Progress on analytical expression of $K\ell_3$ form factors at two loop order

David Greynat[a]

In collaboration with

Roberto Bonciani[a], Christoph Haefeli[a], Roland Kaiser[b],
Antonio Pich[a] and Eduardo de Rafael[b]

a Institut de Física Corpuscular – Valencia
b Centre de Physique Théorique – Marseilles

FlaviaNet Meeting – Orsay
14th - 16th November

Outline

Definitions of the form factors

J. Bijnens and P. Talavera hep-ph/0303103, Nucl. Phys. B669 (2003) 341-362

In the chiral conventions

$$U \doteq \exp\left(\frac{i\sqrt{2}}{F_0}\Phi\right) \qquad \Phi \doteq \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & K^0 \\ K^- & \overline{K^0} & -\frac{2\eta}{\sqrt{6}} \end{pmatrix}$$

We consider here the form factors defined as

$$\left\langle K^+(p) \mid \bar{u}\gamma_\mu s \mid \pi^0(q) \right\rangle \doteq \frac{1}{\sqrt{2}}\left[(p_\mu + q_\mu) f_+^{K\pi}(t) + (p_\mu - q_\mu) f_-^{K\pi}(t)\right] \qquad (1)$$

and

$$f_0^{K\pi}(t) = f_+^{K\pi}(t) + \frac{t}{m_K^2 - m_\pi^2} f_-^{K\pi}(t)$$

with $t \doteq (q-p)^2$.
Specially, we are considering the limit

$$\lim_{t\to 0} f_0^{K\pi}(t) \doteq f_0^{K\pi} = f_+^{K\pi}(0) \qquad (2)$$

Those coefficients are strongly connected to V_{us}.

J. Bijnens' web page http ://www.thep.lu.se/ bijnens/chpt.html

The chiral expansion of $f^{K\pi}$ is given by

$$f = \underbrace{f^{(2)}}_{\doteq 1} + f^{(4)} + f^{(6)} \quad (3)$$

From now, we are taking the two loops expression for $f^{K\pi}$ provided by J. Bijnens.

It involves scalar two loop D-dimensional integrals

$$V \doteq \int \frac{d^D k_1}{(2\pi)^D} \frac{d^D k_2}{(2\pi)^D} \frac{\text{Num.}}{[k_1^2 - m_1^2]\,[(k_1-q)^2 - m_2^2]\,[(k_1+k_2-p)^2 - m_4^2]} \quad (4)$$

at the end 396 scalars integrals !

Necessity of a method to obtain a complete analytical expression.

J. Bijnens' web page http ://www.thep.lu.se/ bijnens/chpt.html

The chiral expansion of $f^{K\pi}$ is given by

$$f = \underbrace{f^{(2)}}_{\doteq 1} + f^{(4)} + f^{(6)} \tag{3}$$

From now, we are taking the two loops expression for $f^{K\pi}$ provided by J. Bijnens.
It involves scalar two loop D-dimensional integrals

$$V \doteq \int \frac{d^D k_1}{(2\pi)^D} \frac{d^D k_2}{(2\pi)^D} \frac{\text{Num.}}{\left[k_1^2 - m_1^2\right]\left[(k_1 - q)^2 - m_2^2\right]\left[(k_1 + k_2 - p)^2 - m_4^2\right]} \tag{4}$$

at the end 396 scalars integrals !

Necessity of a method to obtain a complete analytical expression.

J. Bijnens' web page http ://www.thep.lu.se/ bijnens/chpt.html

The chiral expansion of $f^{K\pi}$ is given by

$$f = \underbrace{f^{(2)}}_{\doteq 1} + f^{(4)} + f^{(6)} \tag{3}$$

From now, we are taking the two loops expression for $f^{K\pi}$ provided by J. Bijnens.
It involves scalar two loop D-dimensional integrals

$$V \doteq \int \frac{d^D k_1}{(2\pi)^D} \frac{d^D k_2}{(2\pi)^D} \frac{\text{Num.}}{\left[k_1^2 - m_1^2\right] \left[(k_1 - q)^2 - m_2^2\right] \left[(k_1 + k_2 - p)^2 - m_4^2\right]} \tag{4}$$

at the end 396 scalars integrals !

Necessity of a method to obtain a complete analytical expression.

