Higgs production at N3LO beyond threshold

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Establishing whether the BEH mechanism and its boson is SM-like will be of outmost importance for the run of the LHC.

Higgs-boson production modes at the LHC:

- Gluon fusion
- TTH
- Higgs strahlung
- VBF

Current status for the total cross section: [D. André @ ICHEP 2014]

\[ \frac{\sigma}{\sigma_{SM}} = 1.00 \pm 0.13 \left[ \pm 0.09^{\text{stat.}} \pm 0.07^{\text{theo.}} \pm 0.07^{\text{syst.}} \right] \]

- Theo. and exp. uncertainties are of the same order.
- Need to improve our theory predictions!
The dominant Higgs production mechanism at the LHC is gluon fusion. Loop-induced process.

For a light Higgs boson, the dimension five operator describing a tree-level coupling of the gluons to the Higgs boson

$$\mathcal{L} = \mathcal{L}_{QCD,5} - \frac{1}{4v} C_1 H G_{\mu\nu}^a G_a^{\mu\nu}$$

Top-mass corrections known at NNLO.

[Harlander, Ozeren; Pak, Rogal, Steinhauser; Ball, Del Duca, Marzani, Forte, Vicini; Harlander, Mantler, Marzani, Ozeren]

In the rest of the talk, I will only concentrate on the effective theory.
The gluon fusion cross section

- The gluon fusion cross section is given in perturbation theory by
  \[
  \sigma(pp \to H + X) = \tau \sum_{ij} \int_{\tau}^{1} dz \mathcal{L}_{ij}(z) \hat{\sigma}_{ij}(\tau/z)
  \]

- The (partonic) cross section depends up to an overall scale only on the ratio
  \[
  \tau = \frac{m^2}{s} \quad \quad \quad \quad z = \frac{m^2}{\hat{s}}
  \]

- The partonic cross section known at NLO and NNLO.

[Dawson; Djouadi, Spira, Zerwas; Harlander, Kilgore; Anastasiou, Melnikov; Ravindran, Smith, van Neerven]

- The inclusive Higgs cross section is known to be ‘plagued’ by large perturbative corrections.
The gluon fusion cross section

\[ \frac{\sigma}{\sigma_{\text{NNLO}}} = \mu = \mu_{\text{R}} = \mu_{\text{F}} \]

As it has already been realized in the literature, smaller scales than the Higgs boson mass lead to a faster convergence of the perturbative expansion [5, 6].

\[ \text{NNLO} \quad \text{NLO} \quad \text{LO} \]

<table>
<thead>
<tr>
<th>( m_H ) (GeV)</th>
<th>200</th>
<th>300</th>
<th>400</th>
<th>500</th>
<th>600</th>
<th>700</th>
<th>800</th>
</tr>
</thead>
<tbody>
<tr>
<td>NNLO</td>
<td>1.3</td>
<td>1.2</td>
<td>1.1</td>
<td>1.0</td>
<td>0.9</td>
<td>0.8</td>
<td>0.7</td>
</tr>
<tr>
<td>NLO</td>
<td>1.0</td>
<td>1.0</td>
<td>0.9</td>
<td>0.9</td>
<td>0.8</td>
<td>0.7</td>
<td>0.6</td>
</tr>
<tr>
<td>LO</td>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
<td>0.8</td>
<td>0.7</td>
<td>0.6</td>
<td>0.5</td>
</tr>
</tbody>
</table>

We estimate the theoretical uncertainty from uncalculated higher order corrections by varying the renormalization and factorization scale in the interval \( \mu \in [m_H^4, m_H^5] \).

In Fig. 2 we present the cross-section at LO, NLO and NNLO in this interval, normalized to the NNLO cross-section at the central scale \( \mu = m_H^2 \). The NNLO and NLO bands overlap largely, corrections beyond NNLO would need to be atypical for our uncertainty estimate to be inaccurate. The cross-section is known to be stable under threshold and other corrections which can be resummed beyond NNLO [23–25].

