INSTITUT D'ASTROPHYSIQUE DE PARIS
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Séminaire du Laboratoire de l'Accélérateur Linéaire

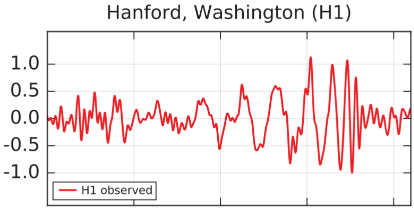
GRAVITATIONAL WAVES AND THE PROBLEM OF MOTION IN GR

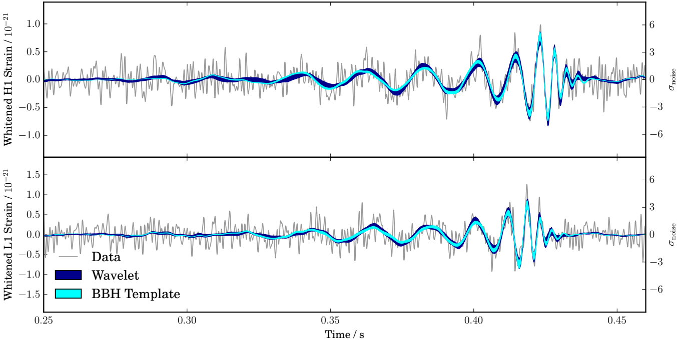
Luc Blanchet

Gravitation et Cosmologie ($\mathcal{G}\mathbb{R}\varepsilon\mathbb{C}\mathcal{O}$)
Institut d'Astrophysique de Paris

8 juillet 2016

Binary black-hole event GW150914 [LIGO/VIRGO collaboration 2016]





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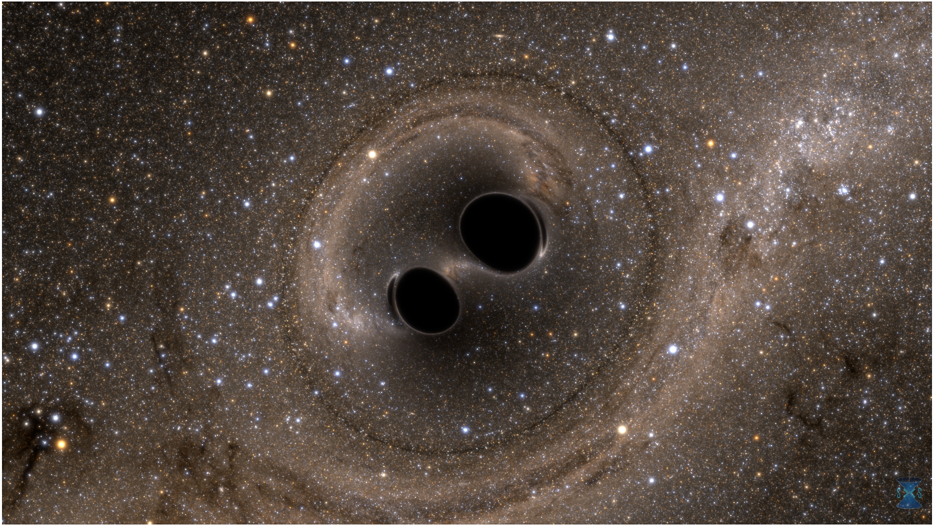

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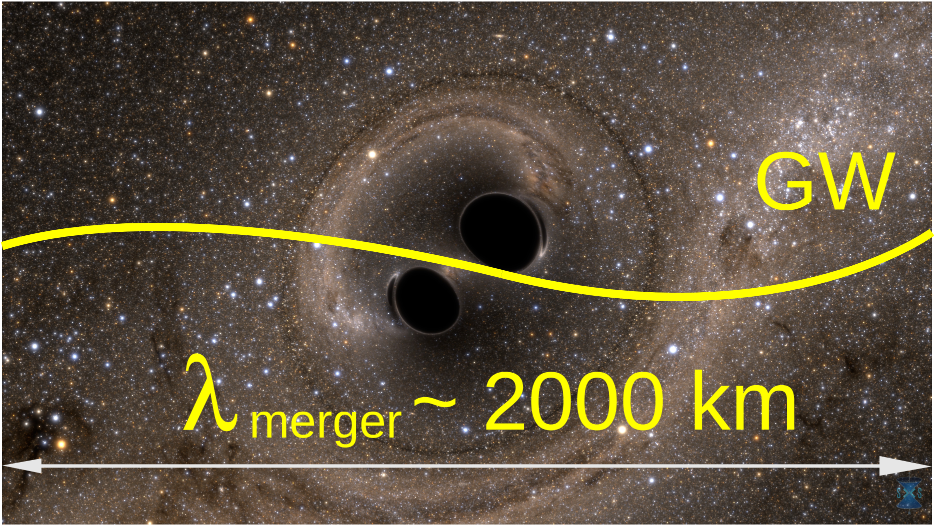
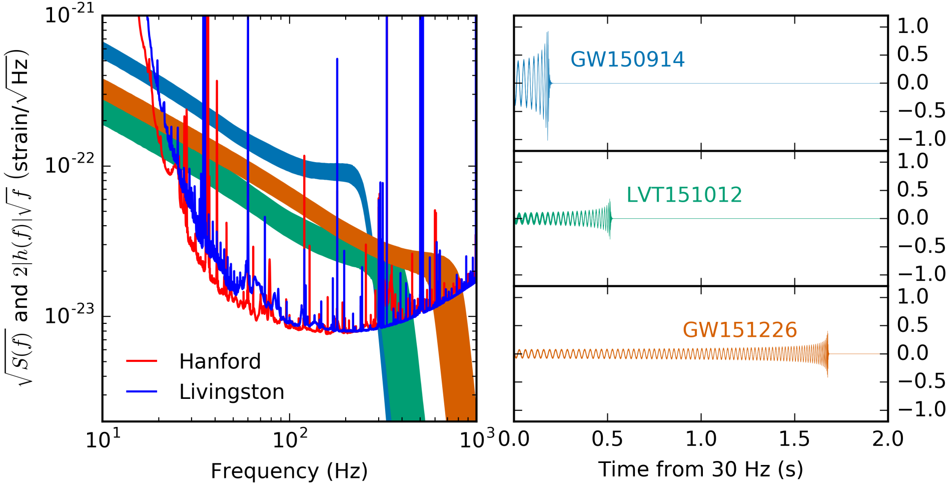


Three gravitational events [LIGO/VIRGO collaboration 2016]



100 years of gravitational radiation [Einstein 1916]

348 DOC. 32 INTEGRATION OF FIELD EQUATIONS

688 Sitzung der physikalisch-mathematischen Klasse vom 22. Juni 1916

Näherungsweise Integration der Feldgleichungen der Gravitation.

Von A. Einstein.

Bei der Behandlung der meisten speziellen (nicht prinzipiellen) Probleme auf dem Gebiete der Gravitationstheorie kann man sich damit begnügen, die $g_{\mu\nu}$ in erster Näherung zu berechnen. Dabei bedient man sich mit Vorteil der imaginären Zeitvariable $x_4 = it$ aus denselben Gründen wie in der speziellen Relativitätstheorie. Unter »erster Näherung« ist dabei verstanden, daß die durch die Gleichung

$$\boxed{g_{\mu\nu} = -\delta_{\mu\nu} + \gamma_{\mu\nu}} \qquad (1)$$

⟸ small perturbation of Minkowski's metric

definierten Größen $\gamma_{\mu\nu}$, welche linearen orthogonalen Transformationen gegenüber Tensorcharakter besitzen, gegen 1 als kleine Größen behandelt werden können, deren Quadrate und Produkte gegen die ersten Potenzen vernachlässigt werden dürfen. Dabei ist $\delta_{\mu\nu} = 1$ bzw. $\delta_{\mu\nu} = 0$, je nachdem $\mu = \nu$ oder $\mu \neq \nu$.

Wir werden zeigen, daß diese $\gamma_{\mu\nu}$ in analoger Weise berechnet werden können wie die retardierten Potentiale der Elektrodynamik.

100 years of gravitational radiation [Einstein 1918]

Einstein's quadrupole formula

mit $4\pi R^2$ multiplizierte S endlich ist der Energieverlust pro Zeiteinheit des mechanischen Systems durch Gravitationswellen. Die Rechnung ergibt

[31]

$$\boxed{4\pi R^2 \bar{S} = \frac{\varkappa}{80\pi}\left[\sum_{\mu\nu} \dddot{\mathfrak{J}}_{\mu\nu}^{2} - \frac{1}{3}\left(\sum_{\mu} \dddot{\mathfrak{J}}_{\mu\mu}\right)^2\right].} \qquad (30)$$

Man sieht an diesem Ergebnis, daß ein mechanisches System, welches dauernd Kugelsymmetrie behält, nicht strahlen kann, im Gegensatz zu dem durch einen Rechenfehler entstellten Ergebnis der früheren Abhandlung.

[32]

Aus (27) ist ersichtlich, daß die Ausstrahlung in keiner Richtung negativ werden kann, also sicher auch nicht die totale Ausstrahlung. Bereits in der früheren Abhandlung ist betont geworden, daß das Endergebnis dieser Betrachtung, welches einen Energieverlust der Körper infolge der thermischen Agitation verlangen würde, Zweifel an der allgemeinen Gültigkeit der Theorie hervorrufen muß. Es scheint, daß eine vervollkommnete Quantentheorie eine Modifikation auch der Gravitationstheorie wird bringen müssen.

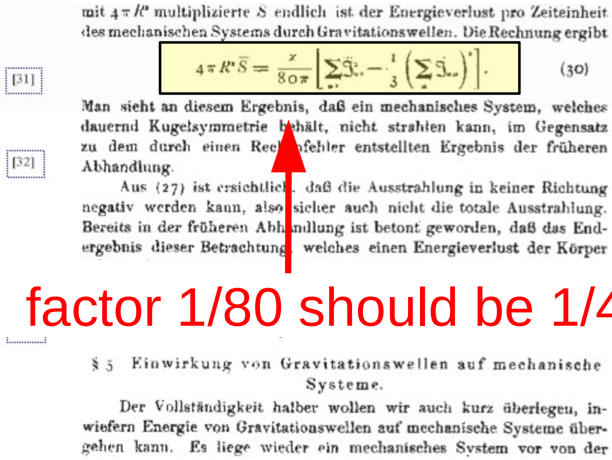
[33]

§ 5 Einwirkung von Gravitationswellen auf mechanische Systeme.