To reduce the number of integrals to calculate we propose the following algorithm :

1. Use the Laporta's Algorithm to reduce the number of integral to a minimal set of Master Integrals.
2. Use the inverse multi-dimensional Converse Mapping theorem to evaluate the analytical expressions of the unknown Master Integrals.
3. Give the analytical expression of the form factors.

To reduce the number of integrals to calculate we propose the following algorithm :

1. Use the Laporta's Algorithm to reduce the number of integral to a minimal set of Master Integrals.
2. Use the inverse multi-dimensional Converse Mapping theorem to evaluate the analytical expressions of the unknown Master Integrals.
3. Give the analytical expression of the form factors.

To reduce the number of integrals to calculate we propose the following algorithm :

1. Use the Laporta's Algorithm to reduce the number of integral to a minimal set of Master Integrals.
2. Use the inverse multi-dimensional Converse Mapping theorem to evaluate the analytical expressions of the unknown Master Integrals.
3. Give the analytical expression of the form factors.

Laporta's Algorithm

S.Laporta, Int. J. Mod. Phys. A 15 (2000) 5087

T. Gehrmann and E. Remiddi, Nucl. Phys. B 580 (2000) 485

R. Bonciani, P. Mastrolia and E. Remiddi, Nucl. Phys. B 661 (2003) 289

Every two loops amplitudes obey to the following form

$$\mathcal{I} = \int \frac{d^D k_1}{(2\pi)^D} \frac{d^D k_2}{(2\pi)^D} \frac{S_1^{n_1} \cdots S_N^{n_N}}{D_1^{\ell_1} \cdots D_L^{\ell_L}}$$

for S scalar products and D denominators.

1. Use the Integrate by parts relation (Stokes' theorem)

$$\int \frac{d^D k_1}{(2\pi)^D} \frac{d^D k_2}{(2\pi)^D} \frac{\partial}{\partial k_j^\mu} \left[v^\mu \frac{S_1^{n_1} \cdots S_N^{n_N}}{D_1^{\ell_1} \cdots D_L^{\ell_L}} \right] = 0$$

   for $v = k_1, k_2, p, q$.

2. Use Lorentz' invariance and discrete symmetries

Laporta's Algorithm

S.Laporta, Int. J. Mod. Phys. A 15 (2000) 5087

T. Gehrmann and E. Remiddi, Nucl. Phys. B 580 (2000) 485

R. Bonciani, P. Mastrolia and E. Remiddi, Nucl. Phys. B 661 (2003) 289

Every two loops amplitudes obey to the following form

$$\mathcal{I} = \int \frac{d^D k_1}{(2\pi)^D} \frac{d^D k_2}{(2\pi)^D} \frac{S_1^{n_1} \cdots S_N^{n_N}}{D_1^{\ell_1} \cdots D_L^{\ell_L}}$$

for S scalar products and D denominators.

1. Use the Integrate by parts relation (Stokes' theorem)

$$\int \frac{d^D k_1}{(2\pi)^D} \frac{d^D k_2}{(2\pi)^D} \frac{\partial}{\partial k_j^\mu} \left[v^\mu \frac{S_1^{n_1} \cdots S_N^{n_N}}{D_1^{\ell_1} \cdots D_L^{\ell_L}} \right] = 0$$

   for $v = k_1, k_2, p, q$.

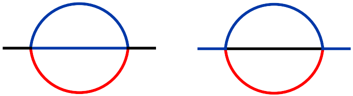
2. Use Lorentz' invariance and discrete symmetries

After the 2 points of the Laporta's Algorithm we obtain a linear system where every amplitudes are linked together.

Finally, using a recursive method and over constraining the generated system, every integrals can be deduced from a small set of Master's integrals.

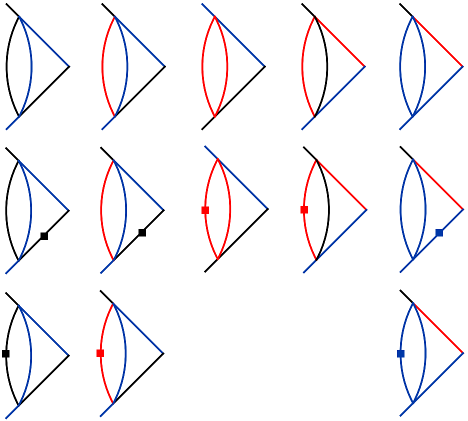
Here, after applying this algorithm, the only analytically unknown topologies of Master Integrals are

- Two points functions :



where m_π, m_K and m_η.

- Three points functions :



m_π, m_K and m_η.

