$\eta \to 3\pi$ at Two Loops

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Various ChPT: http://www.thep.lu.se/∼bijnens/chpt.html

Contents

- Useful proceedings/conferences ETA01 (Uppsala), ETA05 (Cracow), ETA06(Julich), ETA07 (Peniscola)
- Chiral Perturbation Theory (ChPT, CHPT, χPT)
- $\eta \to 3\pi$: Main part of talk
  - Earlier results
  - Some aspects of the two loop calculation
  - Numerical results
- Conclusions

Eta Physics Handbook: ETA01

Physica Scripta, Vol. T99, 2002

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ETA05: Acta Phys.Slov. 56(2006) No 3

Acta Physica Slovaca 56(2006)

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Chiral Perturbation Theory

Degrees of freedom: Goldstone Bosons from Chiral Symmetry Spontaneous Breakdown (without η')

Power counting: Dimensional counting in momenta/masses

Expected breakdown scale: Resonances, so M_ρ or higher depending on the channel

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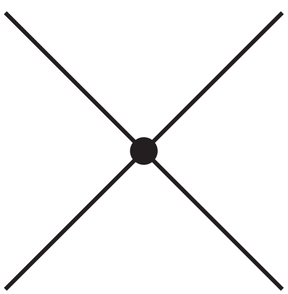
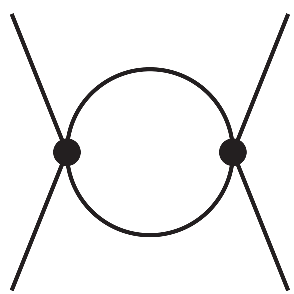
Power counting in momenta: Meson loops

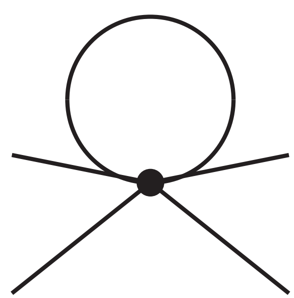


p^2



$(p^2)^2\,(1/p^2)^2\,p^4 = p^4$

$1/p^2$



$(p^2)\,(1/p^2)\,p^4 = p^4$

$\int d^4p$ $\quad$ p^4

Lagrangians

$U(\phi) = \exp(i\sqrt{2}\Phi/F_0)$ parametrizes Goldstone Bosons

$$\Phi(x) = \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & -\frac{2\,\eta_8}{\sqrt{6}} \end{pmatrix} .$$

LO Lagrangian: $\mathcal{L}_2 = \frac{F_0^2}{4}\{\langle D_\mu U^\dagger D^\mu U\rangle + \langle \chi^\dagger U + \chi U^\dagger\rangle\}\,,$

$D_\mu U = \partial_\mu U - i r_\mu U + i U l_\mu\,,$
left and right external currents: $r(l)_\mu = v_\mu + (-) a_\mu$

Scalar and pseudoscalar external densities: $\chi = 2B_0(s + ip)$
quark masses via scalar density: $s = \mathcal{M} + \cdots$

$\langle A\rangle = Tr_F\,(A)$

Lagrangians

$$\mathcal{L}_4 = L_1\langle D_\mu U^\dagger D^\mu U\rangle^2 + L_2\langle D_\mu U^\dagger D_\nu U\rangle\langle D^\mu U^\dagger D^\nu U\rangle$$
$$+L_3\langle D^\mu U^\dagger D_\mu U D^\nu U^\dagger D_\nu U\rangle + L_4\langle D^\mu U^\dagger D_\mu U\rangle\langle \chi^\dagger U + \chi U^\dagger\rangle$$
$$+L_5\langle D^\mu U^\dagger D_\mu U(\chi^\dagger U + U^\dagger \chi)\rangle + L_6\langle \chi^\dagger U + \chi U^\dagger\rangle^2$$
$$+L_7\langle \chi^\dagger U - \chi U^\dagger\rangle^2 + L_8\langle \chi^\dagger U \chi^\dagger U + \chi U^\dagger \chi U^\dagger\rangle$$
$$-iL_9\langle F^R_{\mu\nu} D^\mu U D^\nu U^\dagger + F^L_{\mu\nu} D^\mu U^\dagger D^\nu U\rangle$$
$$+L_{10}\langle U^\dagger F^R_{\mu\nu} U F^{L\mu\nu}\rangle + H_1\langle F^R_{\mu\nu} F^{R\mu\nu} + F^L_{\mu\nu} F^{L\mu\nu}\rangle + H_2\langle \chi^\dagger \chi\rangle$$

L_i: Low-energy-constants (LECs)
H_i: Values depend on definition of currents/densities

These absorb the divergences of loop diagrams: $L_i \to L_i^r$
Renormalization: order by order in the powercounting

Lagrangians

Lagrangian Structure:

	2 flavour		3 flavour		3+3 PQChPT	
p^2	F, B	2	F_0, B_0	2	F_0, B_0	2
p^4	l_i^r, h_i^r	7+3	L_i^r, H_i^r	10+2	$\hat{L}_i^r, \hat{H}_i^r$	11+2
p^6	c_i^r	52+4	C_i^r	90+4	K_i^r	112+3

p^2: Weinberg 1966

p^4: Gasser, Leutwyler 84,85

p^6: JB, Colangelo, Ecker 99,00

- replica method $\Longrightarrow$ PQ obtained from N_F flavour
- All infinities known
- 3 flavour special case of 3+3 PQ: $\hat{L}_i^r, K_i^r \rightarrow L_i^r, C_i^r$
- 53 $\rightarrow$ 52 arXiv:0705.0576 [hep-ph]

Chiral Logarithms

The main predictions of ChPT:

- Relates processes with different numbers of pseudoscalars
- Chiral logarithms

$$m_\pi^2 = 2B\hat{m} + \left(\frac{2B\hat{m}}{F}\right)^2 \left[\frac{1}{32\pi^2}\log\frac{(2B\hat{m})}{\mu^2} + 2l_3^r(\mu)\right] + \cdots$$

$M^2 = 2B\hat{m}$
$B \neq B_0$, $F \neq F_0$ (two versus three-flavour)

LECs and μ

$l_3^r(\mu)$

$$\bar{l}_i = \frac{32\pi^2}{\gamma_i}\, l_i^r(\mu) - \log\frac{M_\pi^2}{\mu^2}\,.$$

Independent of the scale μ.

For 3 and more flavours, some of the $\gamma_i = 0$: $L_i^r(\mu)$

μ :

- m_π, m_K: chiral logs vanish
- pick larger scale
- 1 GeV then $L_5^r(\mu) \approx 0$ large N_c arguments????
- compromise: $\mu = m_\rho = 0.77$ GeV

Expand in what quantities?

