Radiative Corrections in Semileptonic Decays

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(Member of the EEC Network FLAVIAnet)

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OUTLINE

- Motivation
- Anatomy of radiative corrections in the SM
- Conclusions Outlook

MOTIVATION

• Increasing precision reached by ongoing (NA48/2, KLOE, KTeV, ISTRA,...) and future (TRIUF/PIENU, PSI/PEN, J-PARC, CERN/P-326,...) experimental programs for semileptonic decay modes of light pseudoscalar mesons

$$\pi_{\ell 2}/K_{\ell 2}, \quad K_{\ell 3}, \quad K_{\ell 4}, \quad \pi - \beta, \cdots$$

http://www.lnf.it/conference/kaon07

- Appropriate framework : low-energy effective theory

Urech, Nucl. Phys. B 433 (1995)

Neufeld + Rupertsberger, Z. Phys. C 68 (1995); C 71 (1996)

M. Knecht + R. Urech, Nucl. Phys. B 519 (1998)

U.-G. Meißner et al., Phys. Lett. B 406 (1998); B 407 (1998)

M. Knecht et al., Eur. Phys. J. C 12 (2000)

V. Cirigliano et al., Eur. Phys. J. C 23 (2002)

- Issue of low-energy constants

B. Moussallam, Nucl. Phys. B 504 (1997)

S. Descotes-Genon + B. Moussallam (2005)

MOTIVATION

- Increasing precision on hadronic decays of the tau meson, and on semileptonic decays of haevy-light mesons
    - Low-energy effective theory is no longer the appropriate framework

MOTIVATION

- Increasing precision on hadronic decays of the tau meson, and on semileptonic decays of haevy-light mesons
  - Low-energy effective theory is no longer the appropriate framework
  - SM is the appropriate framework

A. Sirlin, Rev. Mod. Phys. 50 (1978)

S. Weinberg, Phys. Rev. D 8 (1973)

G. Preparata + W. I. Weisberger, Phys Rev. 175 (1968)

Abers et al., Phys. Rev. 167 (1968)

MAIN INGREDIENTS

1. Structure of the SM interactions

Interaction between fermions and gauge bosons

$$\mathcal{L}_{\text{int}} = -eA_\mu J^\mu_{(\gamma)} - \frac{\sqrt{g^2+g'^2}}{2} Z_\mu J^\mu_{(Z)} - \frac{g}{\sqrt{2}}\left[W^-_\mu J^\mu_{(W)} + W^+_\mu J^{\mu\dagger}_{(W)}\right],$$

Interaction between fermions and scalars

$$\mathcal{L}_{\text{int}} = -\frac{g}{2M_W}\left[H J_{(H)} + \phi^0 J_{(\phi^0)} + \sqrt{2}(\phi^- J_{(\phi^\pm)} + \phi^+ J^\dagger_{(\phi^\pm)})\right],$$

$$J_{(H)} = \sum_f m_f \overline{\psi_f}\psi_f \quad J_{(\phi^0)} = \partial_\mu J^\mu_{(Z)} \quad J_{(\phi^\pm)} = \partial_\mu J^\mu_{(W)}$$

Interaction between gauge bosons

2. Equal-time commutation relations in QCD between currents

$$\begin{aligned}
[J^0_{(W)}(x)\,,\,J^\mu_{(Z)}(y)]_{x^0=y^0} &= 2\cos^2\theta_w\,J^\mu_{(W)}(x)\,\delta^3(x-y)\\
[J^0_{(W)}(x)\,,\,J^\mu_{(\gamma)}(y)]_{x^0=y^0} &= J^\mu_{(W)}(x)\,\delta^3(x-y)\\
[J^0_{(W)}(x)\,,\,J^{\mu\dagger}_{(W)}(y)]_{x^0=y^0} &= -2\left[\sin^2\theta_w\,J^\mu_{(\gamma)}(x)\,+\,\frac{1}{2}\,J^\mu_{(Z)}(x)\right]\delta^3(x-y)
\end{aligned}$$

and between currents and their divergences

$$\begin{aligned}
[J^0_{(W)}(x)\,,\,J_{(\phi^0)}(y)]_{x^0=y^0} &= J_{(\phi^\pm)}(x)\,\delta^3(x-y)\\
[J^0_{(W)}(x)\,,\,J^\dagger_{(\phi^\pm)}(y)]_{x^0=y^0} &= -\,\frac{1}{2}\,\left[J_{(\phi^0)}(x)\,+\,iJ_{(H)}(x)\right]\delta^3(x-y)\\
[J^0_{(W)}(x)\,,\,J_{(H)}(y)]_{x^0=y^0} &= iJ_{(\phi^\pm)}(x)\,\delta^3(x-y)\\
[J^0_{(Z)}(x)\,,\,J_{(\phi^\pm)}(y)]_{x^0=y^0} &= (\sin^2\theta_w\,-\,\cos^2\theta_w)\,J_{(\phi^\pm)}(x)\,\delta^3(x-y)\\
[J^0_{(\gamma)}(x)\,,\,J_{(\phi^\pm)}(y)]_{x^0=y^0} &= -J_{(\phi^\pm)}(x)\,\delta^3(x-y)
\end{aligned}$$

No Schwinger terms !

