Marián Kolesár, Jiří Novotný

Institute of particle and nuclear physics, MFF UK, Prague

Some aspects of 'Resummed' chiral perturbation theory

A Introduction

B Illustrative example
- $\eta\,\pi^0 \longrightarrow \eta\,\pi^0$ scattering

C Definition of the bare expansion
- analyticity of unitarity corrections
- treatment of the masses inside chiral logarithms

D Remainder treatment
- resonance estimate
- Generalized χPT Lagrangian

E Stability of the chiral series and the Standard approach to NLO

F Summary

EuroFlavour '07, Orsay, Nov 14-16, 2007

A. Introduction

A.1 Phase structure of QCD with varying number of light quarks

$N_f \geq N_f^A = 11/2 N_c$	asymptotic freedom lost	
$N_f^c < N_f < N_f^A$	conformal window	
$N_f = N_f^c$	chiral phase transition	*(Appelquist et al.1998)*
$N_f < N_f^c$	quark confinement, SBχS, hadron spectrum	

Interpretation - counter-play:

- condensating effect of gluon self-interactions
- screening of light quark loop vacuum fluctuations

Indications $N_c = 3$: perturbative methods $N_f^c \sim 10-12$ *(Appelquist et al.1998)*

nonperturbative approaches, lattice $N_f^c \simeq 6$ *(Fischer,Alkofer 2003)* *(Iwasaki et al.2004)*

A.1 Phase structure of QCD with varying number of light quarks

$N_f \geq N_f^A = 11/2 N_c$	asymptotic freedom lost
$N_f^c < N_f < N_f^A$	conformal window
$N_f = N_f^c$	chiral phase transition
$N_f < N_f^c$	quark confinement, SBχS, hadron spectrum

(Appelquist et al.1998)

Interpretation - counter-play:

- condensating effect of gluon self-interactions
- screening of light quark loop vacuum fluctuations

Indications $N_c = 3$: perturbative methods $N_f^c \sim 10{-}12$ *(Appelquist et al.1998)*

nonperturbative approaches, lattice $N_f^c \simeq 6$ *(Fischer,Alkofer 2003)* *(Iwasaki et al.2004)*

"Paramagnetic" inequality: dependence of chiral order parameters on N_f

(Stern et al.2000)

$$F_0(N_f+1) < F_0(N_f), \quad \Sigma(N_f+1) < \Sigma(N_f)$$

$F_0(N_f)$: pseudoscalar decay constant in the chiral limit
$\Sigma(N_f)$: quark condensate in the chiral limit ($\Sigma(N_f) = B_0(N_f) F_0(N_f)^2$)

A.1 Phase structure of QCD with varying number of light quarks

$N_f \geq N_f^A = 11/2 N_c$	asymptotic freedom lost	
$N_f^c < N_f < N_f^A$	conformal window	
$N_f = N_f^c$	chiral phase transition	*(Appelquist et al.1998)*
$N_f < N_f^c$	quark confinement, SBχS, hadron spectrum	

Interpretation - counter-play:

- condensating effect of gluon self-interactions
- screening of light quark loop vacuum fluctuations

Indications $N_c = 3$: perturbative methods $N_f^c \sim 10-12$ *(Appelquist et al.1998)*

nonperturbative approaches, lattice $N_f^c \simeq 6$ *(Fischer,Alkofer 2003)* *(Iwasaki et al.2004)*

"Paramagnetic" inequality: dependence of chiral order parameters on N_f

(Stern et al.2000)

$$F_0(N_f + 1) < F_0(N_f), \quad \Sigma(N_f + 1) < \Sigma(N_f)$$

$F_0(N_f)$: pseudoscalar decay constant in the chiral limit
$\Sigma(N_f)$: quark condensate in the chiral limit ($\Sigma(N_f) = B_0(N_f) F_0(N_f)^2$)

$\longrightarrow$ difference between $SU(2)$ and $SU(3)$ χPT ?

A.2 LEC's connected to suppresion of order paramters

Three flavor χPT: (effect of s-quark vacuum fluctuations)

$$F_0(2)^2 = F_0(3)^2 + 16 m_s B_0 \, L_4^r - 2\bar{\mu}_K + \mathcal{O}(m_s^2)$$

$$\Sigma(2) = \Sigma(3)(1 + \frac{32 m_s B_0}{F_0^2} L_6^r - 2\bar{\mu}_K - \frac{1}{3}\bar{\mu}_\eta) + \mathcal{O}(m_s^2)$$

Large N_c approximation: $N_f/N_c \to 0$ limit

- possible $1/N_c$ and Zweig rule violation?
- L_4, L_6 - Zweig rule and $1/N_c$ suppressed LEC's
- connection to the scalar sector

(Stern et al.2000)

Predictions for L_4^r, L_6^r at M_ρ

- Zweig rule: negative *(Gasser,Leutwyler 1985)*
- Standard χPT to $\mathcal{O}(p^6)$: positive *(Bijnens,Dhonte 2003)*
- Sum rules: positive *(Moussallam 2000)* *(Descotes 2001)*
- Lattice: positive *(MILC Coll.2004,2007)*

A.3 Parameters controlling the suppresion

Convenient parameters relating the order parameters to physical quantities (isospin limit $\hat{m} = (m_u + m_d)/2$)

$$Z(N_f) = \frac{F_0(N_f)^2}{F_\pi^2}, \quad X(N_f) = \frac{2\hat{m}\Sigma(N_f)}{F_\pi^2 M_\pi^2}, \quad Y(N_f) = \frac{X(N_f)}{Z(N_f)} = \frac{m_\pi^2}{M_\pi^2}$$

A.3 Parameters controlling the suppresion

Convenient parameters relating the order parameters to physical quantities
(isospin limit $\hat{m} = (m_u + m_d)/2$)

$$Z(N_f) = \frac{F_0(N_f)^2}{F_\pi^2}, \quad X(N_f) = \frac{2\hat{m}\Sigma(N_f)}{F_\pi^2 M_\pi^2}, \quad Y(N_f) = \frac{X(N_f)}{Z(N_f)} = \frac{m_\pi^2}{M_\pi^2}$$

Experimental results **for the $\pi\pi$ s-wave scattering length (K_{e4}):** *(Stern et al.2002)*

$$X(2) = 0.81 \pm 0.07, \; Z(2) = 0.89 \pm 0.03$$

A.3 Parameters controlling the suppresion

Convenient parameters relating the order parameters to physical quantities (isospin limit $\widehat{m} = (m_u + m_d)/2$)

$$Z(N_f) = \frac{F_0(N_f)^2}{F_\pi^2}, \quad X(N_f) = \frac{2\widehat{m}\Sigma(N_f)}{F_\pi^2 M_\pi^2}, \quad Y(N_f) = \frac{X(N_f)}{Z(N_f)} = \frac{m_\pi^2}{M_\pi^2}$$

Experimental results for the $\pi\pi$ s-wave scattering length (K_{e4}): *(Stern et al.2002)*

$$X(2) = 0.81 \pm 0.07, \ \ Z(2) = 0.89 \pm 0.03$$

Three flavor parameters much less constrained ($r = m_s/\widehat{m}$)

$\pi\pi$ s-wave scattering length (K_{e4}): *(Stern et al.2002)*

$$X(3) \sim 0 - 0.8, \ \ Z(3) \sim 0.3 - 0.9, \ \ r > 14, \ \ Y < 1.2$$

Sum rules ($r \sim 25$): $X(2), Z(2) \sim 0.9, \quad X(3), Z(3) \sim 0.5 - 0.6$ *(Descotes,Stern 2000)*

Recent 'resummed' combined analysis of $\pi\pi$ and πK data: *(Descotes 2007)*

$$X(3) \sim 0 - 0.8, \ \ Z(3) \sim 0.2 - 1, \ \ r > 15, \ \ Y < 1.1$$

A.4 'Resummed' approach to χPT

A.1 $\longrightarrow$ **irregularities in the expansion, possible partial suppression of LO**

Small demonstration: chiral expansion of an observable A:

$$A = A^{(2)} + A^{(4)} + \Delta_A^{(6)}$$

Inverted expansion:

$$\frac{1}{A} = \frac{1}{A^{(2)}} - \frac{A^{(4)}}{{A^{(2)}}^2} + \Delta_{1/A}^{(6)}, \quad \Delta_{1/A}^{(6)} = \frac{1}{A}\left[\left(\frac{A^{(4)}}{A^{(2)}}\right)^2 + \Delta_A^{(6)}\left(\frac{A^{(4)} - A^{(2)}}{{A^{(2)}}^2}\right)\right]$$

A.4 'Resummed' approach to χPT

A.1 $\longrightarrow$ **irregularities in the expansion, possible partial suppression of LO**

Small demonstration: chiral expansion of an observable A:

$$A = A^{(2)} + A^{(4)} + \Delta_A^{(6)}$$

Inverted expansion:

$$\frac{1}{A} = \frac{1}{A^{(2)}} - \frac{A^{(4)}}{A^{(2)^2}} + \Delta_{1/A}^{(6)}, \quad \Delta_{1/A}^{(6)} = \frac{1}{A}\left[\left(\frac{A^{(4)}}{A^{(2)}}\right)^2 + \Delta_A^{(6)}\left(\frac{A^{(4)} - A^{(2)}}{A^{(2)^2}}\right)\right]$$

$\longrightarrow$ even if $\Delta_A^{(6)} \ll A$, if $A^{(2)} \sim A^{(4)}$ $\longrightarrow$

A.4 ‘Resummed’ approach to χPT

A.1 $\longrightarrow$ **irregularities in the expansion, possible partial suppression of LO**

Small demonstration: chiral expansion of an observable A:

$$A = A^{(2)} + A^{(4)} + \Delta_A^{(6)}$$

Inverted expansion:

$$\frac{1}{A} = \frac{1}{A^{(2)}} - \frac{A^{(4)}}{A^{(2)2}} + \Delta_{1/A}^{(6)}, \qquad \Delta_{1/A}^{(6)} = \frac{1}{A}\left[\left(\frac{A^{(4)}}{A^{(2)}}\right)^2 + \Delta_A^{(6)}\left(\frac{A^{(4)} - A^{(2)}}{A^{(2)2}}\right)\right]$$

$\longrightarrow$ even if $\Delta_A^{(6)} \ll A$, if $A^{(2)} \sim A^{(4)}$ $\longrightarrow$ $\Delta_{1/A}^{(6)} \sim 1/A \equiv$ **large higher order remainder**

A.4 'Resummed' approach to χPT

A.1 $\longrightarrow$ **irregularities in the expansion, possible partial suppression of LO**

Small demonstration: chiral expansion of an observable A:

$$A \;=\; A^{(2)} + A^{(4)} + \Delta_A^{(6)}$$

Inverted expansion:

$$\frac{1}{A} \;=\; \frac{1}{A^{(2)}} - \frac{A^{(4)}}{{A^{(2)}}^2} + \Delta_{1/A}^{(6)}, \quad \Delta_{1/A}^{(6)} \;=\; \frac{1}{A}\left[\left(\frac{A^{(4)}}{A^{(2)}}\right)^2 + \Delta_A^{(6)}\left(\frac{A^{(4)} - A^{(2)}}{{A^{(2)}}^2}\right)\right]$$

$\rightarrow$ even if $\Delta_A^{(6)} \ll A$, if $A^{(2)} \sim A^{(4)}$ $\longrightarrow$ $\Delta_{1/A}^{(6)} \sim 1/A \equiv$ **large higher order remainder**

'Resummed' χPT - a special treatment of the chiral expansion

(Descotes-Genon, Fuchs, Girlanda, Stern 2004)

- Standard power counting and form of the effective Lagrangian
- Assumes possible irregularities in the expansion
- Only a limited subset of 'bare' expansions of 'good' observables trusted
- Reparametrizations done in a non-perturbative algebraic way
- Higher order remainders are kept and estimated, treated as sources of error

A.4 'Resummed' approach to χPT

A.1 $\longrightarrow$ **irregularities in the expansion, possible partial suppression of LO**

Small demonstration: chiral expansion of an observable A:

$$A = A^{(2)} + A^{(4)} + \Delta_A^{(6)}$$

Inverted expansion:

$$\frac{1}{A} = \frac{1}{A^{(2)}} - \frac{A^{(4)}}{A^{(2)^2}} + \Delta_{1/A}^{(6)}, \quad \Delta_{1/A}^{(6)} = \frac{1}{A}\left[\left(\frac{A^{(4)}}{A^{(2)}}\right)^2 + \Delta_A^{(6)}\left(\frac{A^{(4)} - A^{(2)}}{A^{(2)^2}}\right)\right]$$

$\longrightarrow$ even if $\Delta_A^{(6)} \ll A$, if $A^{(2)} \sim A^{(4)}$ $\longrightarrow$ $\Delta_{1/A}^{(6)} \sim 1/A \equiv$ **large higher order remainder**

'Resummed' χPT - a special treatment of the chiral expansion

(Descotes-Genon, Fuchs, Girlanda, Stern 2004)

- Standard power counting and form of the effective Lagrangian
- Assumes possible irregularities in the expansion
- Only a limited subset of 'bare' expansions of 'good' observables trusted
- Reparametrizations done in a non-perturbative algebraic way
- Higher order remainders are kept and estimated, treated as sources of error

A.4 'Resummed' approach to χPT

A.1 $\longrightarrow$ **irregularities in the expansion, possible partial suppression of LO**

Small demonstration: chiral expansion of an observable A:

$$A \; = \; A^{(2)} + A^{(4)} + \Delta_A^{(6)}$$

Inverted expansion:

$$\frac{1}{A} \; = \; \frac{1}{A^{(2)}} - \frac{A^{(4)}}{A^{(2)^2}} + \Delta_{1/A}^{(6)}, \quad \Delta_{1/A}^{(6)} \; = \; \frac{1}{A}\left[\left(\frac{A^{(4)}}{A^{(2)}}\right)^2 + \Delta_A^{(6)}\left(\frac{A^{(4)} - A^{(2)}}{A^{(2)^2}}\right)\right]$$

$\longrightarrow$ even if $\Delta_A^{(6)} \ll A$, if $A^{(2)} \sim A^{(4)}$ $\longrightarrow$ $\Delta_{1/A}^{(6)} \sim 1/A \equiv$ **large higher order remainder**

'Resummed' χPT - a special treatment of the chiral expansion

(Descotes-Genon, Fuchs, Girlanda, Stern 2004)

- Standard power counting and form of the effective Lagrangian
- Assumes possible irregularities in the expansion
- Only a limited subset of 'bare' expansions of 'good' observables trusted
- Reparametrizations done in a non-perturbative algebraic way
- Higher order remainders are kept and estimated, treated as sources of error

A.4 'Resummed' approach to χPT

A.1 $\longrightarrow$ **irregularities in the expansion, possible partial suppression of LO**

Small demonstration: chiral expansion of an observable A:

$$A = A^{(2)} + A^{(4)} + \Delta_A^{(6)}$$

Inverted expansion:

$$\frac{1}{A} = \frac{1}{A^{(2)}} - \frac{A^{(4)}}{A^{(2)^2}} + \Delta_{1/A}^{(6)}, \quad \Delta_{1/A}^{(6)} = \frac{1}{A}\left[\left(\frac{A^{(4)}}{A^{(2)}}\right)^2 + \Delta_A^{(6)}\left(\frac{A^{(4)} - A^{(2)}}{A^{(2)^2}}\right)\right]$$

$\rightarrow$ even if $\Delta_A^{(6)} \ll A$, if $A^{(2)} \sim A^{(4)}$ $\rightarrow$ $\Delta_{1/A}^{(6)} \sim 1/A \equiv$ **large higher order remainder**

'Resummed' χPT - a special treatment of the chiral expansion

(Descotes-Genon, Fuchs, Girlanda, Stern 2004)

- Standard power counting and form of the effective Lagrangian
- Assumes possible irregularities in the expansion
- Only a limited subset of 'bare' expansions of 'good' observables trusted
- Reparametrizations done in a non-perturbative algebraic way
- Higher order remainders are kept and estimated, treated as sources of error

A.4 'Resummed' approach to χPT

A.1 $\longrightarrow$ **irregularities in the expansion, possible partial suppression of LO**

Small demonstration: chiral expansion of an observable A:

$$A = A^{(2)} + A^{(4)} + \Delta_A^{(6)}$$

Inverted expansion:

$$\frac{1}{A} = \frac{1}{A^{(2)}} - \frac{A^{(4)}}{A^{(2)2}} + \Delta_{1/A}^{(6)}, \quad \Delta_{1/A}^{(6)} = \frac{1}{A}\left[\left(\frac{A^{(4)}}{A^{(2)}}\right)^2 + \Delta_A^{(6)}\left(\frac{A^{(4)} - A^{(2)}}{A^{(2)2}}\right)\right]$$

$\rightarrow$ even if $\Delta_A^{(6)} \ll A$, if $A^{(2)} \sim A^{(4)}$ $\longrightarrow$ $\Delta_{1/A}^{(6)} \sim 1/A \equiv$ **large higher order remainder**

'Resummed' χPT - a special treatment of the chiral expansion

(Descotes-Genon, Fuchs, Girlanda, Stern 2004)

- Standard power counting and form of the effective Lagrangian
- Assumes possible irregularities in the expansion
- Only a limited subset of 'bare' expansions of 'good' observables trusted
- Reparametrizations done in a non-perturbative algebraic way
- Higher order remainders are kept and estimated, treated as sources of error

A.4 ‘Resummed’ approach to χPT

A.1 $\longrightarrow$ **irregularities in the expansion, possible partial suppression of LO**

Small demonstration: chiral expansion of an observable A:

$$A = A^{(2)} + A^{(4)} + \Delta_A^{(6)}$$

Inverted expansion:

$$\frac{1}{A} = \frac{1}{A^{(2)}} - \frac{A^{(4)}}{A^{(2)2}} + \Delta_{1/A}^{(6)}, \quad \Delta_{1/A}^{(6)} = \frac{1}{A}\left[\left(\frac{A^{(4)}}{A^{(2)}}\right)^2 + \Delta_A^{(6)}\left(\frac{A^{(4)} - A^{(2)}}{A^{(2)2}}\right)\right]$$

$\rightarrow$ even if $\Delta_A^{(6)} \ll A$, if $A^{(2)} \sim A^{(4)}$ $\rightarrow$ $\Delta_{1/A}^{(6)} \sim 1/A \equiv$ **large higher order remainder**

‘Resummed’ χPT - a special treatment of the chiral expansion

(Descotes-Genon, Fuchs, Girlanda, Stern 2004)

- Standard power counting and form of the effective Lagrangian
- Assumes possible irregularities in the expansion
- Only a limited subset of ‘bare’ expansions of ‘good’ observables trusted
- Reparametrizations done in a non-perturbative algebraic way
- Higher order remainders are kept and estimated, treated as sources of error

A.4 'Resummed' approach to χPT - basic steps

Step 1: 'Strict' chiral expansion

- linearly related to a Green function obtained from the generated functional
- expressed strictly in terms of the parameters of the effective Lagrangian
- done formally to all orders, higher orders collected in a remainder

A.4 'Resummed' approach to χPT - basic steps

Step 1: 'Strict' chiral expansion

- linearly related to a Green function obtained from the generated functional
- expressed strictly in terms of the parameters of the effective Lagrangian
- done formally to all orders, higher orders collected in a remainder

Step 2: Definition of the 'bare' expansion

- careful modification of the strict expansion in order to accommodate additional requirements, chiefly:
- unitarity, analytical structure of the S-matrix (physical masses in loops)
- physical masses in all chiral logarithms?

A.4 'Resummed' approach to χPT - basic steps

Step 1: 'Strict' chiral expansion

- linearly related to a Green function obtained from the generated functional
- expressed strictly in terms of the parameters of the effective Lagrangian
- done formally to all orders, higher orders collected in a remainder

Step 2: Definition of the 'bare' expansion

- careful modification of the strict expansion in order to accommodate additional requirements, chiefly:
- unitarity, analytical structure of the S-matrix (physical masses in loops)
- physical masses in all chiral logarithms?

Step 3: Reparametrization of the LEC's

- leading order parameters left free (i.e. r, $F_0 \to Z$,$B_0\hat{m} \to X$ (resp. Y))
- NLO LEC's L_i reparametrized using bare expansions for F_P^2, $F_P^2 M_P^2$
- no additional expansion, done algebraically
- Remainders to masses and decay constants introduced, each reparametrized LEC is replaced by a higher order remainder

A.4 ‘Resummed’ approach to χPT - basic steps

Step 1: ‘Strict’ chiral expansion

- linearly related to a Green function obtained from the generated functional
- expressed strictly in terms of the parameters of the effective Lagrangian
- done formally to all orders, higher orders collected in a remainder

Step 2: Definition of the ‘bare’ expansion

- careful modification of the strict expansion in order to accommodate additional requirements, chiefly:
- unitarity, analytical structure of the S-matrix (physical masses in loops)
- physical masses in all chiral logarithms?

