α_s and the τ hadronic width

Work in process (progress?) with:

Martin Beneke

All results are preliminary!

☞ Investigations of hadronic τ decays already contributed tremendously for fundamental QCD parameters like α_s, the strange mass and non-perturbative condensates.

☞ In particular: (Davier, Höcker, Zhang 2007)

$$\alpha_s(M_\tau) = 0.345 \pm 0.004_{\rm exp} \pm 0.009_{\rm th} \,,$$

leading to

$$\alpha_s(M_Z) = 0.1215 \pm 0.0012 \,.$$

☞ This should be compared to the recent average: (Bethke 2007)

$$\alpha_s(M_Z) = 0.1185 \pm 0.0010 \,,$$

displaying a 2.5σ difference.

Hadronic τ decay rate

Consider the physical quantity R_τ: (Braaten, Narison, Pich 1992)

$$R_\tau \equiv \frac{\Gamma(\tau^- \to \text{hadrons}\, \nu_\tau(\gamma))}{\Gamma(\tau^- \to e^- \bar{\nu}_e \nu_\tau(\gamma))} = 3.640 \pm 0.010 \,.$$

R_τ is related to the QCD correlators $\Pi^{T,L}(z)$: $(z \equiv s/M_\tau^2)$

$$R_\tau = 12\pi \int_0^1 dz\, (1-z)^2 \Big[(1+2z)\, \mathrm{Im}\Pi^T(z) + \mathrm{Im}\Pi^L(z) \Big] \,,$$

with the appropriate combinations

$$\Pi^J(z) = |V_{ud}|^2 \Big[\Pi_{ud}^{V,J} + \Pi_{ud}^{A,J} \Big] + |V_{us}|^2 \Big[\Pi_{us}^{V,J} + \Pi_{us}^{A,J} \Big] \,.$$

Hadronic τ decay moments

Additional information can be inferred from the moments

$$R_\tau^{kl} \equiv \int_0^1 dz\,(1-z)^k z^l \,\frac{dR_\tau}{dz} = R_{\tau,V+A}^{kl} + R_{\tau,S}^{kl}\,.$$

Theoretically, R_τ^{kl} can be expressed as:

$$R_\tau^{kl} = N_c\,S_{\rm EW}\Big\{(|V_{ud}|^2+|V_{us}|^2)\Big[1+\delta^{kl(0)}\Big] + \sum_{D\geq 2}\Big[|V_{ud}|^2\delta_{ud}^{kl(D)}+|V_{us}|^2\delta_{us}^{kl(D)}\Big]\Big\}\,.$$

$\delta_{ud}^{kl(D)}$ and $\delta_{us}^{kl(D)}$ are corrections in the Operator Product Expansion, the most important ones being $\sim m_s^2$ and $m_s\langle\bar{q}q\rangle$.

Perturbative expansion

For R_τ, it is advantageous to work with the Adler function $D(s)$:

$$D(s) \equiv -\, s\,\frac{d}{ds}\,\Pi(s) = \frac{N_c}{12\pi^2}\sum_{n=0}^{\infty} a_\mu^n \sum_{k=1}^{n+1} k\, c_{n,k} L^{k-1}$$

where $a_\mu \equiv \alpha_s(\mu)/\pi$ and $L \equiv \ln(-s/\mu^2)$.

The physical quantity $D(s)$ satisfies a homogeneous RGE:

$$-\,\mu\,\frac{d}{d\mu}\,D(s) = \left[2\,\frac{\partial}{\partial L} + \beta(a)\frac{\partial}{\partial a}\right] D(s) = 0$$

As a consequence, only the coefficients $c_{n,1}$ are independent:

$$c_{0,1} = c_{11} = 1\,, \quad c_{2,1} = 1.640\,, \quad c_{3,1} = 6.371\,,$$

$$c_{4,1} = 49.076\,!!!$$

(Baikov, Chetyrkin, Kühn 2007)

Fixed order perturbation theory amounts to choice $\mu^2 = M_\tau^2$:

$$\delta^{(0)}_{\rm FO} = \sum_{n=0}^{\infty} a^n(M_\tau^2) \sum_{k=1}^{n+1} k\, c_{n,k} J_{k-1}$$

A given perturbative order n depends on all coefficients $c_{m,1}$ with $m \leq n$, and on the coefficients of the QCD β-function.