3. Correlation functions of currents in QCD

$$\int d^4x e^{ik\cdot x} \int d^4y e^{iq\cdot y} \Big\{ \langle F \,|\, T\{ J^{\rho\,\dagger}_{(I)}(x+y) J^{\sigma}_{(I)}(x) J^{\mu}_{(W)}(0) \}_C \,|\, I \,\rangle_{QCD}$$

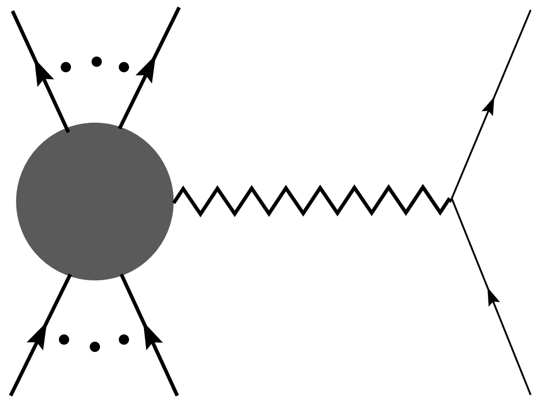
$$\mathcal{T}^{\mu\nu}_{(X)}(q,p) \equiv \int d^4x \, e^{iq\cdot x} \, \langle \, F \,|\, T\{ J^{\mu}_{(X)}(x) J^{\nu}_{(W)}(0) \} \,|\, I \,\rangle_{QCD}$$

$X = \gamma, W, Z$

4. Ward identities that follow from the partial conservation of these currents in QCD

5. Short-distance properties of the above correlators in QCD

Tree level ('tHooft-Feynman gauge)



$$\mathcal{A}^{(0)} = \mathcal{A}^{(0)}_{(W)} + \mathcal{A}^{(0)}_{(\phi^\pm)}$$

$$\mathcal{A}^{(0)}_{(W)} = \left(\frac{-g}{\sqrt{2}}\right)^2 \times i\,\langle\, F \,|\, J^{\mu}_{(W)}(0) \,|\, I \,\rangle_{QCD} \times \frac{(-i)}{p^2 - M_W^2} \times i\, L^{\dagger}_{\mu}$$

$$\mathcal{A}^{(0)}_{(\phi^\pm)} = \left(\frac{-g}{\sqrt{2}M_W}\right)^2 \times i\,\langle\, F \,|\, \partial \cdot J_{(W)}(0) \,|\, I \,\rangle_{QCD} \times \frac{i}{p^2 - M_W^2} \times i(p_\ell - p_{\nu_\ell}) \cdot L^{\dagger}$$

$$L^{\dagger}_{\mu} = \frac{1}{2}\bar{u}_{\nu_\ell}(p_{\nu_\ell})\gamma_\mu(1-\gamma_5)v_\ell(p_\ell), \quad p = P_I - P_F = p_\ell + p_{\nu_\ell}$$

Properties :

- of order $\mathcal{O}(G_{\mathrm{F}})$, $p^2 \ll M_W^2$ (Fermi theory)
- factorization between the leptonic and the hadronic part (form factors)

Properties :

- of order $\mathcal{O}(G_F)$, $p^2 \ll M_W^2$ (Fermi theory)
- factorization between the leptonic and the hadronic part (form factors)

Including radiative corrections :

- factorization no longer holds
- all scales of the SM involved

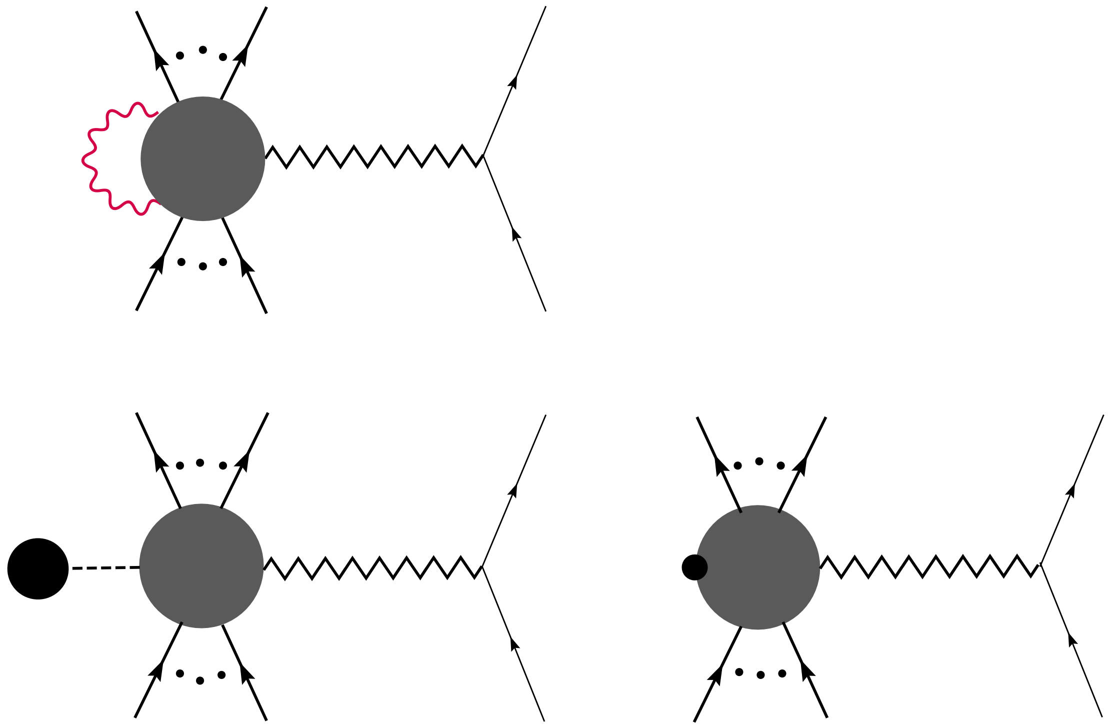
$$m_e, m_\mu, m_u, m_d \ll m_s \ll \Lambda_H \sim 1\,\text{GeV} < m_c, m_\tau < m_b \ll M_W, M_Z, m_t$$

