χPT in the mixed regime

based on the paper F.B. & P. Hernandez arXiv:0707.3887

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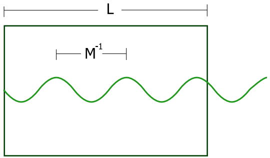
14/11/07

Matching Lattice QCD with χPT...

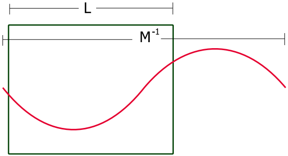
In a finite volume there are two regimes:

- p-regime $(m_q \Sigma V \gg 1,\ L\Lambda_\chi \gg 1)$



  - χPT predicts L,m_q scaling
    - m_q scaling relevant
    - finite size scaling $\sim e^{-LM_\pi}$
    - up to 12 LECs at NLO

- ϵ-regime $(m_q \Sigma V \lesssim 1,\ L\Lambda_\chi \gg 1)$ [Gasser & Leutwyler 1987]



  - χPT predicts L,m_q, ν scaling (ν is the topological sector)
    - L, ν scaling relevant
    - m_q scaling is usually less relevant
    - only F and Σ at NLO

Why consider the mixed regime?

It is necessary to consider the mixed regime: ie. some quarks in the p and some in the ϵ-regime, in the following situations:

1. Split the s and u/d quarks in full theory computations:

$$m_s \Sigma V \gg 1 \qquad m_{u,\,d} \Sigma V \lesssim 1$$

2. Split of valence and sea quarks in Partially Quenched simulations with mixed actions [Baer, Rupak, Shoresh]:

$$m_{sea} \Sigma V \gg 1 \qquad m_{valence} \Sigma V \lesssim 1$$

The ϵ-regime

The Zero Mode contribution to the propagator $(1/M_\pi^2 V)$ explodes as we lower masses keeping the volume fixed...

When $m_q \sim p^4 \sim L^{-4}$

- l-loop diagrams with zero modes circulating are not suppressed $((m_q V)^{-l} \sim 1)$
- l-loop diagrams with non zero modes circulating are suppressed $((V/L^2)^{-l} \sim L^{-2l})$

Solution: factorize zero modes $U = U_0 e^{\frac{2i\xi}{F}}$

- integrate U_0 over all $SU(N_f)$
- $\int dx^4 \xi(x) = 0$

[Gasser & Leutwyler 1987, Hansen 1990]

Non Perturbative integration

One has to consider integrals like ($U \equiv U_0 e^{-\frac{i\theta}{N}}$):

$$\langle f(U) \rangle_\theta = \int_{SU(N)} dU_0 \; f(U) \; \exp\left(\frac{\Sigma V}{2} \mathrm{Tr} \left[\mathcal{M} U + U^\dagger \mathcal{M} \right] \right)$$

Or fixing topology:

$$\langle f(U) \rangle_\nu = \int_{U(N)} dU \; f(U) (\det U)^\nu \; \exp\left(\frac{\Sigma V}{2} \mathrm{Tr} \left[\mathcal{M} U + U^\dagger \mathcal{M} \right] \right)$$

The solution for $f(U) = 1$ is known in term of Bessel function .
More general $f(U)$ are obtained from its functional derivatives.
[Brower et al 1982, Leutwyler & Smilga 1992, Jackson et al. 1996]

Set up of mixed regime 1

Start from:

$$\mathcal{M} = \underbrace{\begin{pmatrix} m_v & 0 \\ 0 & m_s \end{pmatrix}}_{N_v \quad N_s} \qquad m_s \sim L^{-2} \; m_v \sim L^{-4}$$

Then calculating the propagator one sees that some pions are light ($ML \sim 1$) and some others are heavy ($ML \gg 1$). It turns out that only the $SU(N_v)$ generators correspond to light pions.

$$M^2_{ab} = \frac{\Sigma}{F^2}(m_a + m_b) \qquad M^2_{vv} \sim L^{-4} \quad M^2_{vs} \sim M^2_{ss} \sim L^{-2}$$

Set up of mixed regime 2

The parametrization is defined by:

$$U = \begin{pmatrix} U_0 & 0 \\ 0 & Id \end{pmatrix} e^{\frac{2i\xi}{F}} \qquad U_0 \in SU(N_v)$$

$$\int dx^4 Tr[T^a \xi(x)] = 0 \qquad T^a \in SU(N_v)$$

This affects the propagator and the measure through the nontrivial Jacobian $J(\xi)$:

$$d\mu = d\mu_{Haar} d\xi J(\xi)$$

I will analyze the results for the left current correlator. External particles will be restricted to correspond to $SU(N_v)$ generators.