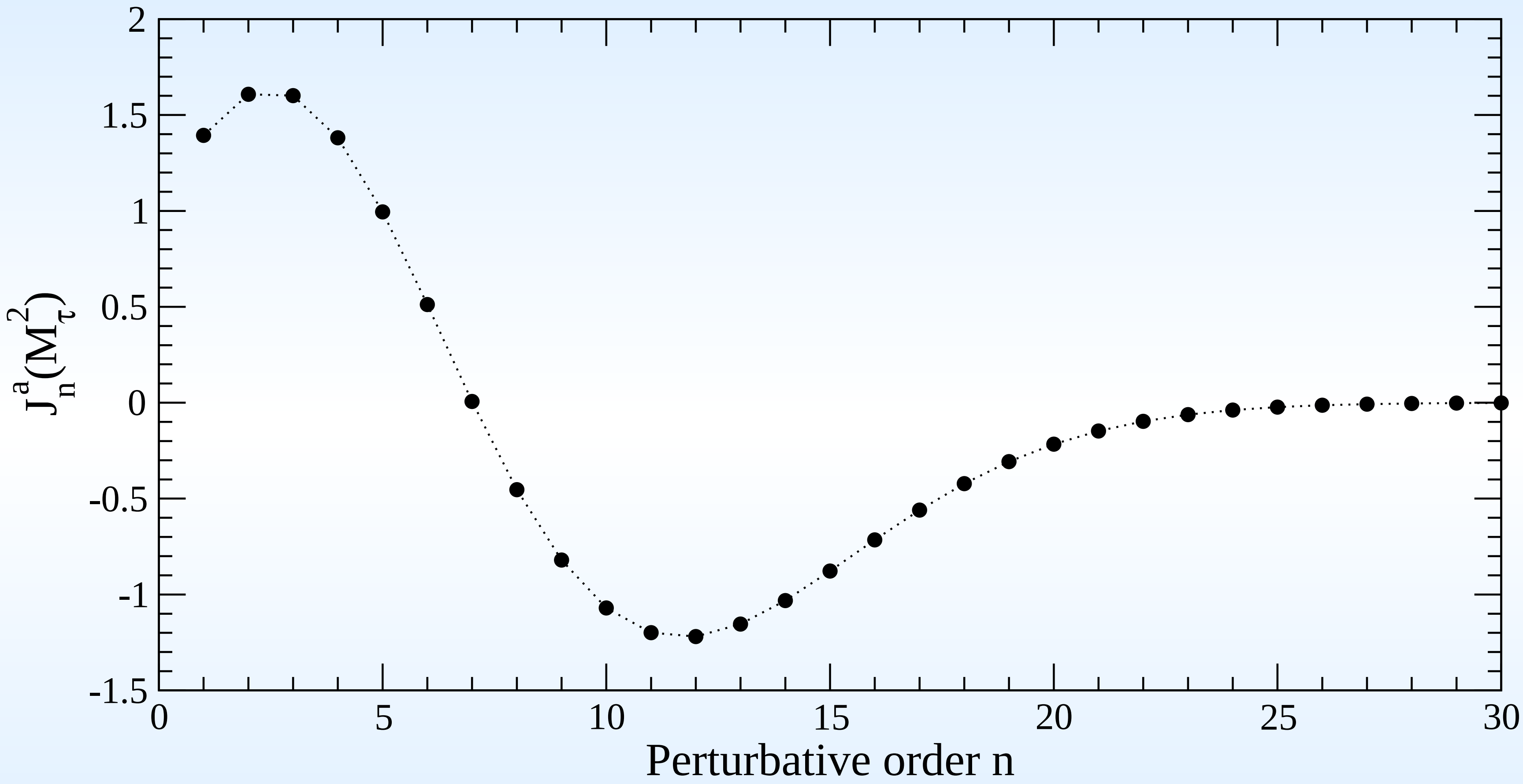
Contour improved perturbation theory employs $\mu^2 = -M_\tau^2 x$:

(Pivovarov; Le Diberder, Pich 1992)

$$\delta^{(0)}_{\rm CI} = \sum_{n=0}^{\infty} c_{n,1} J_n^a(M_\tau^2) \qquad \text{with}$$

$$J_n^a(M_\tau^2) = \frac{1}{2\pi i} \oint\limits_{|x|=1} \frac{dx}{x} (1-x)^3 (1+x) a^n(-M_\tau^2 x)$$

Contour integrals



$$\alpha_s(M_\tau^2) = 0.34\,.$$

Employing $\alpha_s(M_\tau^2) = 0.34$, the numerical analysis results in:

$$
\begin{array}{lccccc}
 & a^1 & a^2 & a^3 & a^4 & a^5 \\
\delta^{(0)}_{\rm FO} = & 0.108 + & 0.061 + & 0.033 + & 0.017 & (+0.009) = 0.220\,(0.229) \\
\delta^{(0)}_{\rm CI} = & 0.148 + & 0.030 + & 0.012 + & 0.009 & (+0.004) = 0.198\,(0.202)
\end{array}
$$

Contour improved PT appears to be better convergent.