The NLO amplitude receives contributions $\mathcal{O}(\alpha G_F)$, but also

$$\mathcal{O}(\alpha G_F \times \frac{m_\ell^2}{M_{W,Z}^2}) \quad \mathcal{O}(\alpha G_F \times \frac{m_q m_{q'}}{M_{W,Z}^2}) \quad \mathcal{O}(\alpha G_F \times \frac{\Lambda_H^2}{M_{W,Z}^2})$$

which are neglected, being $\sim \mathcal{O}(G_F^2)$

Corrections to the hadronic matrix element



- loops of gauge bosons and scalars
- tadpoles and tadpole counterterms
- counterterms

$$\delta \langle F \,|\, J^{\mu}_{(W)}(0) \,|\, I \rangle_{QCD} = \sum_{X} \lim_{k\to 0} iT^{\mu}_{(X)}(k,p)$$

$$X = W, Z, \gamma, \phi^{\pm}, \phi^{0}, H$$

$$\begin{aligned} T^{\mu}_{(X)}(k,p) &= \frac{1}{2} C_X \int \frac{d^4 q}{(2\pi)^4} \frac{1}{q^2 - M_X^2} \left[\eta_{\rho\sigma} - \frac{q_\rho q_\sigma}{q^2 - M_X^2} \right] \\ &\times \int d^4 x e^{ik\cdot x} \int d^4 y e^{iq\cdot y} \Big\{ \langle F \,|\, T\{ J^{\rho\,\dagger}_{(X)}(x+y) J^{\sigma}_{(X)}(x) J^{\mu}_{(W)}(0)\}_C \,|\, I \rangle_{QCD} \\ &\quad + \langle F \,|\, T\{ J^{\rho\,\dagger}_{(X)}(x) J^{\sigma}_{(X)}(x+y) J^{\mu}_{(W)}(0)\}_C \,|\, I \rangle_{QCD} \Big\} \\ &\quad - B^{\mu}_{(X)}(k,p) \end{aligned}$$

$$C_W = \frac{1}{2} g^2, \quad C_Z = \frac{1}{8}(g^2 + g'^2) = \frac{1}{8} g^2 \frac{1}{\cos^2\theta_w}, \quad C_\gamma = \frac{1}{2} g^2 \sin^2\theta_w$$

$B^{\mu}_{(X)}(k,p)$ subtracts the pole at each $(p_{I_i}-k)^2 = M^2_{I_i}$ or each $(p_{F_i}+k)^2 = M^2_{F_i}$

Example:

$$\delta_{(X)}\langle \Omega \,|\, J^{\mu}_{(W)}(0) \,|\, P^{+}(p)\,\rangle_{QCD}$$

$$-B^{\mu}_{(X)} = +\,i\langle \Omega \,|\, J^{\mu}_{(W)}(0) \,|\, P^{+}(p-k)\,\rangle_{QCD} \times \frac{(\delta M^2_{P+})_{(X)}}{(p-k)^2 - M^2_{P+}} \qquad (1)$$

$$\left(\delta M^2_{P+}\right)_{(X)} = -C_X \int \frac{d^4q}{(2\pi)^4}\,\frac{1}{q^2 - M_X^2}\left[\eta_{\rho\sigma} - \frac{q_\rho q_\sigma}{q^2 - M_X^2}\right] \times \int d^4y\, e^{iq\cdot y}\,\langle\, P^{+}(p) \,|\, T\{J^{\rho\,\dagger}_{(X)}(y) J^{\sigma}_{(X)}(0) \,|\, P^{+}(p)\,\rangle_{QCD} \qquad (2)$$

The subtraction of $B^{\mu}_{(X)}(k,p)$ corresponds, at first order in perturbation, to the factor $\sqrt{Z}$ for each external line

L. S. Brown, Phys. Rev. 189 (1969)

Use the Ward identity to express $T^{\mu}_{(X)}(k,p)$ in terms of correlators of two currents

$$
\begin{aligned}
(p-k)_\mu T^{\mu}_{(X)}(k,p) \;=\; \Big\{ & D_{(X)}(k,p) \;-\; (p-k)_\mu B^{\mu}_{(X)}(k,p) \\
& + \int \frac{d^4q}{(2\pi)^4}\,\frac{1}{q^2-M_X^2}\left[\eta_{\rho\sigma} - \frac{q_\rho q_\sigma}{q^2-M_X^2}\right] \times \left[\mathcal{V}^{\rho\sigma}_{(X)}(q+p-k) \;+\; \mathcal{V}^{\rho\sigma}_{(X)}(q+p)\right] \Big\}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{V}^{\rho\sigma}_{(W)}(q) &= -i\,\frac{g^2}{2}\int d^4x e^{iq\cdot x}\,\langle\Omega\,|\,T\{\left[\sin^2\theta_w J^{\rho}_{(\gamma)}(x) + \frac{1}{2}\,J^{\rho}_{(Z)}(x)\right] J^{\sigma}_{(W)}(0)\}\,|\,P^+(p)\,\rangle_{QCD} \\
\mathcal{V}^{\rho\sigma}_{(Z)}(q) &= +i\,\frac{g^2}{4}\int d^4x e^{iq\cdot x}\,\langle\Omega\,|\,T\{J^{\rho}_{(W)}(x) J^{\sigma}_{(Z)}(0)\}\,|\,P^+(p)\,\rangle_{QCD} \\
\mathcal{V}^{\rho\sigma}_{(\gamma)}(q) &= +i\,\frac{g^2}{2}\int d^4x e^{iq\cdot x}\,\sin^2\theta_w\langle\Omega\,|\,T\{J^{\rho}_{(W)}(x) J^{\sigma}_{(\gamma)}(0)\}\,|\,P^+(p)\,\rangle_{QCD}
\end{aligned}
$$