Left currents correlator in mixed-regime

The result coincides with the ϵ-regime one for N_v quarks with a renormalized F up to finite-size effects [F. B. & P. Hernandez 2007]:

$$\mathcal{C}^{\epsilon}(x_0) = \frac{\hat{F}^2}{2T}\left[1 - \frac{N_v}{F^2}\left(-\frac{\beta_1}{\sqrt{V}} + T^2\frac{k_{00}}{V}\right) + \frac{2T^2}{F^2 V}\mu_v\sigma_\nu(\mu_v)h_1(\frac{x_0}{T})\right] + O(e^{-M_{ss}L})$$

$$\hat{F}^2 = F^2\left[1 - \frac{N_s}{F^2}\left(G_\infty\left(0, \frac{M_{ss}^2}{2}\right) - 8L_4 M_{ss}^2\right)\right] \quad \sigma_\nu(\mu) \equiv \frac{1}{2N_v}\langle Tr[U+U^\dagger]\rangle_\nu^{U(N_v)}$$

Where $\mu_v = \Sigma V m_v$. This is in agreement with the Decoupling Theorem:[Appelquist & Carazzone 1975]. Since $M_{vv},\ L^{-1} \ll M_{ss}$ the s-quarks decouple leaving their inprint only in the dependence of the LECs on the heavy mass scale .

Q & PQ χPT

[Sharpe, Bernard & Goltermann 1990]

1. SUSY method to quench quarks (add quarks of opposite statistics):

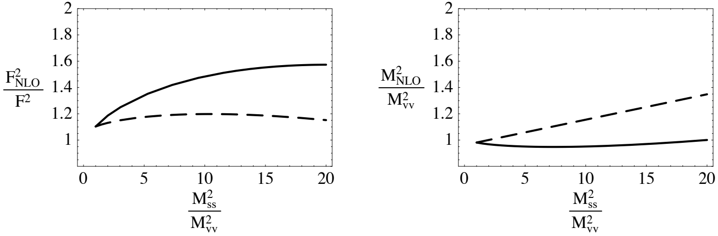
$$\mathcal{Z}_E[J] = \int [dA_\mu] \frac{[\det(\gamma_\mu D_\mu + m_s)]^{N_s} [\det(\gamma_\mu D_\mu + m_v + J)]^{N_v}}{[\det(\gamma_\mu D_\mu + m_v)]^{N_v}} e^{-S_g[A_\mu]}$$

- Perturbative χPT: $SU(N_s + N_v) \rightarrow SU(N_s + N_v | N_v)$
- NP integrations of ϵ-regime over Riemannian maximal submanifold of $\mathcal{G}l(N_s + N_v | N_v)$.

[Zirnbauer, Damgaard et al 1998]

2. Replica Method: $\lim_{N_v \to 0} \Pi_i \frac{\delta}{\delta J_i} \log \mathcal{Z}_E[J]$
   - Based on: Contributions coming from internal quark loops $\propto N_f$
   - Shown to be equivalent to SUSY to all orders in perturbation theory [Splittorff & Damgaard 1992]

m_s dependence of M_{NLO} and F in p-regime (PQ versus Full)



- Full lines: full theory with $m_s = m_v$ and $N_s + N_v = 2$. L_i are taken equal to typical values from [Gasser & Leutwyler 1984].
- Dashed lines: PQ theory with $N_s = 2$. Note the linear dependence on m_s of the pion squared mass.

What happens in the Partially Quenched theory? (Summary)

- in the p and ϵ-regimes it is possible to take the limit $N_v \to 0$ as long as $N_s \neq 0$
- in the mixed regime the limit $N_v \to 0$ is singular even if $N_s \neq 0$

What is the problem of PQ in the mixed regime

The pion η is the one associated to the generator:

$$T_\eta = \sqrt{\frac{N_v N_s}{2(N_s + N_v)}} diag\{\underbrace{\frac{1}{N_v}, \ldots, \frac{1}{N_v}}_{N_v}, \underbrace{-\frac{1}{N_s}, \ldots, -\frac{1}{N_s}}_{N_s}\}$$

- It has an N_v dependent mass: $M_\eta^2 = \frac{N_v \overbrace{M_{ss}^2}^{\sim L^{-2}} + N_s \overbrace{M_{vv}^2}^{\sim L^{-4}}}{N}$
- In the PQ theory ($N_v = 0$), $M_\eta^2 \sim M_{vv}^2$ and is therefore a light field $\Rightarrow$ his zero modes must be treated non perturbatively

Analogies with η' problem in Quenched theory

The η is a singlet of $SU(N_v)$ and plays effectively the role of the η' in the quenched theory: the mixed PQ chiral theory can be matched onto a quenched effective theory as $m_s \to \infty$ with:

$$\frac{\overbrace{\alpha}^{fieldstrength}}{2N_c} = \frac{1}{N_s} \qquad \frac{\overbrace{m_0^2}^{M^2_{\eta'}}}{2N_C} = \frac{M^2_{ss}}{N_s}$$

In contrast with the real η' in the fully quenched case, there are no new low-energy couplings!