Der Vollständigkeit halber wollen wir auch kurz überlegen, inwiefern Energie von Gravitationswellen auf mechanische Systeme übergehen kann. Es liege wieder ein mechanisches System vor von der

100 years of gravitational radiation [Einstein 1918]

Einstein's quadrupole formula



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factor 1/80 should be 1/40

§ 5 Einwirkung von Gravitationswellen auf mechanische Systeme.

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Quadrupole moment formalism [Einstein 1918; Landau & Lifchitz 1947]

1. First quadrupole formula

$$\boxed{h_{ij}^{\mathsf{TT}} = \frac{2G}{c^4 D}\left\{\frac{\mathrm{d}^2 Q_{ij}}{\mathrm{d}t^2}\left(t-\frac{D}{c}\right)+\mathcal{O}\left(\frac{v}{c}\right)\right\}^{\mathsf{TT}}+\mathcal{O}\left(\frac{1}{D^2}\right)}$$

2. Einstein quadrupole formula

$$\boxed{\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)^{\mathsf{GW}} = \frac{G}{5c^5}\left\{\frac{\mathrm{d}^3 Q_{ij}}{\mathrm{d}t^3}\frac{\mathrm{d}^3 Q_{ij}}{\mathrm{d}t^3}+\mathcal{O}\left(\frac{v}{c}\right)^2\right\}}$$

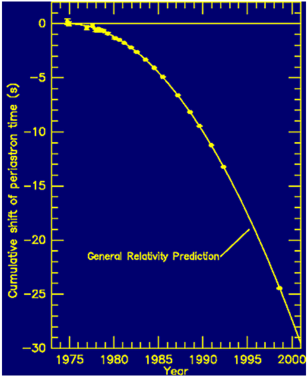
3. Radiation reaction formula [Chandrasekhar & Esposito 1970; Burke & Thorne 1970]

$$\boxed{F_i^{\mathsf{reac}} = -\frac{2G}{5c^5}\rho\, x^j\frac{\mathrm{d}^5 Q_{ij}}{\mathrm{d}t^5}+\mathcal{O}\left(\frac{v}{c}\right)^7}$$

which is a 2.5PN $\sim (v/c)^5$ effect in the source's equations of motion

The quadrupole formula works for the binary pulsar

[Taylor & Weisberg 1982]



$$\boxed{\dot{P} = -\frac{192\pi}{5c^5}\nu\left(\frac{2\pi G\, M}{P}\right)^{5/3}\frac{1+\frac{73}{24}e^2+\frac{37}{96}e^4}{(1-e^2)^{7/2}} \approx -2.4\times 10^{-12}}$$

[Peters & Mathews 1963, Esposito & Harrison 1975, Wagoner 1975, Damour & Deruelle 1983]

The quadrupole formula works also for GW150914 !

1. The GW frequency is given in terms of the chirp mass $\mathcal{M} = \mu^{3/5} M^{2/5}$ by

$$f = \frac{1}{\pi} \left[\frac{256}{5} \frac{G\mathcal{M}^{5/3}}{c^5} (t_\mathsf{f} - t) \right]^{-3/8}$$

2. Therefore the chirp mass is directly measured as

$$\boxed{\mathcal{M} = \left[\frac{5}{96} \frac{c^5}{G\pi^{8/3}} f^{-11/3} \dot{f} \right]^{3/5}}$$

   which gives $\mathcal{M} = 30 M_\odot$ thus $M \geqslant 70 M_\odot$

3. The GW amplitude is predicted to be

$$h_\mathsf{eff} \sim 4.1 \times 10^{-22} \left(\frac{\mathcal{M}}{M_\odot} \right)^{5/6} \left(\frac{100\,\mathsf{Mpc}}{D} \right) \left(\frac{100\,\mathsf{Hz}}{f_\mathsf{merger}} \right)^{-1/6} \sim 1.6 \times 10^{-21}$$

4. The distance $D = 400\,\mathsf{Mpc}$ is measured from the signal itself

Total energy radiated by GW150914

1. The ADM energy of space-time is constant and reads (at any t)

$$E_{\mathsf{ADM}} = (m_1 + m_2)c^2 - \frac{G m_1 m_2}{2r} + \frac{G}{5c^5} \int_{-\infty}^{t} \mathrm{d}t' \left(Q_{ij}^{(3)}\right)^2 (t')$$

2. Initially $E_{\mathsf{ADM}} = (m_1 + m_2)c^2$ while finally (at time t_{f})

$$E_{\mathsf{ADM}} = M_{\mathsf{f}} c^2 + \frac{G}{5c^5} \int_{-\infty}^{t_{\mathsf{f}}} \mathrm{d}t' \left(Q_{ij}^{(3)}\right)^2 (t')$$

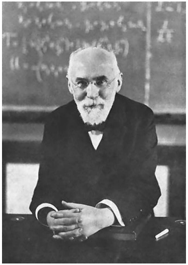
3. The total energy radiated in GW is

$$\boxed{\Delta E^{\mathsf{GW}} = (m_1 + m_2 - M_{\mathsf{f}})c^2 = \frac{G}{5c^5} \int_{-\infty}^{t_{\mathsf{f}}} \mathrm{d}t' \left(Q_{ij}^{(3)}\right)^2 (t') = \frac{G m_1 m_2}{2 r_{\mathsf{f}}}}$$

4. The measured power released is

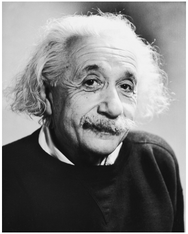
$$\boxed{P^{\mathsf{GW}} \sim \frac{3 M_\odot c^2}{0.2\,\mathsf{s}} \sim 10^{49}\,\mathsf{W} \sim 10^{-3}\,\frac{c^5}{G}}$$

The 1PN equations of motion [Lorentz & Droste 1917]



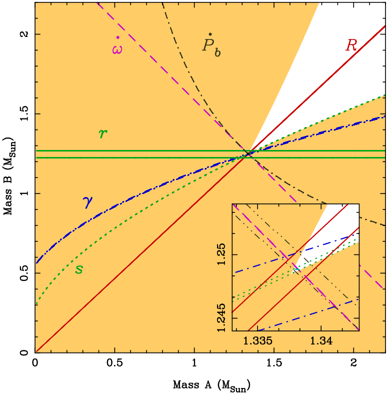
- Obtain the equations of motion of N bodies at the 1PN $\sim (v/c)^2$ order and even derive the 1PN Lagrangian!
- This work published in Dutch has been largely unrecognized untill an English translation was published in 1937

The 1PN equations of motion [Einstein, Infeld & Hoffmann 1938]



$$\boxed{\begin{aligned}\frac{\mathrm{d}^2\boldsymbol{r}_A}{\mathrm{d}t^2} = -\sum_{B\neq A}\frac{Gm_B}{r_{AB}^2}\boldsymbol{n}_{AB}\Bigg[1 - 4\sum_{C\neq A}\frac{Gm_C}{c^2 r_{AC}} - \sum_{D\neq B}\frac{Gm_D}{c^2 r_{BD}}\left(1-\frac{\boldsymbol{r}_{AB}\cdot\boldsymbol{r}_{BD}}{r_{BD}^2}\right)\\ + \frac{1}{c^2}\left(\boldsymbol{v}_A^2 + 2\boldsymbol{v}_B^2 - 4\boldsymbol{v}_A\cdot\boldsymbol{v}_B - \frac{3}{2}(\boldsymbol{v}_B\cdot\boldsymbol{n}_{AB})^2\right)\Bigg]\\ + \sum_{B\neq A}\frac{Gm_B}{c^2 r_{AB}^2}\boldsymbol{v}_{AB}[\boldsymbol{n}_{AB}\cdot(3\boldsymbol{v}_B - 4\boldsymbol{v}_A)] - \frac{7}{2}\sum_{B\neq A}\sum_{D\neq B}\frac{G^2 m_B m_D}{c^2 r_{AB} r_{BD}^3}\boldsymbol{n}_{BD}\end{aligned}}$$

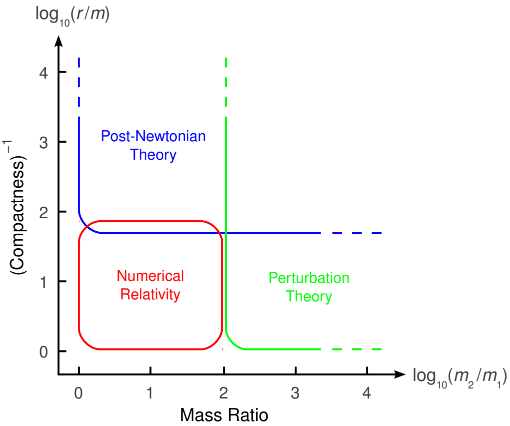
Relativistic effects in binary pulsars [e.g. Stairs 2003]



1PN order $\begin{cases} \dot\omega \text{ relativistic advance of periastron} \\ \gamma \text{ gravitational red-shift and second-order Doppler effect} \\ r \text{ and } s \text{ range and shape of the Shapiro time delay} \end{cases}$

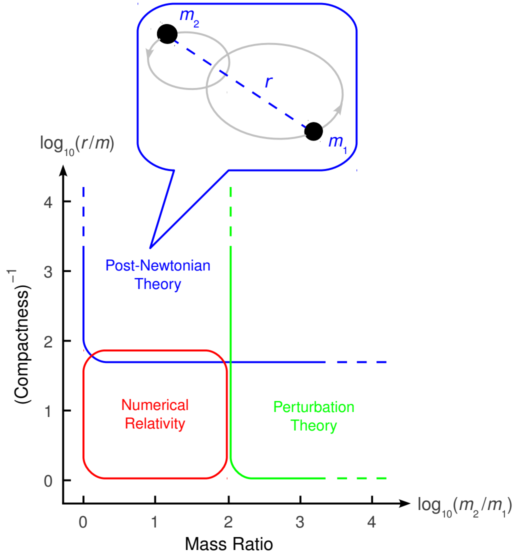
2.5PN order $\{\ \dot P$ secular decrease of orbital period