The corrections to the amplitude induced by the contributions of the two-current correlators are

$$
\begin{aligned}
\mathcal{V}_{(Z)}(p) &= -\,\frac{ig^4}{8}\times\frac{1}{p^2-M_W^2}\times L^{\dagger}_{\mu}\int\frac{d^4q}{(2\pi)^4}\,\frac{1}{q^2-M_Z^2}\,\frac{\partial}{\partial q_\mu}\,\mathcal{T}^{\rho\sigma}_{(Z)}(-q,p)\eta_{\rho\sigma} \\
\mathcal{V}_{(\gamma)}(p) &= -\,\frac{ig^4}{4}\,\sin^2\theta_w\times\frac{1}{p^2-M_W^2}\times L^{\dagger}_{\mu}\int\frac{d^4q}{(2\pi)^4}\,\frac{1}{q^2}\,\frac{\partial}{\partial q_\mu}\,\mathcal{T}^{\rho\sigma}_{(\gamma)}(-q,p)\eta_{\rho\sigma}
\end{aligned}
$$

and

$$\mathcal{V}_{(W)}(p) = \mathcal{V}^{(1)}_{(W)}(p) + \mathcal{V}^{(2)}_{(W)}(p)$$

with

$$\begin{aligned}
\mathcal{V}^{(1)}_{(W)}(p) &= +\frac{ig^4}{8} \times \frac{1}{p^2 - M_W^2} \times L^\dagger_\mu \int \frac{d^4q}{(2\pi)^4} \frac{1}{q^2 - M_W^2} \frac{\partial}{\partial q_\mu} \mathcal{T}^{\rho\sigma}_{(Z)}(q+p,p)\eta_{\rho\sigma} \\
\mathcal{V}^{(2)}_{(W)}(p) &= +\frac{ig^4}{4} \sin^2\theta_w \times \frac{1}{p^2 - M_W^2} \times L^\dagger_\mu \int \frac{d^4q}{(2\pi)^4} \frac{1}{q^2 - M_W^2} \frac{\partial}{\partial q_\mu} \mathcal{T}^{\rho\sigma}_{(\gamma)}(q+p,p)\eta_{\rho\sigma}
\end{aligned}$$

Decompose the photon propagator as

$$\frac{1}{q^2} = \frac{1}{q^2 - M_W^2} - \frac{M_W^2}{q^2 - M_W^2}\frac{1}{q^2}\,, \tag{3}$$

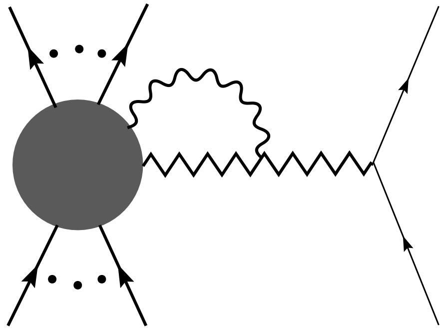
and write accordingly

$$T^\mu_{(\gamma)}(0,p) = T^\mu_{(\gamma>)}(0,p) + T^\mu_{(\gamma<)}(0,p)\,, \tag{4}$$

$$\mathcal{V}_{(\gamma)}(p) = \mathcal{V}_{(\gamma>)}(p) + \mathcal{V}_{(\gamma<)}(p)\,. \tag{5}$$

There remain residual three-point function contributions, which can be absorbed into the counterterms, up to corrections that are or order $\mathcal{O}(G_\mathrm{F}^2)$

The Z boson and photon exchange contributions



$$\mathcal{U}_{(Z)}(p) = \left(-i\,\frac{g}{\sqrt{2}}\right)^2 \times \frac{(-i)}{p^2 - M_W^2} \times L^{\mu\dagger} \times \int \frac{d^4q}{(2\pi)^4}\,\frac{(-i)}{q^2 - M_Z^2}\,\frac{(-i)}{(q-p)^2 - M_W^2}$$
$$\times\,(ig\cos\theta_w)\,V_{\mu\nu\rho}(q,p) \times (-i)\,\frac{\sqrt{g^2 + g'^2}}{2}\,\mathcal{T}^{\nu\rho}_{(Z)}(q,p)$$

$$\mathcal{U}_{(\gamma)}(p) = \left(-i\,\frac{g}{\sqrt{2}}\right)^2 \times \frac{(-i)}{p^2 - M_W^2} \times L^{\mu\dagger} \times \int \frac{d^4q}{(2\pi)^4}\,\frac{(-i)}{q^2}\,\frac{(-i)}{(q-p)^2 - M_W^2}$$
$$\times\,(ig\sin\theta_w)\,V_{\mu\nu\rho}(q,p) \times (-i)\,g\sin\theta_w\,\mathcal{T}^{\nu\rho}_{(\gamma)}(q,p)$$

$$V_{\mu\nu\rho}(q,p) = (2q-p)_\mu\eta_{\nu\rho} + (2p-q)_\nu\eta_{\mu\rho} - (p+q)_\rho\eta_{\mu\nu} \qquad (6)$$

Using the Ward idetities satisfied by the currents in QCD, leads to

$$\begin{aligned}\mathcal{U}_{(Z)}(p) &= -ig^4 \times \frac{1}{p^2 - M_W^2} \cos^2\theta_w \langle F \,|\, J^\mu_{(W)}(0) \,|\, I \rangle_{QCD} \times L^\dagger_\mu \times I_{(Z)}(p) \\ &\quad -i\,\frac{g^4}{2} \times \frac{1}{p^2 - M_W^2} \times L^\dagger_\mu \times \int \frac{d^4q}{(2\pi)^4} \frac{1}{(q^2 - M_Z^2)[(q-p)^2 - M_W^2]} \times q^\mu \eta_{\nu\rho} \mathcal{T}^{\nu\rho}_{(Z)}(q,p) \\ &\quad + \mathcal{O}(\Lambda_H^2 G_{\mathsf{F}}^2, m_\ell^2 G_{\mathsf{F}}^2)\end{aligned}$$