Step 3: Reparametrization of the LEC’s

- leading order parameters left free (i.e. r, $F_0 \to Z$,$B_0\hat{m} \to X$ (resp. Y))
- NLO LEC’s L_i reparametrized using bare expansions for F_P^2, $F_P^2 M_P^2$
- no additional expansion, done algebraically
- Remainders to masses and decay constants introduced, each reparametrized LEC is replaced by a higher order remainder

Step 4: Remainders

- higher order remainders not neglected, explicitly present in the formulas, which are valid to all orders
- generated from the bare expansions of the observable in question and reparameterization of LEC’s
- source of theoretical error, effect has to be estimated

A.4 'Resummed' approach to χPT - basic steps

Step 1: 'Strict' chiral expansion

- linearly related to a Green function obtained from the generated functional
- expressed strictly in terms of the parameters of the effective Lagrangian
- done formally to all orders, higher orders collected in a remainder

Step 2: Definition of the 'bare' expansion

- careful modification of the strict expansion in order to accommodate additional requirements, chiefly:
- unitarity, analytical structure of the S-matrix (physical masses in loops)
- physical masses in all chiral logarithms?

Step 3: Reparametrization of the LEC's

- leading order parameters left free (i.e. r, $F_0 \to Z$,$B_0\hat{m} \to X$ (resp. Y))
- NLO LEC's L_i reparametrized using bare expansions for F_P^2, $F_P^2 M_P^2$
- no additional expansion, done algebraically
- Remainders to masses and decay constants introduced, each reparametrized LEC is replaced by a higher order remainder

Step 4: Remainders

- higher order remainders not neglected, explicitly present in the formulas, which are valid to all orders
- generated from the bare expansions of the observable in question and reparameterization of LEC's
- source of theoretical error, effect has to be estimated

B. Illustrative example: $\eta\,\pi^0 \to \eta\,\pi^0$ scattering

B.1 $\eta\,\pi^0 \to \eta\,\pi^0$ scattering: observables

4-point Green function $G_{\pi\eta}(s,t,u) = F_\pi^2 F_\eta^2 \mathcal{A}_{fi}(s,t,u)$ to NLO

$$G_{\pi\eta}(s,t,u) = G_{pol}(s,t,u) + G^{(4)}_{unit}(s,t,u)|_{J^r_{PQ}(0)=0} + \Delta_G$$

$$G_{pol}(s,t,u) = \alpha + \beta t + \gamma t^2 + \omega (s-u)^2$$

α, β, γ, ω ... 'good' observables

'Bad' observables not linearly related to $G_{\pi\eta}(s,t,u)$

- subthreshold parameters c_{00}, c_{10}, c_{20}, c_{01}
- scattering lengths a_0, a_1

Expansions of 'bad' observables are avoided

- they are calculated as nonlinear functions of expansions of 'good' observables
- specifically quantities in the denominator are not expanded

B.1 $\eta\,\pi^0 \to \eta\,\pi^0$ scattering: observables

4-point Green function $G_{\pi\eta}(s,t,u) = F_\pi^2 F_\eta^2 \mathcal{A}_{fi}(s,t,u)$ to NLO

$$G_{\pi\eta}(s,t,u) = G_{pol}(s,t,u) + G^{(4)}_{unit}(s,t,u)|_{J^r_{PQ}(0)=0} + \Delta_G$$

$$G_{pol}(s,t,u) = \alpha + \beta t + \gamma t^2 + \omega (s-u)^2$$

α, β, γ, ω ... 'good' observables

'Bad' observables not linearly related to $G_{\pi\eta}(s,t,u)$

- subthreshold parameters c_{00}, c_{10}, c_{20}, c_{01}
- scattering lengths a_0, a_1

Expansions of 'bad' observables are avoided

- they are calculated as nonlinear functions of expansions of 'good' observables
- specifically quantities in the denominator are not expanded

B.1 $\eta\,\pi^0 \to \eta\,\pi^0$ scattering: observables

4-point Green function $G_{\pi\eta}(s,t,u) = F_\pi^2 F_\eta^2 \mathcal{A}_{fi}(s,t,u)$ to NLO

$$G_{\pi\eta}(s,t,u) = G_{pol}(s,t,u) + G^{(4)}_{unit}(s,t,u)|_{J^r_{PQ}(0)=0} + \Delta_G$$

$$G_{pol}(s,t,u) = \alpha + \beta t + \gamma t^2 + \omega (s-u)^2$$

α, β, γ, ω ... 'good' observables

'Bad' observables not linearly related to $G_{\pi\eta}(s,t,u)$

- subthreshold parameters c_{00}, c_{10}, c_{20}, c_{01}
- scattering lengths a_0, a_1

Expansions of 'bad' observables are avoided

- they are calculated as nonlinear functions of expansions of 'good' observables
- specifically quantities in the denominator are not expanded

B.2 $\eta\,\pi^0 \rightarrow \eta\,\pi^0$ scattering: strict chiral expansion

(Bernard et al.1991)

Exact renormalization scale independence

$$G^{(2)}(s,t,u) = \frac{F_0^2}{3}\,m_\pi^2$$

$$\begin{aligned}
G_{ct}^{(4)}(s,t,u) = {} & 8(L_1^r(\mu) + \frac{1}{6}L_3^r(\mu))(t-2M_\pi^2)(t-2M_\eta^2) \\
& +4(L_2^r(\mu) + \frac{1}{3}L_3^r(\mu))[(s-M_\pi^2-M_\eta^2)^2 + (u-M_\pi^2-M_\eta^2)^2] \\
& +8L_4^r(\mu)[(t-2M_\pi^2)\,m_\eta^2 + (t-2M_\eta^2)\,m_\pi^2] - \frac{8}{3}L_5^r(\mu)(M_\pi^2+M_\eta^2)\,m_\pi^2 \\
& +8L_6^r(\mu)\,m_\pi^2(\,m_\pi^2 + 5\,m_\eta^2) + 32L_7^r(\mu)(\,m_\pi^2 - \,m_\eta^2)\,m_\pi^2 + \frac{64}{3}L_8^r(\mu)\,m_\pi^4
\end{aligned}$$

$$G_{tad}^{(4)}(s,t,u) = -\frac{F_0^2}{3}\,m_\pi^2\left(3\,\mu_\pi + 2\,\mu_K + \frac{1}{3}\,\mu_\eta\right)$$

$$\begin{aligned}
G_{unit}^{(4)}(s,t,u) = {} & \frac{1}{9}\,m_\pi^4[\,J_{\pi\eta}^r(s) + \,J_{\pi\eta}^r(u)] \\
& +\frac{3}{8}[s-M_\pi^2-M_\eta^2+\frac{2}{3}\,m_\pi^2]^2\,J_{KK}^r(s) + \frac{3}{8}[u-M_\pi^2-M_\eta^2+\frac{2}{3}\,m_\pi^2]^2\,J_{KK}^r(u) \\
& +\frac{1}{3}\,m_\pi^2[t-2M_\pi^2+\frac{3}{2}\,m_\pi^2]\,J_{\pi\pi}^r(t) + \frac{2}{9}\,m_\pi^2(\,m_\eta^2 - \frac{1}{4}\,m_\pi^2)\,J_{\eta\eta}^r(t) \\
& +\frac{1}{8}[t-2M_\pi^2+2\,m_\pi^2][3t-6M_\eta^2+4\,m_\eta^2-\frac{2}{3}\,m_\pi^2]\,J_{KK}^r(t)
\end{aligned}$$

B.2 $\eta\,\pi^0 \to \eta\,\pi^0$ scattering: strict chiral expansion

in, out lines - on mass shell

$$G^{(2)}(s,t,u) = \frac{F_0^2}{3}\,m_\pi^2$$

$$\begin{aligned}G^{(4)}_{ct}(s,t,u) = \;& 8(L_1^r(\mu)+\frac{1}{6}L_3^r(\mu))(t-2M_\pi^2)(t-2M_\eta^2)\\ &+4(L_2^r(\mu)+\frac{1}{3}L_3^r(\mu))[(s-M_\pi^2-M_\eta^2)^2+(u-M_\pi^2-M_\eta^2)^2]\\ &+8L_4^r(\mu)[(t-2M_\pi^2)\,m_\eta^2+(t-2M_\eta^2)\,m_\pi^2]-\frac{8}{3}L_5^r(\mu)(M_\pi^2+M_\eta^2)\,m_\pi^2\\ &+8L_6^r(\mu)\,m_\pi^2(\,m_\pi^2+5\,m_\eta^2)+32L_7^r(\mu)(\,m_\pi^2-\,m_\eta^2)\,m_\pi^2+\frac{64}{3}L_8^r(\mu)\,m_\pi^4\end{aligned}$$

$$G^{(4)}_{tad}(s,t,u) = -\frac{F_0^2}{3}\,m_\pi^2\left(3\,\mu_\pi+2\,\mu_K+\frac{1}{3}\,\mu_\eta\right)$$

$$\begin{aligned}G^{(4)}_{unit}(s,t,u) = \;& \frac{1}{9}\,m_\pi^4[\,J^r_{\pi\eta}(s)+\,J^r_{\pi\eta}(u)]\\ &+\frac{3}{8}[s-M_\pi^2-M_\eta^2+\frac{2}{3}\,m_\pi^2]^2\,J^r_{KK}(s)+\frac{3}{8}[u-M_\pi^2-M_\eta^2+\frac{2}{3}\,m_\pi^2]^2\,J^r_{KK}(u)\\ &+\frac{1}{3}\,m_\pi^2[t-2M_\pi^2+\frac{3}{2}\,m_\pi^2]\,J^r_{\pi\pi}(t)+\frac{2}{9}\,m_\pi^2(\,m_\eta^2-\frac{1}{4}\,m_\pi^2)\,J^r_{\eta\eta}(t)\\ &+\frac{1}{8}[t-2M_\pi^2+2\,m_\pi^2][3t-6M_\eta^2+4\,m_\eta^2-\frac{2}{3}\,m_\pi^2]\,J^r_{KK}(t)\end{aligned}$$

B.2 $\eta\,\pi^0 \to \eta\,\pi^0$ scattering: strict chiral expansion

$$m_\pi^2 = 2B_0\hat{m},\;\; m_K^2 = B_0\hat{m}(r+1),\;\; m_\eta^2 = \tfrac{2}{3}B_0\hat{m}(2r+1)$$

$$\begin{aligned}
G^{(2)}(s,t,u) &= \frac{F_0^2}{3} m_\pi^2 \\
G^{(4)}_{ct}(s,t,u) &= 8(L_1^r(\mu) + \frac{1}{6}L_3^r(\mu))(t-2M_\pi^2)(t-2M_\eta^2) \\
&\quad +4(L_2^r(\mu) + \frac{1}{3}L_3^r(\mu))[(s-M_\pi^2-M_\eta^2)^2 + (u-M_\pi^2-M_\eta^2)^2] \\
&\quad +8L_4^r(\mu)[(t-2M_\pi^2)m_\eta^2 + (t-2M_\eta^2)m_\pi^2] - \frac{8}{3}L_5^r(\mu)(M_\pi^2+M_\eta^2)m_\pi^2 \\
&\quad +8L_6^r(\mu)m_\pi^2(m_\pi^2+5m_\eta^2) + 32L_7^r(\mu)(m_\pi^2-m_\eta^2)m_\pi^2 + \frac{64}{3}L_8^r(\mu)m_\pi^4 \\
G^{(4)}_{tad}(s,t,u) &= -\frac{F_0^2}{3} m_\pi^2 \left(3\,\mu_\pi + 2\,\mu_K + \frac{1}{3}\,\mu_\eta\right) \\
G^{(4)}_{unit}(s,t,u) &= \frac{1}{9} m_\pi^4 [\,J^r_{\pi\eta}(s) + \,J^r_{\pi\eta}(u)] \\
&\quad +\frac{3}{8}[s-M_\pi^2-M_\eta^2+\frac{2}{3}m_\pi^2]^2\,J^r_{KK}(s) + \frac{3}{8}[u-M_\pi^2-M_\eta^2+\frac{2}{3}m_\pi^2]^2\,J^r_{KK}(u) \\
&\quad +\frac{1}{3}m_\pi^2[t-2M_\pi^2+\frac{3}{2}m_\pi^2]\,J^r_{\pi\pi}(t) + \frac{2}{9}m_\pi^2(m_\eta^2-\frac{1}{4}m_\pi^2)\,J^r_{\eta\eta}(t) \\
&\quad +\frac{1}{8}[t-2M_\pi^2+2m_\pi^2][3t-6M_\eta^2+4m_\eta^2-\frac{2}{3}m_\pi^2]\,J^r_{KK}(t)
\end{aligned}$$

B.2 $\eta\,\pi^0 \to \eta\,\pi^0$ scattering: strict chiral expansion

$$\mu_P = m_P^2/32\pi^2 F_0^2 \ \ln[m_P^2/\mu^2]$$

$$G^{(2)}(s,t,u) = \frac{F_0^2}{3}\,m_\pi^2$$

$$\begin{aligned}
G^{(4)}_{ct}(s,t,u) = {} & 8(L_1^r(\mu)+\frac{1}{6}L_3^r(\mu))(t-2M_\pi^2)(t-2M_\eta^2) \\
& +4(L_2^r(\mu)+\frac{1}{3}L_3^r(\mu))[(s-M_\pi^2-M_\eta^2)^2+(u-M_\pi^2-M_\eta^2)^2] \\
& +8L_4^r(\mu)[(t-2M_\pi^2)\,m_\eta^2+(t-2M_\eta^2)\,m_\pi^2]-\frac{8}{3}L_5^r(\mu)(M_\pi^2+M_\eta^2)\,m_\pi^2 \\
& +8L_6^r(\mu)\,m_\pi^2(\,m_\pi^2+5\,m_\eta^2)+32L_7^r(\mu)(\,m_\pi^2-\,m_\eta^2)\,m_\pi^2+\frac{64}{3}L_8^r(\mu)\,m_\pi^4
\end{aligned}$$

$$G^{(4)}_{tad}(s,t,u) = -\frac{F_0^2}{3}\,m_\pi^2\left(3\,\mu_\pi+2\,\mu_K+\frac{1}{3}\,\mu_\eta\right)$$

$$\begin{aligned}
G^{(4)}_{unit}(s,t,u) = {} & \frac{1}{9}\,m_\pi^4[\,J^r_{\pi\eta}(s)+\,J^r_{\pi\eta}(u)] \\
& +\frac{3}{8}[s-M_\pi^2-M_\eta^2+\frac{2}{3}\,m_\pi^2]^2\,J^r_{KK}(s)+\frac{3}{8}[u-M_\pi^2-M_\eta^2+\frac{2}{3}\,m_\pi^2]^2\,J^r_{KK}(u) \\
& +\frac{1}{3}\,m_\pi^2[t-2M_\pi^2+\frac{3}{2}\,m_\pi^2]\,J^r_{\pi\pi}(t)+\frac{2}{9}\,m_\pi^2(\,m_\eta^2-\frac{1}{4}\,m_\pi^2)\,J^r_{\eta\eta}(t) \\
& +\frac{1}{8}[t-2M_\pi^2+2\,m_\pi^2][3t-6M_\eta^2+4\,m_\eta^2-\frac{2}{3}\,m_\pi^2]\,J^r_{KK}(t)
\end{aligned}$$

B.2 $\eta\,\pi^0 \to \eta\,\pi^0$ scattering: strict chiral expansion

Loop functions J^r_{PQ} contain LO masses as well

$$G^{(2)}(s,t,u) = \frac{F_0^2}{3}\, m_\pi^2$$

$$\begin{aligned}G^{(4)}_{ct}(s,t,u) = {} & 8(L_1^r(\mu) + \frac{1}{6}L_3^r(\mu))(t-2M_\pi^2)(t-2M_\eta^2) \\ & +4(L_2^r(\mu)+\frac{1}{3}L_3^r(\mu))[(s-M_\pi^2-M_\eta^2)^2+(u-M_\pi^2-M_\eta^2)^2] \\ & +8L_4^r(\mu)[(t-2M_\pi^2)\,m_\eta^2+(t-2M_\eta^2)\,m_\pi^2] - \frac{8}{3}L_5^r(\mu)(M_\pi^2+M_\eta^2)\,m_\pi^2 \\ & +8L_6^r(\mu)\,m_\pi^2(\,m_\pi^2+5\,m_\eta^2)+32L_7^r(\mu)(\,m_\pi^2-\,m_\eta^2)\,m_\pi^2+\frac{64}{3}L_8^r(\mu)\,m_\pi^4\end{aligned}$$

$$G^{(4)}_{tad}(s,t,u) = -\frac{F_0^2}{3}\,m_\pi^2\left(3\,\mu_\pi+2\,\mu_K+\frac{1}{3}\,\mu_\eta\right)$$

$$\begin{aligned}G^{(4)}_{unit}(s,t,u) = {} & \frac{1}{9}\,m_\pi^4[\boxed{J^r_{\pi\eta}(s)}+\boxed{J^r_{\pi\eta}(u)}] \\ & +\frac{3}{8}[s-M_\pi^2-M_\eta^2+\frac{2}{3}\,m_\pi^2]^2\boxed{J^r_{KK}(s)}+\frac{3}{8}[u-M_\pi^2-M_\eta^2+\frac{2}{3}\,m_\pi^2]^2\boxed{J^r_{KK}(u)} \\ & +\frac{1}{3}\,m_\pi^2[t-2M_\pi^2+\frac{3}{2}\,m_\pi^2]\boxed{J^r_{\pi\pi}(t)}+\frac{2}{9}\,m_\pi^2(\,m_\eta^2-\frac{1}{4}\,m_\pi^2)\boxed{J^r_{\eta\eta}(t)} \\ & +\frac{1}{8}[t-2M_\pi^2+2\,m_\pi^2][3t-6M_\eta^2+4\,m_\eta^2-\frac{2}{3}\,m_\pi^2]\boxed{J^r_{KK}(t)}\end{aligned}$$

B.3 Reparametrization of LEC's

Decay constant and mass strict chiral expansions: *(Descotes et al.2004)*

$$F_\pi^2 = F_0^2(1 - 4\mu_\pi - 2\mu_K) + 16B_0\widehat{m}(\,L_4^r(r+2) + \,L_5^r) + \,\Delta_{F_\pi}^{(4)}$$

$$F_K^2 = F_0^2(1 - \frac{3}{2}\mu_\pi - 3\mu_K - \frac{3}{2}\mu_\eta) + 16B_0\widehat{m}(\,L_4^r(r+2) + \frac{1}{2}\,L_5^r(r+1)) + \,\Delta_{F_K}^{(4)}$$

$$F_\pi^2 M_\pi^2 = 2B_0\widehat{m}F_0^2(1 - 3\mu_\pi - 2\mu_K - \frac{1}{3}\mu_\eta + \frac{32B_0\widehat{m}}{F_0^2}(\,L_8^r + \,L_6^r(r+2)) + \,\Delta_{M_\pi}^{(6)}$$

$$F_K^2 M_K^2 = B_0\widehat{m}F_0^2(r+1)(1 - \frac{3}{2}\mu_\pi - 3\mu_K - \frac{5}{6}\mu_\eta +$$

$$+\frac{16B_0\widehat{m}}{F_0^2}(\,L_8^r(r+1) + 2\,L_6^r(r+2)) + \,\Delta_{M_K}^{(6)}$$

$$F_\eta^2 M_\eta^2 = \frac{2}{3}B_0\widehat{m}F_0^2((2r+1) - 3\mu_\pi - 2(4r+1)\mu_K - \frac{1}{3}(8r+1)\mu_\eta +$$

$$+\frac{32B_0\widehat{m}}{F_0^2}(L_6^r(2r^2+5r+2) + 2\,L_7(r-1)^2 + \,L_8^r(2r^2+1))) + \,\Delta_{M_\eta}^{(6)}$$