Methods to compute GW templates



[courtesy Alexandre Le Tiec]

Methods to compute GW templates



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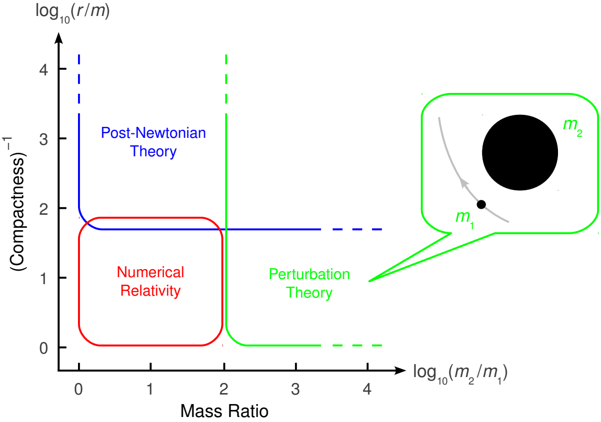
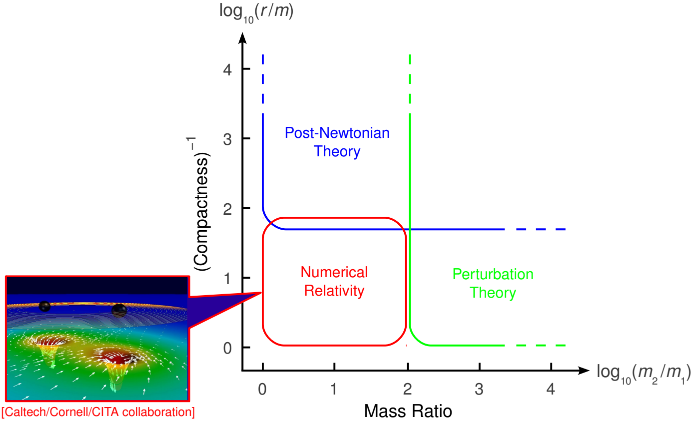


[courtesy Alexandre Le Tiec]

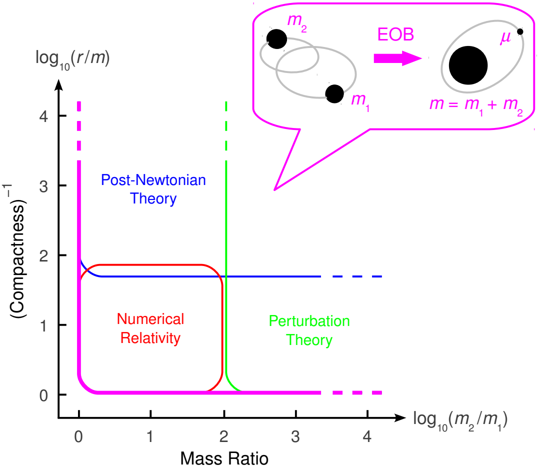
Methods to compute GW templates



[courtesy Alexandre Le Tiec]

Methods to compute GW templates

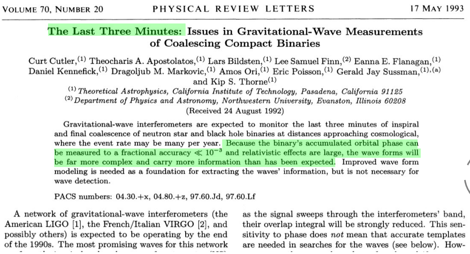
[Buonanno & Damour 1998]



[courtesy Alexandre Le Tiec]

Inspiralling binaries require high-order PN modelling

[Caltech "3mn paper" 1992; Blanchet & Schäfer 1993]



VOLUME 70, NUMBER 20 PHYSICAL REVIEW LETTERS 17 MAY 1993

The Last Three Minutes: Issues in Gravitational-Wave Measurements of Coalescing Compact Binaries

Curt Cutler,[1] Theocharis A. Apostolatos,[1] Lars Bildsten,[1] Lee Samuel Finn,[2] Eanna E. Flanagan,[1] Daniel Kennefick,[1] Dragoljub M. Markovic,[1] Amos Ori,[1] Eric Poisson,[1] Gerald Jay Sussman,[1],[a] and Kip S. Thorne[1]

[1] *Theoretical Astrophysics, California Institute of Technology, Pasadena, California 91125*
[2] *Department of Physics and Astronomy, Northwestern University, Evanston, Illinois 60208*

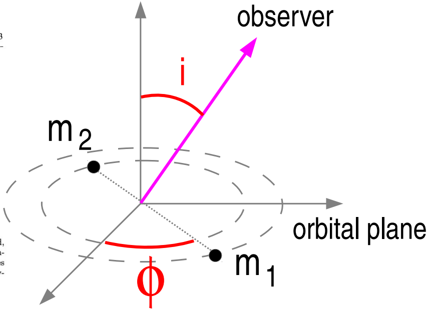
(Received 24 August 1992)

Gravitational-wave interferometers are expected to monitor the last three minutes of inspiral and final coalescence of neutron star and black hole binaries at distances approaching cosmological, where the event rate may be many per year. Because the binary's accumulated orbital phase can be measured to a fractional accuracy $\ll 10^{-3}$ and relativistic effects are large, the wave forms will be far more complex and carry more information than has been expected. Improved wave form modeling is needed as a foundation for extracting the waves' information, but is not necessary for wave detection.

PACS numbers: 04.30.+x, 04.80.+z, 97.60.Jd, 97.60.Lf

A network of gravitational-wave interferometers (the American LIGO [1], the French/Italian VIRGO [2], and possibly others) is expected to be operating by the end of the 1990s. The most promising waves for this network

as the signal sweeps through the interferometers' band, their overlap integral will be strongly reduced. This sensitivity to phase does *not* mean that accurate templates are needed in searches for the waves (see below). How-



$$\phi(t) = \phi_0 \underbrace{- \frac{M}{\mu}\left(\frac{GM\omega}{c^3}\right)^{-5/3}}_{\text{quadrupole formalism}} \Bigg\{ 1 \underbrace{+ \frac{1\text{PN}}{c^2} + \frac{1.5\text{PN}}{c^3} + \cdots + \frac{3\text{PN}}{c^6} + \cdots}_{\text{needs to be computed with 3PN precision at least}} \Bigg\}$$

Here 3PN means 5.5PN as a radiation reaction effect !

The intermediate binary black hole problem

PHYSICAL REVIEW D, VOLUME 58, 061501

Computing the merger of black-hole binaries: The IBBH problem

Patrick R. Brady, Jolien D. E. Creighton, and Kip S. Thorne
Theoretical Astrophysics, California Institute of Technology, Pasadena, California 91125

(Received 22 April 1998; published 26 August 1998)

Gravitational radiation arising from the inspiral and merger of binary black holes (BBH's) is a promising candidate for detection by kilometer-scale interferometric gravitational wave observatories. This Rapid Communication discusses a serious obstacle to searches for such radiation and to the interpretation of any observed waves: the inability of current computational techniques to evolve a BBH through its last $\sim$10 orbits of inspiral ($\sim$100 radians of gravitational-wave phase). A new set of numerical-relativity techniques is proposed for solving this ''intermediate binary black hole'' (IBBH) problem: (i) numerical evolutions performed in coordinates co-rotating with the BBH, in which the metric coefficients evolve on the long timescale of inspiral, and (ii) techniques for mathematically freezing out gravitational degrees of freedom that are not excited by the waves. [S0556-2821(98)50218-4]

PACS number(s): 04.25.Dm, 04.30.Db, 04.70.−s

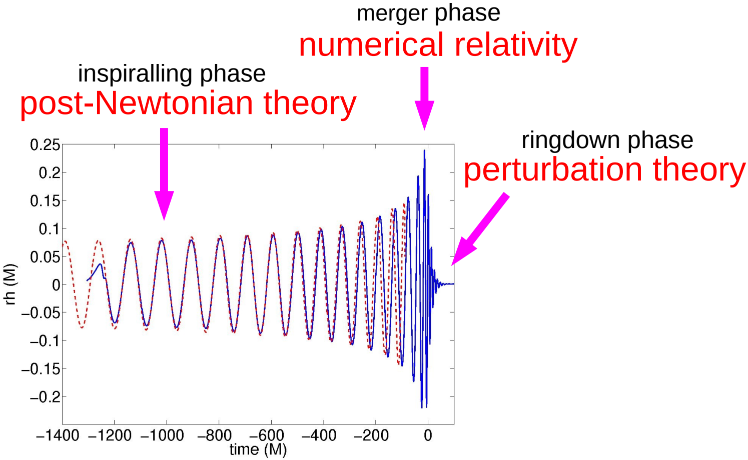
I. MOTIVATION

Among all gravitational wave sources that theorists have considered, the one most likely to be detected first is the final inspiral and merger of binary black holes (BBH's) with

that, in the next several years, this approach will be able to evolve a BBH through the gap for the required $\gtrsim$1200 dynamical time scales. This motivates exploring alternative procedures for computing the evolution and waves during the IBBH phase.

