Cusps in $K \to 3\pi$ decays

Andreas Fuhrer

University of Bern

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In collaboration with M. Bissegger, J. Gasser, B. Kubis and A. Rusetsky

Outline

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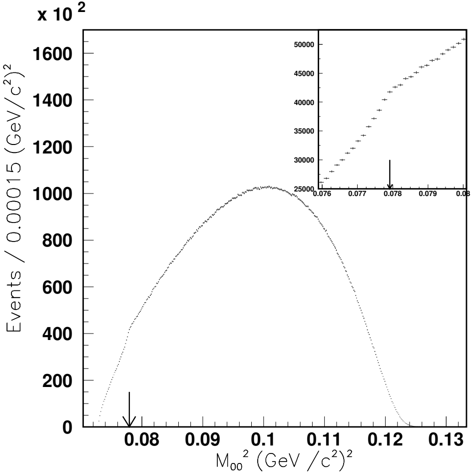
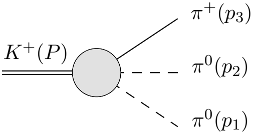
Motivation



$\approx 10^8$ measured K^+ decays



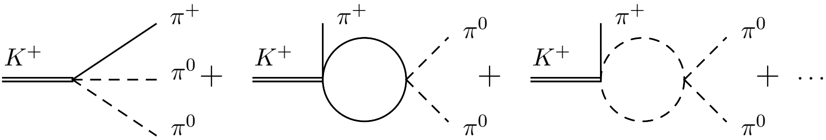
$$M_{00}^2 = s_3 = (P - p_3)^2$$

Figure from NA48/2 2005

The cusp

Theory of the cusp:

Wigner 1948, Meissner, Müller, Steininger 1997



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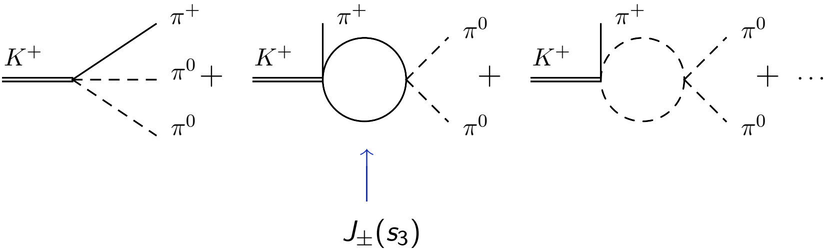

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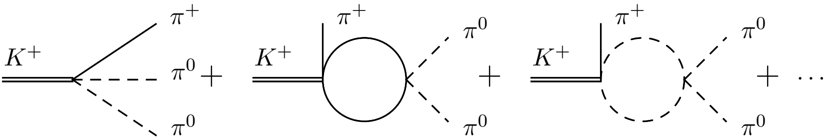

Decompose $J_\pm(s_3)$ as

$$J_\pm(s_3) = R(s_3) + \begin{cases} -\frac{1}{16\pi}\sqrt{\frac{4M_\pi^2}{s_3} - 1} & : \quad s_3 < 4M_\pi^2 \\ \frac{i}{16\pi}\sqrt{1 - \frac{4M_\pi^2}{s_3}} & : \quad s_3 > 4M_\pi^2 \end{cases}$$

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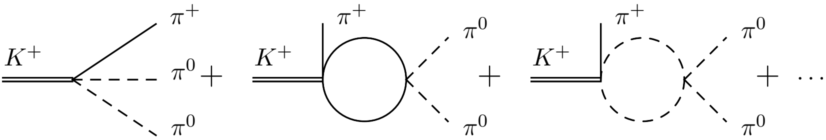

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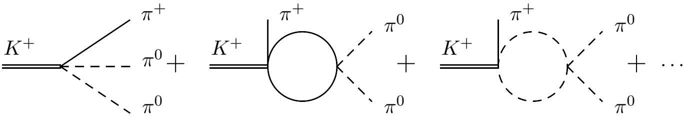
$$s_3^{\text{min}} = 4M_{\pi^0}^2 < 4M_\pi^2$$