B.3 Reparametrization of LEC's

Simple linear equation system for $L_5 \dots L_8$

$$F_\pi^2 = F_0^2(1 - 4\mu_\pi - 2\mu_K) + 16 B_0 \widehat{m}(L_4^r (r+2) + L_5^r) + \Delta_{F_\pi}^{(4)}$$

$$F_K^2 = F_0^2(1 - \frac{3}{2}\mu_\pi - 3\mu_K - \frac{3}{2}\mu_\eta) + 16 B_0 \widehat{m}(L_4^r (r+2) + \frac{1}{2} L_5^r (r+1)) + \Delta_{F_K}^{(4)}$$

$$F_\pi^2 M_\pi^2 = 2 B_0 \widehat{m} F_0^2 (1 - 3\mu_\pi - 2\mu_K - \frac{1}{3}\mu_\eta + \frac{32 B_0 \widehat{m}}{F_0^2}(L_8^r + L_6^r (r+2)) + \Delta_{M_\pi}^{(6)}$$

$$F_K^2 M_K^2 = B_0 \widehat{m} F_0^2 (r+1)(1 - \frac{3}{2}\mu_\pi - 3\mu_K - \frac{5}{6}\mu_\eta + \\ + \frac{16 B_0 \widehat{m}}{F_0^2}(L_8^r (r+1) + 2 L_6^r (r+2)) + \Delta_{M_K}^{(6)}$$

$$F_\eta^2 M_\eta^2 = \frac{2}{3} B_0 \widehat{m} F_0^2 ((2r+1) - 3\mu_\pi - 2(4r+1)\mu_K - \frac{1}{3}(8r+1)\mu_\eta + \\ + \frac{32 B_0 \widehat{m}}{F_0^2}(L_6^r (2r^2 + 5r + 2) + 2 L_7 (r-1)^2 + L_8^r (2r^2+1))) + \Delta_{M_\eta}^{(6)}$$

B.3 Reparametrization of LEC's

NLO LEC's expressed in terms of physical observables and remainders

$$F_\pi^2 = F_0^2(1 - 4\mu_\pi - 2\mu_K) + 16 B_0 \hat{m}(L_4^r(r+2) + L_5^r) + \Delta_{F_\pi}^{(4)}$$

$$F_K^2 = F_0^2(1 - \frac{3}{2}\mu_\pi - 3\mu_K - \frac{3}{2}\mu_\eta) + 16 B_0 \hat{m}(L_4^r(r+2) + \frac{1}{2} L_5^r(r+1)) + \Delta_{F_K}^{(4)}$$

$$F_\pi^2 M_\pi^2 = 2 B_0 \hat{m} F_0^2(1 - 3\mu_\pi - 2\mu_K - \frac{1}{3}\mu_\eta + \frac{32 B_0 \hat{m}}{F_0^2}(L_8^r + L_6^r(r+2)) + \Delta_{M_\pi}^{(6)}$$

$$F_K^2 M_K^2 = B_0 \hat{m} F_0^2 (r+1)(1 - \frac{3}{2}\mu_\pi - 3\mu_K - \frac{5}{6}\mu_\eta + $$

$$+ \frac{16 B_0 \hat{m}}{F_0^2}(L_8^r(r+1) + 2 L_6^r(r+2)) + \Delta_{M_K}^{(6)}$$

$$F_\eta^2 M_\eta^2 = \frac{2}{3} B_0 \hat{m} F_0^2((2r+1) - 3\mu_\pi - 2(4r+1)\mu_K - \frac{1}{3}(8r+1)\mu_\eta +$$

$$+ \frac{32 B_0 \hat{m}}{F_0^2}(L_6^r(2r^2+5r+2) + 2 L_7(r-1)^2 + L_8^r(2r^2+1))) + \Delta_{M_\eta}^{(6)}$$

C. Definition of the bare expansion

C.1 Analyticity and unitarity

Strict form of the expansion does not have the correct analytical structure

Solutions:

- exchange LO masses with physical ones in $\overline{J}_{PQ}$ by hand
- use dispersion relations

C.1 Analyticity and unitarity

Strict form of the expansion does not have the correct analytical structure

Solutions:

- exchange LO masses with physical ones in $\overline{J}_{PQ}$ by hand *(Descotes 2007)*
- use dispersion relations

1. Redefinition of the strict expansion into a bare one - by hand

Original strict form:

Exchange $m_P \longrightarrow M_P$ inside $\overline{J}_{PQ}$

$$G^{strict}_{\pi\eta}(s,t,u) = G_{pol}(s,t,u) + G^{(4)}_{unit}(s,t,u)|_{J^r_{PQ}(0)=0} + \Delta_G$$

$$G^{(4)}_{unit}(s,t,u)|_{J^r_{PQ}(0)=0} = \frac{1}{9}m_\pi^4[\overline{J}_{\pi\eta}(s)] + \frac{3}{8}[s - M_\pi^2 - M_\eta^2 + \frac{2}{3}m_\pi^2]^2\,\overline{J}_{KK}(s)$$

$$+(s \leftrightarrow u) + \frac{1}{3}m_\pi^2[t - 2M_\pi^2 + \frac{3}{2}m_\pi^2]\,\overline{J}_{\pi\pi}(t) + \frac{2}{9}m_\pi^2(m_\eta^2 - \frac{1}{4}m_\pi^2)\,\overline{J}_{\eta\eta}(t)$$

$$+\frac{1}{8}[t - 2M_\pi^2 + 2m_\pi^2][3t - 6M_\eta^2 + 4m_\eta^2 - \frac{2}{3}m_\pi^2]\,\overline{J}_{KK}(t)$$

C.1 Analyticity and unitarity

Strict form of the expansion does not have the correct analytical structure

Solutions:

- exchange LO masses with physical ones in $\overline{J}_{PQ}$ by hand
- use dispersion relations

1. Redefinition of the strict expansion into a bare one - by hand

Bare form definition:

Exchange $m_P \longrightarrow M_P$ inside $\overline{J}_{PQ}$

$$G_{\pi\eta}^{bare}(s,t,u) = G_{pol}(s,t,u) + G'^{(4)}_{unit}(s,t,u)|_{J^r_{PQ}(0)=0} + \Delta_{G'}$$

$$G'^{(4)}_{unit}(s,t,u)|_{J^r_{PQ}(0)=0} = \frac{1}{9}m_\pi^4[\overline{J}_{\pi\eta}(s)] + \frac{3}{8}[s - M_\pi^2 - M_\eta^2 + \frac{2}{3}m_\pi^2]^2\overline{J}_{KK}(s)$$
$$+(s \leftrightarrow u) + \frac{1}{3}m_\pi^2[t - 2M_\pi^2 + \frac{3}{2}m_\pi^2]\overline{J}_{\pi\pi}(t) + \frac{2}{9}m_\pi^2(m_\eta^2 - \frac{1}{4}m_\pi^2)\overline{J}_{\eta\eta}(t)$$
$$+\frac{1}{8}[t - 2M_\pi^2 + 2m_\pi^2][3t - 6M_\eta^2 + 4m_\eta^2 - \frac{2}{3}m_\pi^2]\overline{J}_{KK}(t)$$

C.1 Analyticity and unitarity

Strict form of the expansion does not have the correct analytical structure

Solutions:

- exchange LO masses with physical ones in $\overline{J}_{PQ}$ by hand
- use dispersion relations

2. Redefinition of the strict expansion into a bare one - dispersive relations

Disp.relations determine the form of the unitarity part of the amplitude $\mathcal{S}_{unit}$

$$G^{strict}_{\pi\eta}(s,t,u) = G_{pol}(s,t,u) + G^{(4)}_{unit}(s,t,u)|_{J^r_{PQ}(0)=0} + \Delta_G \qquad G^{(4)}_{unit} \longrightarrow \mathcal{G}_{unit}$$

How to relate $\mathcal{G}_{unit} \leftrightarrow \mathcal{S}_{unit}$? Two possibilities:

C.1 Analyticity and unitarity

Strict form of the expansion does not have the correct analytical structure

Solutions:

- exchange LO masses with physical ones in $\overline{J}_{PQ}$ by hand
- use dispersion relations

2. Redefinition of the strict expansion into a bare one - dispersive relations

Possibility a) $\mathcal{G}_{unit}(s,t,u) = F_0^4 \mathcal{S}_{unit}(s,t,u)$

$$G^{bare}_{\pi\eta}(s,t,u) = G_{pol}(s,t,u) + \mathcal{G}_{unit}(s,t,u) + \triangle_{\mathcal{G}}$$

$$\mathcal{G}_{unit}(s,t,u) = \frac{1}{9}m_\pi^4 \overline{J}_{\pi\eta}(s) + \frac{3}{8}[s - \tfrac{1}{3}M_\pi^2 - \tfrac{1}{3}M_\eta^2 - \tfrac{2}{3}M_K^2 + \tfrac{2}{9}m_\pi^2 - \tfrac{2}{9}m_K^2]^2 \, \overline{J}_{KK}(s)$$

$$+(s \leftrightarrow u) + \frac{1}{3}m_\pi^2 [t - \tfrac{4}{3}M_\pi^2 + \tfrac{5}{6}m_\pi^2] \, \overline{J}_{\pi\pi}(t) + \frac{2}{9}m_\pi^2(m_\eta^2 - \frac{1}{4}m_\pi^2)\overline{J}_{\eta\eta}(t)$$

$$+\frac{1}{8}[t - \tfrac{2}{3}M_\pi^2 - \tfrac{2}{3}M_K^2 + \tfrac{2}{3}m_\pi^2 + \tfrac{2}{3}m_K^2][3t - 2M_K^2 - 2M_\eta^2 + 2m_\eta^2 - \tfrac{2}{3}m_K^2] \, \overline{J}_{KK}(t)$$

terms in front of the loop functions are effected too

C.1 Analyticity and unitarity

Strict form of the expansion does not have the correct analytical structure

Solutions:

- exchange LO masses with physical ones in $\overline{J}_{PQ}$ by hand
- use dispersion relations

2. Redefinition of the strict expansion into a bare one - dispersive relations

Possibility b) $\mathcal{G}_{unit}(s,t,u) = \Pi_{i=1}^{4} F_{P_i} \mathcal{S}_{unit}(s,t,u)$

$$G_{\pi\eta}^{bare}(s,t,u) = G_{pol}(s,t,u) + \mathcal{G}_{unit}(s,t,u) + \Delta_{\mathcal{G}}$$

$$\mathcal{G}_{unit}(s,t,u) = \frac{1}{9}m_\pi^4 \frac{F_0^4}{F_\pi^2 F_\eta^2} \overline{J}_{\pi\eta}(s) + \frac{3}{8}\left[s - \tfrac{1}{3}M_\pi^2 - \tfrac{1}{3}M_\eta^2 - \tfrac{2}{3}M_K^2 + \tfrac{2}{9}m_\pi^2 - \tfrac{2}{9}m_K^2\right]^2 \frac{F_0^4}{F_K^2} \overline{J}_{KK}(s)$$

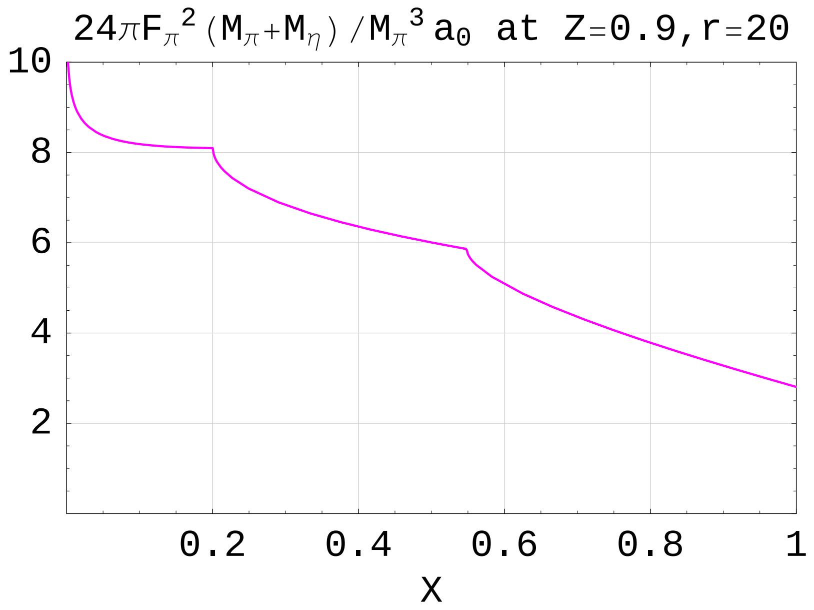
$$+(s \leftrightarrow u) + \frac{1}{3}m_\pi^2 \left[t - \tfrac{4}{3}M_\pi^2 + \tfrac{5}{6}m_\pi^2\right] \frac{F_0^4}{F_\pi^4} \overline{J}_{\pi\pi}(t) + \frac{2}{9}m_\pi^2\left(m_\eta^2 - \frac{1}{4}m_\pi^2\right) \frac{F_0^4}{F_\eta^4} \overline{J}_{\eta\eta}(t)$$

$$+\frac{1}{8}\left[t - \tfrac{2}{3}M_\pi^2 - \tfrac{2}{3}M_K^2 + \tfrac{2}{3}m_\pi^2 + \tfrac{2}{3}m_K^2\right]\left[3t - 2M_K^2 - 2M_\eta^2 + 2m_\eta^2 - \tfrac{2}{3}m_K^2\right] \frac{F_0^4}{F_K^2} \overline{J}_{KK}(t)$$

$\longrightarrow$ perturbative unitarity and exact ren.scale independence

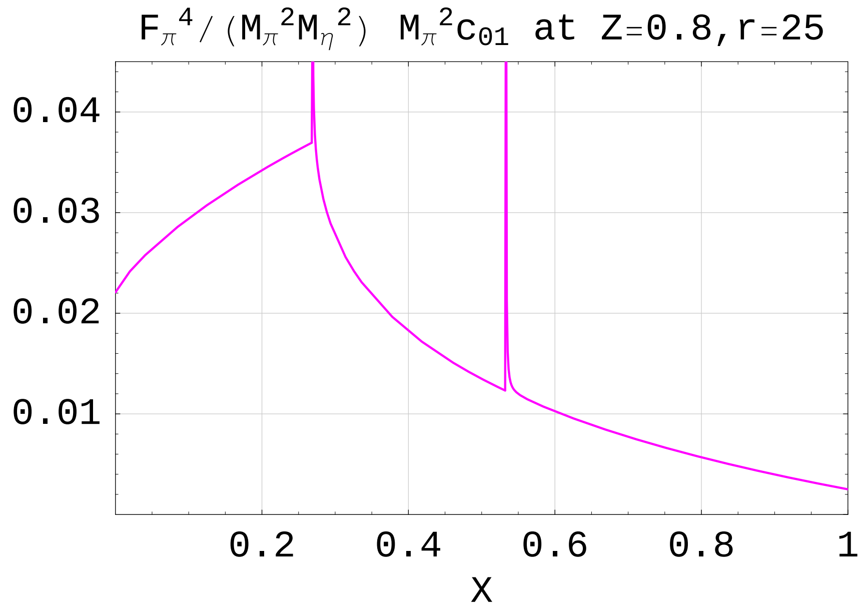
C.1 Definition of the bare expansion - $\eta\pi$ scattering

Central value, remainders neglected



scattering length a_0

solid:	strict form
dotted:	redefinition by hand
dash-dot.:	disp.relations a)
dashed:	disp.relations b)
hor.dashed:	LO value

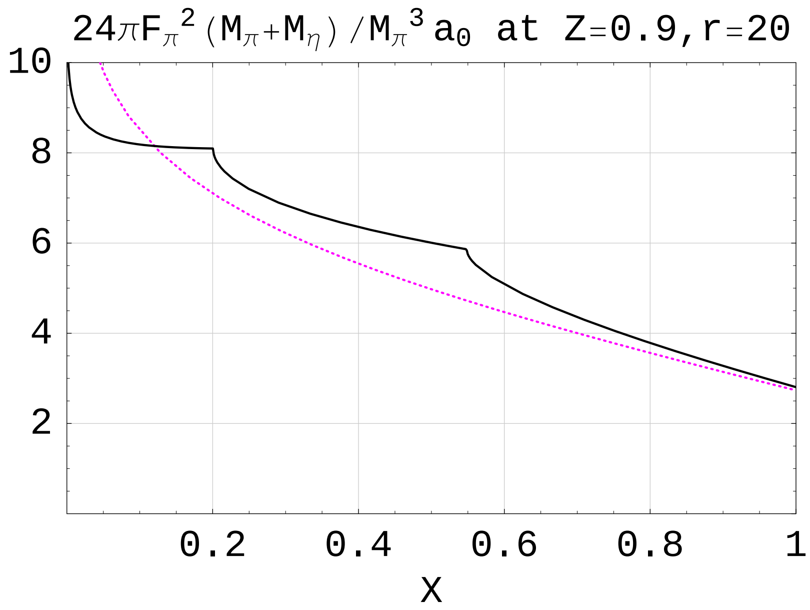


subthreshold parameter c_{01}

solid:	strict form
dotted:	redefinition by hand
dash-dot.:	disp.relations a)
dashed:	disp.relations b)
hor.dashed:	Standard NLO value

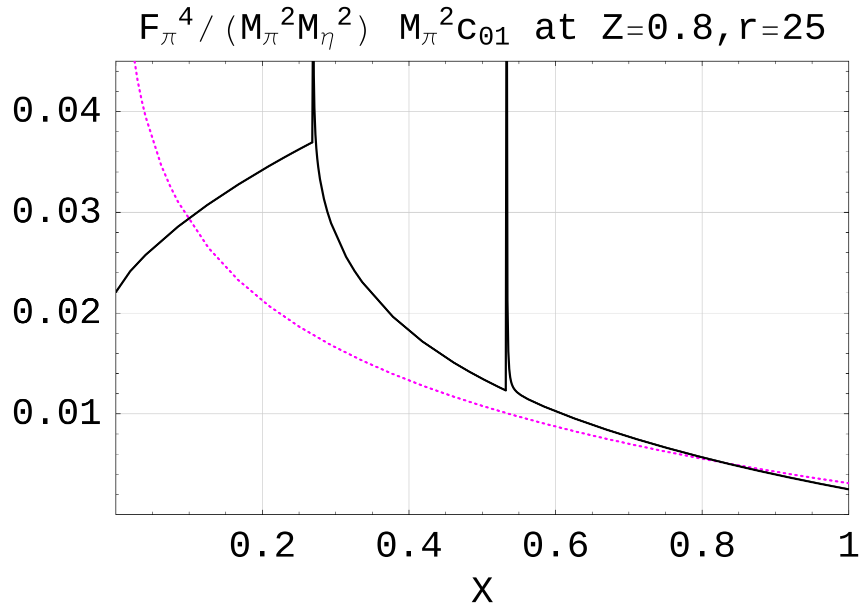
C.1 Definition of the bare expansion - $\eta\pi$ scattering

Central value, remainders neglected



scattering length a_0

solid: strict form
dotted: redefinition by hand
dash-dot.: disp.relations a)
dashed: disp.relations b)
hor.dashed: LO value

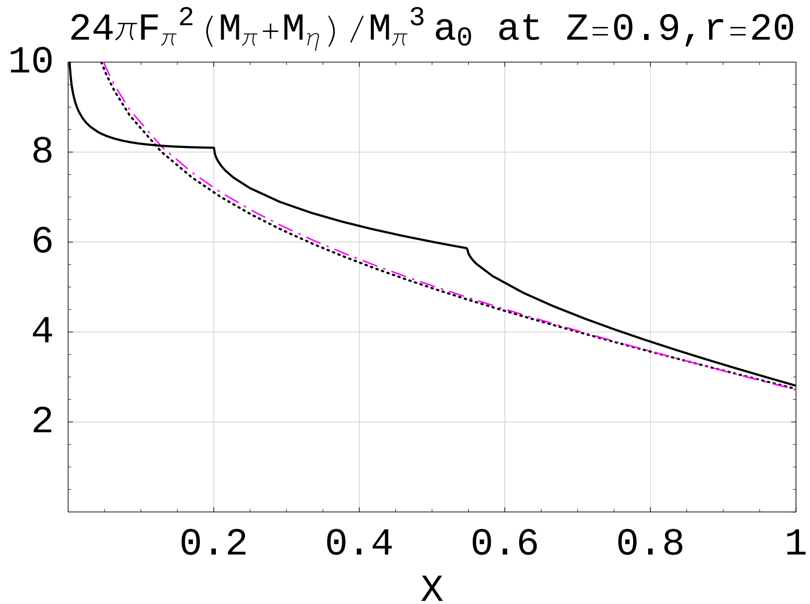


subthreshold parameter c_{01}

solid: strict form
dotted: redefinition by hand
dash-dot.: disp.relations a)
dashed: disp.relations b)
hor.dashed: Standard NLO value

