Dirac Combs and Hurwitz-Zeta Functions in QCD
Evaluation of the HVP contribution to $g_\mu - 2$

Eduardo de Rafael

Aix-Marseille Université, CNRS, CPT, UMR 7332
13288 Marseille, France

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Light and shadow among QCD and QED
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Introduction

Motivation

Study of QCD two-point functions of color singlet local operators (*with possible insertions of soft operators*).
Integrals of these Green's functions over their euclidean momenta *(with appropriate weights)* govern the hadronic contributions to many electromagnetic and weak interaction processes.

Two simple examples *(with no soft insertions)*

- Hadronic Vacuum Polarization two-point function (HVP)

$$\Pi_{\mu\nu}(q) = i\int d^4x\, e^{iq\cdot x}\langle 0|T\left(J_\mu(x)J_\nu(0)\right)|0\rangle = (q_\mu q_\nu - q^2 g_{\mu\nu})\Pi(Q^2)\,,$$

- The Left-Right two-point function (LR) *(in the chiral limit)*

$$\Pi_{LR}^{\mu\nu}(q) = 2i\int d^4x\, e^{iq\cdot x}\langle 0|T\left(L^\mu(x)R^\nu(0)^\dagger\right)|0\rangle = (q^\mu q^\nu - g^{\mu\nu}q^2)\Pi_{LR}(Q^2)\,.$$

They provide excellent theoretical laboratories to test non perturbative approaches.

$\frac{1}{2}(g_\mu - 2)_{\text{Hadrons}} \equiv a_\mu^{\text{HVP}}$ as an example. It is an Integral over the Euclidean HVP Two-Point Function

Euclidean Representation of a_μ^{HVP} (*Lautrup- de Rafael '69*)

$$a_\mu^{\text{HVP}} = \frac{\alpha}{\pi}\int_0^1 dx(1-x)\left[-\Pi\left(\frac{x^2}{1-x}m_\mu^2\right)\right], \quad Q^2 \equiv \frac{x^2}{1-x}m_\mu^2 .$$

This is also the representation used in LQCD (*Blum '03*)

Recall that $\Pi(Q^2)$ (renormalized on shell at $Q^2 = 0$) obeys the Dispersion Relation:

$$-\Pi(Q^2) = \int_0^\infty \frac{dt}{t}\underbrace{\frac{Q^2}{t+Q^2}}\frac{1}{\pi}\text{Im}\Pi(t), \quad Q^2 = -q^2 \geq 0 ,$$

and therefore

$$a_\mu^{\text{HVP}} = \frac{\alpha}{\pi}\int_0^1 dx\,(1-x)\int_0^\infty \frac{dt}{t}\underbrace{\frac{\frac{x^2}{1-x}m_\mu^2}{t+\frac{x^2}{1-x}m_\mu^2}}\frac{1}{\pi}\text{Im}\Pi(t),$$

where $\sigma(t)_{[e^+e^- \to (\gamma) \to \text{Hadrons}]} = \frac{4\pi^2\alpha}{t}\frac{1}{\pi}\text{Im}\Pi(t)$
which is the Standard Phenomenological Representation (*Bouchiat-Michel '61*).

Mellin-Barnes Representation of HVP

There is a representation of $\Pi(Q^2)$ in terms of

$$\mathcal{M}(s) = \underbrace{\int_{4m_\pi^2}^{\infty} \frac{dt}{t} \left(\frac{m_\mu^2}{t}\right)^{1-s} \frac{1}{\pi}\mathrm{Im}\Pi(t)}_{\textit{The Mellin Transform of the Spectral Function}} \qquad \mathrm{Re}\, s < 1\,.$$

From pQCD we know that

$$\mathcal{M}(s) \underset{s\to 1}{\sim} \left(\frac{\alpha}{\pi}\right)\left(\frac{2}{3}\right) N_c \,\frac{1}{3}\,\frac{1}{1-s}\,.$$

The Representation in question is an inverse Mellin-Barnes Integral (*EdeR'14*):

$$\Pi(Q^2) = -\frac{Q^2}{m_\mu^2}\,\frac{1}{2\pi i}\int_{c_s-i\infty}^{c_s+i\infty} ds \left(\frac{Q^2}{m_\mu^2}\right)^{-s} \Gamma(s)\Gamma(1-s)\,\mathcal{M}(s)\,, \quad c_s \equiv \mathrm{Re}(s) \in]0,1[\,.$$

Very useful for expansions of $\Pi(Q^2)$ for Q^2 small (χPT) and large Q^2 (OPE).

