET

Eq. of motion + translational invariance : $\zeta_3^{(b)(n)}(1) = -\Delta E_{1/2}^{(n)} \tau_{1/2}^{(1)(n)}(1)$

$$i\partial_\lambda < D(0^+)(v')|\overline{h}_{v'}^{(c)}\Gamma h_v^{(b)}|\overline{B}(v) > = (\overline{\Lambda} v_\lambda - \overline{\Lambda}^* v'_\lambda)\ 2\tau_{1/2}(w) Tr[P'_+ \Gamma P_+(-\gamma_5)]$$

BPS $< D(0^+)(v')|\overline{h}_{v'}^{(c)}\Gamma i\overrightarrow{\mathcal{D}}_\lambda h_v^{(b)}|\overline{B}(v) > = 0 \ \rightarrow \ \zeta_3^{(b)(n)}(1) = 0$

$\rightarrow \ \tau_{1/2}^{(1)(n)}(1) = 0 \ \rightarrow \ \rho^2 = \frac{3}{4}$ (from Bjorken + Uraltsev SR)

BPS with two derivatives $\rightarrow \ \tau_{3/2}^{(2)(n)}(1) = 0 \ \rightarrow \ \sigma^2 = \frac{15}{16}$

To generalize need to demonstrate $\tau_{L-1/2}^{(L)(n)}(1) = 0$

By induction : $\tau_{1/2}^{(1)(n)}(1) = \tau_{3/2}^{(2)(n)}(1) = 0$, assume $\tau_{L-3/2}^{(L-1)(n)}(1) = 0$

Vector and Axial SR $\rightarrow \ \tau_{L-1/2}^{(L)(n)}(1) = 0 \ \rightarrow (-1)^L \xi^{(L)}(1) = \frac{(2L+1)!!}{2^{2L}}$

Therefore BPS implies the explicit form $\xi(w) = \left(\frac{2}{w+1}\right)^{3/2}$

Example 3 (only the principal series contributes)

$$\xi(w) = \frac{1}{\left[1+\frac{c}{2}(w-1)\right]^2} = \frac{8}{c^2}\int_0^\infty \frac{\rho^2}{sh(\pi\rho)} \frac{sh(\gamma\rho)}{sh(\gamma)} \frac{sin(\rho\tau)}{\rho \ sh(\tau)} \ d\rho$$

$(cos\gamma = \frac{2}{c} - 1)$ valid for any slope $c = \rho_\Lambda^2 \geqslant 1$

Example 4

From the integral representation
if the curvature saturates its lower bound

$$\xi(w) = \frac{sh\left(\tau\sqrt{1-3c}\right)}{sh(\tau)\sqrt{1-3c}} = \frac{sin\left(\tau\sqrt{3c-1}\right)}{sh(\tau)\sqrt{3c-1}}$$

valid for any slope $c = \rho_\Lambda^2 \geqslant 0$

i.e. the lower bound predicted by HQET (Isgur et al.)

This is an irreducible Isgur-Wise function :
One irreducible representation contributes to the integral formula

The Isgur-Wise function is a function of positive type

For any N and any complex numbers a_i and velocities v_i

$\sum_{i,j=1}^{N} a_i^* a_j \ \xi(v_i . v_j) \geq 0$ or, in a covariant form

$\int \frac{d^3\vec{v}}{v^0} \frac{d^3\vec{v}\,'}{v\,'^0} \ \psi(v')^* \ \xi(v.v') \ \psi(v) \geq 0$ for any $\psi(v)$

From the Sum Rule $(w_i = v_i . v', w_j = v_j . v', w_{ij} = v_i . v_j)$

$$\xi(w_{ij}) = \sum_n \sum_L \tau_L^{(n)}(w_i)^* \tau_L^{(n)}(w_j)$$
$$\sum_{0 \leq k \leq L/2} C_{L,k} \ (w_i^2 - 1)^k (w_j^2 - 1)^k (w_i w_j - w_{ij})^{L-2k}$$