The difference between both approaches amounts to 0.022 !

From the uniform convergence of $\delta^{(0)}_{\rm FO}$, and the assumption that the series is not yet asymptotic, one may also infer

$$c_{5,1} = 283 \pm 283\,,$$

leading to a difference of $\delta^{(0)}_{\rm FO} - \delta^{(0)}_{\rm CI} = 0.027$.

To further investigate the difference between CI and FOPT, we propose to model the Borel-transformed Adler function.

$$4\pi^2 D(s) \equiv 1 + R(s) \equiv 1 + \sum_{n=0}^{\infty} r_n \alpha_s(s)^{n+1} \,,$$

where $r_n = c_{n+1,1}/\pi^{n+1}$. The Borel-transform reads:

$$\tilde{R}(\alpha) = \int_0^\infty dt\, e^{-t/\alpha} B[R](t)\,; \quad B[R](t) = \sum_{n=0}^{\infty} r_n \frac{t^n}{n!}\,.$$

Our main model will be a “Padé-type” approximant, which is inspired by the large-β_0 approximation.

$$B[R](u) \;=\; \mathrm{e}^{-Cu}\left\{ d_1^{\mathrm{UV}}\left[\frac{1}{(1+u)^2}+\frac{5}{6}\,\frac{1}{(1+u)}\right]+\frac{d_2^{\mathrm{UV}}}{(2+u)} + \frac{d_1^{\mathrm{IR}}}{(2-u)}+\frac{d_2^{\mathrm{IR}}}{(3-u)}+\frac{d_3^{\mathrm{IR}}}{(4-u)} \right\},$$

where $u=\beta_0 t$.

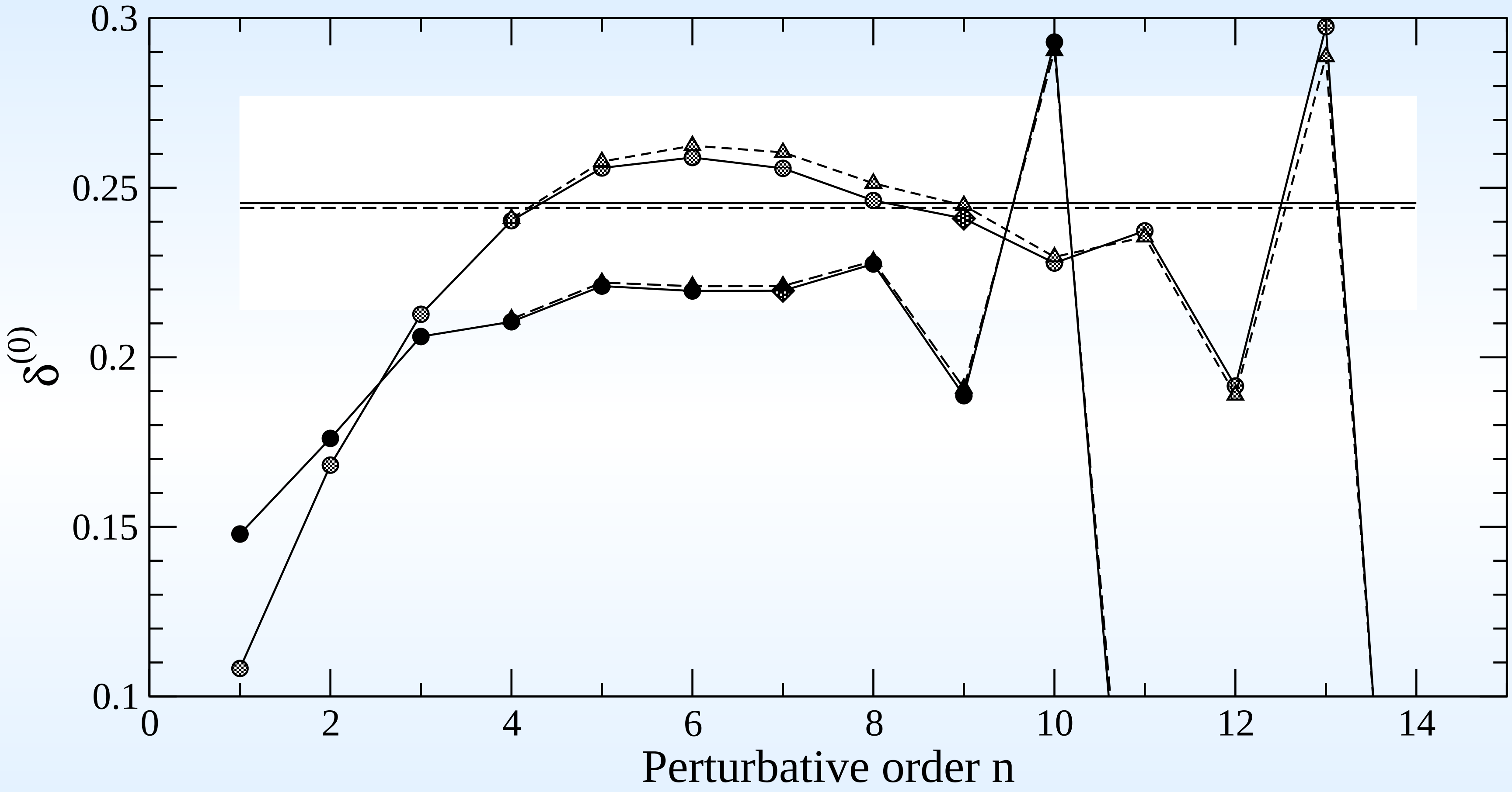
The model incorporates the renormalon pole structure as found in the large-β_0 approximation.