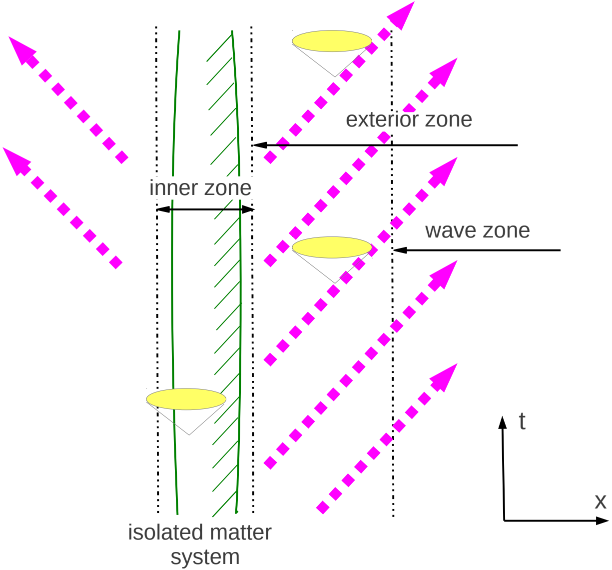
- An alternative solution is to extend the region of validity of the PN approximation by using Padé approximants [Damour, Iyer & Sathyaprakash 1998]
- However the accuracy of the PN approximation for comparable masses turned out to be rather good far into the strong field region [Blanchet 2001]

The gravitational chirp of compact binaries

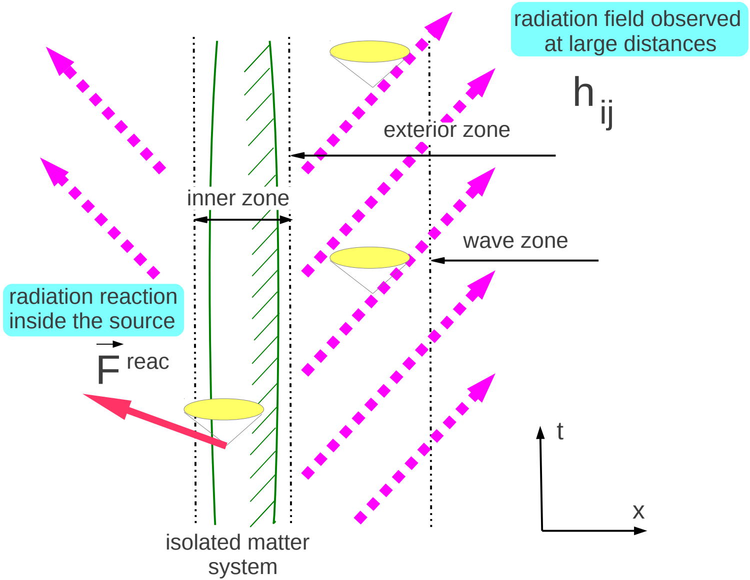


Effective methods such as EOB that interpolate between the PN and NR are very important notably for the data analysis

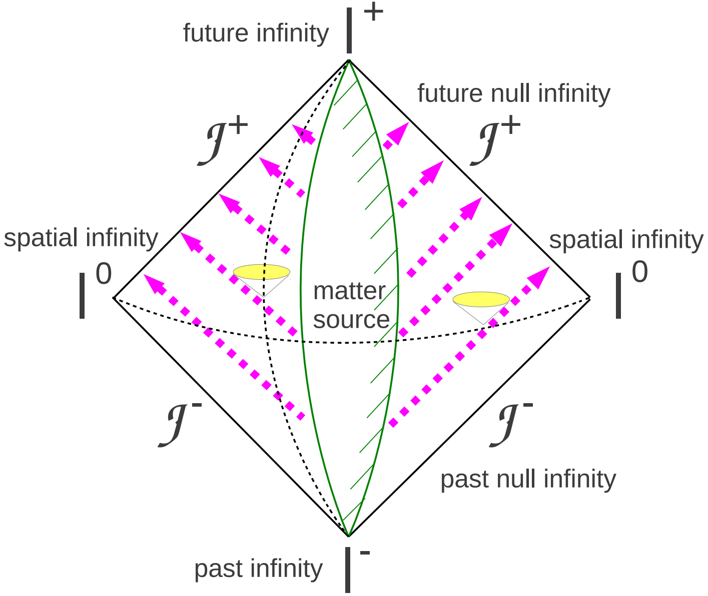
Isolated matter system in general relativity



Isolated matter system in general relativity



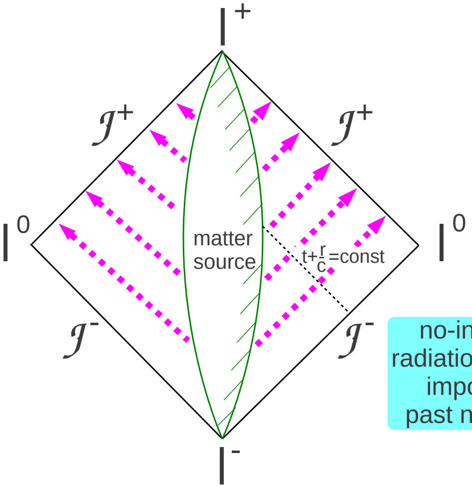
Conformal picture [Penrose 1963]



Asymptotic structure of space-time

1. What is the struture of space-time far away from an isolated matter system?
2. Does a general radiating space-time satisfy rigourous definitions [Penrose 1963, 1965] of asymptotic flatness in general relativity?
3. How to relate the asymptotic structure of space-time [Bondi *et al.* 1962; Sachs 1962] to the matter variable and dynamics of an actual source?
4. How to impose rigourous boundary conditions on the edge of space-time appropriate to an isolated system?

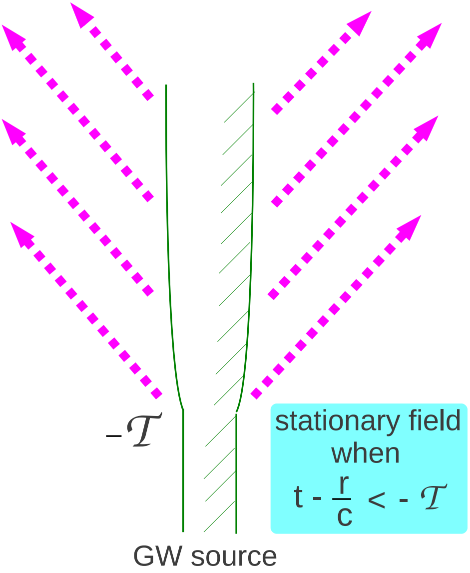
No-incoming radiation condition



no-incoming radiation condition imposed at past null infinity

$$\boxed{\lim_{\substack{r\to+\infty\\ t+\frac{r}{c}=\text{const}}} \left(\frac{\partial}{\partial r}+\frac{\partial}{c\partial t}\right)\left(rh^{\alpha\beta}\right)=0}$$

Hypothesis of stationarity in the remote past



In practice all GW sources observed in astronomy (*e.g.* a compact binary system) will have been formed and started to emit GWs only from a finite instant in the past $-\mathcal{T}$

Post-Minkowskian expansion [e.g. Bertotti & Plebanski 1960]

Appropriate for weakly self-gravitating isolated matter sources

$$\gamma_{\text{PM}} \equiv \frac{GM}{c^2 a} \ll 1 \qquad \begin{cases} M \text{ mass of source} \\ a \text{ size of source} \end{cases}$$

$$\mathfrak{g}^{\alpha\beta} = \eta^{\alpha\beta} + \underbrace{\sum_{n=1}^{+\infty} G^n \, h^{\alpha\beta}_{(n)}}_{G \text{ labels the PM expansion}}$$

$$\boxed{\begin{aligned} &\Box h^{\alpha\beta}_{(n)} = \frac{16\pi G}{c^4} |g| T^{\alpha\beta}_{(n)} + \overbrace{\Lambda^{\alpha\beta}_{(n)}[h_{(1)}, \cdots, h_{(n-1)}]}^{\text{know from previous iterations}} \\ &\partial_\mu h^{\alpha\mu}_{(n)} = 0 \end{aligned}}$$

Very difficult approximation to implement in practice for general sources at high PM orders [Thorne & Kovàcs 1975]

Linearized multipolar vacuum solution [Thorne 1980]

General solution of linearized vacuum field equations in harmonic coordinates

$$\boxed{\Box h^{\alpha\beta}_{(1)} = \partial_\mu h^{\alpha\mu}_{(1)} = 0}$$

$$h^{00}_{(1)} = -\frac{4}{c^2}\sum_{\ell=0}^{+\infty}\frac{(-)^\ell}{\ell!}\partial_L\left(\frac{1}{r}M_L(u)\right)$$

$$h^{0i}_{(1)} = \frac{4}{c^3}\sum_{\ell=1}^{+\infty}\frac{(-)^\ell}{\ell!}\left\{\partial_{L-1}\left(\frac{1}{r}M^{(1)}_{iL-1}(u)\right) + \frac{\ell}{\ell+1}\epsilon_{iab}\partial_{aL-1}\left(\frac{1}{r}S_{bL-1}(u)\right)\right\}$$

$$h^{ij}_{(1)} = -\frac{4}{c^4}\sum_{\ell=2}^{+\infty}\frac{(-)^\ell}{\ell!}\left\{\partial_{L-2}\left(\frac{1}{r}M^{(2)}_{ijL-2}(u)\right) + \frac{2\ell}{\ell+1}\partial_{aL-2}\left(\frac{1}{r}\epsilon_{ab(i}S^{(1)}_{j)bL-2}(u)\right)\right\}$$

- multipole moments $M_L(u)$ and $S_L(u)$ arbitrary functions of $u = t - r/c$
- mass $M = \text{const}$, center-of-mass position $X_i \equiv M_i/M = \text{const}$, linear momentum $P_i \equiv M^{(1)}_i = 0$, angular momentum $S_i = \text{const}$

Multipolar-post-Minkowskian expansion

[Blanchet & Damour 1986, 1988, 1992; Blanchet 1987, 1993, 1998]

1. The linearized solution is the starting point of an explicit MPM algorithm

$$h^{\alpha\beta}_{\mathrm{MPM}} = \sum_{n=1}^{+\infty} G^n \, h^{\alpha\beta}_{(n)}$$

2. Hierarchy of perturbation equations is solved by induction over n

$$\Box h^{\alpha\beta}_{(n)} = \Lambda^{\alpha\beta}_{(n)}[h_{(1)}, h_{(2)}, \dots, h_{(n-1)}]$$
$$\partial_\mu h^{\alpha\mu}_{(n)} = 0$$

3. A regularization is required in order to cope with the divergency of the multipolar expansion when $r \to 0$

Multipolar-post-Minkowskian expansion

[Blanchet & Damour 1986, 1988, 1992; Blanchet 1987, 1993, 1998]