One dimensional Mellin's Transform and *Converse Mapping theorem*

The Mellin's transform of a function f and its inverse transform are defined as

$$\mathcal{M}[f(x)](s) \doteq \int_0^{\infty} dx\, x^{s-1} f(x) \quad \longleftrightarrow \quad f(x) = \int_{c-i\infty}^{c+i\infty} \frac{ds}{2i\pi}\, x^{-s} \mathcal{M}[f(x)](s)$$

If and only if

$c \doteq \mathrm{Re}\, s \in]\alpha, \beta[$ written $\langle \alpha, \beta \rangle$ *Fundamental strip*

It corresponds to the behaviours

$$f(x) \underset{x \to 0^+}{=} \mathcal{O}(x^{-\alpha}) \quad \& \quad f(x) \underset{x \to +\infty}{=} \mathcal{O}(x^{-\beta})$$

Examples :

f		$\mathcal{M}[f]$	
e^{-x}	$\longleftrightarrow$	$\Gamma(s)$	$\langle 0, \infty \rangle$
$(1+x)^{-\nu}$	$\longleftrightarrow$	$\dfrac{\Gamma(\nu - s)\Gamma(s)}{\Gamma(\nu)}$	$\langle 0, \mathrm{Re}\, \nu \rangle$
$\ln(1+x)$	$\longleftrightarrow$	$\dfrac{\pi}{s \sin \pi s}$	$\langle -1, 0 \rangle$

One dimensional Mellin's Transform and *Converse Mapping theorem*

The Mellin's transform of a function f and its inverse transform are defined as

$$\mathcal{M}[f(x)](s) \doteq \int_0^\infty dx\, x^{s-1} f(x) \quad \longleftrightarrow \quad f(x) = \int_{c-i\infty}^{c+i\infty} \frac{ds}{2i\pi}\, x^{-s} \mathcal{M}[f(x)](s)$$

If and only if

$$c \doteq \mathrm{Re}\, s \in]\alpha, \beta[\qquad \text{written} \qquad \langle \alpha, \beta \rangle \qquad \textit{Fundamental strip}$$

It corresponds to the behaviours

$$f(x) \underset{x \to 0^+}{=} \mathcal{O}(x^{-\alpha}) \qquad \& \qquad f(x) \underset{x \to +\infty}{=} \mathcal{O}(x^{-\beta})$$

Examples :

f		$\mathcal{M}[f]$	
e^{-x}	$\longleftrightarrow$	$\Gamma(s)$	$\langle 0, \infty \rangle$
$(1+x)^{-\nu}$	$\longleftrightarrow$	$\dfrac{\Gamma(\nu - s)\Gamma(s)}{\Gamma(\nu)}$	$\langle 0, \mathrm{Re}\,\nu \rangle$
$\ln(1+x)$	$\longleftrightarrow$	$\dfrac{\pi}{s \sin \pi s}$	$\langle -1, 0 \rangle$

Converse Mapping theorem

Flajolet et al. (1994)

Friot, Greynat and de Rafael (2005)

Idea : The singularities in the complex Mellin's plan manage completely the asymptotic behaviour of the associated function

Exemple :

$$\Gamma(s) \asymp \sum_{p=0}^{\infty} \frac{(-1)^p}{p!} \frac{1}{s+p} \qquad \longleftrightarrow \qquad e^{-x} \underset{x\to 0}{\sim} \sum_{p=0}^{\infty} \frac{(-1)^p}{p!} x^p$$

Converse Mapping Theorem

$$\mathcal{M}\left[f(x)\right]_{\mathrm{right}}(s) \asymp \sum_{p>\beta,n} c_{p,n} \frac{1}{(s-p)^n} \quad \leftrightarrow \quad f(x) \underset{x\to+\infty}{\sim} -\sum_{p>\beta,n} c_{n,p}\, x^{-p} \frac{(-1)^{n-1}}{(n-1)!} \ln^{n-1} x$$

$$\mathcal{M}\left[f(x)\right]_{\mathrm{left}}(s) \asymp \sum_{p<\alpha,n} d_{p,n} \frac{1}{(s+p)^n} \quad \leftrightarrow \quad f(x) \underset{x\to 0}{\sim} \sum_{p<\alpha,n} d_{n,p}\, x^{p} \frac{(-1)^{n-1}}{(n-1)!} \ln^{n-1} x$$

Converse Mapping theorem

Flajolet et al. (1994)