$$\begin{aligned}\mathcal{U}_{(\gamma)}(p) &= -ig^4 \times \frac{1}{p^2 - M_W^2} \sin^2\theta_w \langle F \,|\, J^\mu_{(W)}(0) \,|\, I \rangle_{QCD} \times L^\dagger_\mu \times I_{(\gamma)}(p) \\ &\quad -i\,g^4 \sin^2\theta_w \times \frac{1}{p^2 - M_W^2} \times L^\dagger_\mu \times \int \frac{d^4q}{(2\pi)^4} \frac{1}{q^2[(q-p)^2 - M_W^2]} \times q^\mu \eta_{\nu\rho} \mathcal{T}^{\nu\rho}_{(Z)}(q,p) \\ &\quad + \mathcal{O}(\Lambda_H^2 G_{\mathsf{F}}^2, m_\ell^2 G_{\mathsf{F}}^2)\end{aligned}$$

$$I_{(Z)}(p) = i \int \frac{d^4q}{(2\pi)^4} \frac{1}{(q^2 - M_Z^2)[(q-p)^2 - M_W^2]} \qquad I_{(\gamma)}(p) = i \int \frac{d^4q}{(2\pi)^4} \frac{1}{q^2[(q-p)^2 - M_W^2]}$$

Consider then the two sums,

$$\mathcal{U}_{(\gamma)} + \mathcal{V}_{(\gamma>)} + \mathcal{V}^{(2)}_{(W)} \quad \text{and} \quad \mathcal{U}_{(Z)} + \mathcal{V}_{(Z)} + \mathcal{V}^{(1)}_{(W)}$$

which combine

- corrections affecting the hadronic part, *i.e.* corrections to the strong matrix element induced by exchanges of virtual γ's, Z's and W's between the quarks

- with corrections of the hadron$-W$ (or $\bar{q} - q - W$) vertex, given by $\mathcal{U}_{(\gamma)}$ and $\mathcal{U}_{(Z)}$

One obtains a result which, besides the universal divergences represented by $I_{(Z)}(p)$ and $I_{(\gamma)}(p)$, is finite

One obtains a result which, besides the universal divergences represented by $I_{(Z)}(p)$ and $I_{(\gamma)}(p)$, is finite

$$
\begin{aligned}
\mathcal{V}_{(Z)}(p) + \mathcal{V}^{(1)}_{(W)}(p) + \mathcal{U}_{(Z)}(p) \;=\;& +i\,\frac{g^4}{4} \times \frac{1}{p^2 - M_W^2} \times L^\dagger_\mu \times \int \frac{d^4 q}{(2\pi)^4}\, q^\mu \eta_{\nu\rho} \mathcal{T}^{\nu\rho}_{(Z)}(q,p) \\
& \times \left[\frac{1}{q^2 - M_Z^2} - \frac{1}{(q-p)^2 - M_W^2} \right]^2 \\
& - i g^4 \times \frac{1}{p^2 - M_W^2} \cos^2\theta_w \,\langle F \,|\, J^\mu_{(W)}(0) \,|\, I\,\rangle_{QCD} \times L^\dagger_\mu \times I_{(Z)}(p) \\
& + \mathcal{O}(\Lambda_H^2 G_{\mathsf{F}}^2, m_\ell^2 G_{\mathsf{F}}^2)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{V}_{(\gamma>)}(p) + \mathcal{V}^{(2)}_{(W)}(p) + \mathcal{U}_{(\gamma)}(p) \;=\;& -i g^4 \sin^2\theta_w \times \frac{1}{p^2 - M_W^2} \times L^\dagger_\mu \times \int \frac{d^4 q}{(2\pi)^4}\, q^\mu \eta_{\nu\rho} \mathcal{T}^{\nu\rho}_{(\gamma)}(q,p) \\
& \times \left[\frac{1}{(q^2 - M_W^2)^2} - \frac{1}{[(q-p)^2 - M_W^2]\,q^2} \right] \\
& - i g^4 \sin^2\theta_w \times \frac{1}{p^2 - M_W^2} \,\langle F \,|\, J^\mu_{(W)}(0) \,|\, I\,\rangle_{QCD} \times L^\dagger_\mu \times I_{(\gamma)}(p) \\
& + \mathcal{O}(\Lambda_H^2 G_{\mathsf{F}}^2, m_\ell^2 G_{\mathsf{F}}^2)
\end{aligned}
$$

The result only requires the knowledge of the leading short-distance behaviour of the QCD correlators $\mathcal{T}^{\nu\rho}_{(Z)}(q,p)$ and $\mathcal{T}^{\nu\rho}_{(\gamma)}(q,p)$, which is given by free-field theory, up to (perturbative, hence calculable) corrections of order $\mathcal{O}(\alpha_s)$

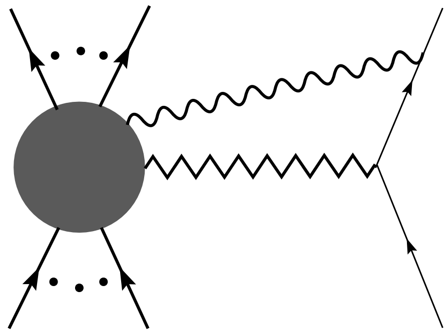
$$\mathcal{V}_{(Z)}(p) + \mathcal{V}^{(1)}_{(W)}(p) + \mathcal{U}_{(Z)}(p) = \mathcal{A}^{(0)} \times \left(\frac{\alpha}{4\pi}\right) \times \left[\delta^{\text{fin}}_{(Z)} + \delta^{\text{div}}_{(Z)}\right] + \mathcal{O}(\Lambda_H^2 G_{\text{F}}^2, m_\ell^2 G_{\text{F}}^2)$$

$$\begin{aligned}\delta^{\text{fin}}_{(Z)} &= \frac{1}{2}\cot^2\theta_w \left[2 + \frac{M_Z^2 + M_W^2}{M_Z^2 - M_W^2}\ln\frac{M_Z^2}{M_W^2} + \Delta_{(Z)}(\alpha_s)\right] \\ \delta^{\text{div}}_{(Z)} &= -32\pi^2\cot^2\theta_w \times I_{(Z)}(p)\end{aligned}$$