Theorem 1:
The MPM solution is the most general solution of Einstein's vacuum equations outside an isolated matter system

Theorem 2:
The general structure of the PN expansion is

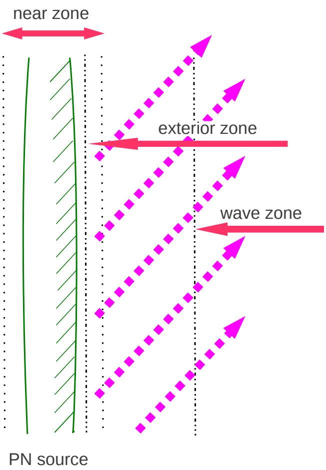
$$\boxed{h^{\alpha\beta}_{\mathsf{PN}}(\mathbf{x},t,c) = \sum_{\substack{p\geqslant 2\\ q\geqslant 0}} \frac{(\ln c)^q}{c^p} h^{\alpha\beta}_{p,q}(\mathbf{x},t)}$$

Theorem 3:
The MPM solution is asymptotically simple at future null infinity in the sense of Penrose [1963, 1965] and agrees with the Bondi-Sachs [1962] formalism

$$\begin{aligned}\underbrace{M_{\mathsf{B}}(u)}_{\text{Bondi mass}} &= \underbrace{M}_{\text{ADM mass}} - \frac{G}{5c^5}\int_{-\infty}^{u} \mathrm{d}\tau M^{(3)}_{ij}(\tau) M^{(3)}_{ij}(\tau)\\ &+ \quad \text{higher multipoles and higher PM computable to any order}\end{aligned}$$

The MPM-PN formalism

A multipolar post-Minkowskian (MPM) expansion in the exterior zone is matched to a general post-Newtonian (PN) expansion in the near zone



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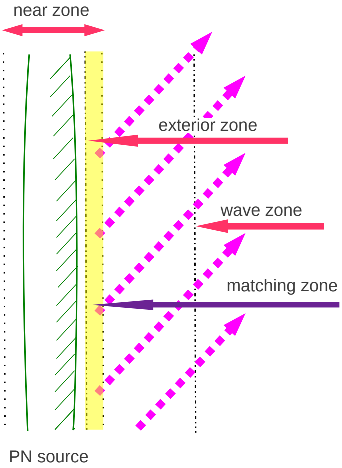


The matching equation

1. This is a variant of the theory of matched asymptotic expansions
   [Kates 1980; Anderson *et al.* 1982; Blanchet 1998]

$$\text{match}\quad\left\{\begin{array}{c}\text{the multipole expansion}\quad \mathcal{M}(h^{\alpha\beta}) \equiv h^{\alpha\beta}_{\mathsf{MPM}}\\ \text{with}\\ \text{the PN expansion}\quad \bar{h}^{\alpha\beta} \equiv h^{\alpha\beta}_{\mathsf{PN}}\end{array}\right.$$

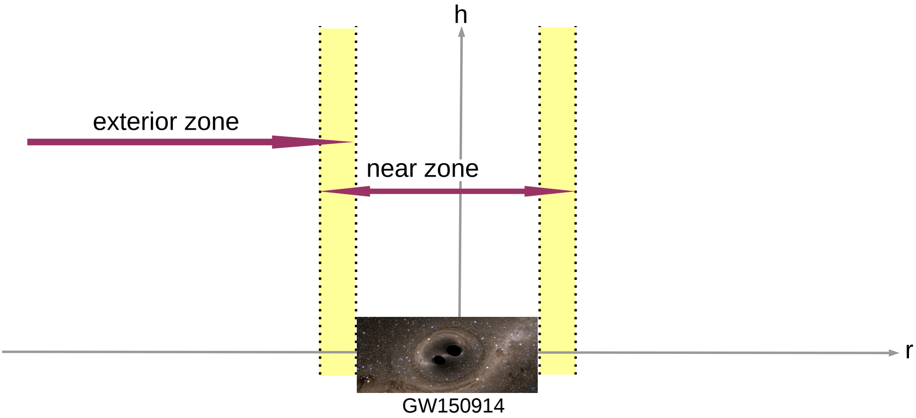
$$\boxed{\overline{\mathcal{M}(h^{\alpha\beta})} = \mathcal{M}(\bar{h}^{\alpha\beta})}$$

   - Left side is the NZ expansion ($r \to 0$) of the exterior MPM field
   - Right side is the FZ expansion ($r \to \infty$) of the inner PN field

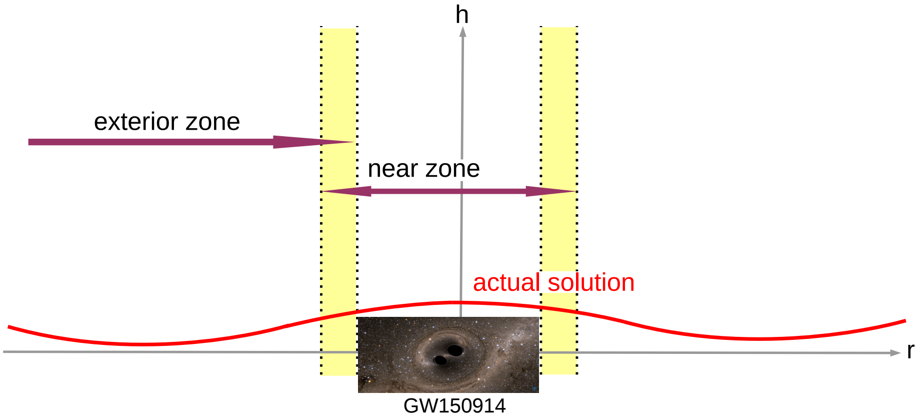
2. The matching equation has been implemented at any post-Minkowskian order in the exterior field and any PN order in the inner field

3. It gives a unique (formal) multipolar-post-Newtonian solution valid everywhere inside and outside the source

The matching equation



The matching equation



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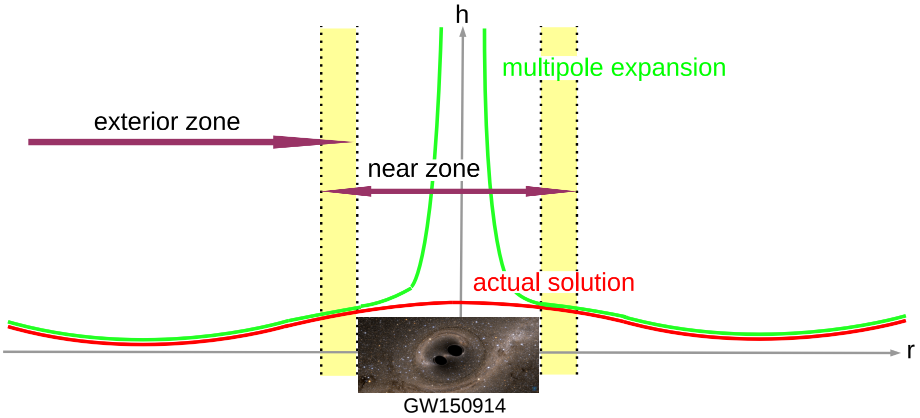

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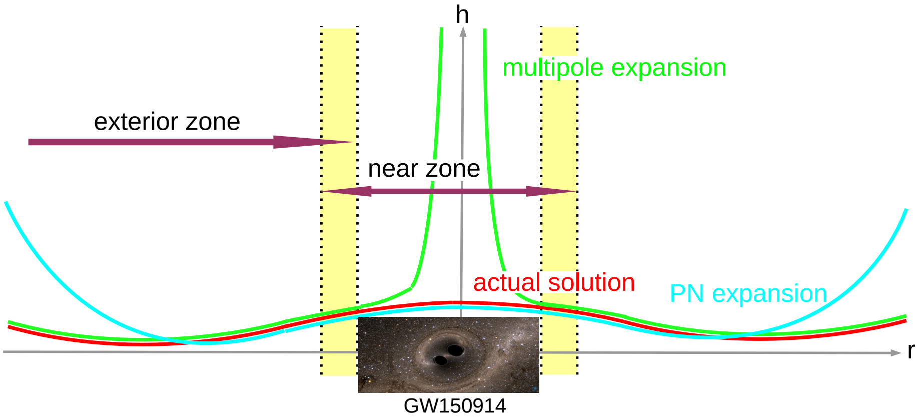
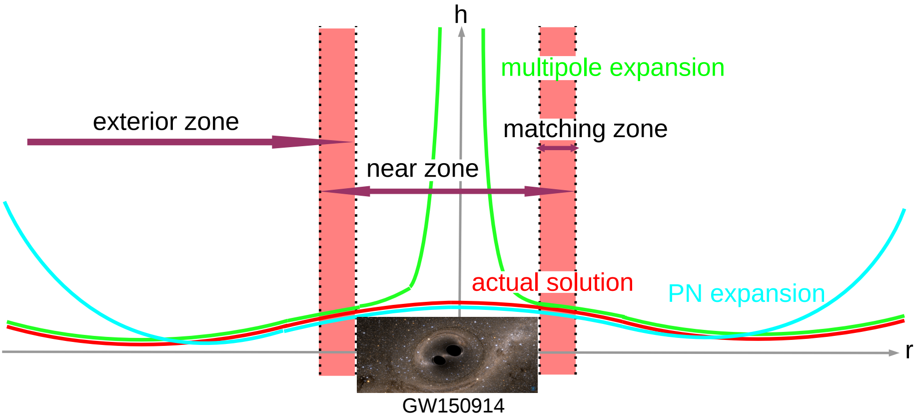


The matching equation



General solution for the multipolar field [Blanchet 1995, 1998]

$$\mathcal{M}(h^{\mu\nu}) = \mathrm{FP}\Box^{-1}_{\text{ret}}\mathcal{M}(\Lambda^{\mu\nu}) + \underbrace{\sum_{\ell=0}^{+\infty}\partial_L\left\{\frac{M_L^{\mu\nu}(t-r/c)}{r}\right\}}_{\text{homogeneous retarded solution}}$$

where $$M_L^{\mu\nu}(t) = \mathrm{FP}\int \mathrm{d}^3\mathbf{x}\,\hat{x}_L\int_{-1}^{1}\mathrm{d}z\,\delta_\ell(z)\;\underbrace{\bar{\tau}^{\mu\nu}(\mathbf{x},t-zr/c)}_{\text{PN expansion of the pseudo-tensor}}$$