(For QED applications see Friot-Greynat-de Rafael'08, Aguilar-Greynat-de Rafael'12)

Ramanujan's Master Theorem

Expansion For Q^2-Small

$$-\frac{m_\mu^2}{Q^2}\Pi(Q^2) \underset{Q^2\to 0}{\sim} \left\{\mathcal{M}(0) - \frac{Q^2}{m_\mu^2}\mathcal{M}(-1) + \left(\frac{Q^2}{m_\mu^2}\right)^2 \mathcal{M}(-2) - \left(\frac{Q^2}{m_\mu^2}\right)^3 \mathcal{M}(-3) + \cdots \right\},$$

where ($n = 0, 1, 2 \cdots$)

$$\mathcal{M}(-n) = \int_{4m_\pi^2}^{\infty} \frac{dt}{t}\left(\frac{m_\mu^2}{t}\right)^{1+n} \frac{1}{\pi}\mathrm{Im}\Pi(t) = \frac{(-1)^{n+1}}{(n+1)!}(m_\mu^2)^{n+1}\left(\frac{\partial^{n+1}}{(\partial Q^2)^{n+1}}\Pi(Q^2)\right)_{Q^2=0}.$$

Ramanujan's Theorem:

$$\int_0^\infty d\left(\frac{Q^2}{m_\mu^2}\right)\left(\frac{Q^2}{m_\mu^2}\right)^{s-1}\left\{\mathcal{M}(0) - \frac{Q^2}{m_\mu^2}\mathcal{M}(-1) + \left(\frac{Q^2}{m_\mu^2}\right)^2 \mathcal{M}(-2) + \cdots \right\} = \Gamma(s)\Gamma(1-s)\mathcal{M}(s),$$

Guarantees the convergence of discrete moments $\mathcal{M}(-n)$ to the full Mellin transform $\mathcal{M}(s)$.

Recall that LQCD has access to the discreet $\mathcal{M}(-n)$ (at least for low n)

Mellin-Barnes Representation of the Muon Anomaly (EdeR, P.L.'14)

Integrating over x, i.e. $Q^2 = \frac{x^2}{1-x} m_\mu^2$ results in:

Integral Representation of a_μ^{HVP} (Model Independent)

$$a_\mu^{\mathrm{HVP}} = \left(\frac{\alpha}{\pi}\right) \frac{1}{2\pi i} \int\limits_{c-i\infty}^{c+i\infty} ds\, \mathcal{F}(s)\, \underbrace{\mathcal{M}(s)}, \qquad \mathrm{Re}\ c \in]0, +1[$$

$$\mathcal{F}(s) = -\Gamma(3-2s)\Gamma(-3+s)\Gamma(1+s)$$

$$\mathcal{M}(s) = \underbrace{\int_{4m_\pi^2}^{\infty} \frac{dt}{t} \left(\frac{m_\mu^2}{t}\right)^{1-s} \frac{1}{\pi} \mathrm{Im}\Pi(t)}_{\textit{Mellin Transform of the Spectral Function}}$$

$\mathcal{M}(s)$ is finite for $s < 1$ and singular at $s = 1$:

$$\mathcal{M}_{\mathrm{pQCD}}(s) \underset{s \to 1}{\sim} \left(\frac{\alpha}{\pi}\right) \left(\frac{2}{3}\right) N_c \frac{1}{3} \frac{1}{1-s}\,.$$

Ramanujan's Theorem Limitations

- The nice feature of Ramanujan's Theorem:

  Guarantees the convergence to the full Mellin transform $\mathcal{M}(s)$

- Unfortunately, it does not tell us which *Interpolation Function* to use when we only know a few discrete $\mathcal{M}(-n)$.