3. Parton density function comparison

The Higgs boson cross-section requires parton density functions (PDF) as input. In Higgsix we employ all PDF sets that allow for NNLO evolution (MSTW08 [27], JR09 [28], NNPDF).

- LO 9.6 pb \( \sim 25\% \)
- NLO 16.7 pb \( \sim 20\% \)
- N2LO 19.6 pb \( \sim 7 - 9\% \)
- N3LO ??? \( \sim 4 - 8\% \)

[Fixed order only]

[Plot from Anastasiou, Bühler, Herzog, Lazopoulos]

[Results for 8 TeV]
The gluon fusion cross section

- We need one more order in the perturbative expansion, N3LO.
- So far no complete computation is available.
  - Scale variation at N3LO is known.
    - [Moch, Vogt; Ball, Bonvini, Forte, Marzani, Ridolfi; Bühler, Lazopoulos]
  - Several approximate N3LO results exist.
    - [Ball, Bonvini, Forte, Marzani, Ridolfi; de Florian, Mazzitelli, Moch, Vogt]
    - How good are these approximations..?  
    - Only full computation can tell…
- **Challenge:** Never has an N3LO computation been performed so far...
  - Uncharted territory!
  - New conceptual challenges.
Outline

- Higgs production at N3LO
- The soft-virtual cross-section at N3LO.
- Approximate cross-sections at N3LO.
- Going beyond the soft-virtual approximation.
Higgs production at N3LO
The gluon fusion cross section

- At NLO, there are two contributions (~1991):
  [Dawson; Djouadi, Spira, Zerwas]

Virtual corrections (‘loops’)  Real emission

- Both contributions are individually divergent:
  - UV divergences are handled by renormalization.
  - IR divergences cancelled by PDF counterterms.
The gluon fusion cross section

- At NNLO, there are three contributions (2002):
  
  [Harlander, Kilgore; Anastasiou, Melnikov; Ravindran, Smith, van Neerven]

- Double virtual
- Real-virtual
- Double real
The gluon fusion cross section

- At N3LO, there are five contributions:
  - Triple virtual
  - Real-virtual squared
  - Double virtual real
  - Double real virtual
  - Triple real
The triple virtual corrections are directly related to the QCD form factor.

The QCD form factor is known

- at one loop. [Gonsalves; Kramer, Lampe; Gehrmann, Huber, Maître]
- at two loops.
- at three loops. [Baikov, Chetyrkin, Smirnov, Smirnov, Steinhauser; Gehrmann, Glover, Huber, Ikizlerli, Studerus]

It is not the loops that are the problem!
Unitarity

- **Optical theorem:**

  \[
  \text{Im} \quad \frac{1}{p^2 - m^2 + i\varepsilon} \rightarrow \delta_+ (p^2 - m^2) = \delta(p^2 - m^2) \theta(p^0)
  \]

- Discontinuities of loop integrals are given by Cutkosky’s rule:

- These relations are at the heart of all the unitarity-based approaches to loop computations.
Reverse-unitarity

- Optical theorem:

\[ \text{Im} \quad = \quad \int d\Phi \]

- We can read the optical theorem ‘backwards’ and write inclusive phase space integrals as unitarity cuts of loop integrals. [Anastasiou, Melnikov; Anastasiou, Dixon, Melnikov, Petriello]

⇒ Rather than computing phase-space integrals, we can compute loop integrals with cuts!

⇒ Makes inclusive phase space integrals accessible to all the technology developed for multi-loop computations!
  › Integration-by-parts & differential equations.
### Reverse-unitarity @ N3LO

**Growth in complexity for real emission**

<table>
<thead>
<tr>
<th>Order</th>
<th>Diagrams</th>
<th>Integrals</th>
</tr>
</thead>
<tbody>
<tr>
<td>LO</td>
<td>1 diagram</td>
<td>1 integral</td>
</tr>
<tr>
<td>NLO</td>
<td>10 diagrams</td>
<td>1 integral</td>
</tr>
<tr>
<td>NNLO</td>
<td>381 diagrams</td>
<td>18 integrals</td>
</tr>
<tr>
<td>N3LO</td>
<td>26565 diagrams</td>
<td>~500 integrals</td>
</tr>
</tbody>
</table>
The threshold expansion

- ~ 500 master integrals only for triple real double real NNLO).
  