(Beneke 1993; Broadhurst 1993)

C is a scheme-dependent constant. ($C=-5/3$ in large-β_0.)

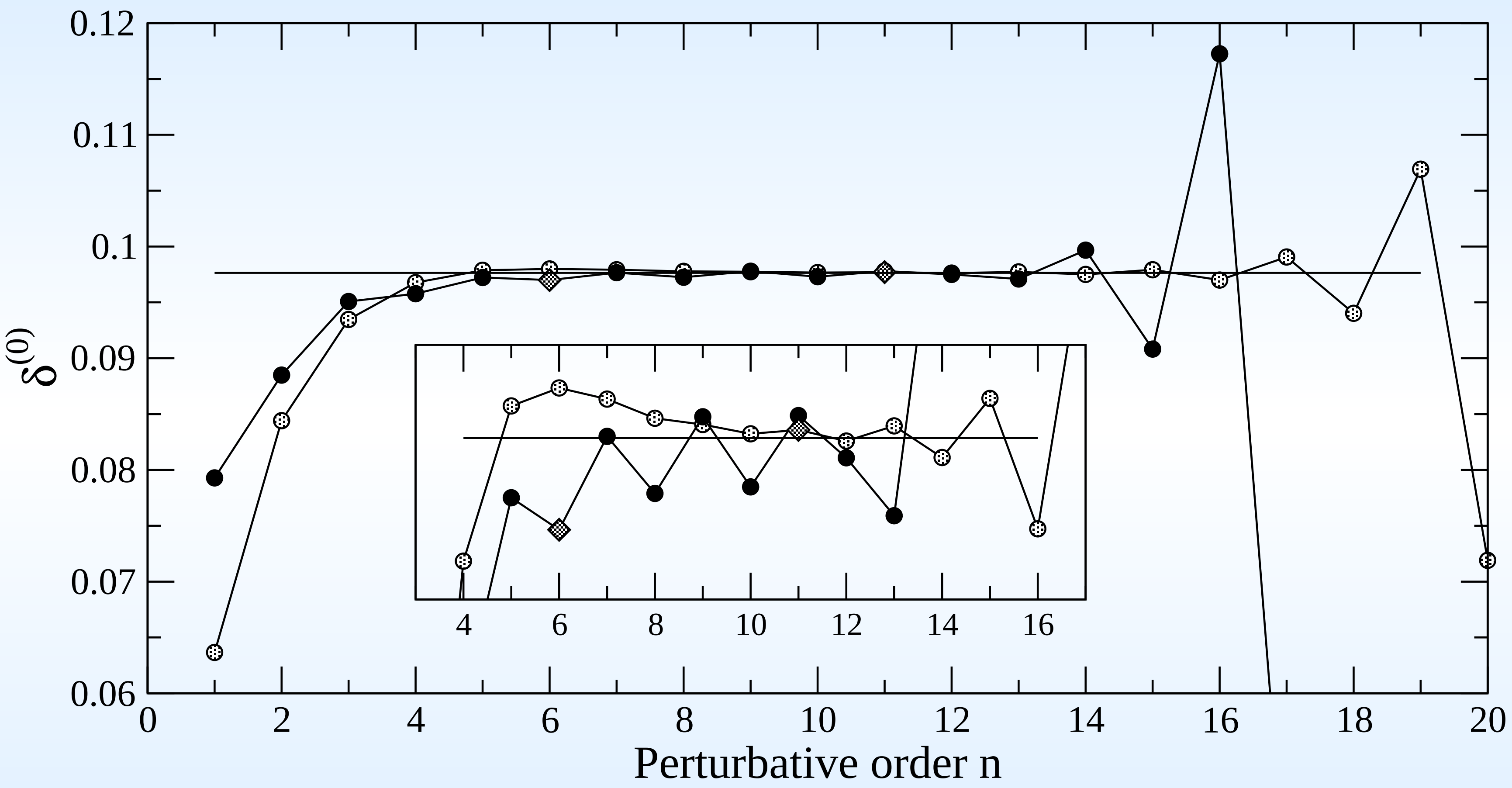
With a definite prescription of how to treat the poles, also the Borel-resummation can be defined. (Principal Value Prescr.)