- Expansion is in momenta and masses
- But is not unique: relations between masses (Gell-Mann–Okubo) exists
- Express orders in terms of physical masses and quantities (F_π, F_K)?
- Express orders in terms of lowest order masses?
- E.g. $s+t+u = 2m_\pi^2 + 2m_K^2$ in πK scattering
- Relative sizes of order p^2, p^2, p^4,... can vary considerably

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- Relative sizes of order p^2, p^2, p^4,... can vary considerably
- I prefer physical masses
- Thresholds correct
- Chiral logs are from physical particles propagating

LECs

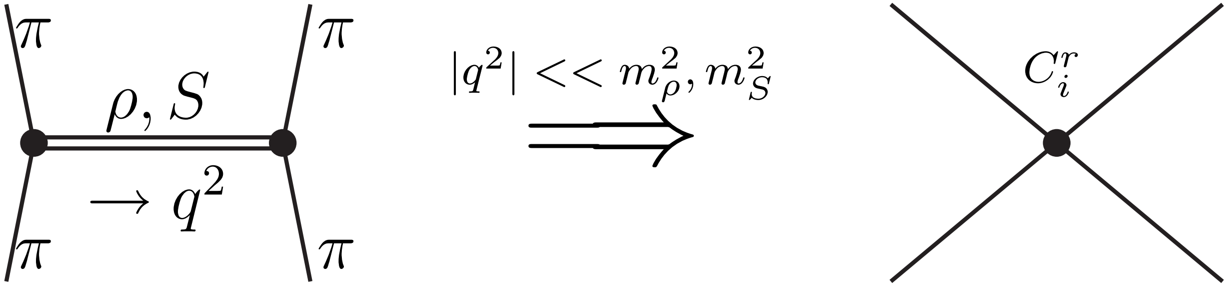
Some combinations of order p^6 LECs are known as well: curvature of the scalar and vector formfactor, two more combinations from $\pi\pi$ scattering (implicit in b_5 and b_6)

General observation:

- Obtainable from kinematical dependences: known
- Only via quark-mass dependence: poorely known

C_i^r

Most analysis use:
C_i^r from (single) resonance approximation



Motivated by large N_c: large effort goes in this

Ananthanarayan, JB, Cirigliano, Donoghue, Ecker, Gamiz, Golterman, Kaiser, Knecht, Peris, Pich, Prades, Portoles, de Rafael,. . .

C_i^r

$$\mathcal{L}_V = -\frac{1}{4}\langle V_{\mu\nu}V^{\mu\nu}\rangle + \frac{1}{2}m_V^2\langle V_\mu V^\mu\rangle - \frac{f_V}{2\sqrt{2}}\langle V_{\mu\nu}f_+^{\mu\nu}\rangle$$
$$-\frac{ig_V}{2\sqrt{2}}\langle V_{\mu\nu}[u^\mu,u^\nu]\rangle + f_\chi\langle V_\mu[u^\mu,\chi_-]\rangle$$
$$\mathcal{L}_A = -\frac{1}{4}\langle A_{\mu\nu}A^{\mu\nu}\rangle + \frac{1}{2}m_A^2\langle A_\mu A^\mu\rangle - \frac{f_A}{2\sqrt{2}}\langle A_{\mu\nu}f_-^{\mu\nu}\rangle$$
$$\mathcal{L}_S = \frac{1}{2}\langle\nabla^\mu S\nabla_\mu S - M_S^2S^2\rangle + c_d\langle Su^\mu u_\mu\rangle + c_m\langle S\chi_+\rangle$$
$$\mathcal{L}_{\eta'} = \frac{1}{2}\partial_\mu P_1\partial^\mu P_1 - \frac{1}{2}M_{\eta'}^2P_1^2 + i\tilde{d}_m P_1\langle\chi_-\rangle\,.$$

$f_V = 0.20$, $f_\chi = -0.025$, $g_V = 0.09$, $c_m = 42$ MeV, $c_d = 32$ MeV, $\tilde{d}_m = 20$ MeV,

$m_V = m_\rho = 0.77$ GeV, $m_A = m_{a_1} = 1.23$ GeV, $m_S = 0.98$ GeV, $m_{P_1} = 0.958$ GeV

f_V, g_V, f_χ, f_A: experiment

c_m and c_d from resonance saturation at $\mathcal{O}(p^4)$

C_i^r

Problems:

- Weakest point in the numerics
- However not all results presented depend on this
- Unknown so far: C_i^r in the masses/decay constants and how these effects correlate into the rest
- No μ dependence: obviously only estimate

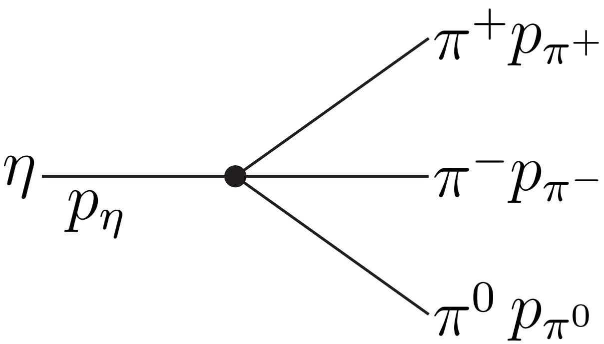
What we do/did about it:

- Vary resonance estimate by factor of two
- Vary the scale μ at which it applies: 600-900 MeV
- Check the estimates for the measured ones
- Again: kinematic can be had, quark-mass dependence difficult

$\eta \to 3\pi$

Reviews: JB, Gasser, Phys.Scripta T99(2002)34 [hep-ph/0202242]

JB, Acta Phys. Slov. 56(2005)305 [hep-ph/0511076]



$$\begin{aligned} s &= (p_{\pi^+} + p_{\pi^-})^2 = (p_\eta - p_{\pi^0})^2 \\ t &= (p_{\pi^-} + p_{\pi^0})^2 = (p_\eta - p_{\pi^+})^2 \\ u &= (p_{\pi^+} + p_{\pi^0})^2 = (p_\eta - p_{\pi^-})^2 \end{aligned}$$

$$s + t + u = m_\eta^2 + 2m_{\pi^+}^2 + m_{\pi^0}^2 \equiv 3s_0 \,.$$

$$\langle \pi^0 \pi^+ \pi^- \mathbf{out} | \eta \rangle = i\,(2\pi)^4\; \delta^4\left(p_\eta - p_{\pi^+} - p_{\pi^-} - p_{\pi^0}\right)\; A(s,t,u)\,.$$

$$\langle \pi^0 \pi^0 \pi^0 \mathbf{out} | \eta \rangle = i\,(2\pi)^4\; \delta^4\left(p_\eta - p_1 - p_2 - p_3\right)\; \overline{A}(s_1,s_2,s_3)$$

$$\overline{A}(s_1,s_2,s_3) = A(s_1,s_2,s_3) + A(s_2,s_3,s_1) + A(s_3,s_1,s_2)\,,$$