$$\mathcal{V}_{(\gamma>)}(p) + \mathcal{V}^{(2)}_{(W)}(p) + \mathcal{U}_{(\gamma)}(p) = \mathcal{A}^{(0)} \times \left(\frac{\alpha}{4\pi}\right) \times \left[\delta^{\text{fin}}_{(\gamma)} + \delta^{\text{div}}_{(\gamma)}\right] + \mathcal{O}(\Lambda_H^2 G_{\text{F}}^2, m_\ell^2 G_{\text{F}}^2)$$

$$\begin{aligned}\delta^{\text{fin}}_{(\gamma)} &= \frac{1}{2}\left[1 + \Delta_{(\gamma)}(\alpha_s)\right] \\ \delta^{\text{div}}_{(\gamma)} &= -32\pi^2 \times I_{(\gamma)}(p)\end{aligned}$$

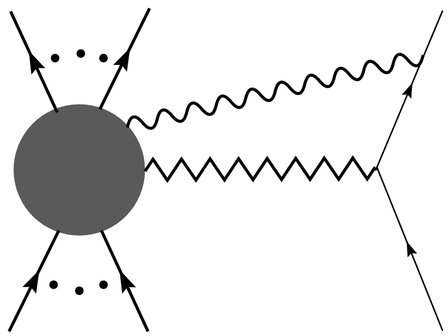
The Z-boson box contribution



$$\mathcal{A}^{\ell\text{-box}}_{(Z)}(p) = \left(-\frac{ig}{\sqrt{2}}\right)^2 \times \left(-\frac{i\sqrt{g^2+g'^2}}{2}\right)^2 \times \int \frac{d^4q}{(2\pi)^4} \frac{(-i)}{q^2-M_Z^2} \frac{(-i)}{(q-p)^2-M_W^2}$$
$$\times\, \bar{u}_{\nu_\ell}(p_{\nu_\ell})\gamma_\rho \left(\frac{1-\gamma_5}{2}\right) \frac{i}{\not{q} - \not{p}_\ell - m_\ell} \gamma_\nu \left[T^3_\ell(1-\gamma_5) - 2Q_\ell \sin^2\theta_w\right] v_\ell(p_\ell)\, \mathcal{T}^{\nu\rho}_{(Z)}(q,p)$$

$$\mathcal{A}^{\nu_\ell\text{-box}}_{(Z)}(p) = \left(-\frac{ig}{\sqrt{2}}\right)^2 \times \left(-\frac{i\sqrt{g^2+g'^2}}{2}\right)^2 \times \int \frac{d^4q}{(2\pi)^4} \frac{(-i)}{q^2-M_Z^2} \frac{(-i)}{(q-p)^2-M_W^2}$$
$$\times\, \bar{u}_{\nu_\ell}(p_{\nu_\ell})\gamma_\nu\, T^3_{\nu_\ell}(1-\gamma_5) \frac{i}{\not{p}_{\nu_\ell} - \not{q}} \gamma_\rho \left(\frac{1-\gamma_5}{2}\right) v_\ell(p_\ell)\, \mathcal{T}^{\nu\rho}_{(Z)}(q,p)$$

The photon box contribution



$$\mathcal{A}^{\text{box}}_{(\gamma)}(p) = \left(-\frac{ig}{\sqrt{2}}\right)^2 \times (-ie) \times (-ieQ_\ell) \times \int \frac{d^4q}{(2\pi)^4} \frac{(-i)}{q^2} \frac{(-i)}{(q-p)^2 - M_W^2}$$
$$\times \bar{u}_{\nu_\ell}(p_{\nu_\ell})\gamma_\rho \left(\frac{1-\gamma_5}{2}\right) \frac{i}{\not{q} - \not{p}_\ell - m_\ell} \gamma_\nu \, v_\ell(p_\ell) \, (\mathcal{T}^{\nu\rho}_{\gamma)}(q,p)$$

In the case of the Z-box diagrams, both contributions are convergent.

Owing to the presence of two massive propagators, only the terms with the highest power of the loop momentum q and the leading asymptotic behaviour of $\mathcal{T}^{\nu\rho}_{(Z)}(q,p)$ will lead to contributions of order $\mathcal{O}(\alpha G_{\mathrm{F}})$.

The remaining contributions, being of order $\mathcal{O}(\Lambda_H^2 G_{\mathrm{F}}^2)$, are not considered. This also includes the dependence on the momenta p and p_ℓ in the heavy gauge boson propagators, since, for instance,

$$1/[(q-p)^2 - M_W^2] = (q^2 - M_W^2)^{-1} \times [1 + 2(p \cdot q)/(q^2 - M_W^2) + \cdots]$$

Finally, one can replace the propagator $1/(q - p_{\nu_\ell})^2$ in $\mathcal{A}^{\nu_\ell\text{-box}}_{(Z)}(p)$ by $1/[(q-p_\ell)^2 - m_\ell^2]$, since the difference generates terms of order $\mathcal{O}(m_\ell^2 G_{\mathrm{F}}^2)$.