  ➡ Tough nut to crack!

- The gluon fusion cross section depends on one single parameter:
  \[ z = \frac{m^2}{s}, \quad \bar{z} = 1 - z \]

- Close to threshold (\(z \sim 1\)), we can approximate the triple real cross section by a power series:
  \[ \hat{\sigma}(z) = \sigma_{-1} + \sigma_0 + (1 - z)\sigma_1 + \mathcal{O}(1 - z)^2 \]

- Goal: Compute cross section as a series around threshold!
The soft-virtual cross section at N3LO
The soft-virtual approximation

- The
  \[ \hat{\sigma}(z) = \sigma_{-1} + \sigma_0 + (1 - z) \sigma_1 + \mathcal{O}(1 - z)^2 \]

- The soft-virtual term receives contributions from a 'pole' at \( z \sim 1 \):
  \[ (1 - z)^{-1 + n\epsilon} = \frac{\delta(1 - z)}{n\epsilon} + \left[ \frac{1}{1 - z} \right]_+ + n\epsilon \left[ \frac{\log(1 - z)}{1 - z} \right]_+ + \mathcal{O}(\epsilon^2) \]

- Plus-distribution terms already known. [Moch, Vogt]

- Complete three-loop corrections are contained the delta function term.
  - The soft-virtual term contains the complete three-loop corrections plus the correction from the emission of up to three soft gluons.
The soft-virtual approximation

- At NLO and NNLO, the soft-virtual term reads \( \mu_R = \mu_F = m_H \)

\[
\hat{\sigma}_{gg}^{SV}(z) = \frac{\pi C(\mu^2)^2}{v^2 (N^2 - 1)^2} \sum_{k=0}^{\infty} \left( \frac{\alpha_s}{\pi} \right)^k \hat{\eta}^{(k)}(z)
\]

\[
\hat{\eta}^{(0)}(z) = \delta(1 - z) \quad \hat{\eta}^{(1)}(z) = 2 C_A \zeta_2 \delta(1 - z) + 4 C_A \left[ \frac{\log(1 - z)}{1 - z} \right] + \\
\hat{\eta}^{(2)}(z) = \delta(1 - z) \left\{ C_A^2 \left( \frac{67}{18} \zeta_2 - \frac{55}{12} \zeta_3 - \frac{1}{8} \zeta_4 + \frac{93}{16} \right) + N_F \left[ C_F \left( \zeta_3 - \frac{67}{48} \right) - C_A \left( \frac{5}{9} \zeta_2 + \frac{1}{6} \zeta_3 + \frac{5}{3} \right) \right] \right\} + \\
\left[ \frac{1}{1 - z} \right] + C_A^2 \left( \frac{11}{3} \zeta_2 + \frac{39}{2} \zeta_3 - \frac{101}{27} \right) + N_F C_A \left( \frac{14}{27} - \frac{2}{3} \zeta_2 \right) + \\
\left[ \frac{\log(1 - z)}{1 - z} \right] + C_A^2 \left( \frac{67}{9} - 10 \zeta_2 \right) - \frac{10}{9} C_A N_F + \\
\left[ \frac{\log^2(1 - z)}{1 - z} \right] + \left( \frac{2}{3} C_A N_F - \frac{11}{3} C_A^2 \right) + \left[ \frac{\log^3(1 - z)}{1 - z} \right] + 8 C_A^2.
\]
N3LO status: soft-virtual

✓ Triple virtual
✓ Real-virtual squared
✓ Double virtual real
✓ Double real virtual
✓ Triple real
✓ +
The soft-virtual approximation

- The computation of the first term has been completed!
  [Anastasiou, CD, Dulat, Furlan, Gehrmann, Herzog, Mistlberger]