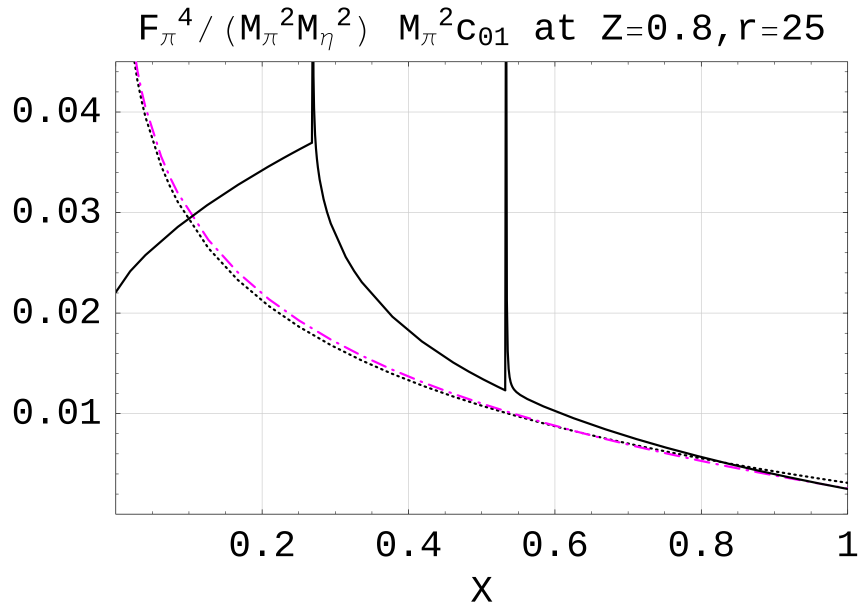
C.1 Definition of the bare expansion - $\eta\pi$ scattering

Central value, remainders neglected



scattering length a_0

solid:	strict form
dotted:	redefinition by hand
dash-dot.:	disp.relations a)
dashed:	disp.relations b)
hor.dashed:	LO value

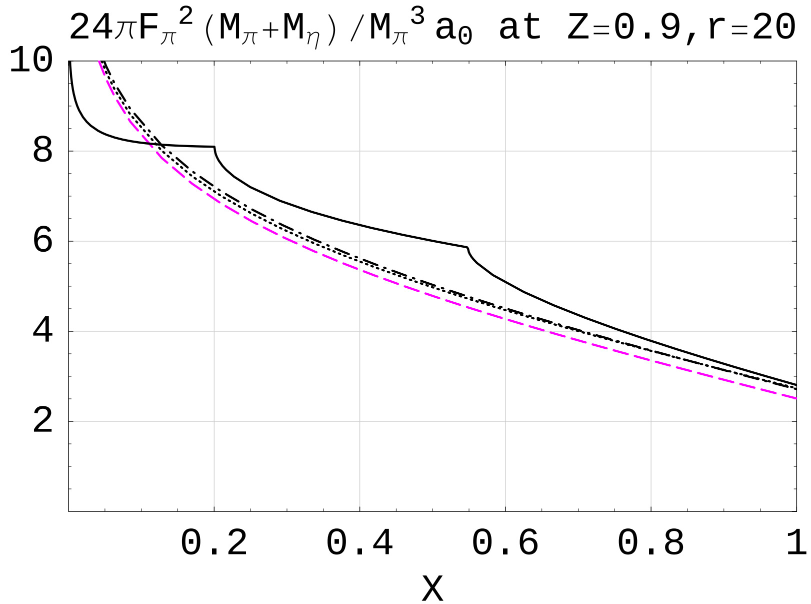


subthreshold parameter c_{01}

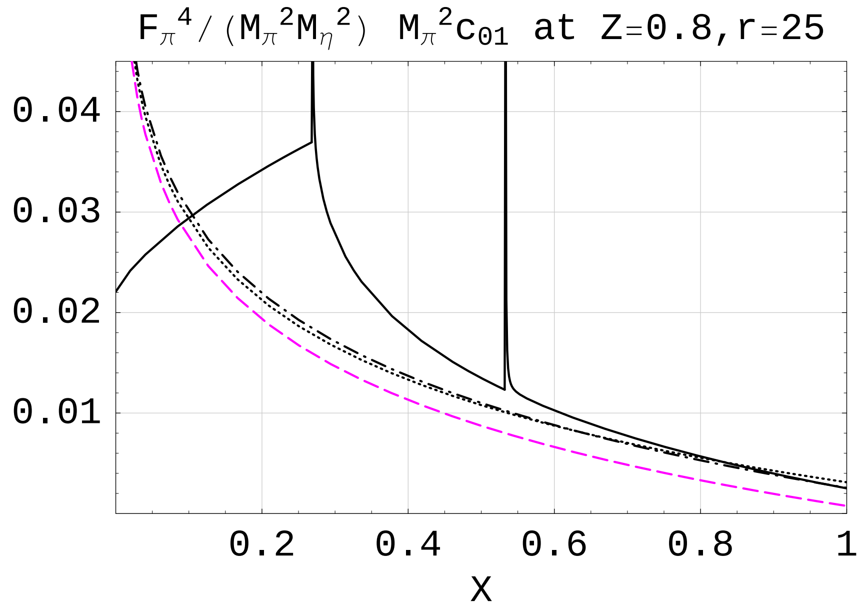
solid:	strict form
dotted:	redefinition by hand
dash-dot.:	disp.relations a)
dashed:	disp.relations b)
hor.dashed:	Standard NLO value

C.1 Definition of the bare expansion - $\eta\pi$ scattering

Central value, remainders neglected



scattering length a_0



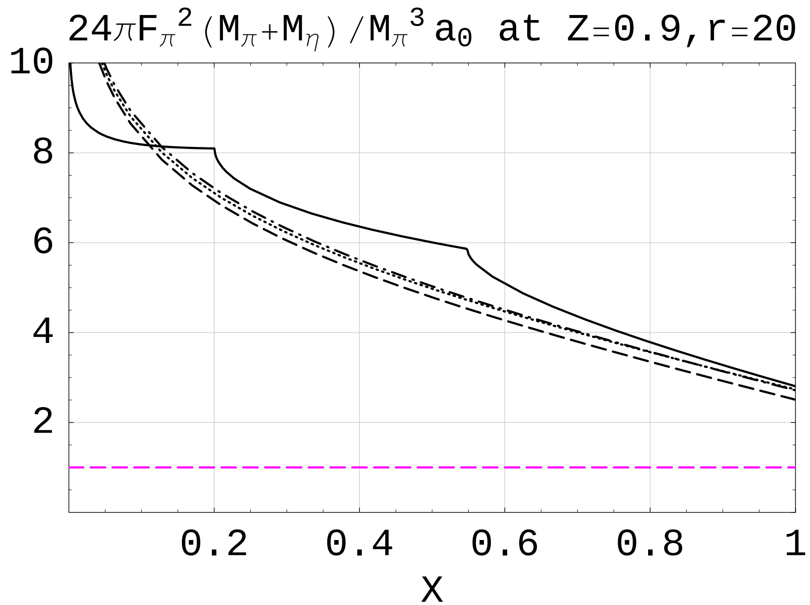
subthreshold parameter c_{01}

solid:	strict form
dotted:	redefinition by hand
dash-dot.:	disp.relations a)
dashed:	disp.relations b)
hor.dashed:	LO value

solid:	strict form
dotted:	redefinition by hand
dash-dot.:	disp.relations a)
dashed:	disp.relations b)
hor.dashed:	Standard NLO value

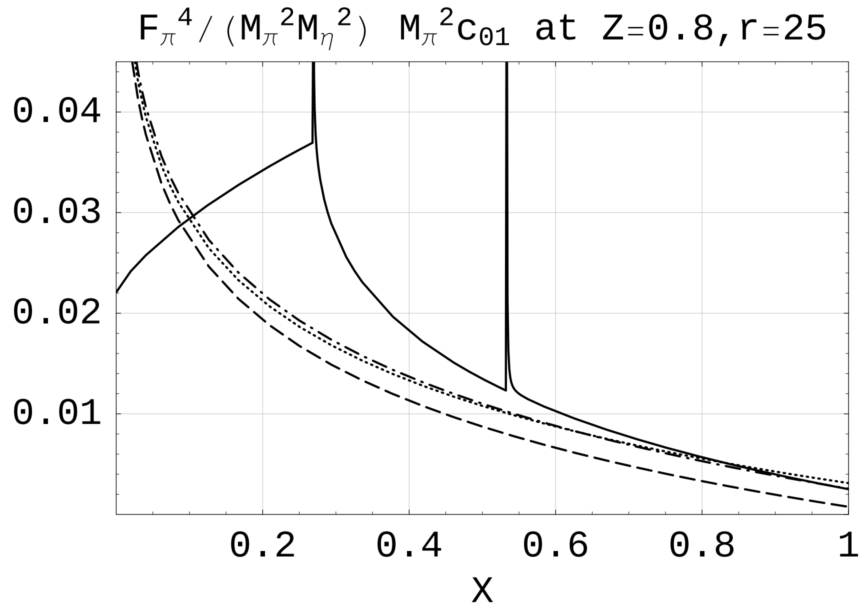
C.1 Definition of the bare expansion - $\eta\pi$ scattering

Central value, remainders neglected



scattering length a_0

solid:	strict form
dotted:	redefinition by hand
dash-dot.:	disp.relations a)
dashed:	disp.relations b)
hor.dashed:	LO value



subthreshold parameter c_{01}

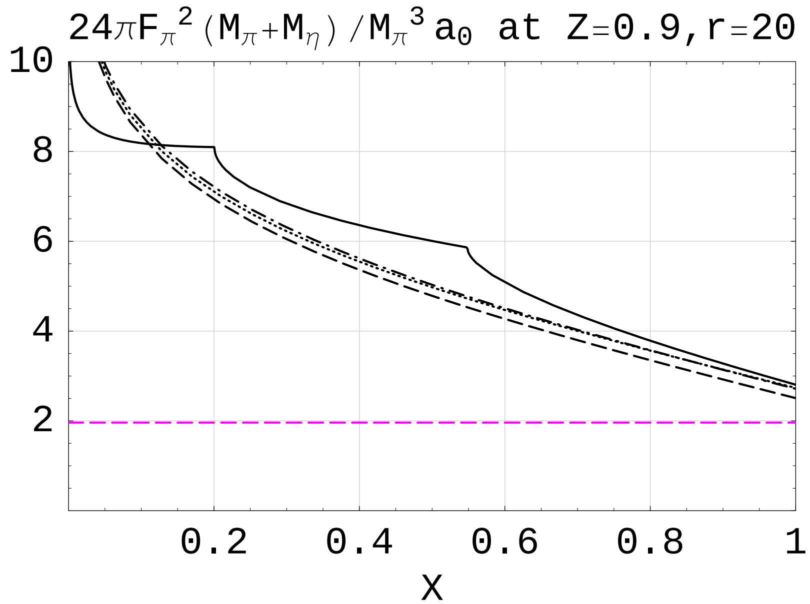
solid:	strict form
dotted:	redefinition by hand
dash-dot.:	disp.relations a)
dashed:	disp.relations b)
hor.dashed:	Standard NLO value

Difference between treatments:

up to 30% of Standard LO value

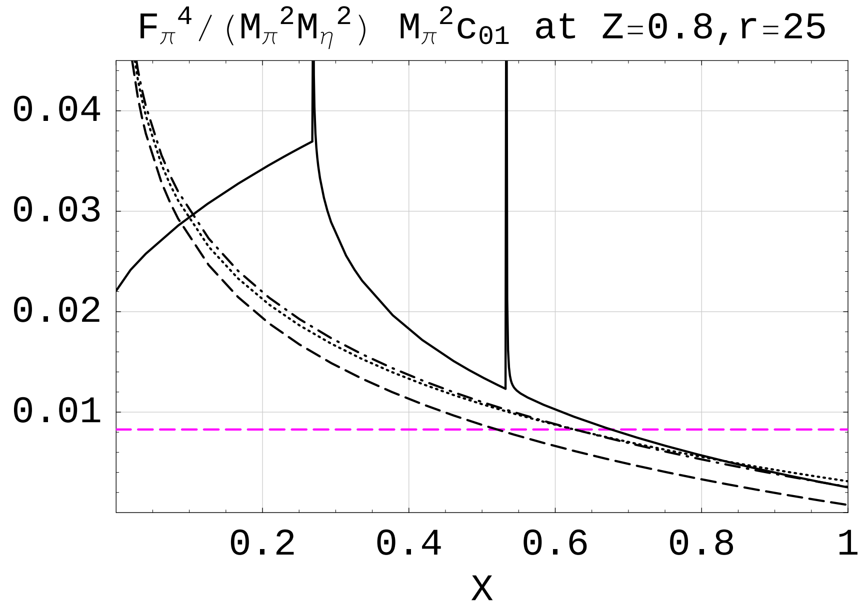
C.1 Definition of the bare expansion - $\eta\pi$ scattering

Central value, remainders neglected



scattering length a_0

solid:	strict form
dotted:	redefinition by hand
dash-dot.:	disp.relations a)
dashed:	disp.relations b)
hor.dashed:	Standard NLO value



subthreshold parameter c_{01}

solid:	strict form
dotted:	redefinition by hand
dash-dot.:	disp.relations a)
dashed:	disp.relations b)
hor.dashed:	Standard NLO value

Difference between treatments:

up to 15% of Standard NLO value

up to 40% of Standard NLO value

C.2 Treatment of the chiral logarithms

Do not influence the analytical structure of the amplitude

Exchange LO masses with physical ones?

$m_\pi^2 = Y M_\pi^2$

$$? \quad \ln(m_P^2/\mu^2) \to \ln(M_P^2/\mu^2) \quad ?$$

C.2 Treatment of the chiral logarithms

Do not influence the analytical structure of the amplitude

Exchange LO masses with physical ones?

$m_\pi^2 = Y M_\pi^2$

$$? \quad \ln(m_P^2/\mu^2) \to \ln(M_P^2/\mu^2) \quad ?$$

Illustrative example - polynomial parameter β:

$$G_{\pi\eta}^{bare}(s,t,u) = \alpha + \beta t + \gamma t^2 + \omega(s-u)^2 + \mathcal{G}_{unit}(s,t,u) + \Delta_{\mathcal{G}}$$

$$\beta = 2\,(M_\eta^2 + M_\pi^2)[\frac{3}{128\pi^2}(\ln\frac{m_K^2}{\mu^2}+1) - 8(L_1^r(\mu) + \frac{1}{6}L_3^r(\mu))] + 8(m_\eta^2 + m_\pi^2)L_4^r(\mu)$$
$$-\frac{1}{32\pi^2}m_\eta^2(\ln\frac{m_K^2}{\mu^2}+1) - \frac{1}{48\pi^2}m_\pi^2(\ln\frac{m_\pi^2}{\mu^2}+1) - \frac{1}{96\pi^2}m_\pi^2(\ln\frac{m_K^2}{\mu^2}+1) + \Delta_\beta$$

Two types of chiral logarithms

C.2 Treatment of the chiral logarithms

Do not influence the analytical structure of the amplitude

Exchange LO masses with physical ones?

$m_\pi^2 = Y M_\pi^2$

$$? \quad \ln(m_P^2/\mu^2) \to \ln(M_P^2/\mu^2) \quad ?$$

Illustrative example - polynomial parameter β:

$$G_{\pi\eta}^{bare}(s,t,u) = \alpha + \beta t + \gamma t^2 + \omega(s-u)^2 + \mathcal{G}_{unit}(s,t,u) + \Delta_{\mathcal{G}}$$

$$\beta = 2\,(M_\eta^2 + M_\pi^2)[\frac{3}{128\pi^2}(\ln\frac{m_K^2}{\mu^2}+1) - 8(L_1^r(\mu) + \frac{1}{6}L_3^r(\mu))] + 8(m_\eta^2 + m_\pi^2)L_4^r(\mu)$$
$$-\frac{1}{32\pi^2}m_\eta^2(\ln\frac{m_K^2}{\mu^2}+1) - \frac{1}{48\pi^2}m_\pi^2(\ln\frac{m_\pi^2}{\mu^2}+1) - \frac{1}{96\pi^2}m_\pi^2(\ln\frac{m_K^2}{\mu^2}+1) + \Delta_\beta$$

Type 1: $M_p^2 \ln m_P^2$ - only from unitarity corrections

Diverge for $Y \to 0$! Have to be treated.

Definite solution: reparametrization of all NLO LEC's including $L_1 \ldots L_3$.

C.2 Treatment of the chiral logarithms

Do not influence the analytical structure of the amplitude

Exchange LO masses with physical ones?

$m_\pi^2 = Y M_\pi^2$

$$? \quad \ln(m_P^2/\mu^2) \to \ln(M_P^2/\mu^2) \quad ?$$

Illustrative example - polynomial parameter β:

$$G_{\pi\eta}^{bare}(s,t,u) = \alpha + \beta t + \gamma t^2 + \omega(s-u)^2 + \mathcal{G}_{unit}(s,t,u) + \Delta_{\mathcal{G}}$$

$$\beta = 2\,(M_\eta^2 + M_\pi^2)[\frac{3}{128\pi^2}(\ln\frac{m_K^2}{\mu^2}+1) - 8(L_1^r(\mu) + \frac{1}{6}L_3^r(\mu))] + 8(m_\eta^2 + m_\pi^2)L_4^r(\mu)$$
$$-\frac{1}{32\pi^2} m_\eta^2(\ln\frac{m_K^2}{\mu^2}+1) - \frac{1}{48\pi^2} m_\pi^2(\ln\frac{m_\pi^2}{\mu^2}+1) - \frac{1}{96\pi^2} m_\pi^2(\ln\frac{m_K^2}{\mu^2}+1) + \Delta_\beta$$

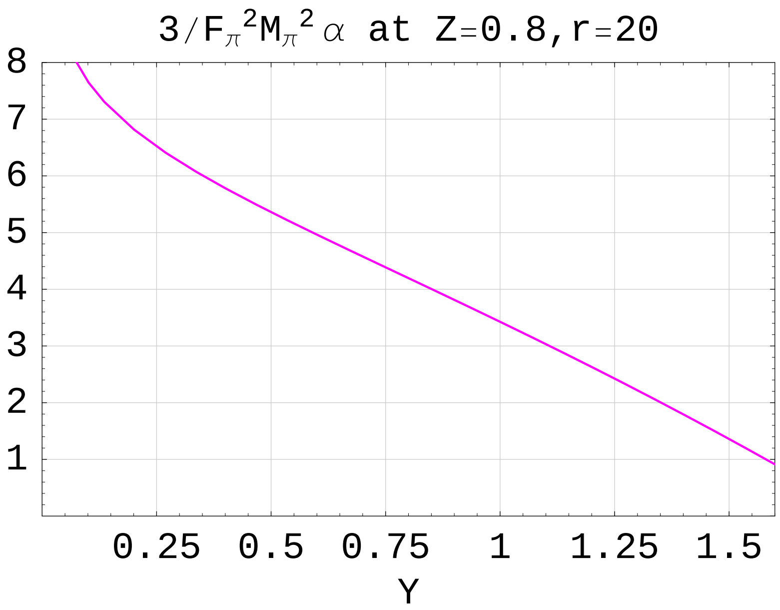
Type 2: $m_p^2 \ln m_P^2$ - both from tadpoles and unitarity corrections

Argued $\ln(m_P^2/\mu^2) \to \ln(M_P^2/\mu^2)$ should not have a large numerical effect:
(Descotes 2007)

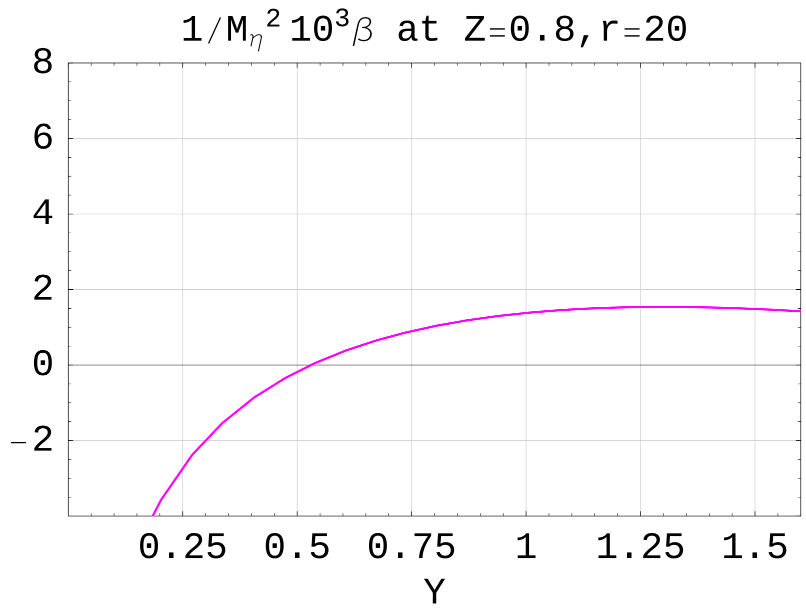
$Y \ll 1$: $m_p^2 \ln m_P^2 \to 0$, $Y \sim 1$: $m_p^2 \to M_P^2$