Friot, Greynat and de Rafael (2005)

Idea : The singularities in the complex Mellin's plan manage completely the asymptotic behaviour of the associated function

Exemple :

$$\Gamma(s) \asymp \sum_{p=0}^{\infty} \frac{(-1)^p}{p!} \frac{1}{s+p} \qquad \longleftrightarrow \qquad e^{-x} \underset{x \to 0}{\sim} \sum_{p=0}^{\infty} \frac{(-1)^p}{p!} x^p$$

Converse Mapping Theorem

$$\mathcal{M}\left[f(x)\right]_{\rm right}(s) \asymp \sum_{p>\beta,n} c_{p,n} \frac{1}{(s-p)^n} \quad \leftrightarrow \quad f(x) \underset{x \to +\infty}{\sim} -\sum_{p>\beta,n} c_{n,p}\, x^{-p} \frac{(-1)^{n-1}}{(n-1)!} \ln^{n-1} x$$

$$\mathcal{M}\left[f(x)\right]_{\rm left}(s) \asymp \sum_{p<\alpha,n} d_{p,n} \frac{1}{(s+p)^n} \quad \leftrightarrow \quad f(x) \underset{x \to 0}{\sim} \sum_{p<\alpha,n} d_{n,p}\, x^{p} \frac{(-1)^{n-1}}{(n-1)!} \ln^{n-1} x$$

Converse Mapping theorem

Flajolet et al. (1994)

Friot, Greynat and de Rafael (2005)

Idea : The singularities in the complex Mellin's plan manage completely the asymptotic behaviour of the associated function

Exemple :

$$\Gamma(s) \asymp \sum_{p=0}^{\infty} \frac{(-1)^p}{p!} \frac{1}{s+p} \qquad \longleftrightarrow \qquad \mathrm{e}^{-x} \underset{x \to 0}{\sim} \sum_{p=0}^{\infty} \frac{(-1)^p}{p!} x^p$$

Converse Mapping Theorem

$$\mathcal{M}\left[f(x)\right]_{\mathrm{right}}(s) \asymp \sum_{p>\beta,n} c_{p,n} \frac{1}{(s-p)^n} \leftrightarrow f(x) \underset{x \to +\infty}{\sim} -\sum_{p>\beta,n} c_{n,p}\, x^{-p} \frac{(-1)^{n-1}}{(n-1)!} \ln^{n-1} x$$

$$\mathcal{M}\left[f(x)\right]_{\mathrm{left}}(s) \asymp \sum_{p<\alpha,n} d_{p,n} \frac{1}{(s+p)^n} \leftrightarrow f(x) \underset{x \to 0}{\sim} \sum_{p<\alpha,n} d_{n,p}\, x^{p} \frac{(-1)^{n-1}}{(n-1)!} \ln^{n-1} x$$

Converse Mapping theorem

Flajolet et al. (1994)

Friot, Greynat and de Rafael (2005)

Idea : The singularities in the complex Mellin's plan manage completely the asymptotic behaviour of the associated function

Exemple :

$$\Gamma(s) \asymp \sum_{p=0}^{\infty} \frac{(-1)^p}{p!} \frac{1}{s+p} \qquad \longleftrightarrow \qquad \mathrm{e}^{-x} \underset{x \to 0}{\sim} \sum_{p=0}^{\infty} \frac{(-1)^p}{p!} x^p$$

Converse Mapping Theorem

$$\mathcal{M}\left[f(x)\right]_{\text{right}}(s) \asymp \sum_{p>\beta,n} c_{p,n} \frac{1}{(s-p)^n} \quad \leftrightarrow \quad f(x) \underset{x \to +\infty}{\sim} -\sum_{p>\beta,n} c_{n,p}\, x^{-p} \frac{(-1)^{n-1}}{(n-1)!} \ln^{n-1} x$$

$$\mathcal{M}\left[f(x)\right]_{\text{left}}(s) \asymp \sum_{p<\alpha,n} d_{p,n} \frac{1}{(s+p)^n} \quad \leftrightarrow \quad f(x) \underset{x \to 0}{\sim} \sum_{p<\alpha,n} d_{n,p}\, x^{p} \frac{(-1)^{n-1}}{(n-1)!} \ln^{n-1} x$$

Converse Mapping theorem

Flajolet et al. (1994)

Friot, Greynat and de Rafael (2005)

Idea : The singularities in the complex Mellin's plan manage completely the asymptotic behaviour of the associated function