- Many different contributions are needed:
  - 22 three-loop.
    [Baikov, Chetyrkin, Smirnov, Smirnov, Steinhauser; Gehrmann, Glover, Huber, Ikizlerli, Studerus]
  - 3 double-virtual-real.
    [CD Gehrmann, Li, Zhu]
  - 7 real-virtual-squared.
    [Anastasiou, CD, Dulat, Herzog, Mistlberger; Kilgore]
  - 10 double-real-virtual.
    [Anastasiou, CD, Dulat, Furlan, Herzog, Mistlberger; Li, von Manteuffel, Schabinger, Zhu]
  - 8 triple real.
    [Anastasiou, CD, Dulat, Mistlberger]
  - three-loop splitting functions.
    [Moch, Vermaseren, Vogt]
  - three-loop beta function.
    [Tarasov, Vladimirov, Zharkov; Larin, Vermaseren]
  - three-loop Wilson coefficient.
    [Chetyrkin, Kniehl, Steinhauser; Schroeder, Steinhauser; Chetyrkin, Kuhn, Sturm]
The integrals
\[
\hat{\eta}^{(3)}(z) = \delta(1-z) \left\{ C_A^3 \left( -\frac{2003}{48} \zeta_6 + \frac{413}{6} \zeta_3^2 - \frac{7579}{144} \zeta_5 + \frac{979}{24} \zeta_2 \zeta_3 - \frac{15257}{864} \zeta_4 - \frac{819}{16} \zeta_3 + \frac{16151}{1296} \zeta_2 + \frac{215131}{5184} \right) \\
+ N_F \left[ C_A^2 \left( \frac{869}{72} \zeta_5 - \frac{125}{12} \zeta_3 \zeta_2 + \frac{2629}{432} \zeta_4 + \frac{1231}{216} \zeta_3 - \frac{70}{81} \zeta_2 - \frac{98059}{5184} \right) \\
+ C_A C_F \left( \frac{5}{2} \zeta_5 + 3 \zeta_3 \zeta_2 + \frac{11}{72} \zeta_4 + \frac{13}{2} \zeta_3 - \frac{71}{36} \zeta_2 - \frac{63991}{5184} \right) + C_F^2 \left( -5 \zeta_5 + \frac{37}{12} \zeta_3 + \frac{19}{18} \right) \right] \\
+ N_F^2 \left[ C_A \left( \frac{19}{36} \zeta_4 + \frac{43}{108} \zeta_3 - \frac{133}{324} \zeta_2 + \frac{2515}{1728} \right) + C_F \left( -\frac{1}{36} \zeta_4 - \frac{7}{6} \zeta_3 - \frac{23}{72} \zeta_2 + \frac{4481}{2592} \right) \right] \right\} \\
+ \left[ \frac{1}{1-z} \right] \left\{ C_A^3 \left( 186 \zeta_5 - \frac{725}{6} \zeta_3 \zeta_2 + \frac{253}{24} \zeta_4 + \frac{8941}{108} \zeta_3 + \frac{8563}{324} \zeta_2 - \frac{297029}{23328} \right) + N_F^2 C_A \left( \frac{5}{27} \zeta_3 + \frac{10}{27} \zeta_2 - \frac{58}{729} \right) \\
+ N_F \left[ C_A^2 \left( -\frac{17}{12} \zeta_4 - \frac{475}{36} \zeta_3 - \frac{2173}{324} \zeta_2 + \frac{31313}{11664} \right) + C_A C_F \left( -\frac{1}{3} \zeta_4 - \frac{19}{18} \zeta_3 - \frac{1}{2} \zeta_2 + \frac{1711}{864} \right) \right] \right\} \\
+ \left[ \frac{\log(1-z)}{1-z} \right] \left\{ C_A^3 \left( -77 \zeta_4 - \frac{352}{3} \zeta_3 - \frac{152}{3} \zeta_2 + \frac{30569}{648} \right) + N_F^2 C_A \left( -\frac{4}{9} \zeta_2 + \frac{25}{81} \right) \\
+ N_F \left[ C_A^2 \left( \frac{46}{3} \zeta_3 + \frac{94}{9} \zeta_2 - \frac{4211}{324} \right) + C_A C_F \left( 6 \zeta_3 - \frac{63}{8} \right) \right] \right\} \\
+ \left[ \frac{\log^2(1-z)}{1-z} \right] \left\{ C_A^3 \left( 181 \zeta_3 + \frac{187}{3} \zeta_2 - \frac{1051}{27} \right) + N_F \left[ C_A^2 \left( -\frac{34}{3} \zeta_2 + \frac{457}{54} \right) + \frac{1}{2} C_A C_F \right] - \frac{10}{27} N_F^2 C_A \right\} \\
+ \left[ \frac{\log^3(1-z)}{1-z} \right] \left\{ C_A^3 \left( -56 \zeta_2 + \frac{925}{27} \right) - \frac{164}{27} N_F C_A^2 + \frac{4}{27} N_F^2 C_A \right\} \\
+ \left[ \frac{\log^4(1-z)}{1-z} \right] \left\{ \frac{20}{9} N_F C_A^2 - \frac{110}{9} C_A^3 \right\} + \left[ \frac{\log^5(1-z)}{1-z} \right] \left\{ \frac{1}{1-z} \right\} + 8 C_A^3. \right\} \\
\left[ \text{Anastasiou, CD, Dulat, Furlan, Gehrmann, Herzog, Mistlberger} \right]
\]
Caveat!