Set up of mixed regime for PQ χPT

The solution is to factorize the zero modes of the η:

$$U = \begin{pmatrix} e^{i\frac{\eta}{N_v}} U_0 & 0 \\ 0 & e^{-i\frac{\eta}{N_s}} \end{pmatrix} e^{\frac{2i\xi}{F}} \qquad U_0 \in SU(N_v)$$

$$\int dx^4 Tr[T^a \xi(x)] = 0 \qquad T^a \in SU(N_v) \cup T_\eta$$

This affects the propagator and the measure through the nontrivial Jacobian $J(\xi)_{PQ}$:

$$d\mu = d\mu_{Haar} d\xi J(\xi)_{PQ}$$

Is there a problem with the η NP integration?

It is found that the perturbative and the NP modes couple through the mass term. However we have shown that they decouple at NLO and at fixed topological sector.

Let us absorb these parameters like that:

$$\mathcal{M} \equiv \begin{pmatrix} m_v & 0 \\ 0 & m_s e^{\frac{i\theta}{N_s}} \end{pmatrix} \qquad \overline{U}_0 \equiv U_0 e^{\frac{i\eta}{N_v}}$$

All this must be substituted in the $\mathcal{L}_\chi$ and...

$\mathcal{Z}_\nu$ in mixed regime for PQ at NLO

$$\mathcal{Z}_\nu = \int d\xi \int d\bar{\theta}\ e^{i\nu\bar{\theta}}\ e^{\cos\left(\frac{\bar{\theta}}{N_s}\right)(\mu_s + A(\xi))} \int_{U(N_v)} d\bar{U}_0 \ldots$$

$$A(\xi) \equiv -M_{ss}^2 \int d^4x Tr\left[P_s \xi^2\right] + \mathcal{O}(\epsilon^2) \qquad \overline{\theta} \equiv \theta - \eta$$

Since $\mu_s \sim L^2$ configurations with large $\overline{\theta}$ are suppressed and $\overline{\theta}$ can be treated perturbatively:

$$\overline{\theta} \sim L^{-1}$$

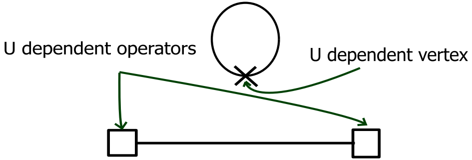
factorization of zero modes

The saddle point approximation leads to the factorization of zero and nonzero modes at NLO:

$$\mathcal{Z}_\nu \simeq C_\nu \int d\xi \; J(\xi)_{PQ} e^{-\int d^4x \left(Tr\left[\partial_\mu \xi(x) \partial_\mu \xi(x)\right] + M_{ss}^2 Tr\left[P_s \xi^2\right]\right)} \int_{U(N_v)} d\bar{U}_0 \det(\bar{U}_0)^\nu e^{\frac{\Sigma V}{2} Tr\left[P_v \left(M\bar{U}_0 + \bar{U}_0^\dagger M\right)\right]} \left(1 - \frac{A(\xi)}{2\mu_s} + B(\xi, \bar{U}_0)\right)$$

In the case of the left current correlator, these changes are irrelevant (the PQ result is just the $N_v \to 0$ limit of the full one).

- There are changes in correlation functions receiving the contribution of the η zero modes
- In the ϵ or mixed regimes "disconnected in ξ" graphs may not exponentiate due to the NP integration



- So easy examples are the vector, the axial, the scalar, the pseudoscalar correlator where graphs as above contribute renormalizing Σ

Vector and Axial correlators in mixed regime

$$\mathcal{V}_\pm(x_0) = \frac{\hat{F}^2}{2T}\left\{\mathcal{J}^0_\mp + \frac{N_v}{\hat{F}^2}\left(\frac{\beta_1}{\sqrt{V}}\mathcal{J}^0_\mp \mp T^2\frac{k_{00}}{V}\mathcal{J}^0_\pm\right) + \overbrace{\frac{4\mu_v\sigma_\nu(\mu_v)}{\hat{F}^2}\frac{T^2}{V}h_1\left(\frac{x_0}{T}\right)}^{\text{only in } \mathcal{V}_-}\right\} + O(e^{-M_{ss}L})$$

$$\mathcal{J}^0_\pm = 1 \mp \frac{1}{N_v^2-1} \pm \frac{N_v}{N_v^2-1}\left[\frac{d\sigma_\nu(\tilde{\mu_v})}{d\tilde{\mu_v}} + N_v\left(\sigma_\nu(\tilde{\mu_v})\right)^2 + \frac{1}{\tilde{\mu_v}}\sigma_\nu(\tilde{\mu_v}) - N_v\frac{\nu^2}{\tilde{\mu_v}^2}\right]$$