- *Padé Approximants* to $\Pi(Q^2)$ at low-Q^2 *cannot be the answer*, because they don't reproduce the pQCD behaviour at $s = 1$ of $\mathcal{M}(s)$.
- *Padé Approximants* to $\Pi(Q^2)$ at low-Q^2 plus pQCD for high-Q^2 values (*the favored LQCD practice at present*) *has not been proved to be the best possible interpolation*, and my claim is that *it is not.*

- In fact there is even *no proof* that the *Padé Approximants* to $\Pi(Q^2)$ at low-Q^2 *plus* pQCD for high-Q^2 values *satisfy Ramanujan's Theorem*.

Replace Physical Spectral Function by Infinite sum of Distributions

$$\frac{1}{\pi}\mathrm{Im}\Pi(t) \;\Rightarrow\; \mathcal{P}(t) \equiv \sum_{n=0}^{\infty} \Big\{ \mathcal{N}\,\sigma^2\delta(t - M^2 - n\sigma^2) + \mathcal{N}_2\,\sigma^2\delta^{(1)}(t - M^2 - n\sigma^2) +$$
$$\mathcal{N}_4\,\sigma^2\delta^{(2)}(t - M^2 - n\sigma^2) + \mathcal{N}_6\,\sigma^2\delta^{(3)}(t - M^2 - n\sigma^2) + \cdots \Big\}\,.$$

WHY?

Because the Mellin transform of Dirac Combs and their derivatives are Hurwitz-Zeta Functions:

$$\int_0^\infty \frac{dt}{t}\left(\frac{m_\mu^2}{t}\right)^{1-s} \mathcal{N}\,\sigma^2\delta(t - M^2 - n\sigma^2) \;=\; \mathcal{N}\left(\frac{m_\mu^2}{\sigma^2}\right)^{1-s} \zeta\left(2 - s, v \equiv \frac{M^2}{\sigma^2}\right)\,,$$

$$\int_0^\infty \frac{dt}{t}\left(\frac{m_\mu^2}{t}\right)^{1-s} \mathcal{N}_2\,\sigma^2\delta^{(1)}(t - M^2 - n\sigma^2) \;=\; \mathcal{N}_2\left(\frac{m_\mu^2}{\sigma^2}\right)^{1-s} (2-s)\,\zeta\left(3 - s, v \equiv \frac{M^2}{\sigma^2}\right)\,,$$

$$\int_0^\infty \frac{dt}{t}\left(\frac{m_\mu^2}{t}\right)^{1-s} \mathcal{N}_4\,\sigma^2\delta^{(1)}(t - M^2 - n\sigma^2) \;=\; \mathcal{N}_4\left(\frac{m_\mu^2}{\sigma^2}\right)^{1-s} (2-s)(3-s)\,\zeta\left(4 - s, v \equiv \frac{M^2}{\sigma^2}\right)\,,$$

Hurwitz-Zeta Functions have the desired QCD singularity structure

Properties of the Hurwitz-Zeta Function

- The Hurwitz-Zeta function is defined by the Dirichlet Series

$$\zeta(s,v) = \sum_{n=0}^{\infty} \frac{1}{(n+v)^s}\,, \quad \mathrm{Re}\, s > 1 \quad \text{and} \quad \mathrm{Re}\, v \neq -n\,.$$

- Integral Representation (which provides basis for analytic continuation)

$$\zeta(s,v) = \frac{1}{\Gamma(s)} \int_0^{\infty} dx x^{s-1} \frac{e^{-vx}}{1-e^{-x}}\,, \quad \mathrm{Re}\, s > 1\,, \mathrm{Re}\, v > 0\,,$$