One obtains

$$\mathcal{A}^{\text{box}}_{(Z)}(p) \;=\; \mathcal{A}^{(0)} \times \left(\frac{\alpha}{4\pi}\right) \times \delta^{\text{box}}_{(Z)}$$

$$\delta^{\text{box}}_{(Z)} \;=\; \frac{M_W^2}{M_Z^2 - M_W^2} \ln \frac{M_Z^2}{M_W^2} \left[\left(2 + \frac{1}{2}\right) \cot^2\theta_w \;+\; 3\overline{Q_q}\tan^2\theta_w \;+\; \Delta^{\text{box}}_{(Z)}(\alpha_s) \right]$$

$\overline{Q_q}$ is the average value of the charges in a quark doublet, in units of the positron charge e, $\overline{Q_q} = (1/2) \times [(2/3) + (-1/3)] = 1/6$ (nonuniversal correction)

In the case of the photon-box diagram, although the integrals are convergent, they contain only one propagator with a heavy mass

Therefore, the knowledge of the asymptotic behaviours of $(\mathcal{T}_{(\gamma)}^{P^+})^{\mu\nu}(q,p)$ and of $(\mathcal{T}_{(\gamma)}^{P^+})^{\mu}(q,p)$ alone is not sufficient in order to evaluate this contribution

Neglecting the dependence on the momentum p in the W boson propagator, one obtains

$$\begin{aligned}
\mathcal{A}^{\text{box}}_{(\gamma)}(p) \quad = \quad & -\frac{ig^2e^2}{2M_W^2} \times L_\mu^\dagger \times \int \frac{d^4q}{(2\pi)^4}\,\frac{1}{q^2}\,\frac{M_W^2}{M_W^2-q^2}\,\frac{1}{(q-p_\ell)^2-m_\ell^2}\,\mathcal{T}^{\nu\rho}_{(\gamma)}(q,p) \\
& \times[p_\rho\delta^\mu_\nu - q^\mu\eta_{\nu\rho} - 2\delta^\mu_\rho p_{\ell\nu} - i\,\epsilon^\mu{}_{\nu\rho\tau}q^\tau] \\
& + \frac{ig^2e^2}{2M_W^2} \times L_\mu^\dagger \times \int \frac{d^4q}{(2\pi)^4}\,\frac{1}{q^2}\,\frac{M_W^2}{M_W^2-q^2}\,\frac{1}{(q-p_\ell)^2-m_\ell^2} \\
& \times\left[2i\,\langle F\,|\,J^\mu_{(W)}(0)\,|\,I\,\rangle_{QCD} - \mathcal{T}^\mu_{(\gamma)}(q,p)\right]\,. \qquad (7)
\end{aligned}$$

Other corrections

- Residual three-current correlation functions
- W-propagator self-energy corrections
- Corrections from the Higgs sector
- Leptonic wave-function and vertex corrections
- Real photon emission

Putting it all together

$$
\begin{aligned}
\Gamma(P^+ \to \ell^+ \nu_\ell(\gamma)) \;=\; & \Gamma^{(0)}(P^+ \to \ell^+ \nu_\ell) \times \Bigg\{ 1 + \left(\frac{\alpha}{2\pi}\right) \Big[\delta_{\text{IB}} + \delta_{\text{SD}} + \delta_{\text{INT}} \\
& + \delta_{(\gamma<)} + \delta^{\text{res. 3-pt.}}_{(\gamma<)} + \delta^{\text{box}}_{(\gamma)} + \delta^{\text{w.f.}}_{(\gamma<)} \\
& + \delta^{\text{div}}_{(\gamma)} + \delta^{\text{fin}}_{(\gamma)} + \delta^{\text{div}}_{(Z)} + \delta^{\text{fin}}_{(Z)} + \delta^{\text{box}}_{(Z)} + \delta^{\text{self}} + \delta^{\text{vertex}} + \delta^{\text{w.f. res.}} \Big] \Bigg\}.
\end{aligned}
$$

A similar formula can be established for the muon decay

$$
\begin{aligned}
\Gamma(\mu^+ \to e^+ \nu_\ell \bar{\nu}_\mu(\gamma)) \;=\; & \Gamma^{(0)}(\mu^+ \to e^+ \nu_\ell \bar{\nu}_\mu) \\
& \times \Bigg\{ 1 + \left(\frac{\alpha}{2\pi}\right) \Big[\widetilde{\delta}_{\text{IB}} + \widetilde{\delta}_{(\gamma<)} + \widetilde{\delta}^{\text{res. 3-pt.}}_{(\gamma<)} + \widetilde{\delta}^{\text{box}}_{(\gamma)} + \widetilde{\delta}^{\text{w.f.}}_{(\gamma<)} \\
& + \widetilde{\delta}^{\text{div}}_{(\gamma)} + \widetilde{\delta}^{\text{fin}}_{(\gamma)} + \widetilde{\delta}^{\text{div}}_{(Z)} + \widetilde{\delta}^{\text{fin}}_{(Z)} + \widetilde{\delta}^{\text{box}}_{(Z)} + \widetilde{\delta}^{\text{self}} + \widetilde{\delta}^{\text{vertex}} + \widetilde{\delta}^{\text{w.f. res.}} \Big] \Bigg\}.
\end{aligned}
$$

The universality structure of the weak interactions induces a certain number of relations among the corrections occuring in the two decays

$$\begin{aligned}
\delta^{\text{div}}_{(\gamma)} &= \widetilde{\delta}^{\text{div}}_{(\gamma)} \\
\delta^{\text{div}}_{(Z)} &= \widetilde{\delta}^{\text{div}}_{(Z)} \\
\delta^{\text{self}} &= \widetilde{\delta}^{\text{self}} \\
\delta^{\text{vertex}} &= \widetilde{\delta}^{\text{vertex}} \\
\delta^{\text{w.f. res.}} &= \widetilde{\delta}^{\text{w.f. res.}}
\end{aligned}$$

The fact that the divergent pieces are identical allows to define the same renormalized weak coupling constant from both processes, and reflects the renormalizability of the standard model. The presence of the strong interactions induces α_s dependent violations of universality in the finite pieces

$$\begin{aligned}
\delta^{\text{fin}}_{(Z)} &= \widetilde{\delta}^{\text{fin}}_{(Z)} + \frac{1}{2}\cot^2\theta_w \times \Delta_{(Z)}(\alpha_s) \\
\delta^{\text{fin}}_{(\gamma)} &= \widetilde{\delta}^{\text{fin}}_{(\gamma)} + \frac{1}{2} \times \Delta_{(\gamma)}(\alpha_s)
\end{aligned}$$