$\eta \to 3\pi$: Lowest order (LO)

Pions are in $I = 1$ state $\Longrightarrow$ $A \sim (m_u - m_d)$ or α_{em}

- α_{em} effect is small (but large via $m_{\pi^+} - m_{\pi^0}$)
- $\eta \to \pi^+\pi^-\pi^0\gamma$ needs to be included directly

$\eta \to 3\pi$: Lowest order (LO)

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ChPT:Cronin 67: $A(s,t,u) = \frac{B_0(m_u - m_d)}{3\sqrt{3}F_\pi^2}\left\{1 + \frac{3(s - s_0)}{m_\eta^2 - m_\pi^2}\right\}$

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with $Q^2 \equiv \frac{m_s^2 - \hat{m}^2}{m_d^2 - m_u^2}$ or $R \equiv \frac{m_s - \hat{m}}{m_d - m_u}$ $\qquad \hat{m} = \frac{1}{2}(m_u + m_d)$

$$A(s,t,u) = \frac{1}{Q^2}\frac{m_K^2}{m_\pi^2}(m_\pi^2 - m_K^2)\,\frac{\mathcal{M}(s,t,u)}{3\sqrt{3}F_\pi^2}\,,$$

$$A(s,t,u) = \frac{\sqrt{3}}{4R}\,M(s,t,u)$$

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LO: $\mathcal{M}(s,t,u) = \frac{3s - 4m_\pi^2}{m_\eta^2 - m_\pi^2}$ $\qquad M(s,t,u) = \frac{1}{F_\pi^2}\left(\frac{4}{3}m_\pi^2 - s\right)$

$\eta \longrightarrow 3\pi$ beyond p^4: p^2 and p^4

$\Gamma\left(\eta \to 3\pi\right) \propto |A|^2 \propto Q^{-4}$ allows a PRECISE measurement

$\boxed{Q \approx 24}$ gives lowest order $\Gamma(\eta \to \pi^+\pi^-\pi^0) \approx 66$ eV .

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Other Source from $m_{K^+}^2 - m_{K^0}^2 \sim Q^{-2} \Longrightarrow \boxed{Q = 20.0 \pm 1.5}$

Lowest order prediction $\Gamma(\eta \to \pi^+\pi^-\pi^0) \approx 140$ eV .

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At order p^4 Gasser-Leutwyler 1985: $\dfrac{\int dLIPS|A_2 + A_4|^2}{\int dLIPS|A_2|^2} = 2.4\,,$

($LIPS$=Lorentz invariant phase-space)

Major source: large S-wave final state rescattering

Experiment: 295 ± 17 eV (PDG 2006)

$\eta \to 3\pi$ beyond p^4: Dispersive

Try to resum the S-wave rescattering:

Anisovich-Leutwyler (AL), Kambor,Wiesendanger,Wyler (KWW)

Different method but similar approximations

Up to p^8: No absorptive parts from $\ell \geq 2$

$\Longrightarrow M(s,t,u) =$

$$M_0(s) + (s-u)M_1(t) + (s-t)M_1(t) + M_2(t) + M_2(u) - \frac{2}{3}M_2(s)$$

M_I: “roughly” contributions with isospin 0,1,2

$\eta \to 3\pi$ beyond p^4: Dispersive

3 body dispersive: difficult: keep only 2 body cuts

start from $\pi\eta \to \pi\pi$ ($m_\eta^2 < 3m_\pi^2$) standard dispersive analysis

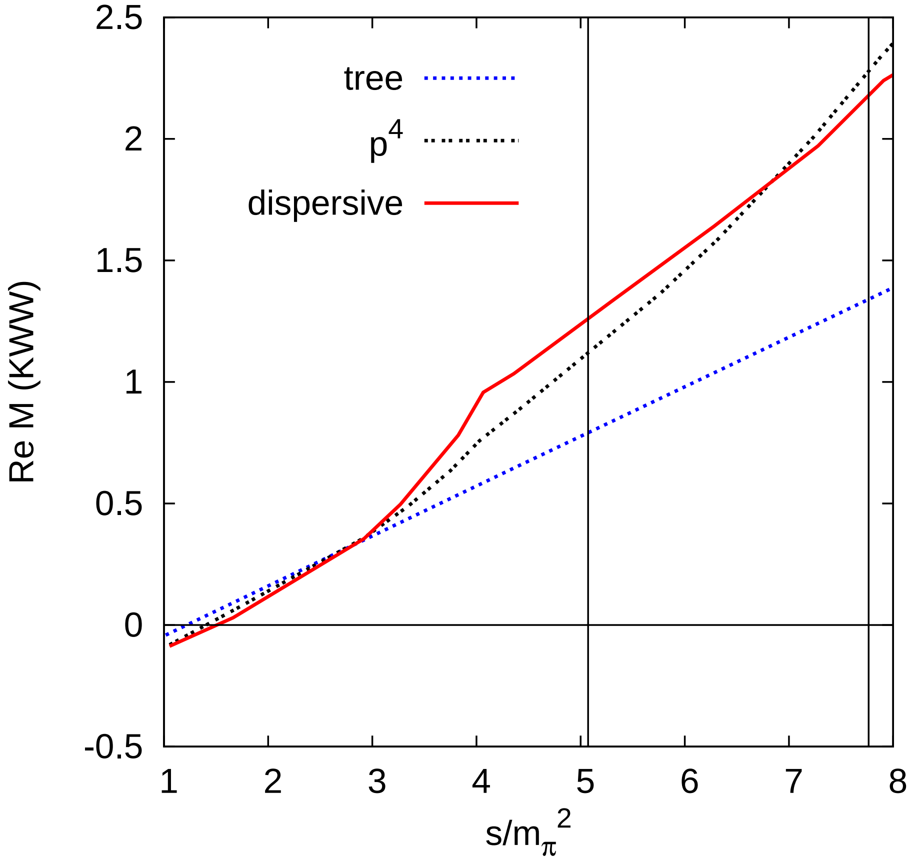
analytically continue to physical m_η^2.