Large-β_0



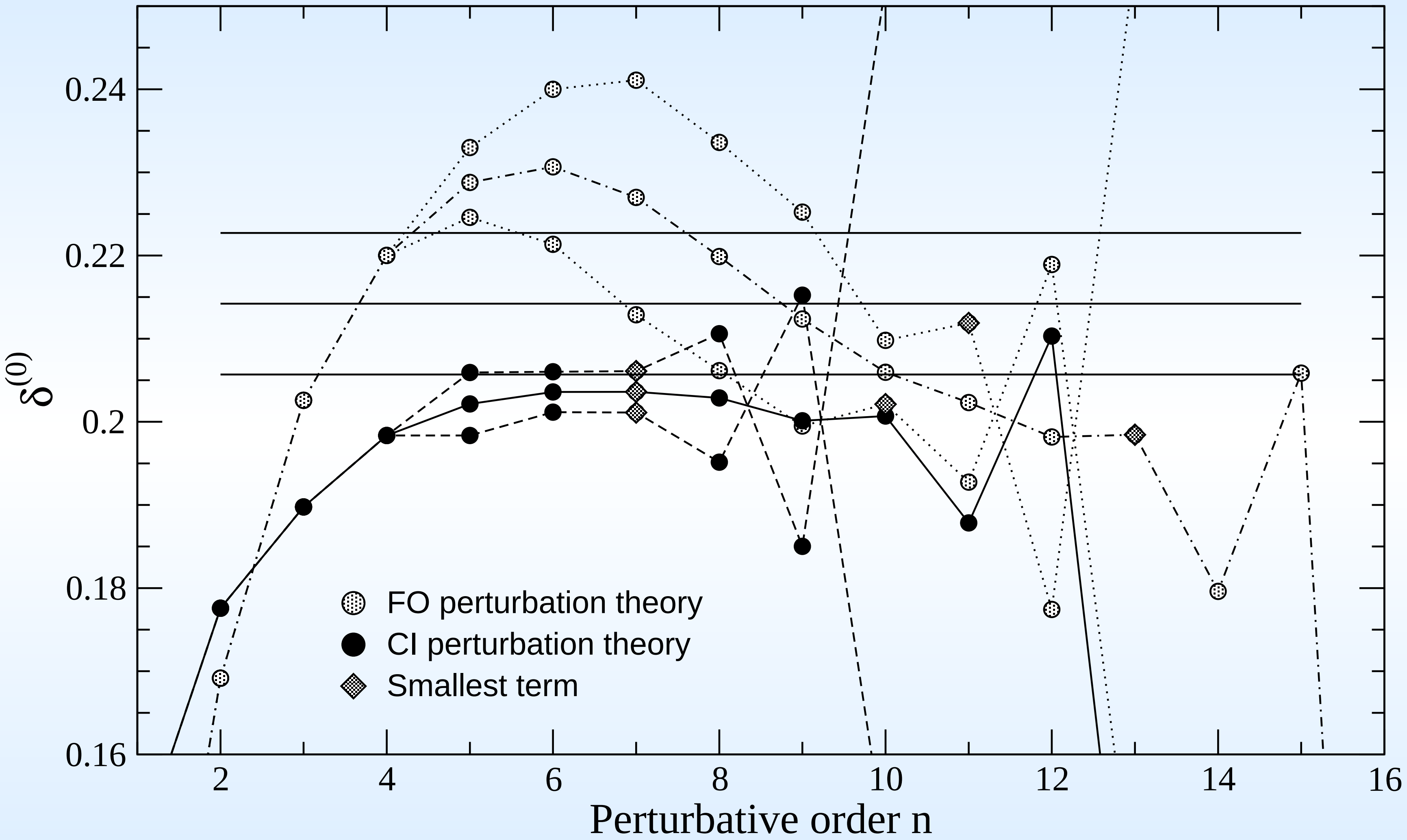
$$\tilde{\delta}^{(0)} = 0.2455 \pm i\,0.0315\,, \quad \delta^{(0)}_{\rm CI} = 0.2197\,, \quad \delta^{(0)}_{\rm FO} = 0.2409\,.$$