Legendre polynomial. Use rest frame $v' = (1, 0, 0, 0)$

$$\sum_{i,j=1}^{N} a_i^* a_j \ \xi(v_i . v_j) = 4\pi \sum_{i,j=1}^{N} \sum_n \sum_L \frac{2^L (L!)^2}{(2L+1)!} \sum_{m=-L}^{m=+L}$$
$$\left[a_i \ \tau_L^{(n)}\left(\sqrt{1 + \vec{v}_i^{\,2}}\right) \mathcal{Y}_L^m(\vec{v}_i)\right]^* \left[a_j \ \tau_L^{(n)}\left(\sqrt{1 + \vec{v}_j^{\,2}}\right) \mathcal{Y}_L^m(\vec{v}_j)\right] \geq 0$$

One example : application to the exponential form

$$\xi(w) = exp\left[-c(w-1)\right]$$

$$I = \int \frac{d^3\vec{v}}{v^0}\frac{d^3\vec{v}\,'}{v\,'^0}\ \phi(|\vec{v}\,'|)^*\ exp\left[-c((v.v')-1)\right]\ \phi(|\vec{v}|)$$

$$= 16\pi^3\ \frac{e^c}{c}\int_{-\infty}^{\infty} K_{i\rho}(c)\ |\tilde{f}(\rho)|^2\ d\rho$$

$$f(\eta) = sh(\eta)\ \phi(sh(\eta))$$

$$K_\nu(z) = \frac{1}{2}\int_{-\infty}^{\infty} exp[-z\ ch(t)]\ e^{\nu t}\ dt \qquad \text{Macdonald function}$$

Whatever the slope $c > 0$, $K_{i\rho}(c)$ *takes negative values*

Asymptotic formula

$$K_{i\rho}(c) \sim \sqrt{\frac{2\pi}{\rho}}\ e^{-\rho\pi/2}\ cos\left[\rho\left(log\left(\frac{2\rho}{c}\right)-1\right)-\frac{\pi}{4}\right] \qquad (\rho >> c)$$

Therefore there a function $\psi(v)$ for which the integral $I < 0$

The exponential form is inconsistent with the Sum Rules

Sum Rule and Lorentz group approaches are equivalent

- The Lorentz group approach implies that $\xi(w)$ is of positive type

$\xi(w) = < U(B_{v'})\psi_0 | U(B_v)\psi_0 >$ $\quad (B_v :$ boost $v_0 \to v)$

$\sum_{i,j=1}^{N} a_i^* a_j \; \xi(v_i.v_j) = \| \sum_{j=1}^{N} a_j U(B_{v_j})\psi_0 \|^2 \geq 0$

- The Sum Rule approach implies the Lorentz group approach

A function $f(\Lambda)$ on the group $SL(2,C)$ is of positive type when

$\sum_{i,j=1}^{N} a_i^* a_j \; f(\Lambda_i^{-1}\Lambda_j) \geq 0$ $\quad (N \geq 1$, complex a_i, $\Lambda_i \in SL(2,C))$

Theorem (Dixmier) : for any function $f(\Lambda)$ of positive type exists a unitary representation $U(\Lambda)$ of $SL(2,C)$ in a Hilbert space $\mathcal{H}$ and an element $\phi_0 \in \mathcal{H} \to f(\Lambda) = < \phi_0 | U(\Lambda)\phi_0 >$

Definition of $f(\Lambda_i^{-1}\Lambda_j) = \xi(v_i.v_j) = \xi(v_0.\Lambda_i^{-1}\Lambda_j v_0)$

Lorentz group in our approach vs. Poincaré group

One can ask the question about which is the relation between the Lorentz group used in our approach and the Poincaré group

• Naïmark : we use the Lorentz group (no translations),
more precisely the orthochronous proper Lorentz group,
more precisely its connected recovering to get half-integer spin
(parity must also be included)

• Wigner : Poincaré group (translations included)
$\rightarrow$ classification of massive and massless particles

• These are two quite different kinds of problems