$$M_I(s) = \frac{1}{\pi}\int_{4m_\pi^2}^{\infty} ds' \,\frac{\mathrm{Im} M_I(s')}{s'-s-i\varepsilon}$$

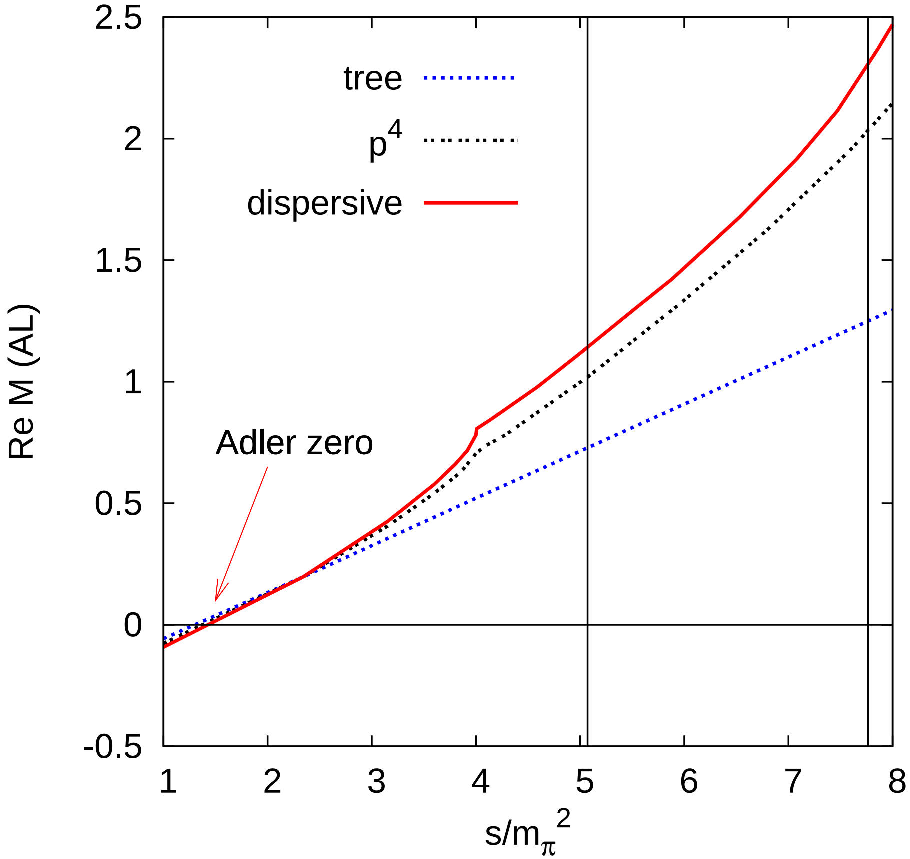
$$\mathrm{Im} M_I(s') \longrightarrow \mathrm{disc} M_I(s) = \frac{1}{2i}\left(M_I(s+i\varepsilon) - M_I(s-i\varepsilon)\right)$$

$$\begin{aligned}
M_0(s) &= a_0 + b_0 s + c_0 s^2 + \frac{s^3}{\pi}\int \frac{ds'}{s'^3}\,\frac{\mathrm{disc} M_0(s')}{s'-s-i\varepsilon}\,,\\
M_1(s) &= a_1 + b_1 s + \frac{s^2}{\pi}\int \frac{ds'}{s'^2}\,\frac{\mathrm{disc} M_1(s')}{s'-s-i\varepsilon}\,,\\
M_2(s) &= a_2 + b_2 s + c_2 s^2 + \frac{s^3}{\pi}\int \frac{ds'}{s'^3}\,\frac{\mathrm{disc} M_2(s')}{s'-s-i\varepsilon}\,.
\end{aligned}$$

$\eta \to 3\pi$ beyond p^4



Along $s = u$ KWW



Along $s = u$ AL

Two Loop Calculation: why

- In $K_{\ell 4}$ dispersive gave about half of p^6 in amplitude
- Same order in ChPT as masses for consistency check on m_u/m_d
- Check size of 3 pion dispersive part
- At order p^4 unitarity about half of correction
- Technology exists:
  - Two-loops: Amorós,JB,Dhonte,Talavera,. . .
  - Dealing with the mixing π^0-η: Amorós,JB,Dhonte,Talavera 01

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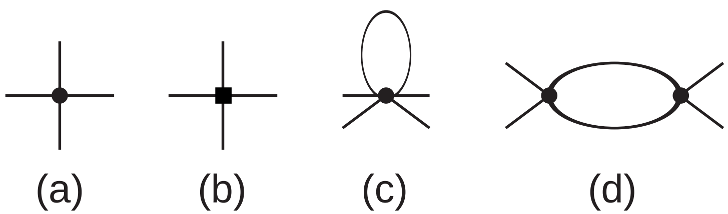
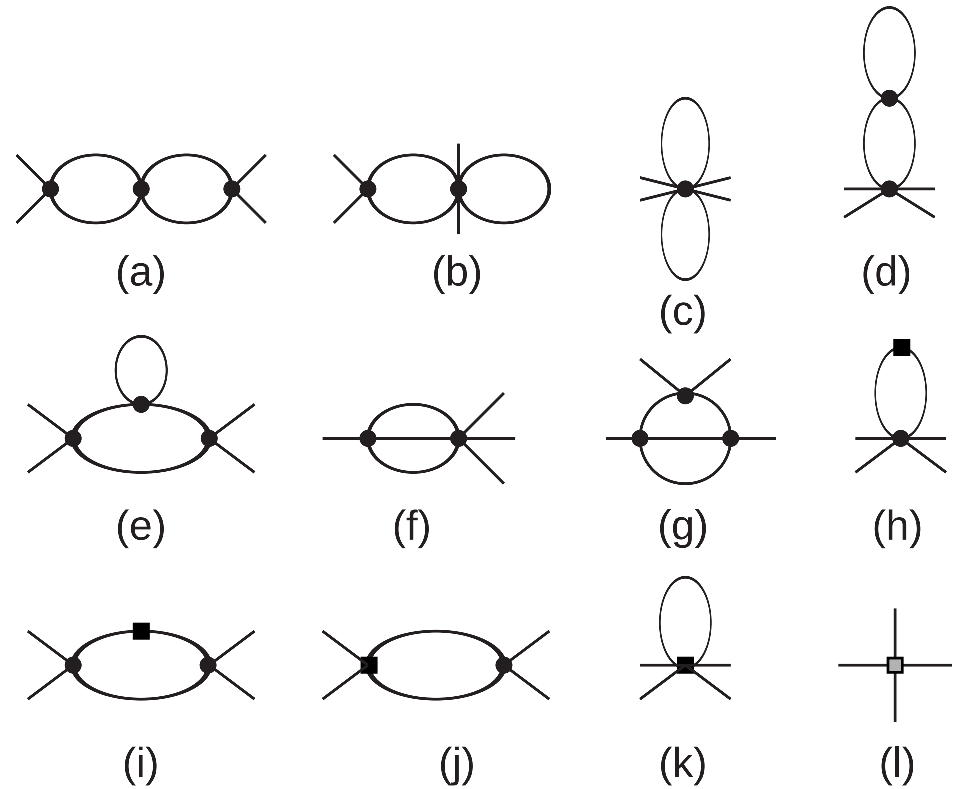
- Done: JB, Ghorbani, arXiv:0709.0230 [hep-ph]
  - Dealing with the mixing π^0-η: extended to $\eta \to 3\pi$