Large-β_0



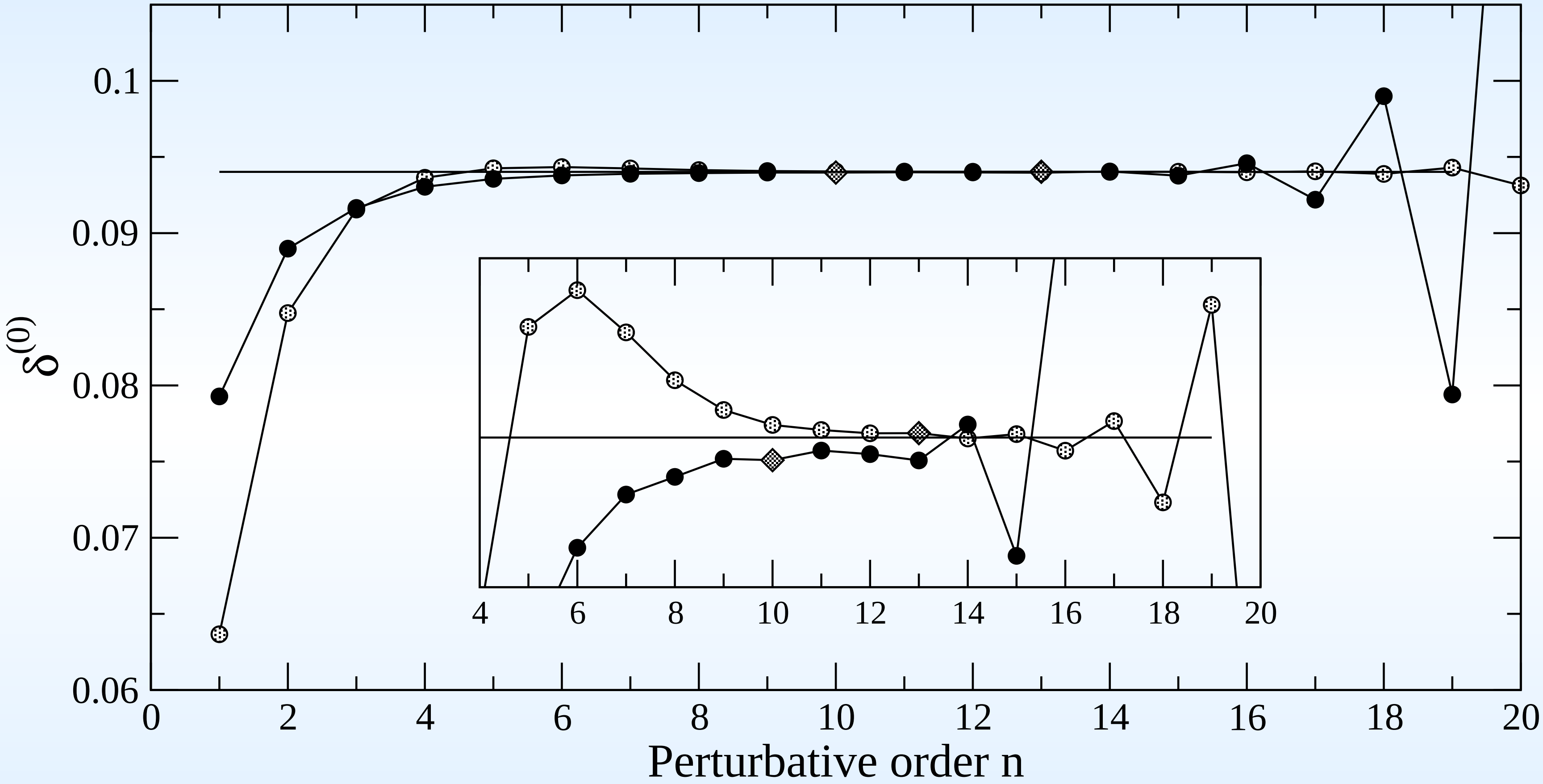
$$\alpha_s(M_\tau^2) = 0.20\,.$$

Model with 2 UV and 3 IR poles



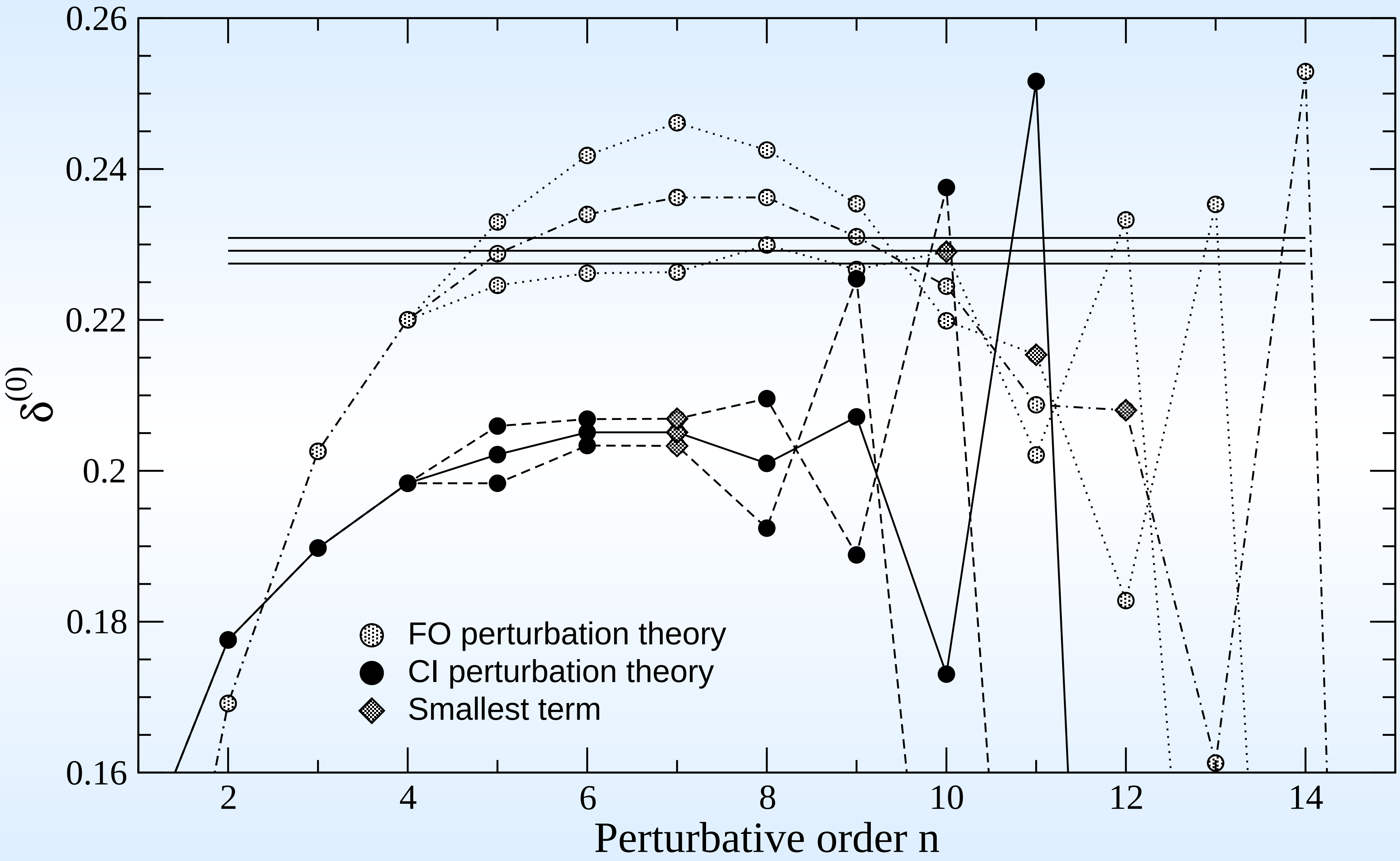
$$\tilde{\delta}^{(0)} = 0.2142 \pm i\, 0.0123\,, \quad \delta^{(0)}_{\rm CI} = 0.2036\,, \quad \delta^{(0)}_{\rm FO} = 0.1984\,.$$