Source of ambiguity:

\[
\int dx_1 dx_2 \left[ f_i(x_1) f_j(x_2) z g(z) \right] \left[ \frac{\hat{\sigma}_{ij}(s, z)}{z g(z)} \right]_{\text{threshold}} \quad \lim_{z \to 1} g(z) = 1
\]
Going beyond soft-virtual

- Can we go beyond the soft-virtual approximation..?
  - More terms in the expansion..?
  - Result in full kinematics..?

- Can we improve the soft-virtual result and do phenomenology..?
  - Recent approximate N3LO results..?
    [Ball, Bonvini, Forte, Marzani, Ridolfi; de Florian, Mazzitelli, Moch, Vogt]
  - How good are these approximations..?
Approximate cross sections at N3LO
Approximate N3LO results

- Recently, approximate results at N3LO have been presented that include terms beyond the soft-virtual approximation (gluons only).

- Ball, Bonvini, Forte, Marzani, Ridolfi:
  - Soft-virtual term at N3LO.
  - High-energy behaviour, including top-mass effects at N3LO.
  - Analyticity.

- de Florian, Mazzitelli, Moch, Vogt:
  - Soft-virtual term at N3LO.
  - First three logarithms from the next term in the expansion, + numerical guesses for the missing logarithms.
Mellin-space vs. z-space

\[ \hat{\sigma}(N) = \int_0^1 dz\, z^{N-1} \hat{\sigma}(z) \quad \hat{\sigma}(z) = \int_{c-i\infty}^{c+i\infty} \frac{dN}{2\pi i} \, z^{-N} \hat{\sigma}(N) \]

- Mellin-space is the natural language for resummation.

<table>
<thead>
<tr>
<th>z-space</th>
<th>Mellin-space</th>
</tr>
</thead>
<tbody>
<tr>
<td>Soft / threshold limit:</td>
<td>( z \to 1 )</td>
</tr>
<tr>
<td>High-energy limit:</td>
<td>( z \to 0 )</td>
</tr>
</tbody>
</table>

- Experience from lower orders: numerical convergence of soft expansion better in Mellin-space.
The high-energy limit

- The leading behaviour of the cross section at small \( N \) is known at N3LO.
  - In the infinite top-mass limit. [Hautmann]
  - Including finite top-mass effects. [Ball, Del Duca, Forte, Marzani, Vicini]

- Infinite top-mass not compatible with the high-energy limit
  - Tension between \( m_t \gg 1 \) and \( s \gg 1 \).