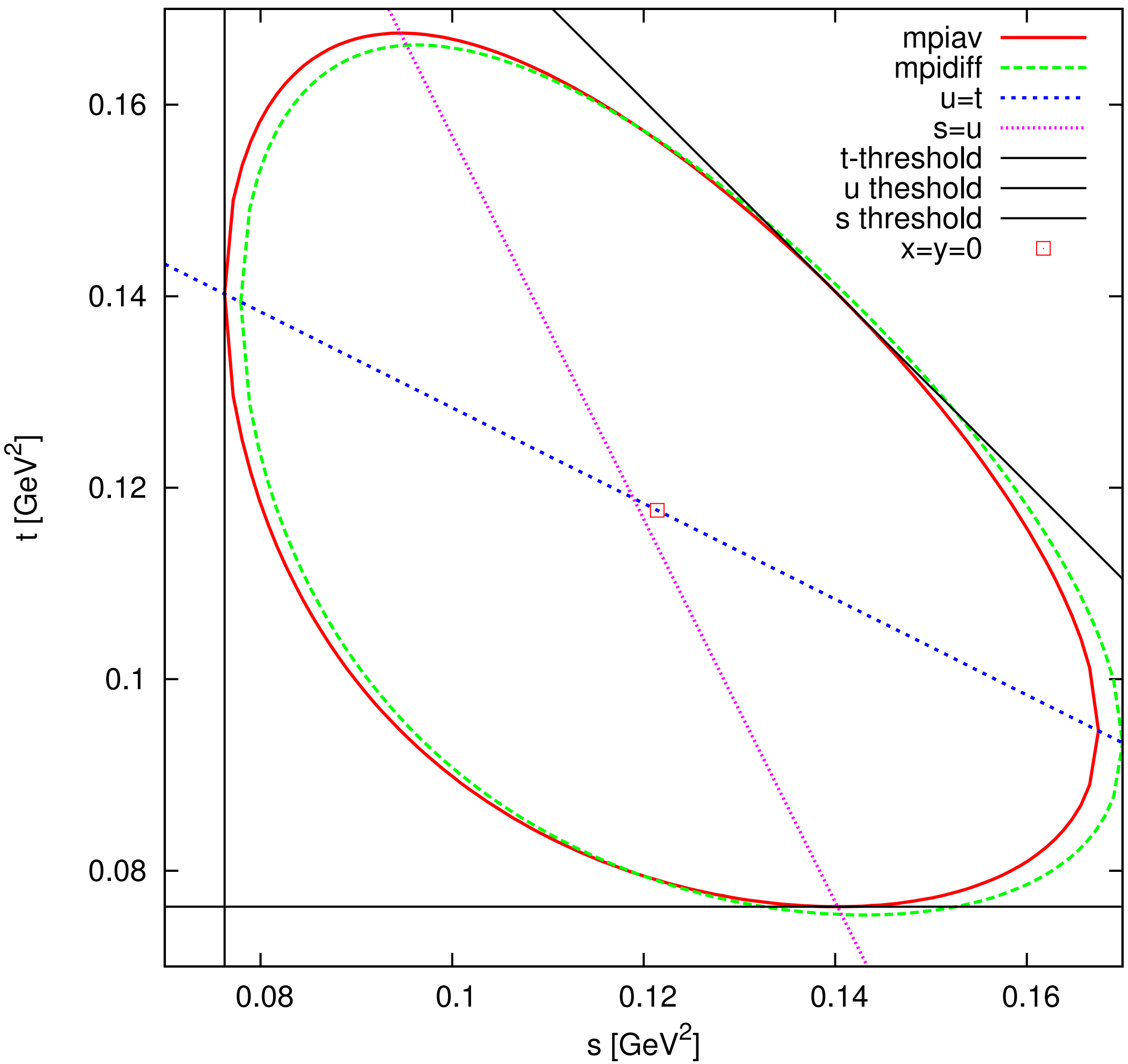
Diagrams





- Include mixing, renormalize, pull out factor $\frac{\sqrt{3}}{4R}$, ...
- Two independent calculations (comparison major amount of work)