- The FP procedure plays the role of an UV regularization in the non-linearity term but an IR regularization in the multipole moments
- From this one obtains the multipole moments of the source at any PN order solving the wave generation problem
- This is a formal PN solution *i.e.* a set of rules for generating the PN series regardless of the exact mathematocal nature of this series

General solution for the inner PN field

[Poujade & Blanchet 2002; Blanchet, Faye & Nissanke 2004]

$$\bar{h}^{\mu\nu} = \text{FP}\Box^{-1}_{\text{ret}}\bar{\tau}^{\mu\nu} + \underbrace{\sum_{\ell=0}^{+\infty}\partial_L\left\{\frac{R_L^{\mu\nu}(t-r/c) - R_L^{\mu\nu}(t+r/c)}{r}\right\}}_{\text{homogeneous antisymmetric solution}}$$

where $\quad R_L^{\mu\nu}(t) = \text{FP}\int \mathrm{d}^3\mathbf{x}\,\hat{x}_L \int_1^\infty \mathrm{d}z\,\gamma_\ell(z)\;\underbrace{\mathcal{M}(\tau^{\mu\nu})(\mathbf{x}, t - zr/c)}_{\text{multipole expansion of the pseudo-tensor}}$

- The radiation reaction effects starting at 2.5PN order appropriate to an isolated system are determined to any order
- In particular nonlinear radiation reaction effects associated with tails are contained in the second term and start at 4PN order

Radiative moments at future null infinity

Correct for the logarithmic deviation of retarded time in harmonic coordinates with respect to the actual null coordinate

$$\underbrace{T - \frac{R}{c}}_{\text{radiative coordinates}} = \underbrace{t - \frac{r}{c}}_{\text{harmonic coordinates}} - \frac{2GM}{c^3}\ln\left(\frac{r}{c\tau_0}\right) + \mathcal{O}\left(\frac{1}{r}\right)$$

Asymptotic waveform is parametrized by radiative moments U_L and V_L [Thorne 1980]

$$\boxed{h_{ij}^{\mathrm{TT}} = \frac{1}{R}\sum_{\ell=2}^{\infty} N_{L-2}\,\underbrace{U_{ijL-2}(T-R/c)}_{\text{mass-type}} + \epsilon_{ab(i}N_{aL-1}\,\underbrace{V_{j)bL-2}(T-R/c)}_{\text{current-type}} + \mathcal{O}\left(\frac{1}{R^2}\right)}$$

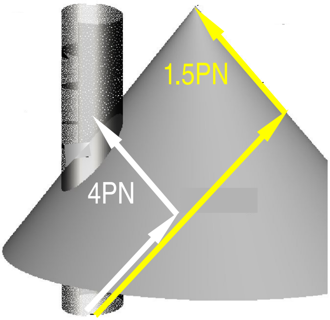
The 3PN radiative quadrupole moment

$$\boxed{\begin{aligned} U_{ij}(t) = {} & M_{ij}^{(2)}(t) + \underbrace{\frac{2GM}{c^3}\int_0^{+\infty} \mathrm{d}\tau\, M_{ij}^{(4)}(t-\tau)\left[\ln\left(\frac{\tau}{2\tau_0}\right)+\frac{11}{12}\right]}_{\text{1.5PN tail integral}} \\ & + \frac{G}{c^5}\left\{\underbrace{-\frac{2}{7}\int_0^{+\infty} \mathrm{d}\tau\, M_{a<i}^{(3)} M_{j>a}^{(3)}(t-\tau)}_{\text{2.5PN memory integral}} + \text{instantaneous terms}\right\} \\ & + \underbrace{\frac{2G^2M^2}{c^6}\int_0^{+\infty} \mathrm{d}\tau\, M_{ij}^{(5)}(t-\tau)\left[\ln^2\left(\frac{\tau}{2\tau_0}\right)+\frac{57}{70}\ln\left(\frac{\tau}{2\tau_0}\right)+\frac{124627}{44100}\right]}_{\text{3PN tail-of-tail integral}} \\ & + \mathcal{O}\left(\frac{1}{c^7}\right) \end{aligned}}$$

The tail-of-tail-of-tail effect arises at 4.5PN order and has been recently computed [Marchand, Blanchet & Faye 2016]

Tails of gravitational waves [Bonnor 1959; Blanchet & Damour 1988, 1992]

Tails are produced by backscatter of GWs on the curvature induced by the matter source's total mass M



$$\delta h_{ij}^{\text{tail}} = \frac{4G}{c^4 r} \underbrace{\frac{G\color{red}{M}}{c^3} \int_{-\infty}^{t} \mathrm{d}t' \, \color{red}{M_{ij}}(t') \ln\left(\frac{t-t'}{\tau_0}\right)}_{\text{The tail is dominantly a 1.5PN effect}} + \cdots$$

Application to compact binary inspiral

1. Apply the previous PN solution to systems of point particles

$$\boxed{T^{\mu\nu}(x) = \sum_A \int_{-\infty}^{+\infty} \mathrm{d}\tau_A\, p_A^{(\mu} u_A^{\nu)} \frac{\delta^{(4)}(x - y_A)}{\sqrt{-g_A}} + \text{(spin contributions)}}$$

2. Suplement the calculation by a self-field regularization
   - Hadamard's regularization
   - Dimensional regularization
3. The self-field regularization should be applied conjointly with the FP regularization, say in the multipole moments

$$\mathrm{FP}_{B\to 0}\left\{\mathrm{AC}_{d\to 3}\int \frac{\mathrm{d}^d\mathbf{x}}{{\ell_0}^{d-3}}\left(\frac{|\mathbf{x}|}{r_0}\right)^B F(\mathbf{x})\right\}$$

4. The IR scale r_0 and UV scale ℓ_0 should disappear at the end of the calculation

Dimensional regularization [t'Hooft & Veltman 1972]

1. Einstein's field equations are solved in d spatial dimensions (with $d \in \mathbb{C}$) with distributional sources. In Newtonian approximation

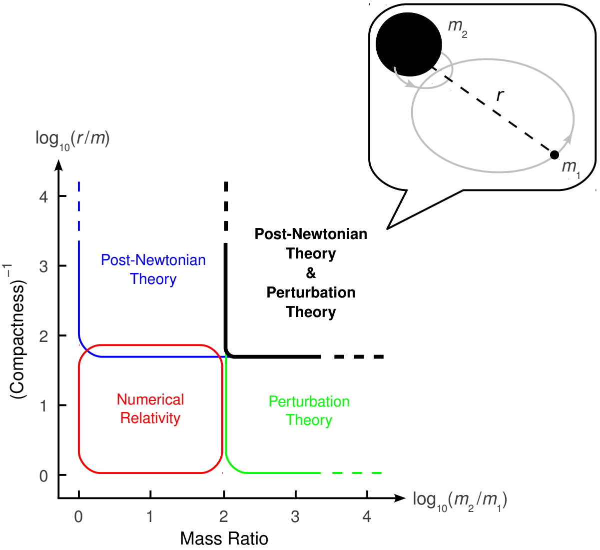
$$\Delta U = -4\pi \frac{2(d-2)}{d-1} G\rho$$

2. For two point-particles $\rho = m_1 \delta_{(d)}(\mathbf{x} - \mathbf{y}_1) + m_2 \delta_{(d)}(\mathbf{x} - \mathbf{y}_2)$ we get

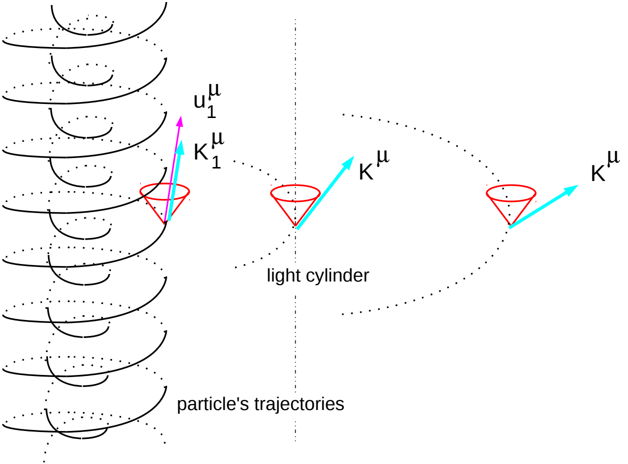
$$\boxed{U(\mathbf{x}, t) = \frac{2(d-2)k}{d-1} \left(\frac{G m_1}{|\mathbf{x} - \mathbf{y}_1|^{d-2}} + \frac{G m_2}{|\mathbf{x} - \mathbf{y}_2|^{d-2}} \right) \quad \text{with} \quad k = \frac{\Gamma\left(\frac{d-2}{2}\right)}{\pi^{\frac{d-2}{2}}}}$$

3. Computations are performed when $\Re(d)$ is a large negative number, and the result is analytically continued for any $d \in \mathbb{C}$ except for isolated poles
4. Dimensional regularization is then followed by a renormalization of the worldline of the particles so as to absorb the poles $\propto (d-3)^{-1}$

Checking the PN machinery with GSF



Looking at the conservative part of the dynamics



$$u_1^\mu = u_1^t K^\mu \quad \text{where} \quad \boxed{u_1^t = \Bigg(- \underbrace{(g_{\mu\nu})_1}_{\text{regularized metric}} \frac{v_1^\mu v_1^\nu}{c^2}\Bigg)^{-1/2}}$$

[Detweiler 2008]