C.2 Treatment of the chiral logarithms

$\eta\pi$ scattering



polynomial parameter α

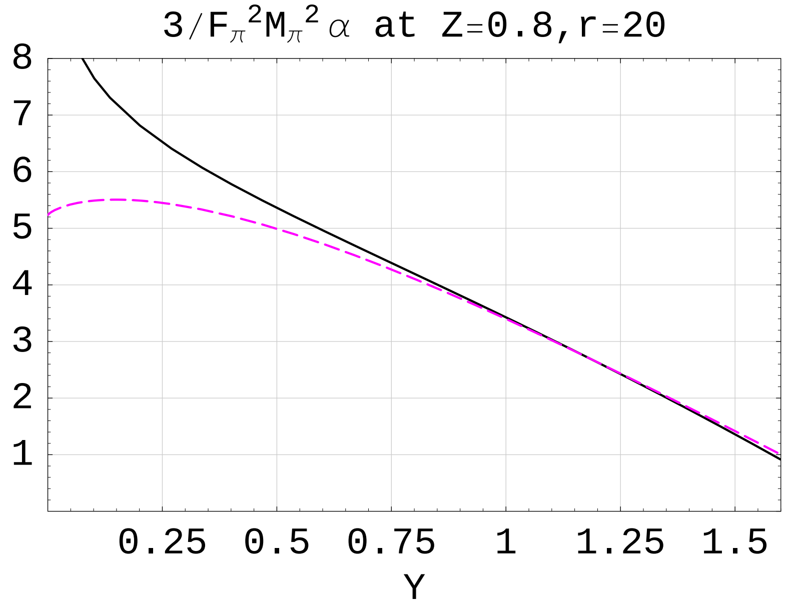


polynomial parameter β

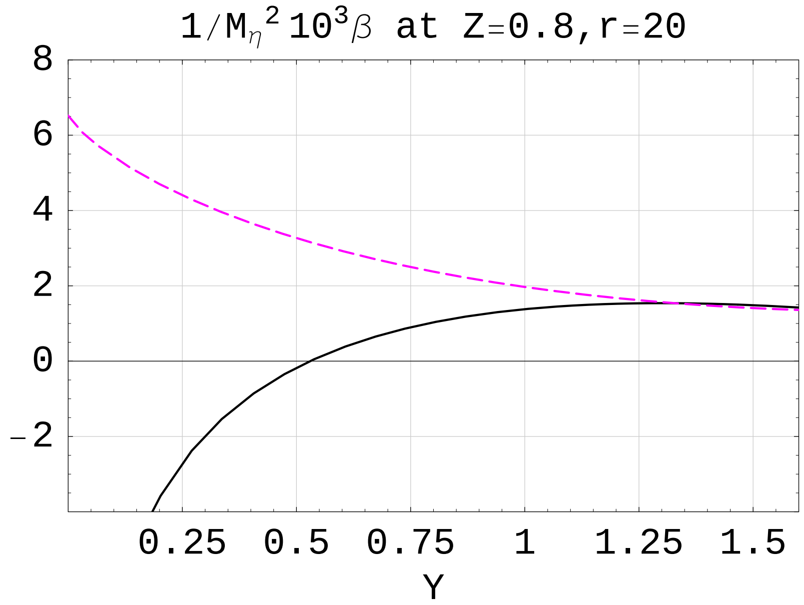
solid:	strict form
dashed:	$\ln(m_P^2/\mu^2) \rightarrow \ln(M_P^2/\mu^2)$ in type 1 logs
dotted:	$\ln(m_P^2/\mu^2) \rightarrow \ln(M_P^2/\mu^2)$ in type 1 and 2
hor.dashed:	LO value

C.2 Treatment of the chiral logarithms

$\eta\pi$ scattering



polynomial parameter α

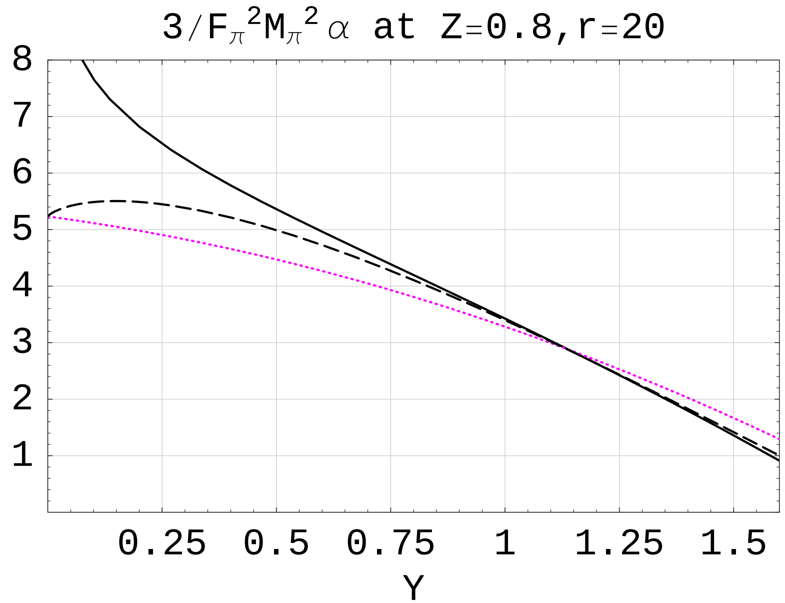


polynomial parameter β

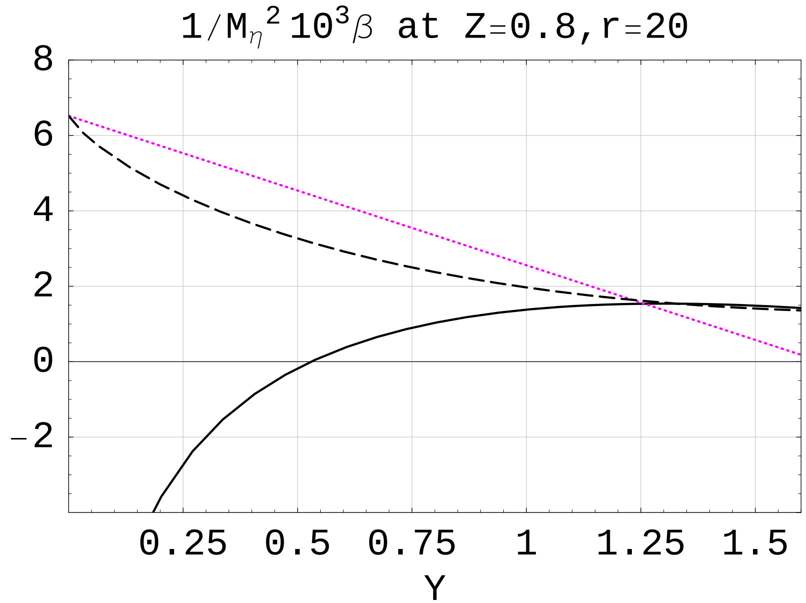
solid:	strict form
dashed:	$\ln(m_P^2/\mu^2) \rightarrow \ln(M_P^2/\mu^2)$ in type 1 logs
dotted:	$\ln(m_P^2/\mu^2) \rightarrow \ln(M_P^2/\mu^2)$ in type 1 and 2
hor.dashed:	LO value

C.2 Treatment of the chiral logarithms

$\eta\pi$ scattering



polynomial parameter α

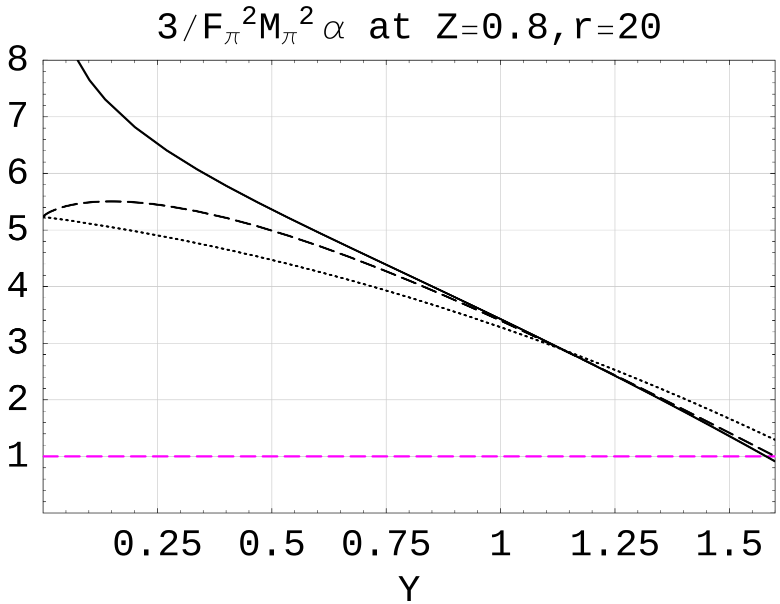


polynomial parameter β

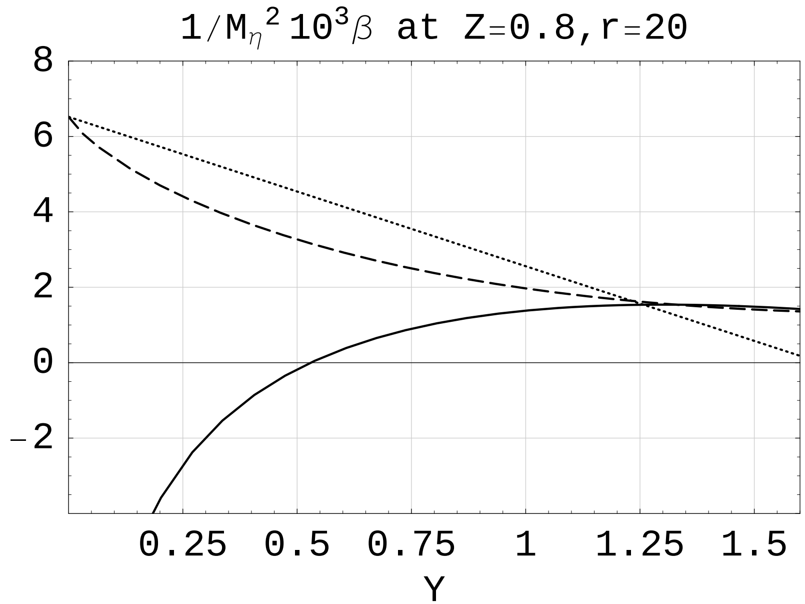
solid:	strict form
dashed:	$\ln(m_P^2/\mu^2)\to\ln(M_P^2/\mu^2)$ in type 1 logs
dotted:	$\ln(m_P^2/\mu^2)\to\ln(M_P^2/\mu^2)$ in type 1 and 2
hor.dashed:	LO value

C.2 Treatment of the chiral logarithms

$\eta\pi$ scattering



polynomial parameter α



polynomial parameter β

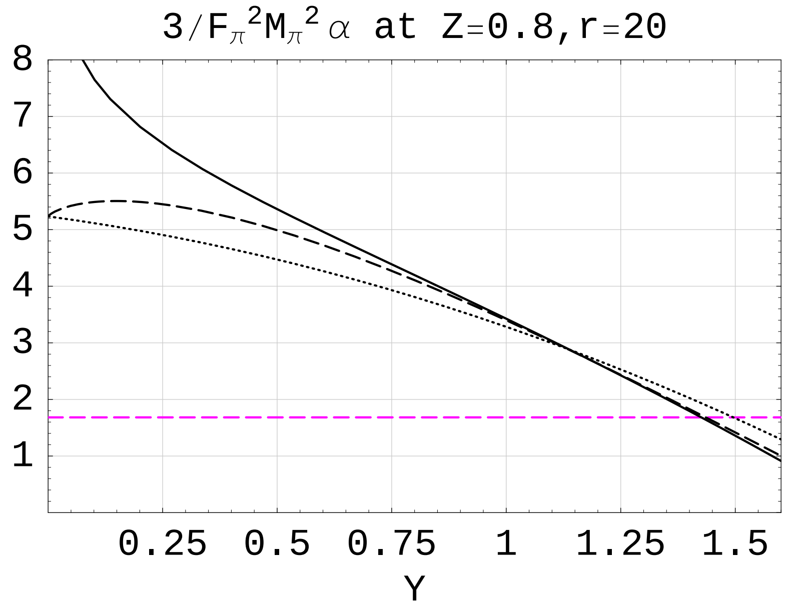
solid:	strict form
dashed:	$\ln(m_P^2/\mu^2) \to \ln(M_P^2/\mu^2)$ in type 1 logs
dotted:	$\ln(m_P^2/\mu^2) \to \ln(M_P^2/\mu^2)$ in type 1 and 2
hor.dashed:	LO value

Difference between treatments:

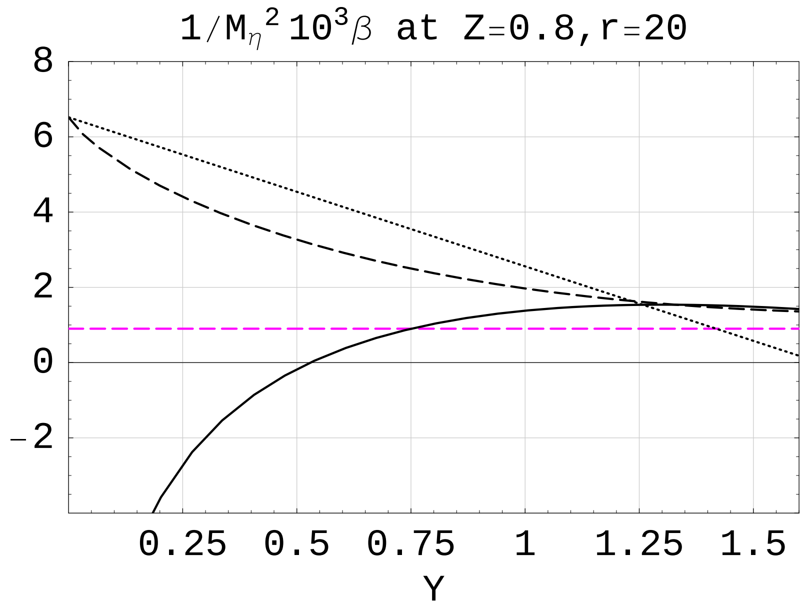
up to 50% of Standard LO value

C.2 Treatment of the chiral logarithms

$\eta\pi$ scattering



polynomial parameter α



polynomial parameter β

solid:	strict form
dashed:	ln(m_P^2/μ^2)→ln(M_P^2/μ^2) in type 1 logs
dotted:	ln(m_P^2/μ^2)→ln(M_P^2/μ^2) in type 1 and 2
hor.dashed:	Standard NLO value

Difference between treatments:

up to 30% of Standard NLO value

up to 1.5x of Standard NLO value

D. Remainder treatment

D.1 Remainder estimates

1. Based on general arguments about the convergence of the chiral series
(Stern et al.2004, Descotes 2007)

$$\Delta_A^{(6)} \sim \pm 0.1 A$$

In principle an assumption.

2. Based on information outside $\mathcal{O}(p^4)$ χPT

The framework of RχPT is well suited to incorporate additional information:

- makes a distinction between the explicitly manageable part of the expansion and the remainder
- consistently distinguishes between both parts and keeps traction of them
- makes the distinction at the right point - the number of LEC's is too large in higher orders to be treated solely within the theory
- the remainder can be estimated in various ways and considered as a source of error

We investigated:

- resonance Lagrangian
- Generalized χPT Lagrangian

D.1 Remainder estimates

1. Based on general arguments about the convergence of the chiral series *(Stern et al.2004, Descotes 2007)*

$$\Delta_A^{(6)} \sim \pm 0.1 A$$

In principle an assumption.

2. Based on information outside $\mathcal{O}(p^4)$ χPT

The framework of RχPT is well suited to incorporate additional information:

- makes a distinction between the explicitly manageable part of the expansion and the remainder
- consistently distinguishes between both parts and keeps traction of them
- makes the distinction at the right point - the number of LEC's is too large in higher orders to be treated solely within the theory
- the remainder can be estimated in various ways and considered as a source of error

We investigated:

- resonance Lagrangian
- Generalized χPT Lagrangian

D.1 Remainder estimates

1. Based on general arguments about the convergence of the chiral series
(Stern et al.2004, Descotes 2007)

$$\Delta_A^{(6)} \sim \pm 0.1 A$$

In principle an assumption.

2. Based on information outside $\mathcal{O}(p^4)$ χPT

The framework of RχPT is well suited to incorporate additional information:

- makes a distinction between the explicitly manageable part of the expansion and the remainder
- consistently distinguishes between both parts and keeps traction of them
- makes the distinction at the right point - the number of LEC's is too large in higher orders to be treated solely within the theory
- the remainder can be estimated in various ways and considered as a source of error

We investigated:

- resonance Lagrangian
- Generalized χPT Lagrangian

D.2 Resonance estimate

Reconstructing an approximation of a more complete theory (RχT)

$$G_{\pi\eta}^{R\chi T}(s,t,u) = G_{\pi\eta}^{\chi PT}(s,t,u) + \Delta G_{\pi\eta}^{R}(s,t,u)$$

Ingredients:

D.2 Resonance estimate

Reconstructing an approximation of a more complete theory (RχT)

$$G^{R\chi T}_{\pi\eta}(s,t,u) = G^{\chi PT}_{\pi\eta}(s,t,u) + \Delta G^{R}_{\pi\eta}(s,t,u)$$

Ingredients:

1.'Resummed' χPT bare expansion

$$G^{bare}_{\pi\eta}(s,t,u) = G_{pol}(s,t,u) + \mathcal{G}_{unit}(s,t,u) + \Delta_{\mathcal{G}}$$

Provides the explicit form to NLO in chiral counting

D.2 Resonance estimate

Reconstructing an approximation of a more complete theory (RχT)

$$G_{\pi\eta}^{R\chi T}(s,t,u) = G_{\pi\eta}^{\chi PT}(s,t,u) + \Delta G_{\pi\eta}^{R}(s,t,u)$$

Ingredients:

1.'Resummed' χPT bare expansion

$$G_{\pi\eta}^{bare}(s,t,u) = G_{pol}(s,t,u) + \mathcal{G}_{unit}(s,t,u) + \Delta_{\mathcal{G}}$$

Provides the explicit form to NLO in chiral counting

2. Resonances *(Ecker et al.1989)*

$$\begin{aligned} G_{\eta\pi}^{R}(s,t,u) &= -\frac{2}{3(s-M_S^2)}\left(c_d(s-M_\pi^2-M_\eta^2)+2c_m m_\pi^2\right)^2 + (s\leftrightarrow u) \\ &+\frac{2}{3(t-M_S^2)}\left(c_d(t-2M_\pi^2)+2c_m m_\pi^2\right)\left(c_d(t-2M_\eta^2)+2c_m(2m_\eta^2-m_\pi^2)\right) \\ &-\frac{4}{t-M_{S_1}^2}\left(\tilde{c}_d(t-2M_\pi^2)+2\tilde{c}_m m_\pi^2\right)\left(\tilde{c}_d(t-2M_\eta^2)+2\tilde{c}_m m_\eta^2\right) \\ &-\frac{4c_m^2}{3M_S^2}m_\pi^2\left(m_\eta^2-m_\pi^2\right)+\frac{4\tilde{c}_m^2}{M_{S_1}^2}m_\pi^2\left(m_\pi^2+m_\eta^2\right)+\frac{16\,\tilde{d}_m^2}{M_{\eta_1}^2-M_\eta^2}m_\pi^2\left(m_\eta^2-m_\pi^2\right) \end{aligned}$$

Expand as chiral series and resum NNLO and all higher order terms

D.2 Resonance estimate

Reconstructing an approximation of a more complete theory (RχT)

$$G_{\pi\eta}^{R\chi T}(s,t,u) = G_{\pi\eta}^{\chi PT}(s,t,u) + \Delta G_{\pi\eta}^{R}(s,t,u)$$

Result:

1.'Resummed' χPT bare expansion

$$G_{\pi\eta}^{bare}(s,t,u) = G_{pol}(s,t,u) + \mathcal{G}_{unit}(s,t,u) + \Delta G_R(s,t,u) + \widetilde{\Delta}_{\mathcal{G}}$$

2. Resonances

$$\begin{aligned}
\Delta G_R(s,t,u) \;=\; & -\frac{2s}{3(s-M_S^2)M_S^2}\left(c_d(s-M_\pi^2-M_\eta^2)+2c_m m_\pi^2\right)^2 + (s\leftrightarrow u)\\
& +\frac{2t}{3(t-M_S^2)M_S^2}\left(c_d(t-2M_\pi^2)+2c_m m_\pi^2\right)\left(c_d(t-2M_\eta^2)+2c_m(2m_\eta^2-m_\pi^2)\right)\\
& -\frac{4t}{(t-M_{S_1}^2)M_{S_1}^2}\left(\widetilde{c}_d(t-2M_\pi^2)+2\widetilde{c}_m m_\pi^2\right)\left(\widetilde{c}_d(t-2M_\eta^2)+2\widetilde{c}_m m_\eta^2\right)\\
& +\frac{16\,\widetilde{d}_m^2 M_\eta^2}{(M_{\eta_1}^2-M_\eta^2)M_{\eta_1}^2} m_\pi^2\left(m_\eta^2-m_\pi^2\right)
\end{aligned}$$