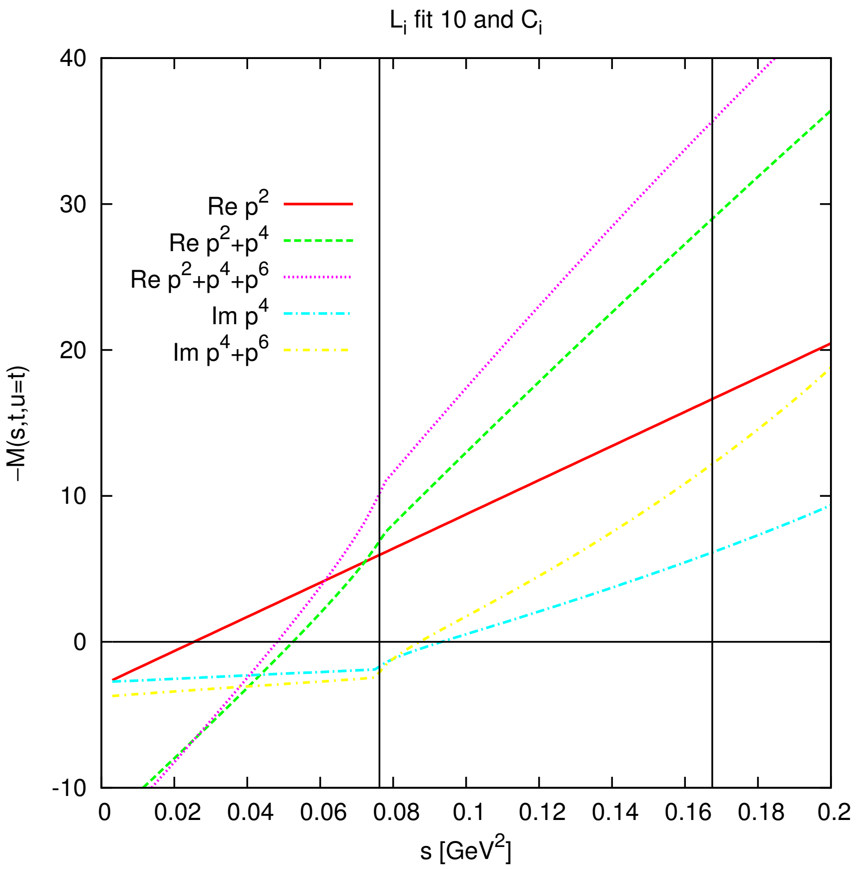
Dalitzplot



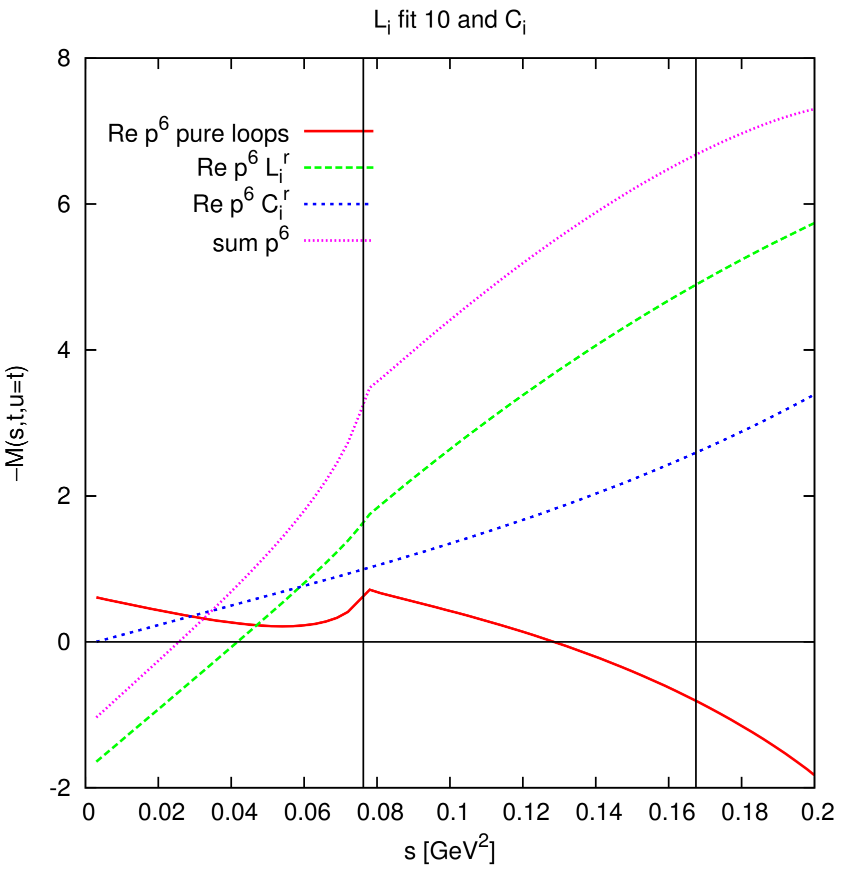
x variation: vertical

y variation: parallel to $t = u$

$\eta \to 3\pi$: $M(s, t = u)$

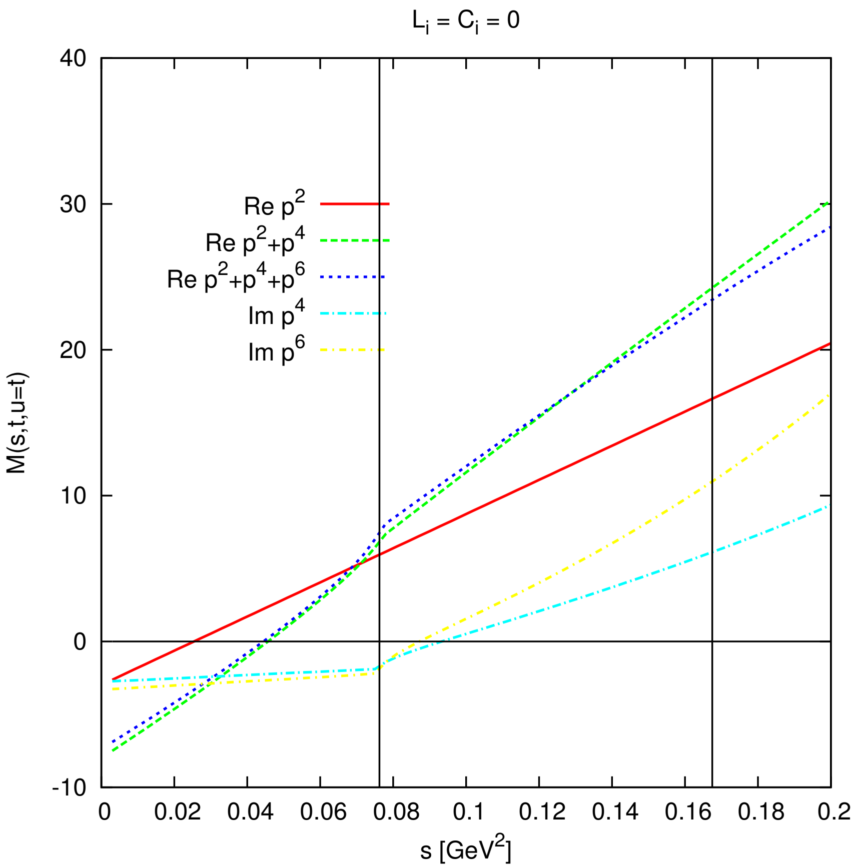


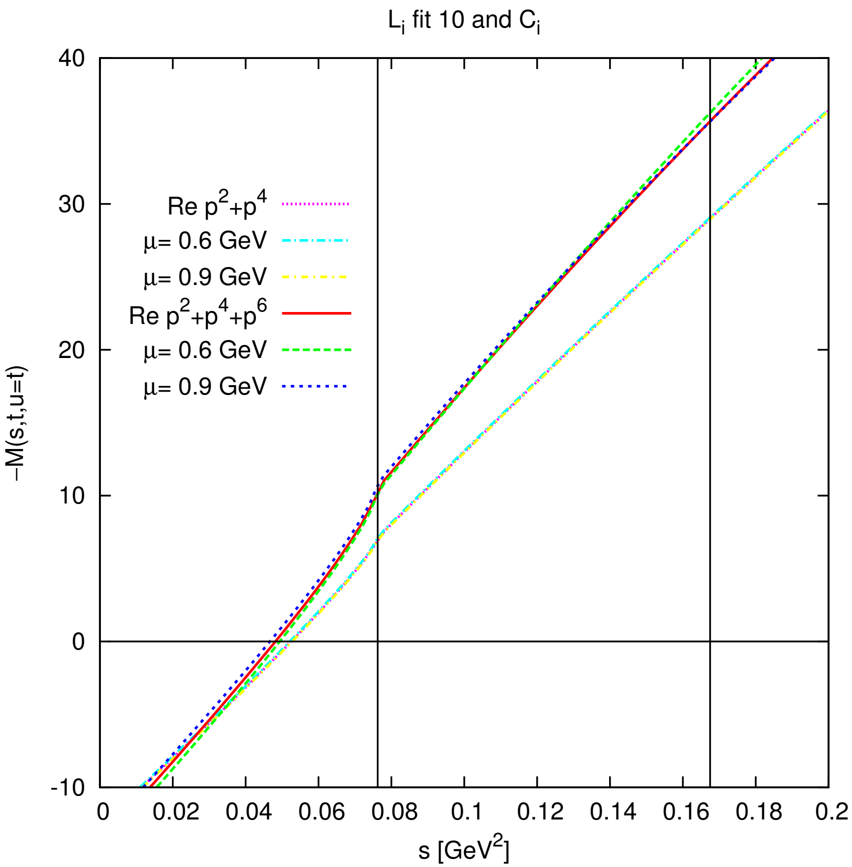
Along $t = u$



Along $t = u$ parts

$\eta \to 3\pi$: $M(s,t=u)$





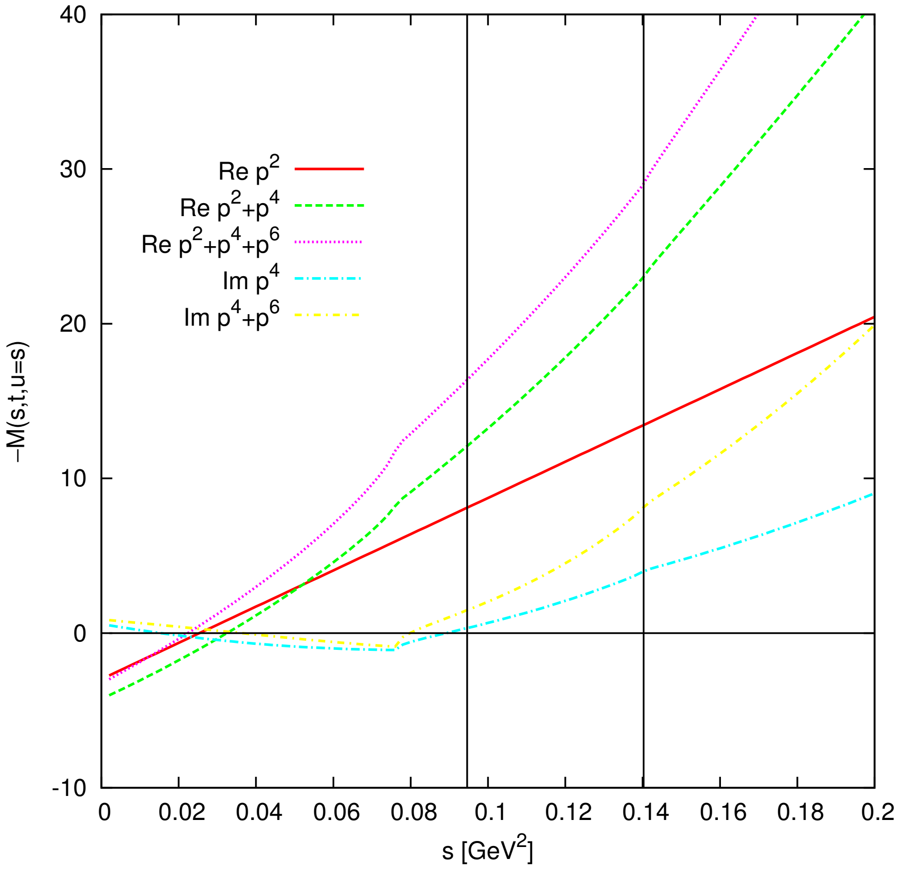
Along $t = u$

$L_i^r = C_i^r = 0$

Along $t = u$: μ dependence

I.e. where $C_i^r(\mu)$ estimated

$\eta \to 3\pi$: $M(s = u, t)$



Along $s = u$

Shape agrees with AL

Correction larger:
20-30% in amplitude

Dalitz plot

$$x = \sqrt{3}\frac{T_+ - T_-}{Q_\eta} = \frac{\sqrt{3}}{2m_\eta Q_\eta}(u-t)$$

$$y = \frac{3T_0}{Q_\eta} - 1 = \frac{3\left((m_\eta - m_{\pi^\circ})^2 - s\right)}{2m_\eta Q_\eta} - 1 \stackrel{\text{iso}}{=} \frac{3}{2m_\eta Q_\eta}(s_0 - s)$$

$$Q_\eta = m_\eta - 2m_{\pi^+} - m_{\pi^0}$$

T^i is the kinetic energy of pion π^i

$$z = \frac{2}{3}\sum_{i=1,3}\left(\frac{3E_i - m_\eta}{m_\eta - 3m_\pi^0}\right)^2$$

E_i is the energy of pion π^i

$$|M|^2 = A_0^2\left(1 + ay + by^2 + dx^2 + fy^3 + gx^2y + \cdots\right)$$

$$|\overline{M}|^2 = \overline{A}_0^2\left(1 + 2\alpha z + \cdots\right)$$

Experiment: charged

Exp.	a	b	d
KLOE	$-1.090 \pm 0.005^{+0.008}_{-0.019}$	$0.124 \pm 0.006 \pm 0.010$	$0.057 \pm 0.006^{+0.007}_{-0.016}$
Crystal Barrel	-1.22 ± 0.07	0.22 ± 0.11	0.06 ± 0.04 (input)
Layter et al.	-1.08 ± 0.014	0.034 ± 0.027	0.046 ± 0.031
Gormley et al.	-1.17 ± 0.02	0.21 ± 0.03	0.06 ± 0.04

KLOE has: $f = 0.14 \pm 0.01 \pm 0.02$.