Exemple :

$$\Gamma(s) \asymp \sum_{p=0}^{\infty} \frac{(-1)^p}{p!} \frac{1}{s+p} \qquad \longleftrightarrow \qquad \mathrm{e}^{-x} \underset{x\to 0}{\sim} \sum_{p=0}^{\infty} \frac{(-1)^p}{p!} x^p$$

Converse Mapping Theorem

$$\mathcal{M}\left[f(x)\right]_{\mathrm{right}}(s) \asymp \sum_{p>\beta,n} c_{p,n} \frac{1}{(s-p)^n} \quad \leftrightarrow \quad f(x) \underset{x\to+\infty}{\sim} -\sum_{p>\beta,n} c_{n,p}\, x^{-p} \frac{(-1)^{n-1}}{(n-1)!} \ln^{n-1} x$$

$$\mathcal{M}\left[f(x)\right]_{\mathrm{left}}(s) \asymp \sum_{p<\alpha,n} d_{p,n} \frac{1}{(s+p)^n} \quad \leftrightarrow \quad f(x) \underset{x\to 0}{\sim} \sum_{p<\alpha,n} d_{n,p}\, x^{p} \frac{(-1)^{n-1}}{(n-1)!} \ln^{n-1} x$$

Multi-dimensional Mellin's Transform and Grothendieck's Residues theory

We define the *n*-dimensional Mellin's transform of function f as

$$\mathcal{M}[f](s_1,\dots,s_n) \doteq \int_0^\infty dx_1 \cdots \int_0^\infty dx_n \;\; x_1^{s_1-1}\cdots x_n^{s_n-1}\, f(x_1,\dots,x_n)$$

and its inverse transformation

$$f(x_1,\dots,x_n) \doteq \int\limits_{c_1+i\mathbb{R}} \frac{ds_1}{2i\pi}\cdots \int\limits_{c_n+i\mathbb{R}} \frac{ds_n}{2i\pi} \;\; x_1^{-s_1}\cdots x_n^{-s_n}\, \mathcal{M}[f](s_1,\dots,s_n)$$

This inversion formula is of course valid in the fundamental polyhedra defined as all the constraints on $\mathbf{c} \doteq {}^T(c_1,\dots,c_n)$ where the Mellin's transform is completely analytical.

If we want to extend the *Converse Mapping* Theorem to the multi-dimensional case we need to introduce "briefly" the Grothendieck's Residue theory.

A few words on Grothendieck's Residues theory

P. Griffiths, J.Harris, *Principles of Algebraic Geometry*, Wyley NYC 1978

A.K. Tsikh *et al.*, hep-th 9609215

N.B. : From now, all vectors in n-dimension are written as $\mathbf{s} = {}^{T}(s_1, \ldots, s_n)$

One way to see the residues in multi-dimensional complex analysis is to consider the quantity (for any h completely analytic)

$$\text{Res.}\left[\frac{h(\mathbf{s})}{f_1(\mathbf{s})\cdots f_n(\mathbf{s})}\right]_{\mathbf{0}} = \oint_{\mathbf{0}} \frac{h(\mathbf{s})}{f_1(\mathbf{s})\cdots f_n(\mathbf{s})} \frac{ds_1}{2i\pi} \wedge \cdots \wedge \frac{ds_n}{2i\pi} \doteq \oint_{\mathbf{0}} \omega$$

All the curves, the divisors, in the 2n-dimension complex space given by – $j \in [\![1, n]\!]$

$$D_j \doteq \left\{\mathbf{s} \in \mathbb{C}^n, f_j(\mathbf{s}) = 0\right\}$$

have intersections points in this space. They provide the calculation of the residue in a summation over

$$\bigcap_{j \in [\![1,n]\!]} D_j$$

via a sequential Cauchy's theorem.