D.2 Resonance estimate

Reconstructing an approximation of a more complete theory (RχT)

$$G^{R\chi T}_{\pi\eta}(s,t,u) = G^{\chi PT}_{\pi\eta}(s,t,u) + \Delta G^{R}_{\pi\eta}(s,t,u)$$

Result:

1.'Resummed' χPT bare expansion

$$G^{bare}_{\pi\eta}(s,t,u) = G_{pol}(s,t,u) + \mathcal{G}_{unit}(s,t,u) + \Delta G_R(s,t,u) + \widetilde{\Delta}_{\mathcal{G}}$$

Remainder 'saturation' instead of usual LEC saturation

2. Resonances

$$\begin{aligned}
\Delta G_R(s,t,u) = & -\frac{2s}{3(s-M_S^2)M_S^2}\left(c_d(s-M_\pi^2-M_\eta^2)+2c_m m_\pi^2\right)^2 + (s\leftrightarrow u) \\
& +\frac{2t}{3(t-M_S^2)M_S^2}\left(c_d(t-2M_\pi^2)+2c_m m_\pi^2\right)\left(c_d(t-2M_\eta^2)+2c_m(2m_\eta^2-m_\pi^2)\right) \\
& -\frac{4t}{(t-M_{S_1}^2)M_{S_1}^2}\left(\widetilde{c}_d(t-2M_\pi^2)+2\widetilde{c}_m m_\pi^2\right)\left(\widetilde{c}_d(t-2M_\eta^2)+2\widetilde{c}_m m_\eta^2\right) \\
& +\frac{16\,\widetilde{d}_m^2 M_\eta^2}{(M_{\eta_1}^2-M_\eta^2)M_{\eta_1}^2} m_\pi^2\left(m_\eta^2-m_\pi^2\right)
\end{aligned}$$

D.2 Resonance estimate

Reconstructing an approximation of a more complete theory (RχT)

$$G_{\pi\eta}^{R\chi T}(s,t,u) = G_{\pi\eta}^{\chi PT}(s,t,u) + \Delta G_{\pi\eta}^{R}(s,t,u)$$

Result:

1.'Resummed' χPT bare expansion

$$G_{\pi\eta}^{bare}(s,t,u) = G_{pol}(s,t,u) + \mathcal{G}_{unit}(s,t,u) + \Delta G_R(s,t,u) + \widetilde{\Delta}_{\mathcal{G}}$$

Ren.scale independent - no need to fix a saturation scale

2. Resonances

$$\begin{aligned}\Delta G_R(s,t,u) = & -\frac{2s}{3(s-M_S^2)M_S^2}\left(c_d(s-M_\pi^2-M_\eta^2)+2c_m m_\pi^2\right)^2 + (s\leftrightarrow u)\\ & +\frac{2t}{3(t-M_S^2)M_S^2}\left(c_d(t-2M_\pi^2)+2c_m m_\pi^2\right)\left(c_d(t-2M_\eta^2)+2c_m(2m_\eta^2-m_\pi^2)\right)\\ & -\frac{4t}{(t-M_{S_1}^2)M_{S_1}^2}\left(\widetilde{c}_d(t-2M_\pi^2)+2\widetilde{c}_m m_\pi^2\right)\left(\widetilde{c}_d(t-2M_\eta^2)+2\widetilde{c}_m m_\eta^2\right)\\ & +\frac{16\,\widetilde{d}_m^2 M_\eta^2}{(M_{\eta_1}^2-M_\eta^2)M_{\eta_1}^2} m_\pi^2\left(m_\eta^2-m_\pi^2\right)\end{aligned}$$

D.2 Resonance estimate

Reconstructing an approximation of a more complete theory (RχT)

$$G_{\pi\eta}^{R\chi T}(s,t,u) = G_{\pi\eta}^{\chi PT}(s,t,u) + \Delta G_{\pi\eta}^{R}(s,t,u)$$

Result:

1.'Resummed' χPT bare expansion

$$G_{\pi\eta}^{bare}(s,t,u) = G_{pol}(s,t,u) + \mathcal{G}_{unit}(s,t,u) + \Delta G_R(s,t,u) + \widetilde{\Delta_{\mathcal{G}}}$$

Ren.scale independent - no need to fix a saturation scale

2. Resonances

$$\begin{aligned}
\Delta G_R(s,t,u) = & -\frac{2s}{3(s-M_S^2)M_S^2}\left(c_d(s-M_\pi^2-M_\eta^2)+2c_m m_\pi^2\right)^2 + (s \leftrightarrow u) \\
& + \frac{2t}{3(t-M_S^2)M_S^2}\left(c_d(t-2M_\pi^2)+2c_m m_\pi^2\right)\left(c_d(t-2M_\eta^2)+2c_m(2m_\eta^2-m_\pi^2)\right) \\
& - \frac{4t}{(t-M_{S_1}^2)M_{S_1}^2}\left(\widetilde{c}_d(t-2M_\pi^2)+2\widetilde{c}_m m_\pi^2\right)\left(\widetilde{c}_d(t-2M_\eta^2)+2\widetilde{c}_m m_\eta^2\right) \\
& + \frac{16\,\widetilde{d}_m^2 M_\eta^2}{(M_{\eta_1}^2-M_\eta^2)M_{\eta_1}^2} m_\pi^2\left(m_\eta^2-m_\pi^2\right)
\end{aligned}$$

To all orders - resonance poles explicitly present

D.3 The GχPT estimate

The resonance estimate only deals with the derivative part of the series

$\longrightarrow$ the expansion in terms of quark masses is not estimated

D.3 The GχPT estimate

The resonance estimate only deals with the derivative part of the series

$\longrightarrow$ **the expansion in terms of quark masses is not estimated**

The Generalized χPT Lagrangian uses an alternative power counting

(Stern et al.1995)

$\longrightarrow$ **Standard $\mathcal{O}(p^6)$ and $\mathcal{O}(p^8)$ terms are present at NLO**

$\longrightarrow$ **the ones suspected to upset the expansion**

D.3 The GχPT estimate

The resonance estimate only deals with the derivative part of the series

$\longrightarrow$ the expansion in terms of quark masses is not estimated

The Generalized χPT Lagrangian uses an alternative power counting
(Stern et al.1995)

$\longrightarrow$ Standard $\mathcal{O}(p^6)$ and $\mathcal{O}(p^8)$ terms are present at NLO

$\longrightarrow$ the ones suspected to upset the expansion

Matching the bare expansions from both versions of power counting

$$G^{bare}(s,t,u) = G_{pol}(s,t,u) + \mathcal{G}_{unit}(s,t,u) + \Delta_{\mathcal{G}}$$

$$G^{bare}(s,t,u) = G_{pol}^{G\chi PT}(s,t,u) + \mathcal{G}_{unit}^{G\chi PT}(s,t,u) + \Delta_{\mathcal{G}}^{G\chi PT}$$

$$\Delta_{\mathcal{G}} \ = \ [G_{pol}^{G\chi PT}(s,t,u) - G_{pol}(s,t,u)] + [\mathcal{G}_{unit}^{G\chi PT}(s,t,u) - \mathcal{G}_{unit}(s,t,u)] + \Delta_{\mathcal{G}}^{G\chi PT}$$

D.3 The GχPT estimate

The resonance estimate only deals with the derivative part of the series

$\longrightarrow$ the expansion in terms of quark masses is not estimated

The Generalized χPT Lagrangian uses an alternative power counting *(Stern et al.1995)*

$\longrightarrow$ Standard $\mathcal{O}(p^6)$ and $\mathcal{O}(p^8)$ terms are present at NLO

$\longrightarrow$ the ones suspected to upset the expansion

Matching the bare expansions from both versions of power counting

$$G^{bare}(s,t,u) = G_{pol}(s,t,u) + \mathcal{G}_{unit}(s,t,u) + \Delta_{\mathcal{G}}$$

$$G^{bare}(s,t,u) = G_{pol}^{G\chi PT}(s,t,u) + \mathcal{G}_{unit}^{G\chi PT}(s,t,u) + \Delta_{\mathcal{G}}^{G\chi PT}$$

$$\Delta_{\mathcal{G}} = [G_{pol}^{G\chi PT}(s,t,u) - G_{pol}(s,t,u)] + [\mathcal{G}_{unit}^{G\chi PT}(s,t,u) - \mathcal{G}_{unit}(s,t,u)] + \Delta_{\mathcal{G}}^{G\chi PT}$$

Sewing the resonance and the GχPT remainder estimates:

$$\Delta G_R^{G\chi PT} = \Delta G_R - \frac{16\tilde{c}_m^2 m_\pi^2 m_\eta^2 t}{M_{S_1}^4} - \frac{16 c_m^2 m_\pi^2}{3M_S^4}(m_\pi^2 M_\eta^2 + m_\pi^2 M_\pi^2 - m_\eta^2 t) - \frac{16\tilde{d}_m^2 M_\eta^2}{M_{\eta_1}^4} m_\pi^2 (m_\eta^2 - m_\pi^2)$$

D.3 The GχPT estimate

Illustrative example - polynomial parameter β:

$$G_{\pi\eta}^{bare}(s,t,u) = \alpha + \beta t + \gamma t^2 + \omega(s-u)^2 + \mathcal{G}_{unit}(s,t,u) + \Delta_{\mathcal{G}}$$

$$\beta = 2\,(M_\eta^2 + M_\pi^2)[\frac{3}{128\pi^2}(\ln\frac{m_K^2}{\mu^2}+1) - 8(L_1^r(\mu) + \frac{1}{6}L_3^r(\mu))] + 8(m_\eta^2 + m_\pi^2)L_4^r(\mu)$$

$$-\frac{1}{32\pi^2}m_\eta^2(\ln\frac{m_K^2}{\mu^2}+1) - \frac{1}{48\pi^2}m_\pi^2(\ln\frac{m_\pi^2}{\mu^2}+1) - \frac{1}{96\pi^2}m_\pi^2(\ln\frac{m_K^2}{\mu^2}+1) + \Delta_\beta$$

D.3 The GχPT estimate

Illustrative example - polynomial parameter β:

$$G_{\pi\eta}^{bare}(s,t,u) = \alpha + \beta t + \gamma t^2 + \omega(s-u)^2 + \mathcal{G}_{unit}(s,t,u) + \Delta_{\mathcal{G}}$$

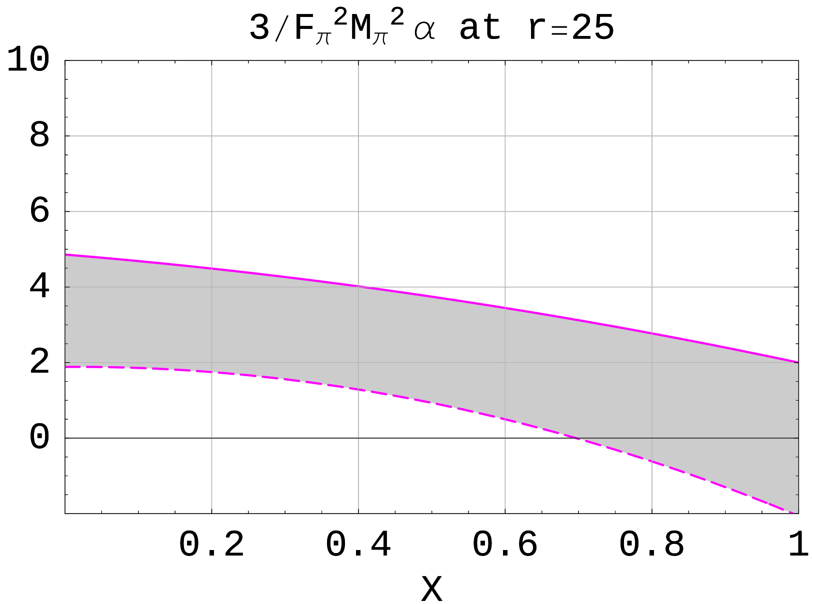
$$\beta = 2\,(M_\eta^2 + M_\pi^2)[\frac{3}{128\pi^2}(\ln\frac{m_K^2}{\mu^2}+1) - 8(L_1^r(\mu)+\frac{1}{6}L_3^r(\mu))] + 8(m_\eta^2+m_\pi^2)L_4^r(\mu)$$
$$-\frac{1}{32\pi^2}m_\eta^2(\ln\frac{m_K^2}{\mu^2}+1) - \frac{1}{48\pi^2}m_\pi^2(\ln\frac{m_\pi^2}{\mu^2}+1) - \frac{1}{96\pi^2}m_\pi^2(\ln\frac{m_K^2}{\mu^2}+1) + \Delta_\beta$$

$$\Delta_\beta = \Delta_\beta^{G\chi PT} + \frac{8}{3}[(C_1^S + D^S)(2r+1) + 2B_4(r^2+2)]$$
$$+\frac{1}{3}\left[\tilde{m}_\pi^2 + 4\hat{m}^2(3A_0 - 4(r-1)Z_0^P + 2(2r+1)Z_0^S) - 2B_0\hat{m}\right] J_{\pi\pi}^r(0)$$
$$-\frac{3}{12}\left[2\tilde{m}_\pi^2 - 8\hat{m}^2(r-1)(A_0+2Z_0^P) - 4B_0\hat{m}\right] J_{KK}^r(0) + \frac{1}{8}[6(\tilde{m}_\eta^2 - M_\eta^2 + \tilde{m}_\pi^2 - M_\pi^2) - \frac{8}{3}\tilde{m}_K^2$$
$$+\frac{8}{3}(r+1)\hat{m}^2(3A_0(r+3)+4Z_0^S(r+5)+2(r-1)Z_0^P) - \frac{8}{3}B_0\hat{m}(2r+5)+6M_\eta^2+6M_\pi^2]J_{KK}^r(0)$$

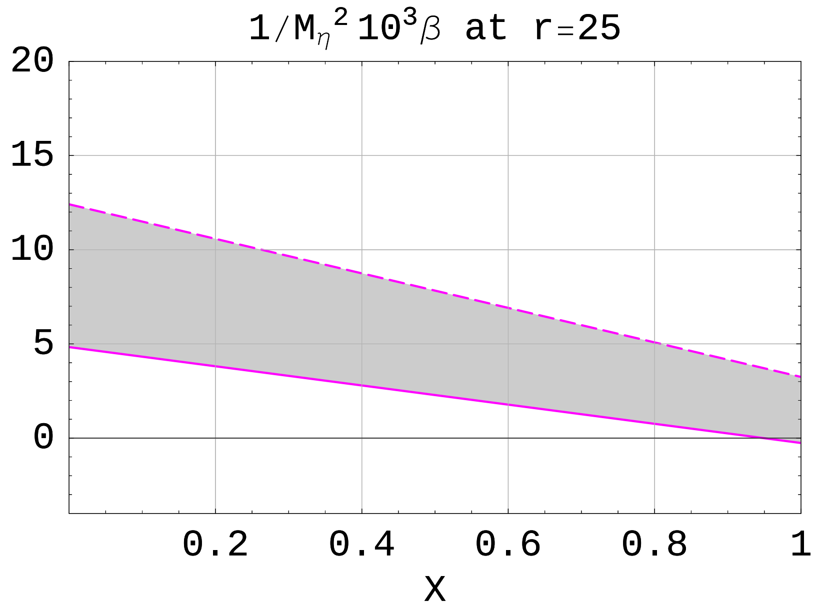
$\tilde{m}_P$ are Generalized LO masses

D.4 Remainder estimates - numerical results

Remainders neglected - parameter range $X \sim$0-1, $Z \sim$0.5-0.9, fixed r=25



polynomial parameter α



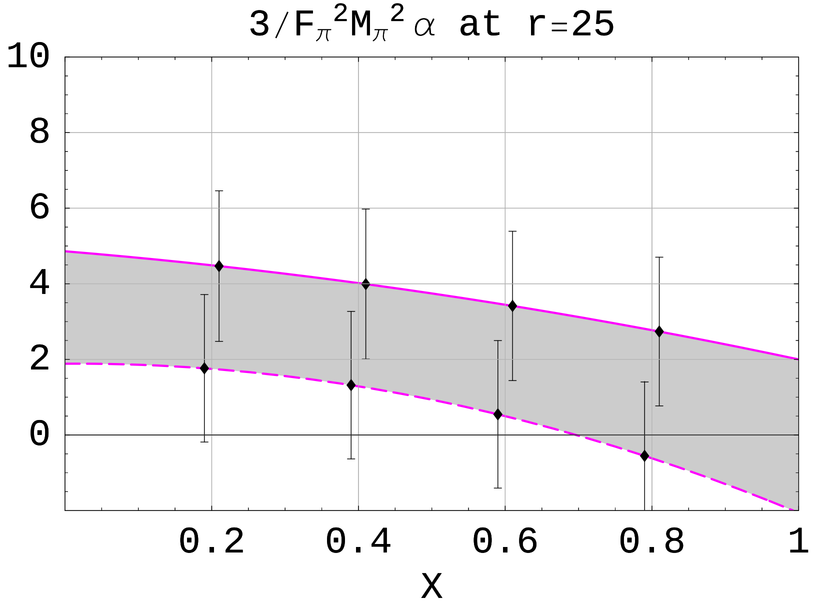
polynomial parameter β

solid: $Z = 0.9$
dashed: $Z = 0.5$
grey: parameter range X ~0-1, Z ~0.5-0.9

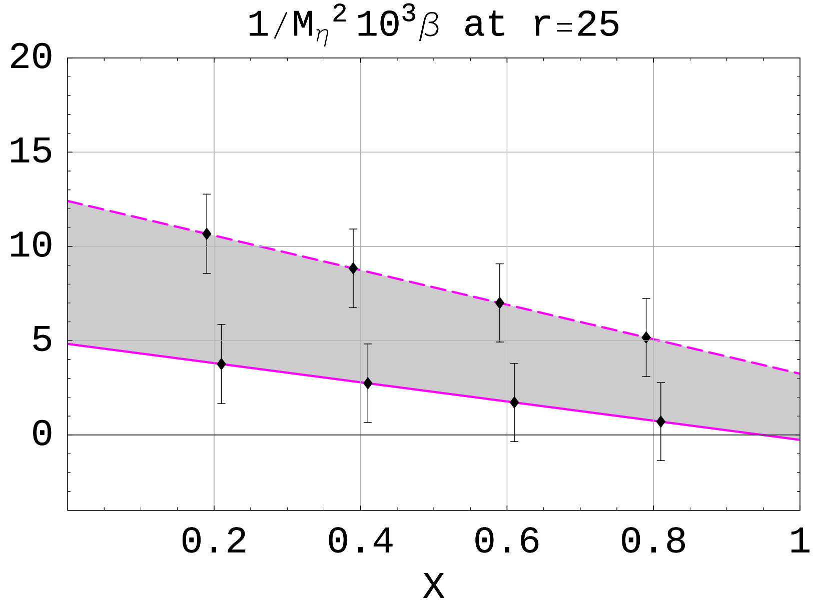
Free parameters of the theory: X, Z, r

D.4 Remainder estimates - numerical results

Remainder estimate - 10% uncertainty



polynomial parameter α



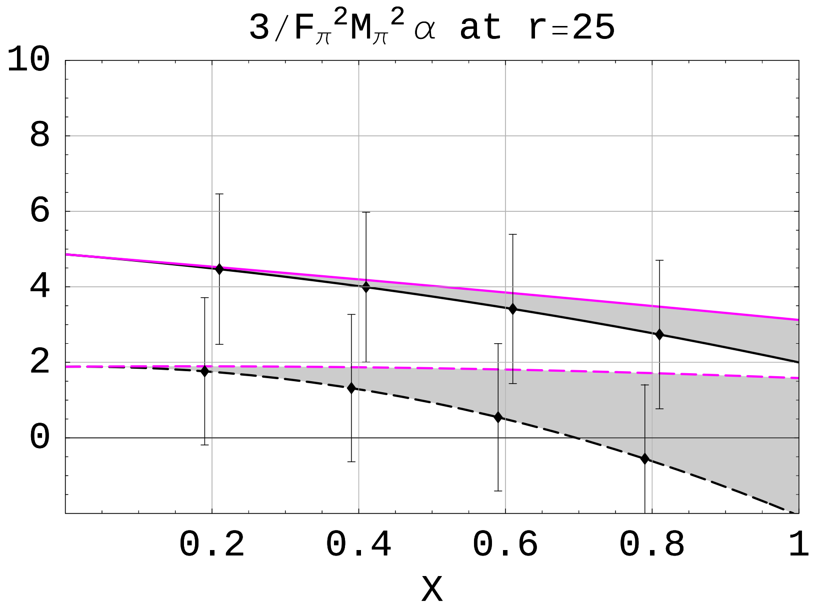
polynomial parameter β

solid:	$Z = 0.9$
dashed:	$Z = 0.5$
Err.bars:	10% uncertainty estimate
grey:	parameter range X ~0-1, Z ~0.5-0.9