An additional difference in the Z-box contributions arises as a consequence of the fact that the average charges in the quark and lepton multiplets do not coincide

$$\delta^{\text{box}}_{(Z)} = \frac{M_W^2}{M_Z^2 - M_W^2} \ln \frac{M_Z^2}{M_W^2} \left[\left(2 + \frac{1}{2}\right) \cot^2\theta_w + 3\overline{Q_q} \tan^2\theta_w + \Delta^{\text{box}}_{(Z)}(\alpha_s) \right], \quad \overline{Q_q} = 1/6$$

$$\widetilde{\delta}^{\text{box}}_{(Z)} = \frac{M_W^2}{M_Z^2 - M_W^2} \ln \frac{M_Z^2}{M_W^2} \left[\left(2 + \frac{1}{2}\right) \cot^2\theta_w + 3\overline{Q_\ell} \tan^2\theta_w \right], \quad \overline{Q_\ell} = -1/2$$

As far as the remaining contributions are concerned, one has ($r_\ell = m_\ell^2/M_{P^+}^2$

$$\delta_{\text{IB}} + \delta^{\text{pt}}_{(\gamma<)} + \delta^{\text{res. 3-pt.}}_{(\gamma<);\,\text{pt}} + \delta^{\text{box}}_{(\gamma);\,\text{pt}} + \delta^{\text{w.f.}}_{(\gamma<)} = \frac{1}{2} H(r_\ell) + \frac{7}{2} - \frac{3}{2} \ln \frac{M_{P^+}^2}{M_W^2}$$

$$H(z) = \frac{23}{2} - \frac{3}{1-z} + 11 \ln z - \frac{2\ln z}{1-z} - \frac{3\ln z}{(1-z)^2} - 8\ln(1-z) - \frac{4(1+z)}{1-z} \ln z \ln(1-z) - \frac{8(1+z)}{1-z} \text{Li}_2(1-z).$$

while in the muon case, the analogous combination

$$\widetilde{\delta}_{\text{IB}} + \widetilde{\delta}_{(\gamma<)} + \widetilde{\delta}^{\text{res. 3-pt.}}_{(\gamma<)} + \widetilde{\delta}^{\text{box}}_{(\gamma)} + \widetilde{\delta}^{\text{w.f.}}_{(\gamma<)} = \frac{25}{4} - \pi^2$$

reproduces the finite result of the local Fermi theory

Introducing a new coupling constant, defined as

$$G_\mu = \frac{g^2}{4\sqrt{2}M_W^2}\left\{1 + \left(\frac{\alpha}{4\pi}\right)\left[\widetilde{\delta}^{\text{div}}_{(\gamma)} + \widetilde{\delta}^{\text{fin}}_{(\gamma)} + \widetilde{\delta}^{\text{div}}_{(Z)} + \widetilde{\delta}^{\text{fin}}_{(Z)} + \widetilde{\delta}^{\text{box}}_{(Z)} + \widetilde{\delta}^{\text{self}} + \widetilde{\delta}^{\text{vertex}} + \widetilde{\delta}^{\text{w.f. res.}}\right]\right\}$$

gives

$$\begin{aligned}\Gamma(P^+ \to \ell^+\nu_\ell(\gamma)) \;=\; & \frac{G_\mu^2|V^*_{\text{CKM}}|^2}{4\pi} M_{P^+}^3 F_{P^+}^2 r_\ell(1-r_\ell)^2\left\{1 + \left(\frac{\alpha}{2\pi}\right)\left[\frac{1}{2}H(r_\ell) + \frac{7}{2} - \frac{3}{2}\ln\frac{M_{P^+}^2}{M_W^2}\right.\right. \\ & + \delta_{\text{SD}} + \delta_{\text{INT}} + \delta^{\text{res}}_{(\gamma<)} + \delta^{\text{res. 3-pt.}}_{(\gamma<);\,\text{res}} + \delta^{\text{box}}_{(\gamma);\,\text{res}} \\ & + \frac{M_W^2}{M_Z^2 - M_W^2}\ln\frac{M_Z^2}{M_W^2}\left[3(\overline{Q_q} - \overline{Q_\ell})\tan^2\theta_w + \Delta^{\text{box}}_{(Z)}(\alpha_s)\right] \\ & \left.\left. + \frac{1}{2}\cot^2\theta_w\,\Delta_{(Z)}(\alpha_s) + \frac{1}{2}\Delta_{(\gamma)}(\alpha_s)\right]\right\}\end{aligned}$$

The same computation in the low-energy effective theory gives

$$\Gamma(P^+ \to \ell^+\nu_\ell(\gamma)) = \frac{G_\mu^2|V_{\text{CKM}}|^2\mathcal{F}_{P^+}^2 m_\ell^2 M_{P^+}}{4\pi}(1-r_\ell)^2\left\{1 + \left(\frac{\alpha}{\pi}\right)\left[\ln\frac{M_Z^2}{\mu^2} + C_{P^+}(\mu) + \frac{1}{4}H(r_\ell)\right]\right\}$$

$$C_{P^+} = 2\pi^2 E^r(\mu) + \text{chiral logs}$$

$$E := \frac{8}{3}K_1 + \frac{8}{3}K_2 + \frac{20}{9}K_5 + \frac{20}{9}K_6 + 4K_{12} - \frac{4}{3}X_1 - 4X_2 + 4X_3 - X_6$$

CONCLUSIONS   OUTLOOK

- SM provides a framework to compute radiative corrections to semileptonic decays of mesons in situations where low-energy effective theory does not apply (hadronic tau decays, semileptonic decays of B and D mesons)

- Result is finite and involves three-current and two-current correlation functions of QCD, whose evaluation requires nonperturbative approaches (lattice, large-N_C)

- In the case of light pseudoscalar mesons, identification of the low-energy constants in terms of these QCD correlators