Multi-dimensional Converse Mapping Theorem

J.-Ph. Aguilar, D. Greynat and E. de Rafael, Work in progress (2008)

Idea : If you combine the calculation of the Grothendieck's residues and the *multi-dimensional Jordan's lemma* you can define sectors in complex plans where the x_j are bigger or smaller than 1 and their relative position and permit to generate the complete asymptotic behaviour in each variables

In the case of ratios of Euler's second function : the Γ function, the *multi-dimensional Converse Mapping* theorem for

$$f(\mathbf{s}) = \int_{\gamma+i\mathbb{R}^n} x_1^{-s_1}\cdots x_n^{-s_n}\, \frac{\prod_{j=1}^{j=p}\Gamma(\mathbf{a}_j\cdot\mathbf{s}+b_j)}{\prod_{k=1}^{k=q}\Gamma(\mathbf{c}_k\cdot\mathbf{s}+d_k)}\, \frac{ds_1}{2i\pi}\wedge\cdots\wedge\frac{ds_n}{2i\pi}$$

the divisors are

$$D_j^{\ell} = \left\{\mathbf{s}\in\mathbb{C}^n,\ \mathbf{a}_j\cdot\mathbf{s}+b_j = -\ell\ ,\ \ell\in\mathbb{N}\right\}$$

J.-Ph. Aguilar, D. Greynat and E. de Rafael, Work in progress (2008)

The multi-dimensional Jordan's lemma provides a sub-set $\mathcal{J}$ of index j to permit the convergence of the series coming from the calculation of the Grothendieck's residue theorem. (we give here the theorem only in the case of simple non-degenerate poles)

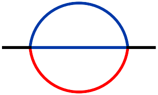
$$f(\mathbf{s}) = \sum_{j\in\mathcal{J}} \text{Res.} \left[\frac{\prod_{k=1}^{k=q} \Gamma(\mathbf{c}_k\cdot\mathbf{s}+d_k)^{-1}}{\prod_{j=1}^{j=p} \Gamma(\mathbf{a}_j\cdot\mathbf{s}+b_j)^{-1}} \right]_{\mathbf{s}\in\cap D_j}$$

$$= \sum_{j\in\mathcal{J}} \frac{(-1)^{|\boldsymbol{\ell}|}}{\boldsymbol{\ell}!\,\det(\mathbf{a}_j)} \frac{\prod\limits_{j\neq\mathcal{J}} \Gamma(\mathbf{a}_j\cdot\mathbf{s}_j^{\boldsymbol{\ell}}+b_j)}{\prod\limits_{k=1}^{k=q} \Gamma(\mathbf{c}_k\cdot\mathbf{s}_j^{\boldsymbol{\ell}}+d_k)} X_1^{-(\mathbf{s}_j^{\boldsymbol{\ell}})_1}\cdots X_n^{-(\mathbf{s}_j^{\boldsymbol{\ell}})_n}$$

Hopefully more clear on the following example...

An example of the calculation on the Master Integrals

We consider the sunrise type integral (m_π, m_K and m_η)



1. Feynman's parametrization $D = 4 - \epsilon$

$$H(m_\eta^2, m_K^2, m_K^2, m_\pi^2)$$
$$= -\frac{\pi^D}{(2\pi)^{2D}}\Gamma(\epsilon-1)\iint_{[0,1]^2} dxdy(1-x)^{\frac{3}{2}\epsilon-2}\left[1-Y+Yx\right]^{\frac{3}{2}\epsilon-3}$$
$$\times(1-Y)^{1-\epsilon}x^{1-\epsilon}(1-x)^{1-\epsilon}\left(m_\pi^2\right)^{1-\epsilon}$$
$$\times\left|1-\frac{1-Y+xY}{(1-x)(1-Y)}\rho_2-\frac{1-Y+xY}{x(1-Y)}\rho_1\right|^{1-\epsilon}$$

where $Y = 1 - y(1-y)$, $\rho_1 = m_K^2/m_\pi^2$ and $\rho_2 = m_\eta^2/m_\pi^2$.

2. Inverse Mellin's representation

Using the general functions inverse Mellin's representation ($c \in <0, \nu>$)

$$|1-x|^{-\nu}\,\text{sign}\,(1-x) = \int_{c+i\,\mathbb{R}} \frac{ds}{2i\pi}\, x^{-s}\Gamma(1-\nu)\left[\frac{\Gamma(s)}{\Gamma(s-\nu+1)} - \frac{\Gamma(\nu-s)}{\Gamma(1-s)}\right] ,$$

And using the polar coordinates we obtain the double inverse Mellin's representation of H :

$$H(m_\eta^2, m_K^2, m_K^2, m_\pi^2) = -\frac{\Gamma(\epsilon-1)\Gamma(1-\epsilon)}{(4\pi)^D}\left(\frac{m_\pi^2}{2}\right)^{1-\epsilon}\sqrt{\pi}$$

$$\times \int_{\mathbf{c}+i\,\mathbb{R}^2} \frac{ds_1\wedge ds_2}{(2i\pi)^2}\,(4\rho_1)^{-s_1}\,\rho_2^{-s_2}\mathrm{M}(s_1,s_2)$$

$$\times\left[\mathrm{h}(s_1,s_2) - \frac{\rho_1}{4}\mathrm{h}(s_1+1,s_2) - \frac{\rho_2}{4}\mathrm{h}(s_1,s_2+1)\right] ,$$