- In particular, for $s = -m$ with $m = 0, 1, 2, \cdots$,

$$\zeta(-m,v) = -\frac{\mathrm{B}_{m+1}(v)}{m+1}\,,$$

where $\mathrm{B}_{m+1}(v)$ are the Bernoulli polynomial of degree $m+1$:

$$\mathrm{B}_1(v) = v - \frac{1}{2}\,, \quad \mathrm{B}_2(v) = v^2 - v + \frac{1}{6} \quad \cdots$$

- Integral Representation implies

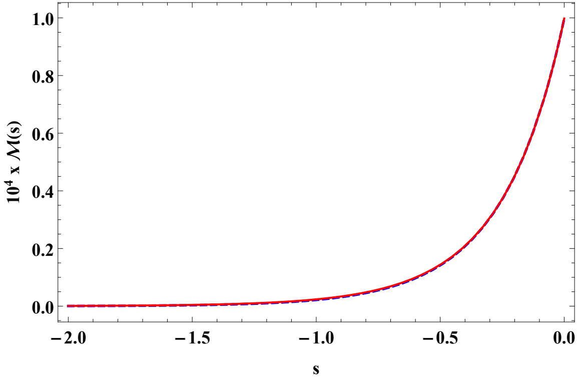
$$\frac{1}{\pi}\mathrm{Im}\Pi(t) \Rightarrow \left\{ \mathcal{N}\frac{\sigma^2}{t} + \mathcal{N}_2\left(\frac{\sigma^2}{t}\right)^2 + \mathcal{N}_4\left(\frac{\sigma^2}{t}\right)^3 + \cdots \right\} \frac{e^{-M^2/t}}{1-e^{-\sigma^2/t}}$$

Which approaches better and better to the shape of the Physical Spectral Function

- First Approximation

$$\mathcal{P}^{(\text{first})}(t) = \frac{\alpha}{\pi}\frac{1}{3}N_c\left\{\left(\frac{2}{3}\right)\sum_{n=0}^{\infty}\sigma^2\delta(t-M^2-n\sigma^2)+\left(\frac{4}{9}\right)\sum_{n=0}^{\infty}\sigma'^2\delta(t-M'^2-n\sigma'^2)\right\},$$

Fix M and σ with $\mathcal{M}(0)$ from Toy Model and the fact that there is no $1/Q^2$ term in OPE



Mellin Transforms of the Spectral Function of the Toy Model (red) and of the first Spectral Function Distribution (blue-dashed).

Continuation of Tests

$$a_\mu^{\mathrm{HVP}}(\mathrm{first}) = \left(\frac{\alpha}{\pi}\right)^2 \frac{1}{3} N_c \, \frac{1}{2} \int_0^1 dx x(2-x) \int_0^\infty dt \frac{m_\mu^2}{\left(t + \frac{x^2}{1-x} m_\mu^2\right)^2} \, \mathcal{P}^{(\mathrm{first})}(t)$$

$$= \left(\frac{\alpha}{\pi}\right)^2 \frac{1}{3} N_c \, \frac{1}{2} \frac{m_\mu^2}{\sigma^2} \int_0^1 dx x(2-x) \times$$

$$\left[\frac{2}{3}\zeta\left(2, v + \frac{x^2}{1-x}\frac{m_\mu^2}{\sigma^2}\right) + \frac{4}{9}\frac{\sigma^2}{\sigma'^2}\zeta\left(2, v' + \frac{x^2}{1-x}\frac{m_\mu^2}{\sigma'^2}\right)\right]$$

$$= 6.726 \times 10^{-8} \, .$$

This result reproduces the Toy Model result at the 1% level !