- If one includes the correct high-energy limit (and requires the correct analytic behaviour in \( z \)-space), we find \(~16\%\) increase compared to NNLO (8 TeV, \( \mu_R = m_H \), gluons only).
  [Ball, Bonvini, Forte, Marzani, Ridolfi]
  - To be compared to \(~6\%\) from expanding resummation to N3LO.
Subleading soft terms

- Recently, the first three next-to-to-soft terms were published:
  \[ \hat{\sigma}(z) = \sigma_{-1} + \sigma_0 + (1 - z) \sigma_1 + \mathcal{O}(1 - z)^2 \]
  
  \[ -512 C_A^3 \ln^5(1 - z) + \left\{ 1728 C_A^3 + \frac{640}{3} C_A^2 \beta_0 \right\} \ln^4(1 - z) \]
  
  \[ + \left\{ \left( -\frac{1168}{3} + 3584 \zeta_2 \right) C_A^3 - \left( \frac{2512}{3} + \frac{\xi_H^{(3)}}{3} \right) C_A^2 \beta_0 - \frac{64}{3} C_A \beta_0^2 \right\} \ln^3(1 - z) \]

  \[ \xi_H^{(3)} \simeq 300 \]  
  [de Florian, Mazzitelli, Moch, Vogt]

- In Mellin-space:
  Estimated/guessed from DY
  
  \[ \ln^5 N + 5.701 \ln^4 N + 18.9 \ln^3 N + 46 \ln^2 N + 18 \ln N + 9 \]

- Leads to an increase of \(~10-13\%\) (14\,TeV, \(\mu_R = m_H\), gluons only).
Validity of approximation

• “... approximation works well at lower orders...”

Does this also hold for the cross section?

Depends on which are the relevant values of $N$

Closer to threshold: Larger $N$
Further from threshold: Smaller $N$

Will fail as we increase the collider energy

The exact result is always between the two approximations for $E<20\text{TeV}$

We will use the SV and $1/N$ results to constrain the N3LO result in that region

[Plots from de Florian, Mazzitelli, Moch, Vogt]
Going beyond the soft-virtual approximation
State of the art at N3LO

- $gg$ Soft-virtual [Moch, Vogt; Anastasiou, CD, Dulat, Furlan, Gehrmann, Herzog, Mistlberger]

  First 3 next-to-soft logs [de Florian, Mazzitelli, Moch, Vogt]

  Full next-to-soft Full first three logs (exact)

- $gq$ First next-to-soft log [Almasy, Lo Presti, Vogt]

  Full next-to-soft Full first three logs (exact)

- $q\bar{q}$ Full first three logs (exact)

- $qq$ Full first three logs (exact)

- $qQ$ Full first three logs (exact)
Towards full kinematics

- We have the full contribution from
  - Emission of one parton at one loop, all channels.
    [Anastasiou, CD, Dulat, Herzog, Mistlberger; Kilgore]
  - Emission of one parton at two loops, all channels.
    [Dulat, Mistlberger; CD, Gehrmann]
  - UV and PDF counterterms, all channels.
    [Höschele, Hoff, Pak, Steinhauser, Ueda; Bühler, Lazopoulos]

- We know that all the poles must cancel when we combine ALL contribution.
  - The knowledge of the previous contributions is enough to fix the first three logarithm in all channels.
    [Anastasiou, CD, Dulat, Furlan, Gehrmann, Herzog, Mistlberger]
Next-To-Soft Contribution (gg)