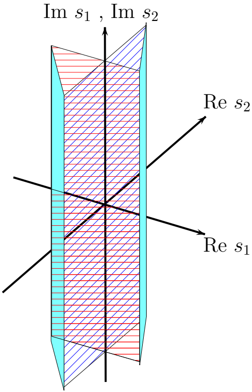
with

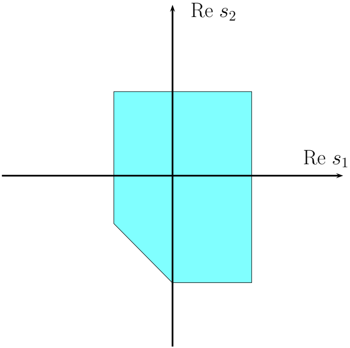
$$\mathrm{M}(s_1,s_2) = \frac{\Gamma(s_1)\Gamma(s_2)}{\Gamma(s_1+s_2)}\left[\frac{\Gamma(s_1+s_2)}{\Gamma(s_1+s_2-\epsilon+1)} - \frac{\Gamma(\epsilon-s_1-s_2)}{\Gamma(1-s_1-s_2)}\right]$$

and

$$\mathrm{h}(s_1,s_2) = \frac{\Gamma(2-\epsilon+s_1)\,\Gamma\left(1-\frac{\epsilon}{2}+s_2\right)\Gamma\left(1-\frac{\epsilon}{2}+s_1\right)}{\Gamma\left(3-\frac{3}{2}\epsilon+s_1+s_2\right)\Gamma\left(\frac{3}{2}-\frac{\epsilon}{2}+s_1\right)}$$

3. Fundamental polyhedra and pertinent sector





We obtain here 6 different 2-forms :

$$\omega_1 \doteq \frac{ds_1 \wedge ds_2}{(2i\pi)^2} \, (4\rho_1)^{-s_1} \, \rho_2^{-s_2} \, \Gamma\left[\begin{matrix} s_1\,, s_2\,, 2-\epsilon+s_1\,, 1-\frac{\epsilon}{2}+s_2\,, 1-\frac{\epsilon}{2}+s_1 \\ 1-\epsilon+s_1+s_2\,, 3-\frac{3}{2}\epsilon+s_1+s_2\,, \frac{3}{2}-\frac{\epsilon}{2}+s_1 \end{matrix}\right]$$

$$\omega_2 \doteq \frac{ds_1 \wedge ds_2}{(2i\pi)^2} \, (4\rho_1)^{-s_1} \, \rho_2^{-s_2} \, \Gamma\left[\begin{matrix} s_1\,, s_2\,, 3-\epsilon+s_1\,, 1-\frac{\epsilon}{2}+s_2\,, 2-\frac{\epsilon}{2}+s_1 \\ 1-\epsilon+s_1+s_2\,, 4-\frac{3}{2}\epsilon+s_1+s_2\,, \frac{5}{2}-\frac{\epsilon}{2}+s_1 \end{matrix}\right]$$

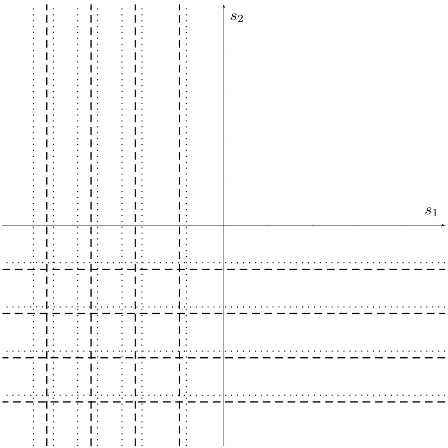
$$\omega_3 \doteq \frac{ds_1 \wedge ds_2}{(2i\pi)^2} \, (4\rho_1)^{-s_1} \, \rho_2^{-s_2} \, \Gamma\left[\begin{matrix} s_1\,, s_2\,, 2-\epsilon+s_1\,, 2-\frac{\epsilon}{2}+s_2\,, 1-\frac{\epsilon}{2}+s_1 \\ 1-\epsilon-s_1-s_2\,, 4-\frac{3}{2}\epsilon+s_1+s_2\,, \frac{3}{2}-\frac{\epsilon}{2}+s_1 \end{matrix}\right]$$