Crystal Barrel: d input, but a and b insensitive to d

Theory: charged

	A_0^2	a	b	d	f
LO	120	-1.039	0.270	0.000	0.000
NLO	314	-1.371	0.452	0.053	0.027
NLO ($L_i^r = 0$)	235	-1.263	0.407	0.050	0.015
NNLO	538	-1.271	0.394	0.055	0.025
NNLOp (y from T^0)	574	-1.229	0.366	0.052	0.023
NNLOq (incl $(x,y)^4$)	535	-1.257	0.397	0.076	0.004
NNLO ($\mu = 0.6$ GeV)	543	-1.300	0.415	0.055	0.024
NNLO ($\mu = 0.9$ GeV)	548	-1.241	0.374	0.054	0.025
NNLO ($C_i^r = 0$)	465	-1.297	0.404	0.058	0.032
NNLO ($L_i^r = C_i^r = 0$)	251	-1.241	0.424	0.050	0.007
dispersive (KWW)	—	-1.33	0.26	0.10	—
tree dispersive	—	-1.10	0.33	0.001	—
absolute dispersive	—	-1.21	0.33	0.04	—
error	18	0.075	0.102	0.057	0.160

NLO to NNLO: Little change

Error on $|M(s,t,u)|^2$:

$|M^{(6)} M(s,t,u)|$

Experiment: neutral

Exp.	α
KLOE 2007	$-0.027 \pm 0.004^{+0.004}_{-0.006}$
KLOE (prel)	$-0.014 \pm 0.005 \pm 0.004$
Crystal Ball	-0.031 ± 0.004
WASA/CELSIUS	$-0.026 \pm 0.010 \pm 0.010$
Crystal Barrel	$-0.052 \pm 0.017 \pm 0.010$
GAMS2000	-0.022 ± 0.023
SND	$-0.010 \pm 0.021 \pm 0.010$

	$\overline{A}_0^2$	α
LO	1090	0.000
NLO	2810	0.013
NLO ($L_i^r = 0$)	2100	0.016
NNLO	4790	0.013
NNLOq	4790	0.014
NNLO ($C_i^r = 0$)	4140	0.011
NNLO ($L_i^r = C_i^r = 0$)	2220	0.016
dispersive (KWW)	—	$-(0.007$—$0.014)$
tree dispersive	—	-0.0065
absolute dispersive	—	-0.007
Borasoy	—	-0.031
error	160	0.032

Note: NNLO ChPT gets a_0^0 in $\pi\pi$ correct

α is difficult

Expand amplitudes and isospin:

$$M(s,t,u) = A\left(1+\tilde{a}(s-s_0)+\tilde{b}(s-s_0)^2+\tilde{d}(u-t)^2+\cdots\right)$$

$$\overline{M}(s,t,u) = A\left(3+\left(\tilde{b}+3\tilde{d}\right)\left((s-s_0)^2+(t-s_0)^2+(u-s_0)^2\right)+$$

Gives relations ($R_\eta = (2m_\eta Q_\eta)/3$)

$$a = -2R_\eta\,\mathrm{Re}(\tilde{a})\,, \quad b = R_\eta^2\left(|\tilde{a}|^2+2\mathrm{Re}(\tilde{b})\right)\,, \quad d = 6R_\eta^2\,\mathrm{Re}(\tilde{d})\,.$$

$$\alpha = \frac{1}{2}R_\eta^2\,\mathrm{Re}\left(\tilde{b}+3\tilde{d}\right) = \frac{1}{4}\left(d+b-R_\eta^2|\tilde{a}|^2\right) \le \frac{1}{4}\left(d+b-\frac{1}{4}a^2\right)$$

equality if Im$(\tilde{a}) = 0$

Large cancellation in α, overestimate of b likely the problem

r and decay rates

$$\sin\epsilon = \frac{\sqrt{3}}{4R} + \mathcal{O}(\epsilon^2)$$

$$\begin{aligned}
\Gamma(\eta \to \pi^+\pi^-\pi^0) &= \sin^2\epsilon \cdot 0.572 \text{ MeV} && \text{LO}\,,\\
& \sin^2\epsilon \cdot 1.59 \text{ MeV} && \text{NLO}\,,\\
& \sin^2\epsilon \cdot 2.68 \text{ MeV} && \text{NNLO}\,,\\
& \sin^2\epsilon \cdot 2.33 \text{ MeV} && \text{NNLO}\, C_i^r = 0\,,\\
\Gamma(\eta \to \pi^0\pi^0\pi^0) &= \sin^2\epsilon \cdot 0.884 \text{ MeV} && \text{LO}\,,\\
& \sin^2\epsilon \cdot 2.31 \text{ MeV} && \text{NLO}\,,\\
& \sin^2\epsilon \cdot 3.94 \text{ MeV} && \text{NNLO}\,,\\
& \sin^2\epsilon \cdot 3.40 \text{ MeV} && \text{NNLO}\, C_i^r = 0\,.
\end{aligned}$$

r and decay rates

$$r \equiv \frac{\Gamma(\eta \to \pi^0\pi^0\pi^0)}{\Gamma(\eta \to \pi^+\pi^-\pi^0)}$$

$$\begin{aligned} r_{\rm LO} &= 1.54 \\ r_{\rm NLO} &= 1.46 \\ r_{\rm NNLO} &= 1.47 \\ r_{{\rm NNLO}\, C_i^r=0} &= 1.46 \end{aligned}$$

PDG 2006

$$\begin{aligned} r &= 1.49 \pm 0.06 \quad \text{our average}\,. \\ r &= 1.43 \pm 0.04 \quad \text{our fit}\,, \end{aligned}$$

Good agreement

R and Q

	LO	NLO	NNLO	NNLO $(C_i^r = 0)$
R (η)	19.1	31.8	42.2	38.7
R (Dashen)	44	44	37	—
R (Dashen-violation)	36	37	32	—
Q (η)	15.6	20.1	23.2	22.2
Q (Dashen)	24	24	22	—
Q (Dashen-violation)	22	22	20	—

LO from $R = \dfrac{m^2_{K^0} + m^2_{K^+} - 2m^2_{\pi^0}}{2\left(m^2_{K^0} - m^2_{K^+}\right)}$ (QCD part only)

NLO and NNLO from masses: Amorós, JB, Talavera 2001

$Q^2 = \dfrac{m_s + \hat{m}}{2\hat{m}} R = 12.7R \qquad (m_s/\hat{m} = 24.4)$

Conclusions

- Precision physics: Q or R
- $NNLO$-calculation of $\eta \to 3\pi$ performed
- Some puzzling differences vs NLO+dispersive
- R from meson masses vs from $\eta \to 3\pi$