Measuring $\pi\pi$ scattering lengths

Cabibbo 2004, Cabibbo and Isidori 2005



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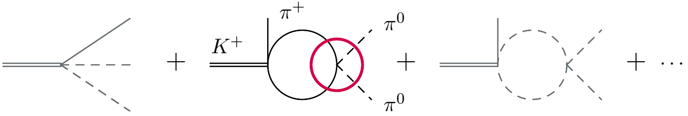

$$A(\pi^+\pi^- \to \pi^0\pi^0) \quad = \quad \frac{32\pi}{3}(a_2 - a_0)\left\{1 + \frac{M_\pi^2 - M_{\pi^0}^2}{3M_\pi^2} + \cdots\right\}$$

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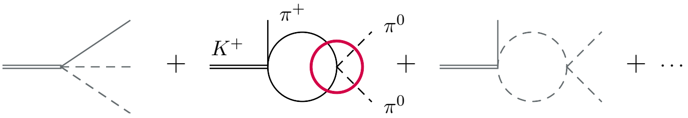

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Method: Measure couplings $\to$ $\pi\pi$ scattering lengths

Outline

Nonrelativistic framework

$$\mathcal{L}_{kin} = \Phi^\dagger(2W)(i\partial_t - W)\Phi, \qquad W = \sqrt{M_\pi^2 - \Delta}$$

$$\frac{1}{M_\pi^2 - p^2} = \underbrace{\frac{1}{2\omega(\vec{p})}\frac{1}{\omega(\vec{p}) - p^0}}_{\text{particles}} + \underbrace{\frac{1}{2\omega(\vec{p})}\frac{1}{\omega(\vec{p}) + p^0}}_{\text{antiparticles}}$$

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$$\mathcal{L}_{int} = C_x(\pi_+^\dagger \pi_-^\dagger \pi_0 \pi_0 + h.c.) + L_0(K_L^\dagger \pi_0 \pi_+ \pi_- + h.c.) + \cdots$$

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- Nonrelativistic theory correctly reproduces the singularity structure of the full theory at small momenta $|\vec{p}| \ll M_\pi$
- Distant singularities are included in the couplings C_i
- Matching: The couplings C_i are adjusted such that the relativistic and the nonrelativistic amplitudes agree at low momenta
  $\longrightarrow C_x \sim a$

Colangelo, Gasser, Kubis, Rusetsky 2006

Perturbative expansion

Momenta	:	$\lvert\vec{p}\rvert/M_\pi = O(\epsilon)$
Kinetic energies	:	$T = \omega(\vec{p}) - M_\pi = O(\epsilon^2)$
Mass difference	:	$M_\pi^2 - M_{\pi^0}^2 = O(\epsilon^2)$
Masses	:	$O(1)$
Pion loops	:	$O(\epsilon)$

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Combined expansion

- in ϵ (momenta)
- and a (scattering lengths/effective ranges/...)

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- Expansion is done directly in terms of scattering lengths in contrast to chiral perturbation theory.

$$a = O(M_\pi^2) + O(M_\pi^4) + O(M_\pi^6) + \cdots$$

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- Lagrangian framework: strictures by unitarity and analyticity are automatically satisfied, inclusion of photons straightforward

Results

$K^\pm \to \pi^\pm \pi^0 \pi^0$

$K^\pm \to \pi^\pm \pi^+ \pi^-$

Colangelo, Gasser, Kubis, Rusetsky 2006

$K_L \to 3\pi^0$
$K_L \to \pi^0 \pi^+ \pi^-$

$\eta \to 3\pi^0$
$\eta \to \pi^0 \pi^+ \pi^-$

Bissegger, Gasser, Kubis, Rusetsky, AF, 2007

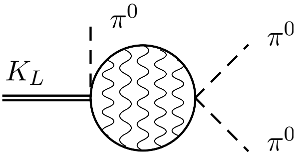
Comments:

- Order $\epsilon^4, a\epsilon^5, a^2\epsilon^2$
- Keep higher orders in ϵ: $p_0 = \sqrt{m^2 + \vec{p}^2}$
- P-waves included

Outline

Pionium

Change of analytical structure at the cusp at $O(\alpha)$



Ionisation energy: $\sim 1.86\,\mathrm{keV}$

Width of ground state: $\sim 0.2\,\mathrm{eV}$

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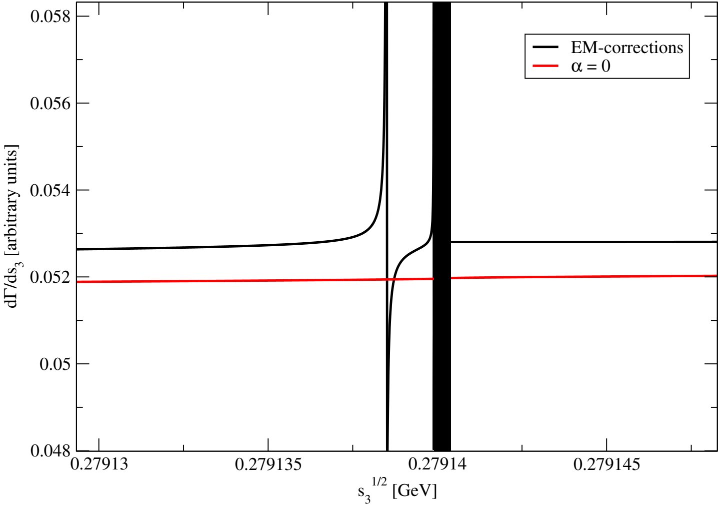

$1.86\,\text{keV} \approx 6.9 \cdot 10^{-3}$ bins

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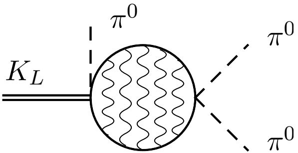

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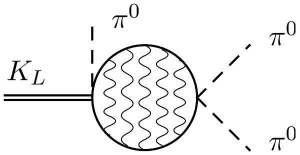

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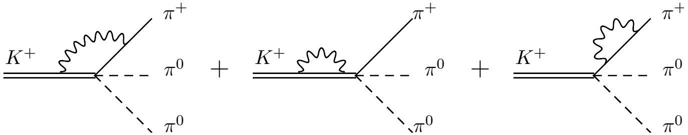
$\longrightarrow$ Radiative corrections for $K_L \to 3\pi^0$ done

Photons in $K^+ \to \pi^+\pi^0\pi^0$ decays

1) Virtual photons: infrared singularities agree with relativistic theory

   Tricky part: threshold expansion $\longrightarrow$ IR and UV divergencies cancel each other

   Pionium: ok



Photons in $K^+ \to \pi^+\pi^0\pi^0$ decays

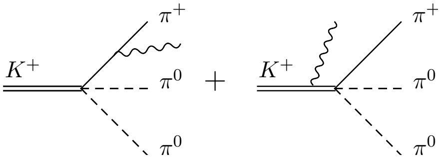
1) Virtual photons: infrared singularities agree with relativistic theory

   Tricky part: threshold expansion $\longrightarrow$ IR and UV divergencies cancel each other

   Pionium: ok

2) Real photons: phase space integrations possible

   Tricky part: four particle phase space integration in d dimensions with cut in E_γ



Phase space factorisation

$$
\begin{aligned}
P_4\left[\,|\mathcal{M}|^2\,, Q^2\right] &= (2\pi)^d \int ds_3 P_3\left[\,P_2\left[|\mathcal{M}|^2, s_3\right]\,, Q^2\right] \\
P_n\left[X, Q^2\right] &= \int d\mu(p_1)\cdots d\mu(p_n)\delta^{(d+1)}(Q - p_1 \cdots - p_n)X \\
d\mu(k) &= \frac{d^d k}{(2\pi)^d 2k^0}
\end{aligned}
$$