$$\omega_4 \doteq -\frac{ds_1 \wedge ds_2}{(2i\pi)^2} \, (4\rho_1)^{-s_1} \, \rho_2^{-s_2} \, \Gamma\left[\begin{matrix} s_1\,, s_2\,, \epsilon-s_1-s_2\,, 2-\epsilon+s_1\,, 1-\frac{\epsilon}{2}+s_2\,, 1-\frac{\epsilon}{2}+s_1 \\ s_1+s_2\,, 1-s_1-s_2\,, 3-\frac{3}{2}\epsilon+s_1+s_2\,, \frac{3}{2}-\frac{\epsilon}{2}+s_1 \end{matrix}\right]$$

$$\omega_5 \doteq -\frac{ds_1 \wedge ds_2}{(2i\pi)^2} \, (4\rho_1)^{-s_1} \, \rho_2^{-s_2} \, \Gamma\left[\begin{matrix} s_1\,, s_2\,, \epsilon-s_1-s_2\,, 3-\epsilon+s_1\,, 1-\frac{\epsilon}{2}+s_2\,, 2-\frac{\epsilon}{2}+s_1 \\ s_1+s_2\,, 1-s_1-s_2\,, 4-\frac{3}{2}\epsilon+s_1+s_2\,, \frac{5}{2}-\frac{\epsilon}{2}+s_1 \end{matrix}\right]$$

$$\omega_6 \doteq -\frac{ds_1 \wedge ds_2}{(2i\pi)^2} \, (4\rho_1)^{-s_1} \, \rho_2^{-s_2} \, \Gamma\left[\begin{matrix} s_1\,, s_2\,, \epsilon-s_1-s_2\,, 2-\epsilon+s_1\,, 2-\frac{\epsilon}{2}+s_2\,, 1-\frac{\epsilon}{2}+s_1 \\ s_1+s_2\,, 1-s_1-s_2\,, 4-\frac{3}{2}\epsilon+s_1+s_2\,, \frac{3}{2}-\frac{\epsilon}{2}+s_1 \end{matrix}\right]$$

The divisors are for example for ω_1 the following lines



Multi-dimensional Converse Mapping theorem implies to sum over intersections in the fourth quadrant.

Obtaining then the following representation

$$\int \omega_1$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(4\rho_1)^n}{n!}\frac{\rho_2^k}{k!}\left(\Gamma\left[\begin{matrix} 2-\epsilon-n\,,\,1-\frac{\epsilon}{2}-k\,,\,1-\frac{\epsilon}{2}-n \\ -n-k-\epsilon+2\,,\,3-\frac{3}{2}\epsilon-n-k\,,\,\frac{3}{2}-\frac{\epsilon}{2}-n\end{matrix}\right]\right.$$

$$\left. + \rho_2^{1-\frac{\epsilon}{2}}\Gamma\left[\begin{matrix} -k-1+\frac{\epsilon}{2}\,,\,2-\epsilon-n\,,\,1-\frac{\epsilon}{2}-n \\ \frac{3}{2}-\frac{\epsilon}{2}-n\,,\,2-\epsilon-n-k\,,\,1-\frac{\epsilon}{2}-n-k\end{matrix}\right] + 4\rho_1\Gamma\left[\begin{matrix} 1-\frac{\epsilon}{2}-k\,,\,1-\frac{\epsilon}{2}-n\,,\,-1+\frac{\epsilon}{2}-n \\ \frac{1}{2}-n\,,\,2-\epsilon-k-n\,,\,1-\frac{\epsilon}{2}-k-n\end{matrix}\right]\right)$$

We doing the same processus for all 6 2-forms ω_j... To obtain finally the following behaviour after an ϵ-expansion

$$\bar{H}(m_\eta^2, m_K^2, m_K^2, m_\pi^2) \sim \frac{1}{\epsilon^2}\left[-\frac{1}{8}\left(2m_K^2 + m_\eta^2\right)\right] + \cdots$$

in agreement with literature

Of course the epsilon-expansion and the cut of the infinite series are not obligatory and in this sense, we have an analytic expansion of the Master Integrals .

M. Caffo *et al.*, Nuovo Cimmento Vol. III, A, N 4 (1998)

CONCLUSIONS

A work in progress... closed to the end...