Standard PN theory agrees with GSF calculations

$$u^t_{\rm SF} = -y - 2y^2 - 5y^3 + \left(-\frac{121}{3} + \frac{41}{32}\pi^2\right) y^4 + \left(-\frac{1157}{15} + \frac{677}{512}\pi^2 - \frac{128}{5}\gamma_{\rm E} - \frac{64}{5}\ln(16y)\right) y^5 - \frac{956}{105}y^6\ln y - \frac{13696\pi}{525}y^{13/2} - \frac{51256}{567}y^7\ln y + \frac{81077\pi}{3675}y^{15/2} + \frac{27392}{525}y^8\ln^2 y + \frac{82561159\pi}{467775}y^{17/2} - \frac{27016}{2205}y^9\ln^2 y - \frac{11723776\pi}{55125}y^{19/2}\ln y - \frac{4027582708}{9823275}y^{10}\ln^2 y + \frac{99186502\pi}{1157625}y^{21/2}\ln y + \frac{23447552}{165375}y^{11}\ln^3 y + \cdots$$

1. Integral PN terms such as 3PN permit checking dimensional regularization [Blanchet, Detweiler, Le Tiec & Whiting 2010]
2. Half-integral PN terms starting at 5.5PN order permit checking the non-linear tail (and tail-of-tail) terms [Blanchet, Faye & Whiting 2014]

3.5PN energy flux of compact binaries

[BDIWW 1995; B 1996, 1998; BFIJ 2002; BDEI 2006]

$$\begin{aligned}\mathcal{F} = \frac{32c^5}{5G}\nu^2 x^5 \Bigg\{ 1 &+ \overbrace{\left(-\frac{1247}{336} - \frac{35}{12}\nu\right) x}^{\text{1PN}} + \overbrace{4\pi x^{3/2}}^{\text{1.5PN tail}} \\ &+ \left(-\frac{44711}{9072} + \frac{9271}{504}\nu + \frac{65}{18}\nu^2\right) x^2 + \overbrace{\left(-\frac{8191}{672} - \frac{583}{24}\nu\right)\pi x^{5/2}}^{\text{2.5PN tail}} \\ &+ \Bigg[\frac{6643739519}{69854400} + \overbrace{\frac{16}{3}\pi^2 - \frac{1712}{105}\gamma_\mathsf{E} - \frac{856}{105}\ln(16\,x)}^{\text{3PN tail-of-tail}} \\ &\qquad + \left(-\frac{134543}{7776} + \frac{41}{48}\pi^2\right)\nu - \frac{94403}{3024}\nu^2 - \frac{775}{324}\nu^3\Bigg] x^3 \\ &+ \underbrace{\left(-\frac{16285}{504} + \frac{214745}{1728}\nu + \frac{193385}{3024}\nu^2\right)\pi x^{7/2}}_{\text{3.5PN tail}} + \mathcal{O}\left(\frac{1}{c^8}\right)\Bigg\}\end{aligned}$$

The 4.5PN coefficient has been obtained recently [Marchand, Blanchet & Faye 2016]

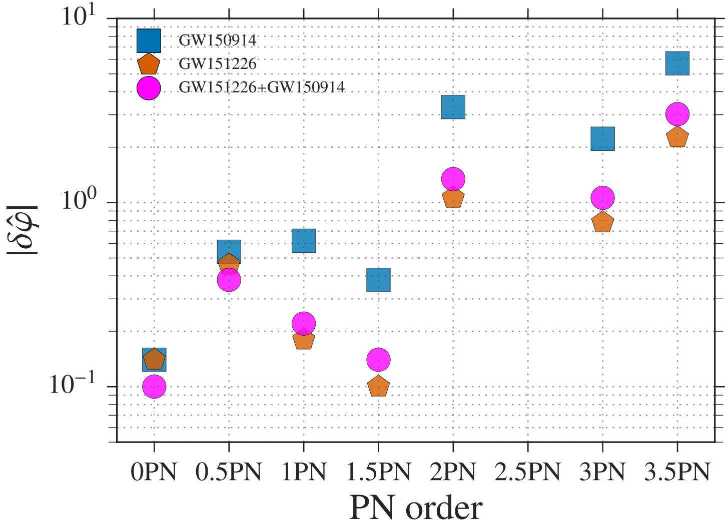
3.5PN energy flux of compact binaries

[BDIWW 1995; B 1996, 1998; BFIJ 2002; BDEI 2006]

$$
\begin{aligned}
\mathcal{F} = \frac{32c^5}{5G}\nu^2 x^5 \Bigg\{ 1 &+ \overbrace{\left(-\frac{1247}{336} - \frac{35}{12}\nu\right) x}^{\text{1PN}} + \overbrace{4\pi x^{3/2}}^{\text{1.5PN tail}} \\
&+ \left(-\frac{44711}{9072} + \frac{9271}{504}\nu + \frac{65}{18}\nu^2\right) x^2 + \overbrace{\left(-\frac{8191}{672} - \frac{583}{24}\nu\right)\pi x^{5/2}}^{\text{2.5PN tail}} \\
&+ \Bigg[\frac{6643739519}{69854400} + \overbrace{\frac{16}{3}\pi^2 - \frac{1712}{105}\gamma_E - \frac{856}{105}\ln(16\,x)}^{\text{3PN tail-of-tail}} \\
&\qquad + \left(-\frac{134543}{7776} + \frac{41}{48}\pi^2\right)\nu - \frac{94403}{3024}\nu^2 - \frac{775}{324}\nu^3\Bigg] x^3 \\
&+ \underbrace{\left(-\frac{16285}{504} + \frac{214745}{1728}\nu + \frac{193385}{3024}\nu^2\right)\pi x^{7/2}}_{\text{3.5PN tail}} + \mathcal{O}\left(\frac{1}{c^8}\right)\Bigg\}
\end{aligned}
$$

The 4.5PN coefficient has been obtained recently [Marchand, Blanchet & Faye 2016]

Measurements of PN parameters [LIGO/VIRGO collaboration 2016]



3.5PN dominant gravitational wave modes

[BIWW 1995; ABIQ 2004; BFIS 2008; FBI 2014]

$$\boxed{\begin{aligned} h_{22} &= \frac{2G\,m\,\nu\,x}{R\,c^2}\sqrt{\frac{16\pi}{5}}\,e^{-2\mathrm{i}\psi}\Bigg\{1 + x\left(-\frac{107}{42}+\frac{55\nu}{42}\right) + 2\pi x^{3/2} \\ &\qquad + x^2\left(-\frac{2173}{1512}-\frac{1069\nu}{216}+\frac{2047\nu^2}{1512}\right) \\ &\qquad + \underbrace{[\cdots]\;x^{5/2}}_{\text{2.5PN}} + \underbrace{[\cdots]\;x^{3}}_{\text{3PN}} + \underbrace{[\cdots]\;x^{7/2}}_{\text{3.5PN}} + \mathcal{O}\left(x^4\right)\Bigg\} \\ h_{33} &= \cdots \\ h_{31} &= \cdots \end{aligned}}$$

Tail contributions in this expression are factorized out in the phase variable

$$\psi = \phi - \frac{2GM\omega}{c^3}\ln\left(\frac{\omega}{\omega_0}\right)$$

4PN spin-orbit effects in the orbital frequency

[Marsat, Bohé, Faye, Blanchet & Buonanno 2013]

$$\frac{\dot{\omega}}{\omega^2} = \frac{96}{5}\,\nu\,x^{5/2}\Bigg\{\overbrace{1 + x\,[\cdots] + x^{3/2}\,[\cdots] + x^2\,[\cdots] + x^{5/2}\,[\cdots] + x^3\,[\cdots]}^{\text{non-spin terms}}$$
$$+ \underbrace{[\cdots]\,x^{3/2}}_{\text{1.5PN SO}} + \underbrace{[\cdots]\,x^2}_{\text{2PN SS}} + \underbrace{[\cdots]\,x^{5/2}}_{\text{2.5PN SO}} + \underbrace{[\cdots]\,x^3}_{\text{3PN SO}_{\text{tail}}\ \&\ \text{SS}}$$
$$+ \underbrace{[\cdots]\,x^{7/2}}_{\text{3.5PN SO}} + \underbrace{[\cdots]\,x^4}_{\text{4PN S0}_{\text{tail}}\ \&\ \text{SS}} + \mathcal{O}\left(x^4\right)\Bigg\}$$

- Leading SO and SS terms due to [Kidder, Will & Wiseman 1993; Kidder 1995]
- Many NL SS terms in EOM computed with the ADM Hamiltonian [Hergt, Steinhoff & Schäfer 2010] and the Effective Field Theory [Porto & Rothstein 2006; Levi 2010]

THE 4PN EQUATIONS OF MOTION

Based on a collaboration with

Laura Bernard, Alejandro Bohé, Guillaume Faye & Sylvain Marsat

4PN equations of motion of compact binaries

$$\boxed{\begin{aligned}\frac{\mathrm{d}v_1^i}{\mathrm{d}t} = &- \frac{Gm_2}{r_{12}^2} n_{12}^i \\ &+ \overbrace{\frac{1}{c^2}\left\{\left[\frac{5G^2 m_1 m_2}{r_{12}^3} + \frac{4G^2 m_2^2}{r_{12}^3} + \cdots\right] n_{12}^i + \cdots\right\}}^{\text{1PN Lorentz-Droste-Einstein-Infeld-Hoffmann term}} \\ &+ \underbrace{\frac{1}{c^4}[\cdots]}_{\text{2PN}} + \underbrace{\frac{1}{c^5}[\cdots]}_{\substack{\text{2.5PN}\\ \text{radiation reaction}}} + \underbrace{\frac{1}{c^6}[\cdots]}_{\text{3PN}} + \underbrace{\frac{1}{c^7}[\cdots]}_{\substack{\text{3.5PN}\\ \text{radiation reaction}}} + \underbrace{\frac{1}{c^8}[\cdots]}_{\substack{\text{4PN}\\ \text{conservative \& radiation tail}}} + \mathcal{O}\left(\frac{1}{c^9}\right)\end{aligned}}$$

2PN:

- [Otha, Okamura, Kimura & Hiida 1973, 1974; Damour & Schäfer 1985] — ADM Hamiltonian
- [Damour & Deruelle 1981; Damour 1983] — Harmonic coordinates
- [Kopeikin 1985; Grishchuk & Kopeikin 1986] — Extended fluid balls
- [Blanchet, Faye & Ponsot 1998] — Direct PN iteration
- [Itoh, Futamase & Asada 2001] — Surface integral method