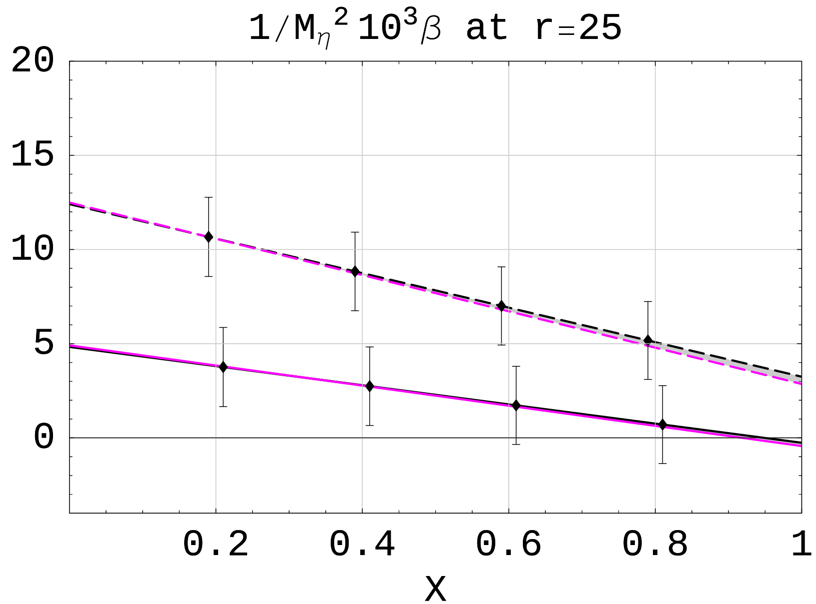
Small reminders might generate significant uncertainty

D.4 Remainder estimates - numerical results

Remainder estimate - resonances



polynomial parameter α



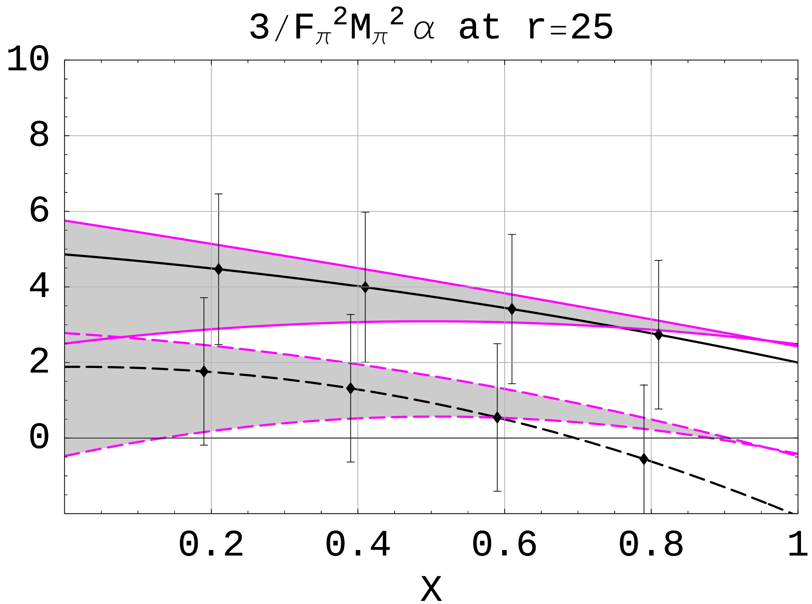
polynomial parameter β

solid:	$Z = 0.9$
dashed:	$Z = 0.5$
Err.bars:	10% uncertainty estimate
grey:	resonance remainder estimate

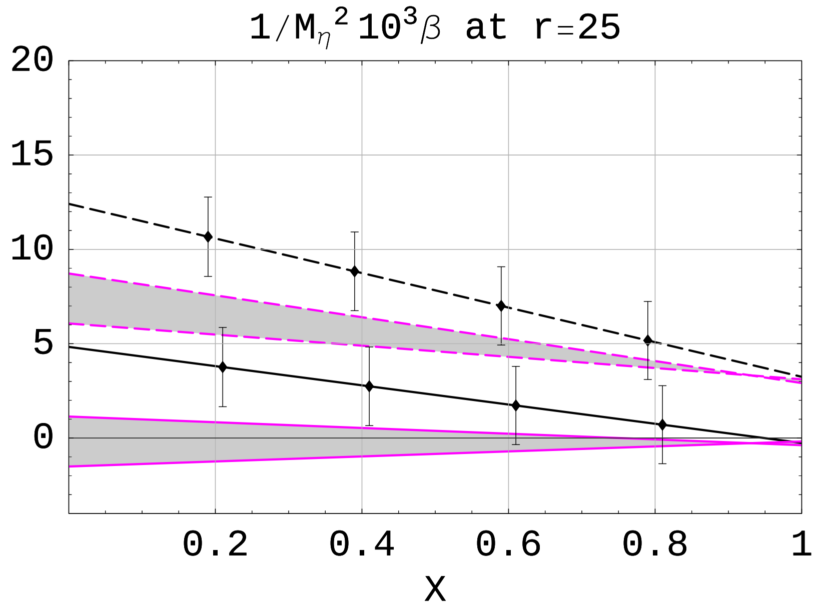
Compatible with 10% remainder magnitude assumption

D.4 Remainder estimates - numerical results

Remainder estimate - GχPT and resonances combined



polynomial parameter α



polynomial parameter β

solid: $Z = 0.9$
dashed: $Z = 0.5$
Err.bars: 10% uncertainty estimate
grey: resonance+GχPT remainder estimate, scale dependence $\mu \sim M_\eta$-M_ρ

α: compatible, β: borderline

E. Stability of the chiral series and the Standard approach to NLO

E.1 Standard approach to NLO

Standard reparametrization - inverted expansions for LO LEC's:

$$F_0^2 = F_\pi^2(1 + 4\mu_\pi + 2\mu_K) - 8M_\pi^2(L_4^r(2+r) + L_5^r)$$

$$2B_0\hat{m} = M_\pi^2(1 - \mu_\pi + \frac{1}{3}\mu_\eta - \frac{8M_\pi^2}{F_\pi^2}(2(L_8^r + (2+r)L_6^r) - (L_5^r + (2+r)L_4^r))$$

$$r = \frac{2M_K^2}{M_\pi^2} - 1 \quad \text{or} \quad r = \frac{3M_\eta^2}{2M_\pi^2} - \frac{1}{2}$$

Next-to-leading order LEC's:

[1] $\mathcal{O}(p^4)$ fit (Bijnens et al.1994)
[2] $\mathcal{O}(p^6)$ fit (Bijnens et al.2000)

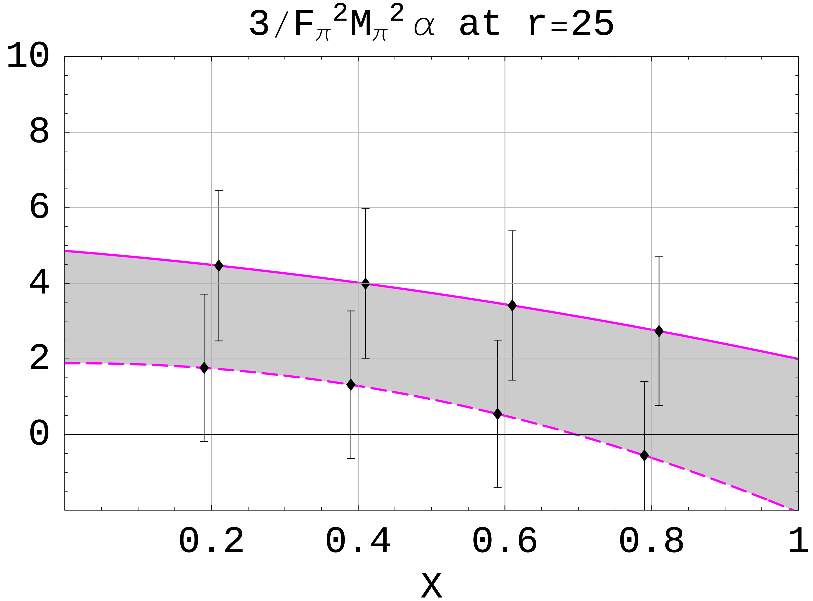
Results:

L_i	α/α^{CA}	$10^3\beta/M_\eta^2$	c_{00}/c_{00}^{CA}	$10^3 c_{10}$	a_0/a_0^{CA}	$10^3 a_1$
[1]	1.68	0.90	1.06	0.91	1.96	0.59
[2]	1.91	-0.68	1.51	-0.67	1.18	-0.60
Δ	2.48	7.49	2.49	7.49	3.21	2.80

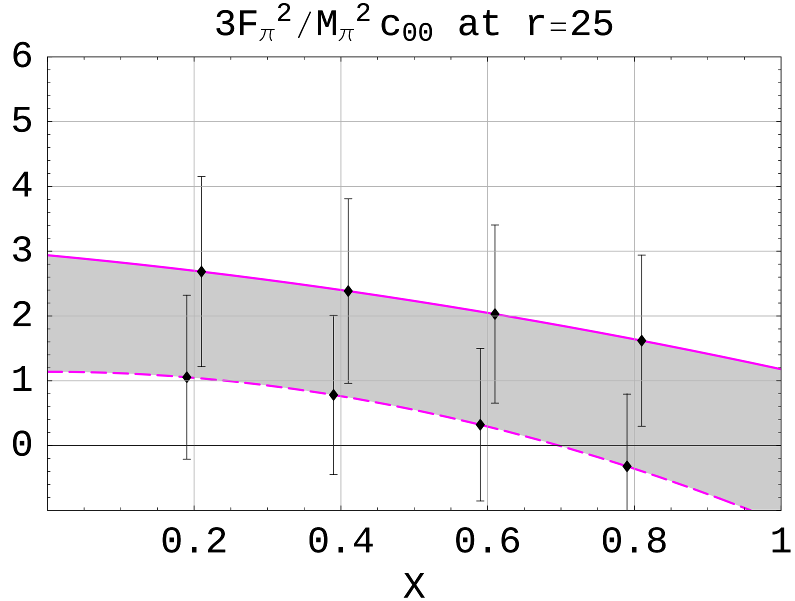
Strong sensitivity to LEC fit $\rightarrow$ suggests large higher order corrections

E.2 Stability of the chiral series - $\eta\pi$ scattering

Parameter range $X\sim$0-1, $Z\sim$0.5-0.9, fixed r=25; 10% remainder estimate



polynomial parameter α



subthreshold parameter c_{00}

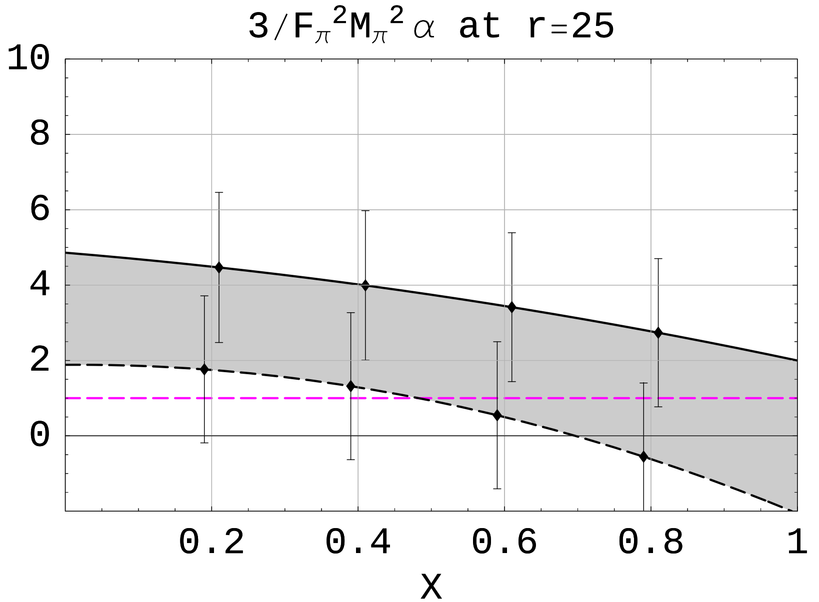
solid:	$Z = 0.9$
dashed:	$Z = 0.5$
grey:	parameter range X ~0-1, Z ~0.5-0.9
Err.bars:	10% uncertainty estimate

Watch out for:

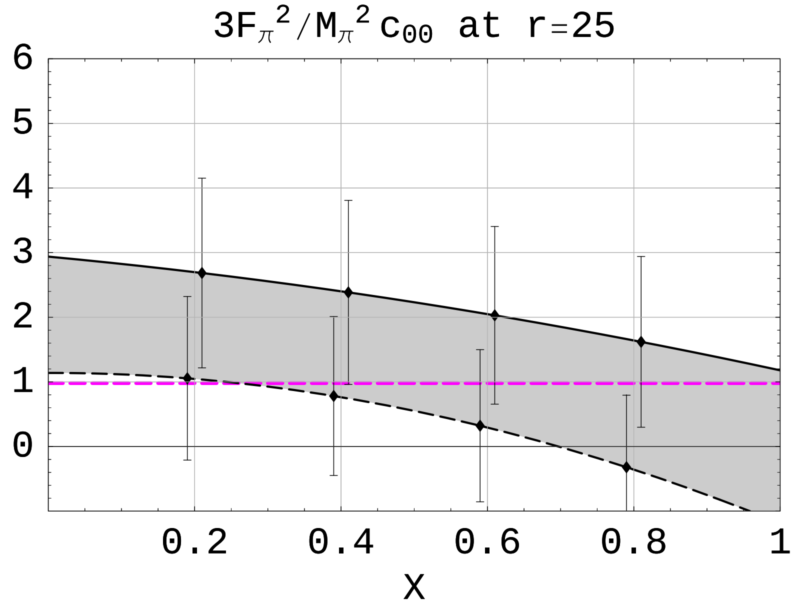
sensitivity to X and Z, the uncertainty generated by small remainders

E.2 Stability of the chiral series - $\eta\pi$ scattering

Comparison with Standard LO value



polynomial parameter α



subthreshold parameter c_{00}

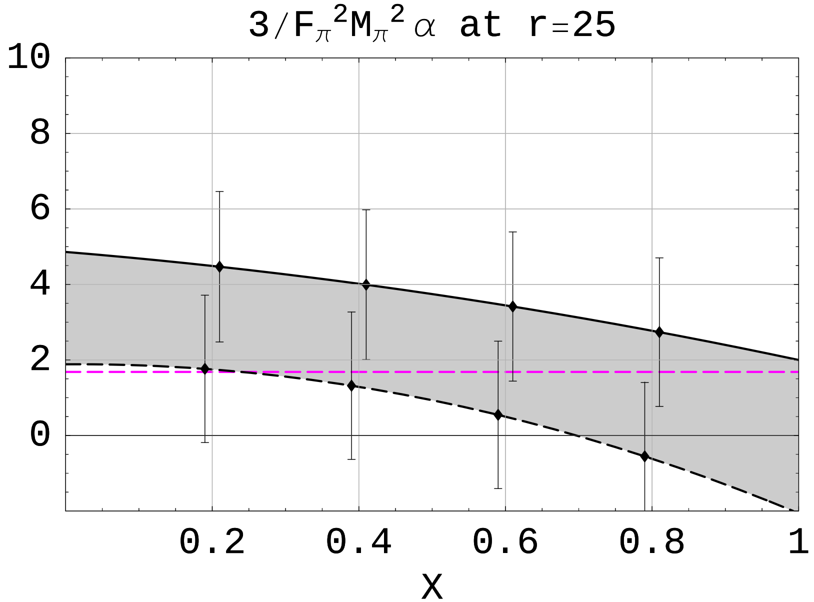
solid:	$Z = 0.9$
dashed:	$Z = 0.5$
grey:	parameter range $X \sim$0-1, $Z \sim$0.5-0.9
Err.bars:	10% uncertainty estimate
hor.dashed	LO value

Watch out for:

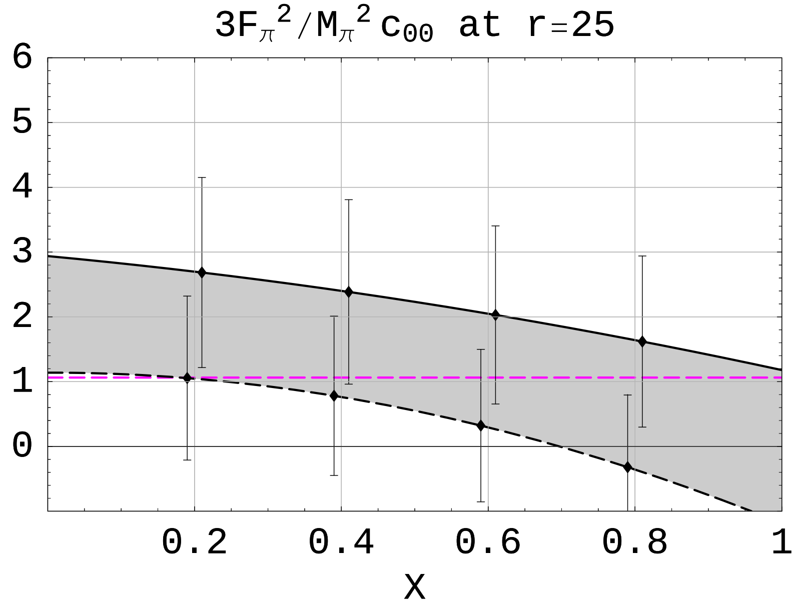
the ratio of LO the to the possible complete result depending on X and Z

E.2 Stability of the chiral series - $\eta\pi$ scattering

Comparison with Standard NLO value



polynomial parameter α



subthreshold parameter c_{00}

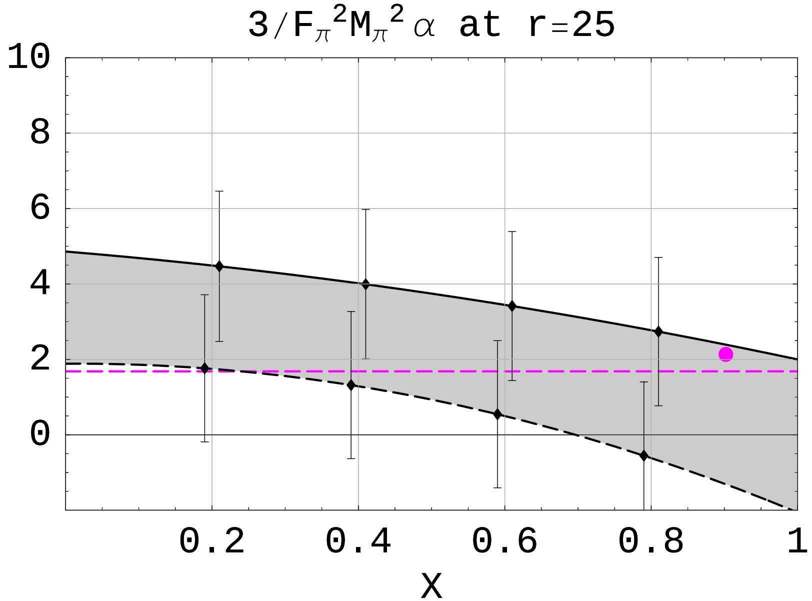
solid:	$Z = 0.9$
dashed:	$Z = 0.5$
grey:	parameter range $X \sim$0-1, $Z \sim$0.5-0.9
Err.bars:	10% uncertainty estimate
hor.dashed	Standard NLO value

Watch out for:

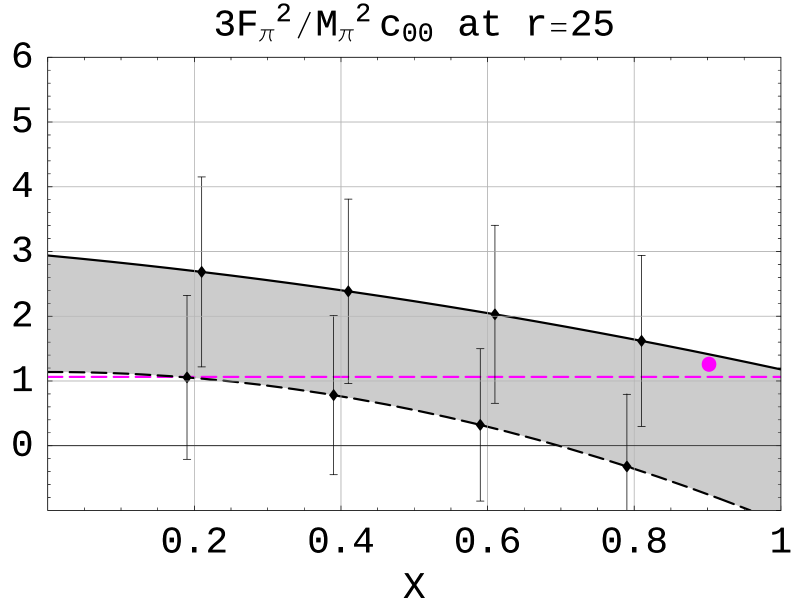
how sufficient is Standard NLO result depending on X and Z

E.2 Stability of the chiral series - $\eta\pi$ scattering

Restoration of the Standard value from RχPT at the point X^{std}, Z^{std}, r^{std}



polynomial parameter α



subthreshold parameter c_{00}

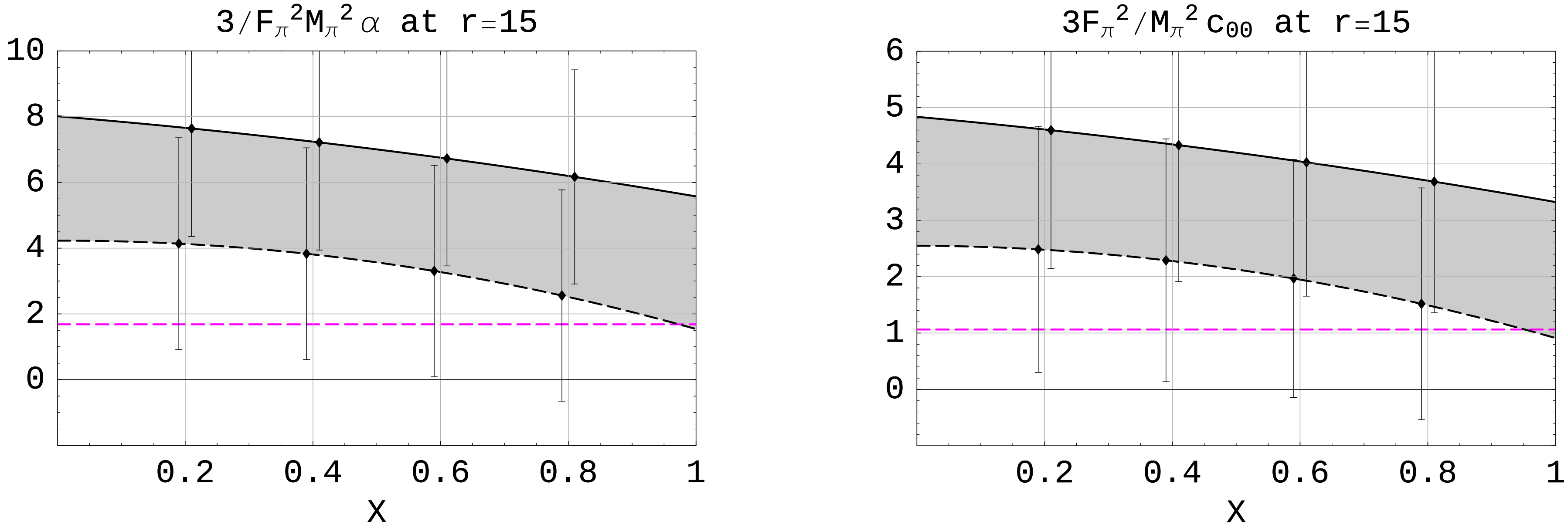
solid:	$Z = 0.9$
dashed:	$Z = 0.5$
grey:	parameter range $X \sim$0-1, $Z \sim$0.5-0.9
Err.bars:	10% uncertainty estimate
hor.dashed	Standard NLO value
point	Standard reference point X^{std}=0.9, Z^{std}=0.87, r^{std}=25

Watch out for:

whether RχPT correctly restores the Standard value, ‘good’ vs.‘bad’ observable

E.2 Stability of the chiral series - $\eta\pi$ scattering

Dependence on r - shift of the range at r=15



polynomial parameter α subthreshold parameter c_{00}

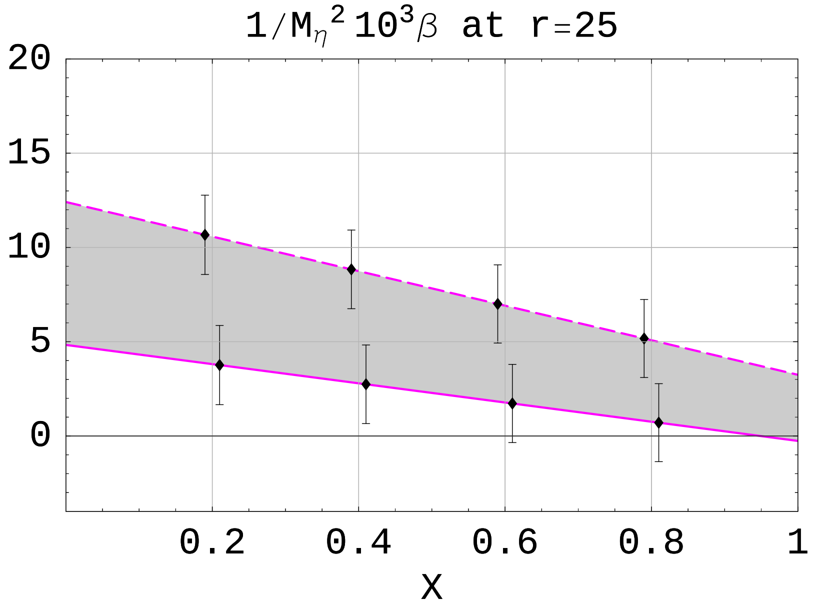
solid:	$Z = 0.9$
dashed:	$Z = 0.5$
grey:	parameter range X ~0-1, Z ~0.5-0.9
Err.bars:	10% uncertainty estimate
hor.dashed	Standard NLO value

Watch out for:

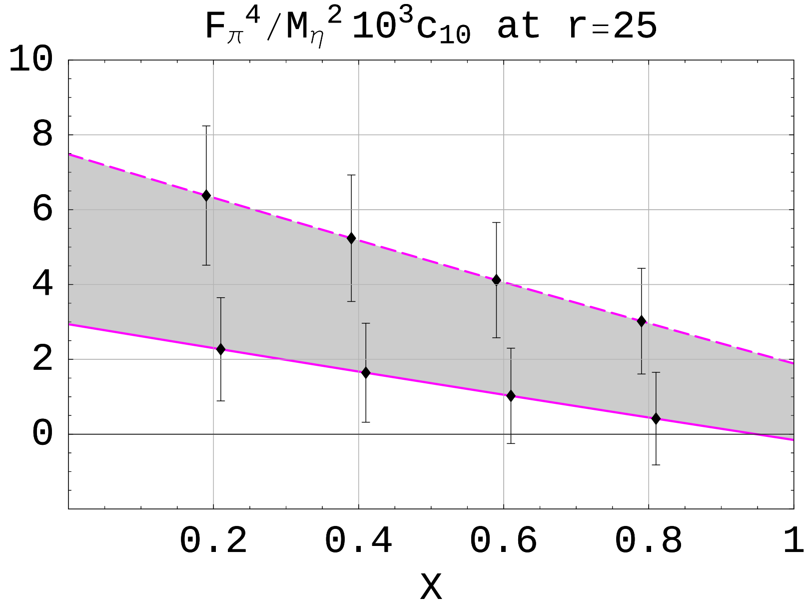
if there is a change with small r

E.2 Stability of the chiral series - $\eta\pi$ scattering

Parameter range X∼0-1, Z∼0.5-0.9, fixed r=25; 10% remainder estimate



polynomial parameter β



subthreshold parameter c_{10}

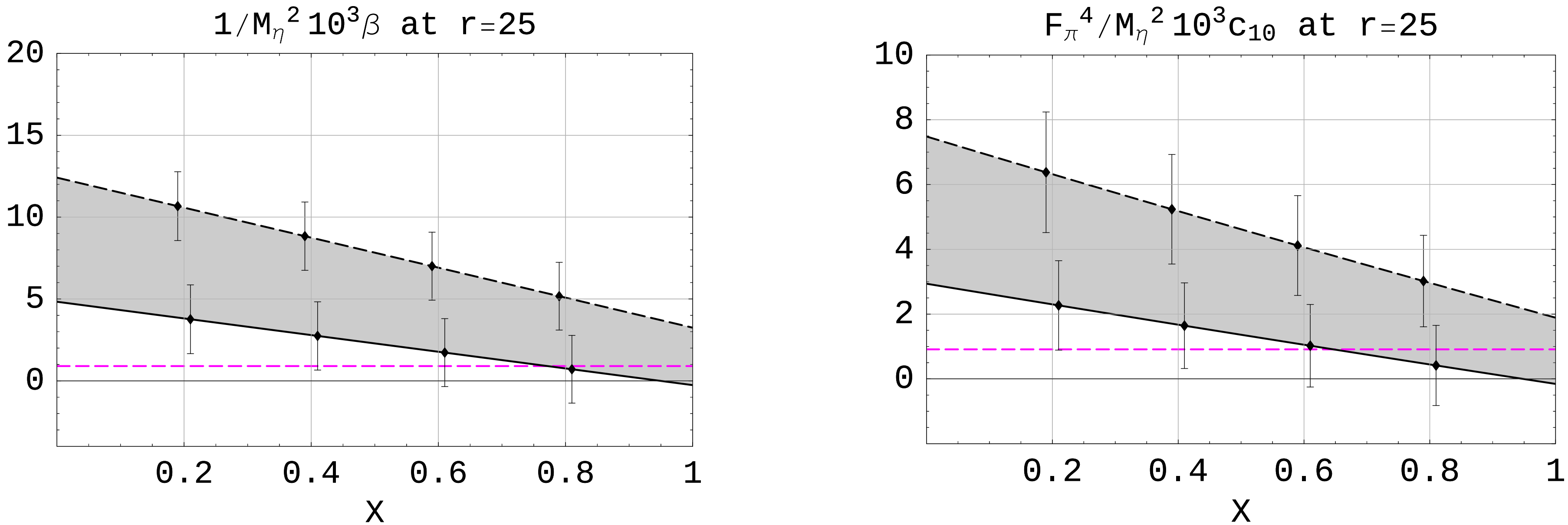
solid:	$Z = 0.9$
dashed:	$Z = 0.5$
grey:	parameter range X ∼0-1, Z ∼0.5-0.9
Err.bars:	10% uncertainty estimate

Watch out for:

sensitivity to X and Z, the uncertainty generated by small remainders

E.2 Stability of the chiral series - $\eta\pi$ scattering

Comparison with Standard NLO value



polynomial parameter β | subthreshold parameter c_{10}

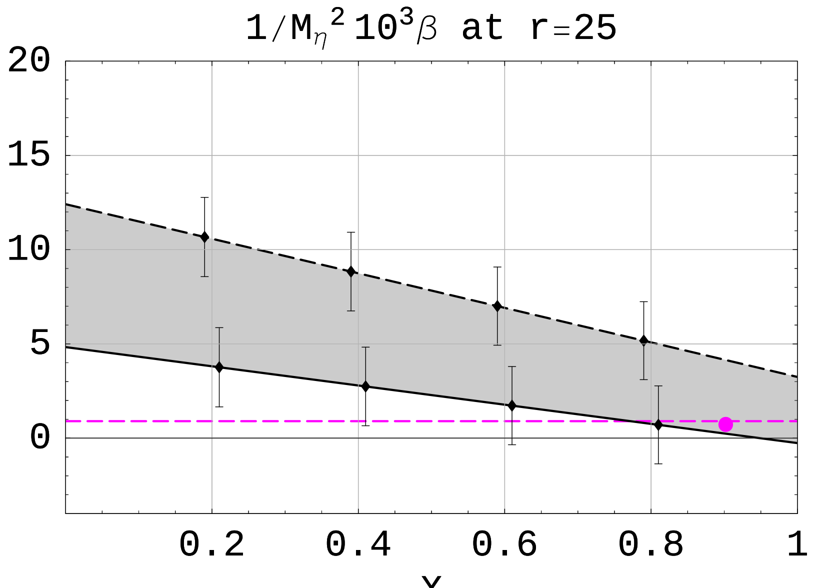
solid:	$Z = 0.9$
dashed:	$Z = 0.5$
grey:	parameter range X ~0-1, Z ~0.5-0.9
Err.bars:	10% uncertainty estimate
hor.dashed	Standard NLO value

Watch out for:

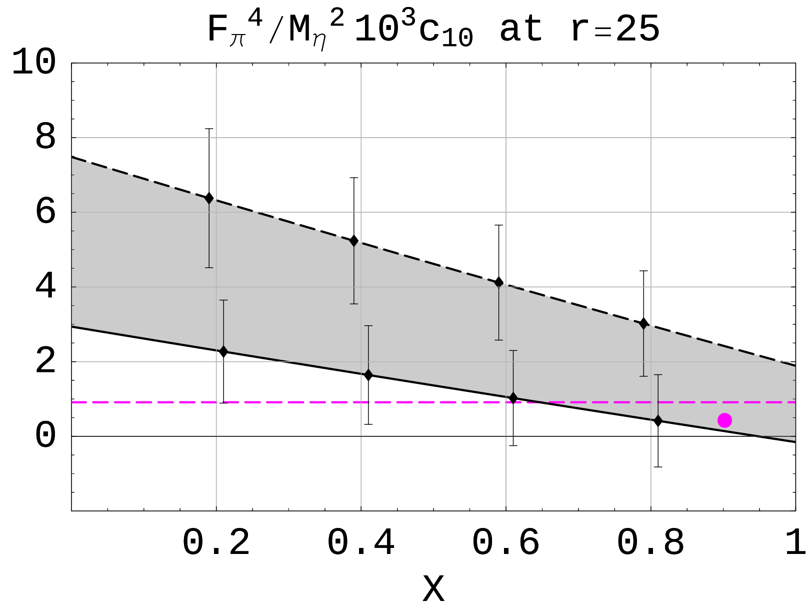
how sufficient is Standard NLO result depending on X and Z

E.2 Stability of the chiral series - $\eta\pi$ scattering

Restoration of the Standard value from RχPT at the point X^{std}, Z^{std}, r^{std}



polynomial parameter β



subthreshold parameter c_{10}

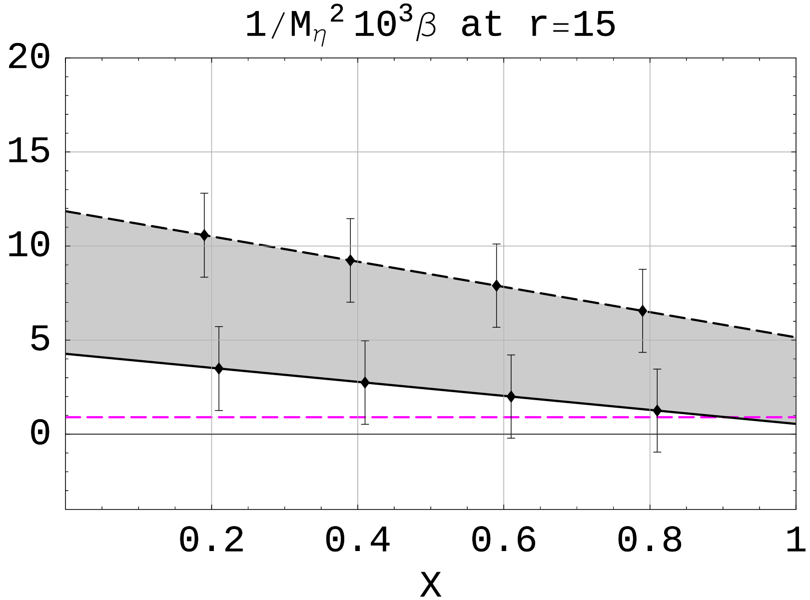
solid:	$Z = 0.9$
dashed:	$Z = 0.5$
grey:	parameter range X ~0-1, Z ~0.5-0.9
Err.bars:	10% uncertainty estimate
hor.dashed	Standard NLO value
point	Standard reference point X^{std}=0.9, Z^{std}=0.87, r^{std}=25

Watch out for:

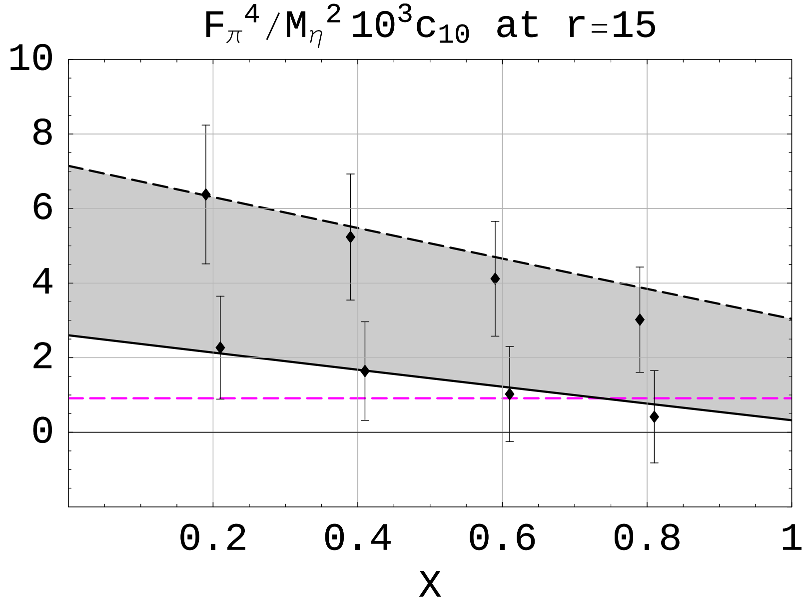
whether RχPT correctly restores the Standard value, ‘good’ vs.‘bad’ observable

E.2 Stability of the chiral series - $\eta\pi$ scattering

Dependence on r - shift of the range at r=15



polynomial parameter β



subthreshold parameter c_{10}

solid:	$Z = 0.9$
dashed:	$Z = 0.5$
grey:	parameter range X ~0-1, Z ~0.5-0.9
Err.bars:	10% uncertainty estimate
hor.dashed	Standard NLO value

Watch out for:

if there is a change with small r

F. Summary

- $\eta\,\pi^0 \longrightarrow \eta\,\pi^0$ scattering appears to be sensitive to effects considered by 'Resummed' χPT. Unfortunately no low energy data is available.

- We have studied the various possibilities of the definition of the bare expansion and have shown that the differences might be significant.

- We have estimated the remainders in several ways, namely incorporated resonances and used the GχPT Lagrangian in order to get a sense of the magnitude of the remainders

F. Summary

- $\eta\,\pi^0 \rightarrow \eta\,\pi^0$ scattering appears to be sensitive to effects considered by 'Resummed' χPT. Unfortunately no low energy data is available.
- We have studied the various possibilities of the definition of the bare expansion and have shown that the differences might be significant.
- We have estimated the remainders in several ways, namely incorporated resonances and used the GχPT Lagrangian in order to get a sense of the magnitude of the remainders

F. Summary

- $\eta\,\pi^0 \longrightarrow \eta\,\pi^0$ scattering appears to be sensitive to effects considered by 'Resummed' χPT. Unfortunately no low energy data is available.

- We have studied the various possibilities of the definition of the bare expansion and have shown that the differences might be significant.

- We have estimated the remainders in several ways, namely incorporated resonances and used the GχPT Lagrangian in order to get a sense of the magnitude of the remainders

F. Summary

- $\eta\,\pi^0 \longrightarrow \eta\,\pi^0$ scattering appears to be sensitive to effects considered by 'Resummed' χPT. Unfortunately no low energy data is available.
- We have studied the various possibilities of the definition of the bare expansion and have shown that the differences might be significant.
- We have estimated the remainders in several ways, namely incorporated resonances and used the GχPT Lagrangian in order to get a sense of the magnitude of the remainders

F. Summary

- $\eta\,\pi^0 \longrightarrow \eta\,\pi^0$ scattering appears to be sensitive to effects considered by 'Resummed' χPT. Unfortunately no low energy data is available.

- We have studied the various possibilities of the definition of the bare expansion and have shown that the differences might be significant.

- We have estimated the remainders in several ways, namely incorporated resonances and used the GχPT Lagrangian in order to get a sense of the magnitude of the remainders

Thank you for your attention!