Model with 2 UV and 3 IR poles



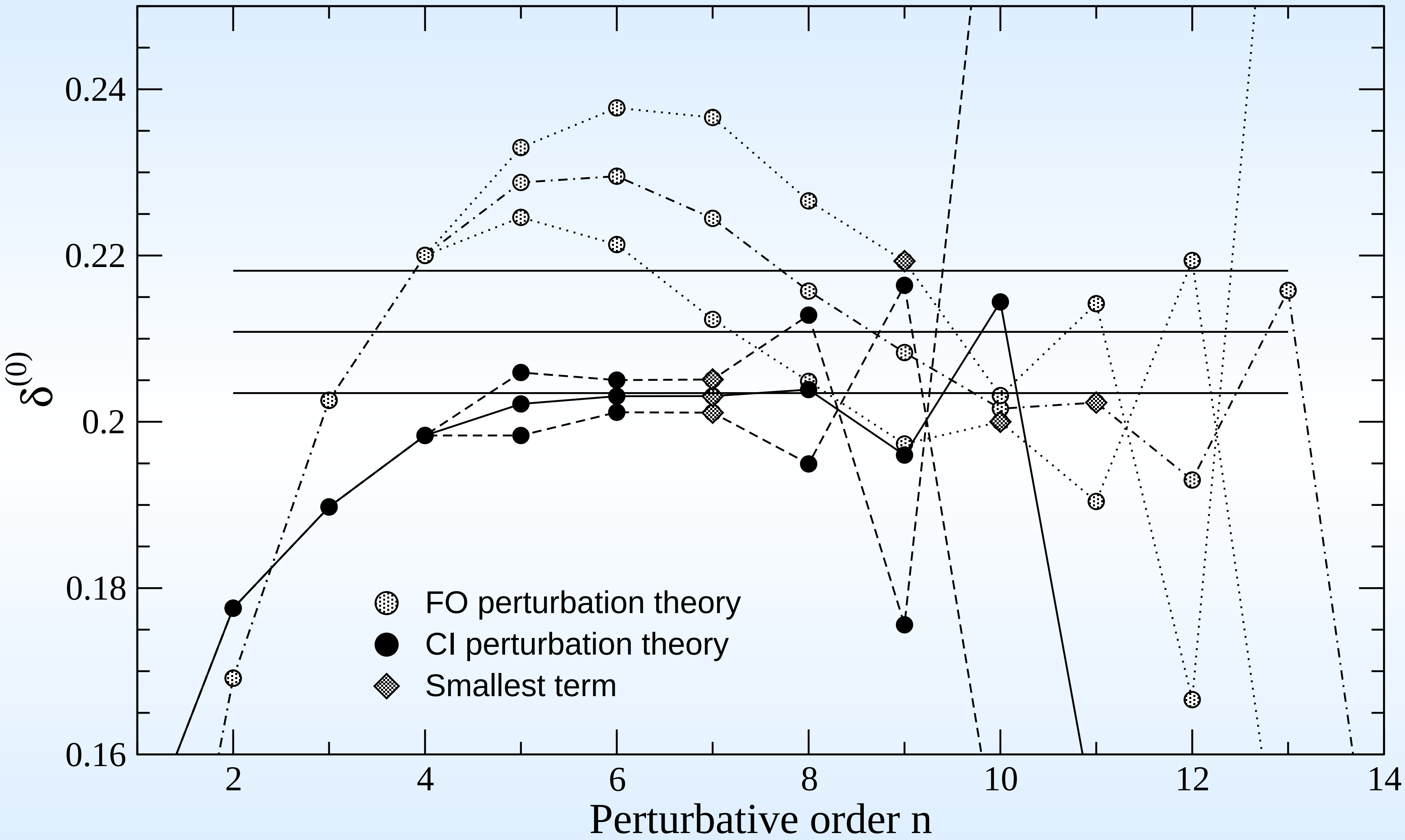
$$\alpha_s(M_\tau^2) = 0.20\,.$$

Leading IR pole fixed to large-β_0



$$\tilde{\delta}^{(0)} = 0.2292 \pm i\,0.0275\,, \quad \delta^{(0)}_{\rm CI} = 0.2051\,, \quad \delta^{(0)}_{\rm FO} = 0.2080\,.$$

Quadratic higher poles



$$\tilde{\delta}^{(0)} = 0.2108 \pm i\, 0.0090\,, \quad \delta^{(0)}_{\rm CI} = 0.2031\,, \quad \delta^{(0)}_{\rm FO} = 0.2023\,.$$

Employing the hadronic decay rate into light quarks

$$R_{\tau,V+A} = N_c \, |V_{ud}|^2 \, S_{\rm EW} \Big[1 + \delta^{(0)} + \delta^{\rm NP}_{V+A} \Big]$$

one finds

$$\delta^{(0)} = \frac{R_{\tau,V+A}}{3|V_{ud}|^2 S_{\rm EW}} - 1 - \delta^{\rm NP}_{V+A} = 0.2032(48)(21)$$

The first uncertainty is due to $R_{\tau,V+A}$, while the remaining error is dominated by $\delta^{\rm NP}_{V+A}$.

Scanning over plausible models and adjusting α_s such as to reproduce $\delta^{(0)}$, one finally obtains

$$\alpha_s(M_\tau) = 0.3293(52)(94) \;\Rightarrow\; \alpha_s(M_Z) = 0.1197(13)$$

Conclusions

For small coupling, FOPT povides the smoother approach to the resummed value $\tilde{\delta}^{(0)}$. At $\alpha_s \approx 0.33$, though CIPT and FOPT turn out compatible, the situation is less clear.

In all studied cases the difference $\tilde{\delta}^{(0)} - \delta_{\rm CI}^{(0)}$ is found to be of the order of the complex ambiguity.

The size of the complex ambiguity is dominated by the size of the residue of the leading IR pole at $u = 2$.

Work on m^2 and scalar contributions in process with F. Schwab.

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Thank You for Your attention !