- The result coincides with the ϵ-regime one [Damgaard et al 2002] for N_v quarks with a renormalized F and up to FS effects of order $\sim e^{-M_{ss}L}$
- What changes between PQ and full theory parametrizations is the condensate at NLO $\tilde{\Sigma}$ ($\tilde{\mu_v} \equiv V\tilde{\Sigma}m_v$)
- I recall that taking the limit $N_v \to 0$ in σ_ν is not trivial. [Zirnbauer 1998]. However $\mathcal{J}^0_\pm \to 1 \pm 1$ if $N_v \to 0$.

Condensate at NLO in PQ and full theory

In the full theory we get:

$$\tilde{\mu} \equiv \mu \left(1 - \frac{1}{F^2}\left[N_v \overline{G}_V(0;0) + N_s G_V\left(0;\frac{M_{ss}^2}{2}\right) - \overline{E}(0;0,0) + \frac{N_s}{N_v^2 V M_{ss}^2} - 16 L_6 N_s M_{ss}^2\right]\right)$$

In the PQ theory we get:

$$\tilde{\mu}_{PQ} \equiv \mu \left(1 - \frac{1}{F^2}\left[N_v \overline{G}_V(0;0) + N_s G_V\left(0;\frac{M_{ss}^2}{2}\right) - \overline{E}_{PQ}(0;0,0) - 16 L_6 N_s M_{ss}^2\right]\right)$$

The difference is the contribution coming from the zero mode of the η.

m_s dependence of M_{NLO} in mixed regime

Also in the mixed regime M^2_{NLO} depends linearly on m_s in the PQ approximation:

$$\frac{M^2_{NLO-PQ}}{M_{LO}} = 1 + \frac{1}{F^2}\left(-\frac{1}{N_s}\frac{\beta_1}{L^2} + \frac{M^2_{ss}}{N_s}\left(\beta_2 + \frac{1}{8\pi^2}\left(\ln\frac{L}{L_0} - c_1\right)\right)\right.$$
$$\left. + 16L_6 N_s M^2_{ss} - 8L_4 M^2_{ss}\right)$$
$$\frac{M^2_{NLO}}{M_{LO}} = 1 + \frac{1}{F^2}\left(-\frac{1}{N_v}\frac{\beta_1}{L^2} - \frac{N_s}{NN_v} G\left(0; \frac{N_v M^2_{ss}}{N}\right) + 16L_6 N_s M^2_{ss} - 8L_4 M^2_{ss}\right)$$

Conclusions

We have studied χPT in a finite volume for non-degenerate masses such that N_v are in the ϵ-regime and N_s in the p-regime

- Full theory results match the ϵ regime ones for N_v quarks with effective low-energy couplings that depend on the heavier scale set by the N_s heavier quark masses (up to exponentially suppressed finite-size effects)
- The PQ theory, when all the light quarks are quenched, requires a different parametrization, because the η zero modes become non-perturbative
- The PQ results match the ϵ-regime ones for the quenched theory with effective low-energy couplings that depend on the mass and number of the quarks in the p-regime (up to exponentially suppressed finite-size effects)

Outlook

- Apply the results to analyze PQ correlation functions from lattice simulations with mixed actions (overlap on the valence and Wilson on the sea)
- Include discretization effects if necessary

Many thanks for your attention!

$\mathcal{Z}_\nu$ in mixed regime for PQ at NLO

$$\mathcal{Z}_\nu = \int d\xi \; J(\xi)_{PQ} e^{-\int d^4x Tr\left[\partial_\mu \xi(x) \partial_\mu \xi(x)\right]} \int d\bar{\theta} \; e^{i\nu\bar{\theta}} \; e^{\cos\left(\frac{\bar{\theta}}{N_s}\right)(\mu_s + A(\xi))}$$
$$\times \int_{U(N_v)} d\bar{U}_0 \det(\bar{U}_0)^\nu e^{\frac{\Sigma V}{2} Tr\left[P_v\left(\mathcal{M}^\dagger \bar{U}_0 + \bar{U}_0^\dagger \mathcal{M}\right)\right]} e^{B(\xi,\bar{U}_0)}$$

$$A(\xi) \equiv -M_{ss}^2 \int d^4x Tr\left[P_s \xi^2\right] + \mathcal{O}(\epsilon^2) \qquad \bar{\theta} \equiv \theta - \eta \qquad B(\xi, \bar{U}_0) \sim L^{-2}$$

Since $\mu_s \sim L^2$ configurations with large $\bar{\theta}$ are suppressed and $\bar{\theta}$ can be treated perturbatively:

$$\bar{\theta} \sim L^{-1}$$