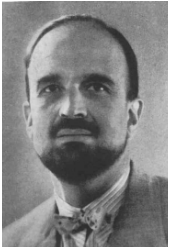
4PN equations of motion of compact binaries

$$\frac{\mathrm{d}v_1^i}{\mathrm{d}t} = -\frac{Gm_2}{r_{12}^2} n_{12}^i + \overbrace{\frac{1}{c^2}\left\{\left[\frac{5G^2m_1m_2}{r_{12}^3} + \frac{4G^2m_2^2}{r_{12}^3} + \cdots\right] n_{12}^i + \cdots\right\}}^{\text{1PN Lorentz-Droste-Einstein-Infeld-Hoffmann term}} + \underbrace{\frac{1}{c^4}[\cdots]}_{\text{2PN}} + \underbrace{\frac{1}{c^5}[\cdots]}_{\substack{\text{2.5PN}\\ \text{radiation reaction}}} + \underbrace{\frac{1}{c^6}[\cdots]}_{\text{3PN}} + \underbrace{\frac{1}{c^7}[\cdots]}_{\substack{\text{3.5PN}\\ \text{radiation reaction}}} + \underbrace{\frac{1}{c^8}[\cdots]}_{\substack{\text{4PN}\\ \text{conservative \& radiation tail}}} + \mathcal{O}\left(\frac{1}{c^9}\right)$$

3PN:
- [Jaranowski & Schäfer 1999; Damour, Jaranowski & Schäfer 2001] — ADM Hamiltonian
- [Blanchet & Faye 2000; de Andrade, Blanchet & Faye 2001] — Harmonic equations of motion
- [Itoh & Futamase 2003; Itoh 2004] — Surface integral method
- [Foffa & Sturani 2011] — Effective field theory

4PN:
- [Jaranowski & Schäfer 2013; Damour, Jaranowski & Schäfer 2014] — ADM Hamiltonian
- [Bernard, Blanchet, Bohé, Faye & Marsat 2015] — Fokker Lagrangian

Fokker action of N particles [Fokker 1929]



1. Gauge-fixed action for a system of N point particles

$$S = \frac{c^3}{16\pi G}\int \mathrm{d}^4x\,\sqrt{-g}\Big[R \underbrace{-\frac{1}{2}g_{\mu\nu}\Gamma^\mu\Gamma^\nu}_{\text{Gauge-fixing term}}\Big]$$

$$- \sum_A \underbrace{m_A c^2 \int \mathrm{d}t\,\sqrt{-(g_{\mu\nu})_A\, v_A^\mu v_A^\nu/c^2}}_{N\text{ point particles}}$$

2. Fokker action is obtained by inserting an explicit PN solution of the Einstein field equations

$$g_{\mu\nu}(\mathbf{x},t) \longrightarrow \overline{g}_{\mu\nu}(\mathbf{x};\boldsymbol{y}_B(t),\boldsymbol{v}_B(t),\cdots)$$

3. The PN equations of motion of the N particles (self-gravitating system) are

$$\boxed{\frac{\delta S_\mathsf{F}}{\delta \boldsymbol{y}_A} \equiv \frac{\partial L_\mathsf{F}}{\partial \boldsymbol{y}_A} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L_\mathsf{F}}{\partial \boldsymbol{v}_A}\right) + \cdots = 0}$$

Fokker action in the PN approximation

- The Fokker action is split into a PN (near-zone) term plus a contribution involving the multipole (far-zone) expansion

$$S^g_{\mathsf{F}} = \underset{B=0}{\mathrm{FP}} \int \mathrm{d}^4x \left(\frac{r}{r_0}\right)^B \overline{\mathcal{L}}_g + \underset{B=0}{\mathrm{FP}} \int \mathrm{d}^4x \left(\frac{r}{r_0}\right)^B \mathcal{M}(\mathcal{L}_g)$$

- The multipole term gives zero for any "instantaneous" term

$$\int \mathrm{d}^4x \left(\frac{r}{r_0}\right)^B \mathcal{M}(\mathcal{L}_g)\Big|_{\mathrm{inst}} = 0$$

thus only "hereditary" terms contribute and they are at least 5.5PN

- Finally we obtain

$$\boxed{S^g_{\mathsf{F}} = \underset{B=0}{\mathrm{FP}} \int \mathrm{d}^4x \left(\frac{r}{r_0}\right)^B \overline{\mathcal{L}}_g}$$

where the constant r_0 represents an IR cut-off scale and plays a crucial role at the 4PN order

Gravitational wave tail effect at the 4PN order

- At 4PN order there is an imprint of gravitational wave tails in the local (near-zone) dynamics of the source
- This leads to a non-local-in-time contribution in the Fokker action

$$\boxed{S_{\mathsf{F}}^{\mathsf{tail}} = \frac{G^2 M}{5c^8} \underset{s_0}{\mathsf{Pf}} \iint \frac{\mathrm{d}t\mathrm{d}t'}{|t-t'|} I_{ij}^{(3)}(t)\, I_{ij}^{(3)}(t')}$$

- The constant s_0 is *a priori* different from the IR scale r_0 but posing

$$s_0 = r_0\, e^{-\alpha}$$

  we find that r_0 finally cancels out so the result is IR finite
- The remaining constant α turns out to be an ambiguity parameter that we fix by requiring that the energy invariant function for circular orbits agrees with gravitational self-force (GSF) calculations at 4PN order

The method "$n+2$"

- Adopt as basic gravitational variables

$$\overline{h} \equiv \left(\overline{h}^{00} + \overline{h}^{ii}, \overline{h}^{0i}, \overline{h}^{ij}\right)$$

- Suppose that $\overline{h}$ is known up to order $1/c^{n+2}$ thus

$$\boxed{\overline{h} = \overline{h}_n + \delta\overline{h}_n \quad \text{where} \quad \delta\overline{h}_n = \mathcal{O}\left(\frac{1}{c^{n+3}}\right)}$$

- Expand the Fokker action around the known solution

$$S_\mathsf{F}[\overline{h}] = S_\mathsf{F}[\overline{h}_n] + \underbrace{\int \mathrm{d}^4x\, \frac{\delta S_\mathsf{F}}{\delta \overline{h}}[\overline{h}_n]\, \delta\overline{h}_n + \mathcal{O}(\delta\overline{h}_n^2)}_{\text{is at least of order } \mathcal{O}(1/c^{2n+2})}$$

- Thus the Fokker action is known up to $1/c^{2n}$ *i.e.* nPN order

Conserved energy for circular orbits at 4PN order

- The energy for circular orbits at the 4PN order in the small mass ratio limit is known from self-force calculations of the redshift variable
- This permits to fix the ambiguity parameter α and to complete the 4PN equations of motion

$$\boxed{\begin{aligned} E^{\text{4PN}} = -\frac{\mu c^2 x}{2}\Bigg\{ & 1 + \left(-\frac{3}{4} - \frac{\nu}{12}\right) x + \left(-\frac{27}{8} + \frac{19}{8}\nu - \frac{\nu^2}{24}\right) x^2 \\ & + \left(-\frac{675}{64} + \left[\frac{34445}{576} - \frac{205}{96}\pi^2\right]\nu - \frac{155}{96}\nu^2 - \frac{35}{5184}\nu^3\right) x^3 \\ & + \left(-\frac{3969}{128} + \left[-\frac{123671}{5760} + \frac{9037}{1536}\pi^2 + \frac{896}{15}\gamma_\mathsf{E} + \frac{448}{15}\ln(16x)\right]\nu \right. \\ & \left. \qquad + \left[-\frac{498449}{3456} + \frac{3157}{576}\pi^2\right]\nu^2 + \frac{301}{1728}\nu^3 + \frac{77}{31104}\nu^4\right) x^4 \Bigg\} \end{aligned}}$$

Conserved energy for circular orbits at 4PN order

- The energy for circular orbits at the 4PN order in the small mass ratio limit is known from self-force calculations of the redshift variable
- This permits to fix the ambiguity parameter α and to complete the 4PN equations of motion

$$\boxed{\begin{aligned}E^{\mathsf{4PN}} = -\frac{\mu c^2 x}{2}\Bigg\{1 &+ \left(-\frac{3}{4}-\frac{\nu}{12}\right)x + \left(-\frac{27}{8}+\frac{19}{8}\nu-\frac{\nu^2}{24}\right)x^2 \\ &+ \left(-\frac{675}{64}+\left[\frac{34445}{576}-\frac{205}{96}\pi^2\right]\nu-\frac{155}{96}\nu^2-\frac{35}{5184}\nu^3\right)x^3 \\ &+ \Bigg(-\frac{3969}{128}+\left[-\frac{123671}{5760}+\frac{9037}{1536}\pi^2+\frac{896}{15}\gamma_{\mathsf{E}}+\frac{448}{15}\ln(16x)\right]\nu \\ &\qquad + \left[-\frac{498449}{3456}+\frac{3157}{576}\pi^2\right]\nu^2+\frac{301}{1728}\nu^3+\frac{77}{31104}\nu^4\Bigg)x^4\Bigg\}\end{aligned}}$$

Conserved energy for circular orbits at 4PN order

1. We did several computations of the energy function $E(\Omega)$. For instance, we can use the associated Hamiltonian formalism [DJS 2014]

$$S_\mathsf{F} = \int_{-\infty}^{+\infty} \left[\sum_A p_A^i v_A^i - H\right] \quad \text{where} \quad H_\mathsf{F}^{\mathsf{tail}} = -L_\mathsf{F}^{\mathsf{tail}}$$

2. With canonical variables r, φ, p_r, p_φ we have to solve for circular orbits

$$\begin{aligned} \frac{\delta H}{\delta r}\left[r^0, p_r^0 = 0, p_\varphi^0\right] &= 0 \\ \frac{\delta H}{\delta p_\varphi}\left[r^0, p_r^0 = 0, p_\varphi^0\right] &= \Omega \end{aligned}$$

3. Our end result for the 4PN equations of motion differs from [DJS 2014]
   - We disagree on their treatment of the non-local action when computing the energy for circular orbits
   - We are improving our IR regularization [BBBFM, in progress]