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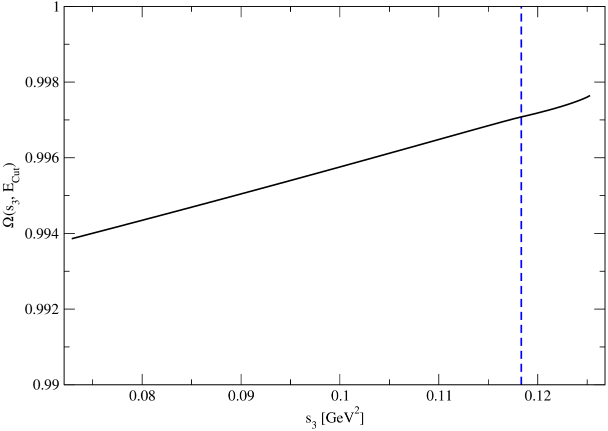
Simplifications for $K^+ \to \pi^+\pi^0\pi^0$:

- differential width is measured
- $|\mathcal{M}|^2$ does not depend on integration variables of P_2

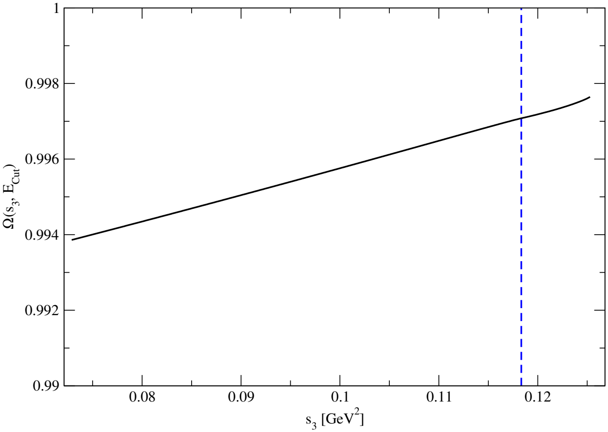
$$\frac{d\Gamma}{ds_3} = (2\pi)^d N P_2\left[1,s_3\right] P_3\left[|\mathcal{M}|^2, Q^2\right]$$

Bremsstrahlung



$$\frac{d\Gamma}{ds_3}(s_3) \quad = \quad \Omega(s_3, E_{Cut}) \left.\frac{d\Gamma}{ds_3}\right|_{\alpha=0}$$

Bremsstrahlung



Comments:

- Preliminary
- Kinematical cusp: E_{Cut}
- Relativistic loop functions
- Charged channel: Isidori 2007

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Summary

- Evaluation of decay amplitdues for $K_L \rightarrow 3\pi$ and for $\eta \rightarrow 3\pi$ at order $\epsilon^4, a\epsilon^5, a^2\epsilon^2$

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- Evaluation of decay amplitdues for $K_L \to 3\pi$ and for $\eta \to 3\pi$ at order $\epsilon^4, a\epsilon^5, a^2\epsilon^2$
- Prescription for treatement of pionium contributions
  - $\longrightarrow$ Radiative corrections for $K_L \to 3\pi^0$ done
- Considerable progress in radiative corrections for $K^+ \to \pi^+\pi^0\pi^0$ decays

SPARES

$$M_K - 3M_{\pi^0} \simeq 90\,\mathrm{MeV}$$

Does the expansion in ϵ converge in the whole decay region?

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$$|\mathcal{M}_{K^+\to\pi^+\pi^0\pi^0}|^2 = 1 + gu + hu^2 + kv^2$$

$$u = (s_3 - s_0)/M_\pi^2 \quad v = (s_2 - s_1)/M_\pi^2 \quad s_i = (P - p_i)^2$$

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Yes!