If we use the BHLS-Model input: $\mathcal{M}(0) = 10.1307 \pm 0.0745$

$$a_\mu^{\mathrm{HVP}}(\mathrm{first}) = (681.85 \pm 4.79) \times 10^{-10} \quad a_\mu^{\mathrm{HVP}}(\mathrm{BHLS}) = (683.50 \pm 4.75) \times 10^{-10}$$

Excellent Agreement (even better than with the toy model)

Continuation of Tests: Second Approximation

Assume that the first two derivatives of $\Pi(Q^2)$ at the origin are known. In the Toy Model: $\mathcal{M}(0) = 0.9979 \times 10^{-4}$ and $\mathcal{M}(-1) = 0.0235 \times 10^{-4}$.

$$\mathcal{P}^{(\mathrm{second})}(t) = \frac{\alpha}{\pi}\frac{1}{3}N_c \sum_{n=0}^{\infty}\left\{\left(\frac{2}{3}\right)\sigma^2\delta(t - M^2 - n\sigma^2) + \beta\sigma^2\delta^{(2)}(t - M^2 - n\sigma^2) + \frac{4}{9}\sigma'^2\delta(t - M^2 - n\sigma^2)\right\}.$$

This fixes the parameter values:

$$\sigma = 0.9775\ \mathrm{GeV}, \quad \beta = 0.00406 \quad \text{and} \quad \mathrm{v} = \frac{1}{2}.$$

$$a_\mu^{\mathrm{HVP}}(\mathrm{second}) = \left(\frac{\alpha}{\pi}\right)^2 \frac{1}{3}N_c\,\frac{1}{2}\int_0^1 dx x(2-x)\int_0^\infty dt\,\frac{m_\mu^2}{\left(t + \frac{x^2}{1-x}m_\mu^2\right)^2}\,\mathcal{P}^{(\mathrm{second})}(t)$$

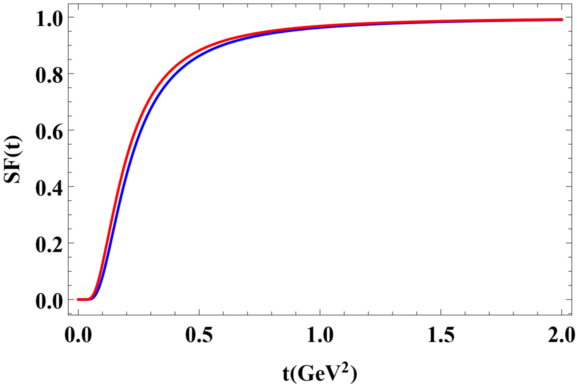
$$= \left(\frac{\alpha}{\pi}\right)^2 \frac{1}{3}N_c\,\frac{1}{2}\,\frac{m_\mu^2}{\sigma^2}\int_0^1 dx x(2-x)\times$$

$$\left[\frac{2}{3}\zeta\left(2, v + \frac{x^2}{1-x}\frac{m_\mu^2}{\sigma^2}\right) + 6\,\beta\,\zeta\left(4, v + \frac{x^2}{1-x}\frac{m_\mu^2}{\sigma^2}\right) + \frac{4}{9}\frac{\sigma^2}{\sigma'^2}\zeta\left(2, v' + \frac{x^2}{1-x}\frac{m_\mu^2}{\sigma'^2}\right)\right]$$

$$= 6.817 \times 10^{-8}.$$

which reproduces the Toy Model result at the 0.4% level !!

Spectral Functions Corresponding to First and Second Approximations



Spectral Function of the 2nd Approximation (red) and of the first Approximation (blue).

- We conclude from these tests that with a precise determination of $\mathcal{M}(0)$ i.e. *with a precise determination of just the slope of the HVP function at the origin from LQCD*, one can already obtain the result for a_{μ}^{HVP} with an *accuracy comparable to the determination using experimental data*.
- We wish to emphasize that the method we propose, besides the eventual determination of $\mathcal{M}(0)$, only uses as other input two well known properties of QCD: *asymptotic freedom* and the fact that in the chiral limit there is *no* $1/Q^2$ *term in the OPE of* $\Pi(Q^2)$.
- As shown, *the method is improvable* with more and more experimental or LQCD input and *Ramanujan guarantees convergence*.