\[
\hat{\eta}^{(3)}_{gg}(z)|_{(1-z)^0} = -8 N^3 \log^5(1 - z) + \left( \frac{353}{9} N^3 - \frac{20}{9} N^2 N_f \right) \log^4(1 - z) \\
+ \left[ \left( 56 \zeta_2 - \frac{3469}{54} \right) N^3 + \frac{205}{18} N^2 N_f - \frac{4}{27} N N_f^2 \right] \log^3(1 - z) \\
+ \left\{ \left( -181 \zeta_3 - \frac{2147}{12} \zeta_2 + \frac{2711}{27} \right) N^3 + \left[ \left( \frac{545}{48} \zeta_2 - \frac{4139}{216} \right) N^2 + \frac{1}{4} \right] N_f \right\} \log^2(1 - z) \\
+ \left[ \left( 77 \zeta_4 + 362 \zeta_3 + \frac{2375}{18} \zeta_2 - \frac{9547}{108} \right) N^3 + \left[ \left( -\frac{223}{12} \zeta_3 - \frac{1813}{72} \zeta_2 + \frac{8071}{324} \right) N^2 \\
+ 3 \zeta_3 + \frac{1}{24} \zeta_2 - \frac{17}{4} \right] N_f + \left( \frac{4}{9} \zeta_2 - \frac{163}{324} \right) N N_f^2 \right\} \log(1 - z) \\
+ \left( -186 \zeta_5 + \frac{725}{6} \zeta_2 \zeta_3 - \frac{821}{12} \zeta_4 - \frac{32849}{216} \zeta_3 - \frac{11183}{162} \zeta_2 + \frac{834419}{23328} \right) N^3 \\
+ \left[ \left( \frac{19}{8} \zeta_4 + \frac{1789}{72} \zeta_3 + \frac{4579}{324} \zeta_2 - \frac{527831}{46656} \right) N^2 - \frac{1}{4} \zeta_4 - \frac{149}{72} \zeta_3 - \frac{5}{24} \zeta_2 + \frac{5065}{1728} \right] N_f \\
+ \left[ \left( -\frac{5}{27} \zeta_3 - \frac{19}{36} \zeta_2 + \frac{49}{729} \right) N N_f^2 \right].
\]

[Anastasiou, CD, Dulat, Furlan, Gehrmann, Herzog, Mistlberger]
Next-To-Soft Contribution

- We can compute the full contribution to the second term in the threshold expansion

\[ \hat{\sigma}(z) = \sigma_{-1} + \sigma_0 + (1 - z) \sigma_1 + O(1 - z)^2 \]

- Receives contribution from both gg and gq channels.

- Needed some rethinking of our technology for double-real emission at one loop.

- There are now contributions from collinear virtual gluons.

- We find full agreement with known results for leading logarithms. [Almasy, Lo Presti, Vogt; de Florian, Mazzitelli, Moch, Vogt]

- In particular \( \xi_H^{(3)} = \frac{896}{3} \approx 298.666 \ldots \)
Ambiguity in $z$-space

- **Ambiguity:**

$$\sigma = r^{1+\alpha} \sum_{ij} \left( f_i^{(\alpha)} \otimes f_j^{(\alpha)} \otimes \hat{\sigma}_{ij}(z) \right)(\tau)$$

$$f_i^{(\alpha)}(x) \equiv \frac{f_i(x)}{x^\alpha}.$$  

→ Full hadronic cross section cross section is independent order-by-order of $\alpha$.

- Truncating the soft expansion introduces a dependence on $\alpha$:

$$\hat{\sigma}_{ij}(z) \approx \hat{\sigma}_{ij}(z)|_{(1-z)^{-1}} + \hat{\sigma}_{ij}(z)|_{(1-z)^0} + \alpha(1 - z) \hat{\sigma}_{ij}(z)|_{(1-z)^{-1}} + \mathcal{O}(1 - z)^1$$

→ Soft-expansion introduces an ambiguity, which can have numerical impact.

- Is this ambiguity also present in Mellin-space..?
Ambiguity in Mellin-space

- Multiplying by $z^\alpha$ in $z$-space corresponds to shifting $N \rightarrow N + \alpha$ in Mellin-space.

$$\hat{\sigma}(N) = \int_0^1 dz \, z^{N-1} \hat{\sigma}(z)$$

- The threshold limit $N \rightarrow \infty$ is obviously insensitive to this!

- In order to quantify the validity of approximate cross sections via threshold expansion, we study the dependence of the result on $\alpha$. 
In this proceeding we study the uncertainty in the case of the gluon fusion Higgs production cross-section at N^3 LO. We consider lower orders in perturbative QCD to study the convergence behaviour of the expansion for the Higgs cross-section and inspect the impact of the ambiguity due to the truncation of the threshold expansion. Furthermore, we demonstrate that the ambiguity for the SV approximation at N^3 LO is large.

Threshold Expansion for the Higgs boson cross-section

The probability distribution of a gluon occurring in a process steeply falls with its energy and suggests the possibility of performing a fast converging threshold expansion for the Higgs cross-section. Already at NNLO a threshold expansion was performed and was shown to be rapidly converging towards the full result. Here we study the strong coupling expansion of the heavy top effective theory. In this note we are interested in the effect complementing existing lower order calculations with a threshold expansion at N^n LO. The threshold approximations and expansions which we will discuss will always contain the full (non-expanded) dependence on terms which enter the result at lower orders in the strong coupling expansion. We will also include full N^n LO dependence on renormalisation and factorisation scales as well as the full dependence on those N^n LO corrections which are generated from higher order corrections to the Wilson coefficient.

Parametrising the expansion with the variable
\[ z = \frac{m^2}{x_1 x_2 s} \]
leads to a series of the partonic cross-section in \((1 - z)^n\).

\[ \hat{\sigma}_{ij}(s, z) = \sigma_{SV} + (1 - z)\sigma^{(0)}_{ij} + (1 - z)\sigma^{(1)}_{ij} + \ldots \]

(2)

If a series expansion is truncated at a given finite order an unavoidable ambiguity is introduced due to missing higher order terms. To study the impact of truncating the threshold expansion of the Higgs boson cross-section we spuriously insert a function \(g(z)\) s.t. \[\sigma = \sum_{i,j} \int dx_1 dx_2 \left[ f_i(x_1) f_j(x_2) z g(z) \right] \hat{\sigma}_{ij}(s, z) z g(z) \]
threshold.

(3)

For all choices of \(g(z)\) the expansion truncated at \(O((1 - z)^n)\) thus leaves all equivalent results up to \(O((1 - z)^{n+1})\).
Figure 3 – The gluon-fusion cross-section at 13 TeV at the LHC as a function of $\mu/m_H$ up to LO (black), NLO (red), NNLO (green) and soft-virtual N3LO (blue). The N3LO SV approximation is modified with different functions $g(z)$.

3 Conclusion

The rapidly increasing experimental precision of Higgs cross-section measurements raises an urgent demand for the improvement of the theoretical prediction for the inclusive Higgs boson cross-section at the LHC. With the recent publication of the first term in the threshold expansion of the N3LO gluon-fusion QCD cross-section an important step in this direction was taken. In this proceedings we have analysed the quality of the threshold expansion. We find that the expansion is converging fast at lower orders in QCD perturbation theory and expect to find similar behaviour at N3LO. We studied the uncertainty introduced due to the truncation of the threshold expansion at NLO, NNLO and N3LO and conclude that at least several terms in the expansion are necessary in order to infer reliable predictions for LHC measurements and improve upon the current status. We conclude that the calculation of further terms in the threshold expansion and even the full Higgs boson cross-section at N3LO is highly desirable.

4 Acknowledgements

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References

## Dependence on the truncation

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[14 TeV, $\mu = m_H$, gluons only]

Preliminary
Dependence on the truncation

\[ n: g(z) = z^n \]
Conclusion

• The computation of the Higgs cross section at N3LO moves forward at a steady pace!
  ➡ Soft-virtual contribution known.
  ➡ Next-to-soft contribution known (noth gg & gQ).
  ➡ First three logs known exactly for all channels.
  ➡ Contribution form single-real emission fully known.

• Approximate results should be taken with a grain of salt!
  ➡ Only full result for N3LO cross section will be the final judge!

• Stay tuned!