Generalized Parton Distributions and their covariant extension

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Outline

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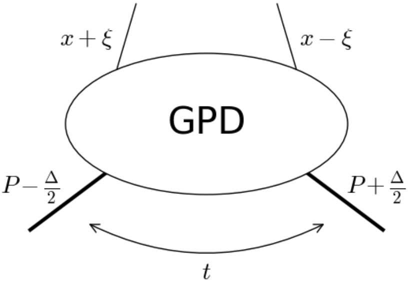
1. Introduction to Generalized Parton Distributions
   - Definition and properties
   - Experimental access
2. Modeling Generalized Parton Distributions
   - Overlap of Light-cone wave functions
   - Double Distributions
   - Inversion of Incomplete Radon Transform
   - Results
3. Conclusion

Definition of GPDs

- Quark GPD (twist-2, spin-0 hadron): (Müller et al., 1994; Radyushkin, 1996; Ji, 1997)

$$H^q(x,\xi,t) = \frac{1}{2}\int \frac{\mathrm{d}z^-}{2\pi}\, e^{i\,x\,P^+ z^-} \left\langle P+\frac{\Delta}{2}\left|\bar{q}(-z)\,\gamma^+ q(z)\right| P-\frac{\Delta}{2}\right\rangle\Big|_{z^+=0,\,z_\perp=0} . \tag{1}$$



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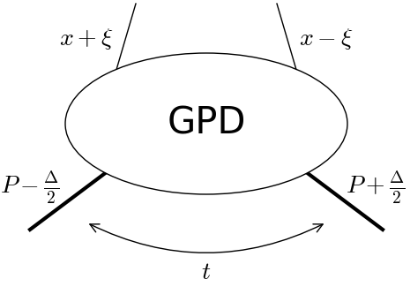

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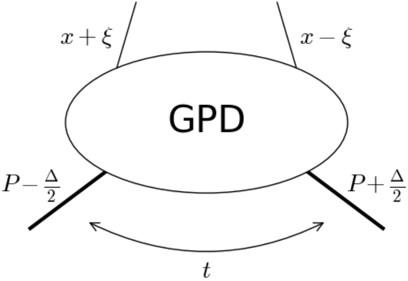

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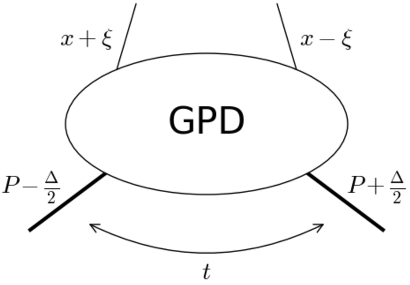

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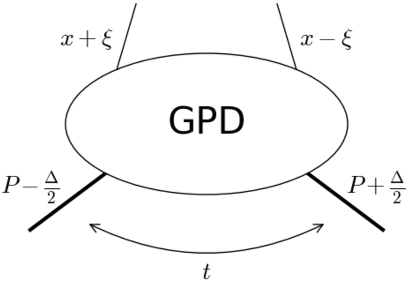

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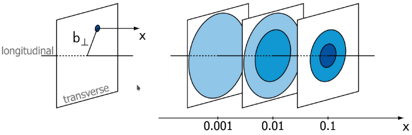

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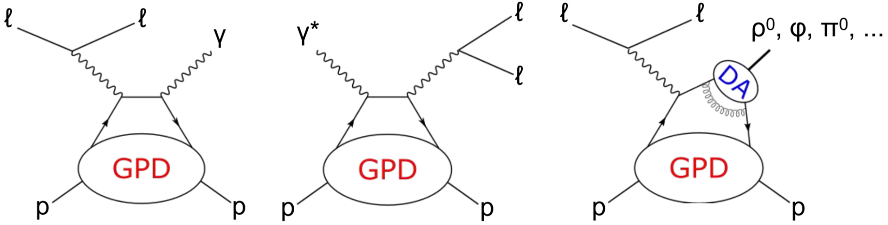

DVCS TCS DVMP

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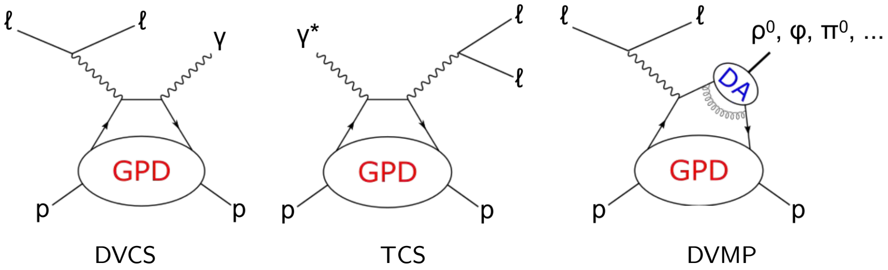

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$$\mathcal{F}\left(\xi, t, Q^2\right) = \int_{-1}^{1} \mathrm{d}x\, C\left(x, \xi, \alpha_S\left(\mu_F\right), \frac{Q}{\mu_F}\right) F\left(x, \xi, t, \mu_F\right). \quad (7)$$

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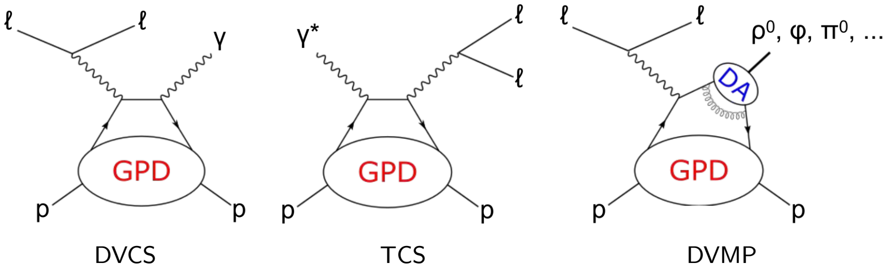

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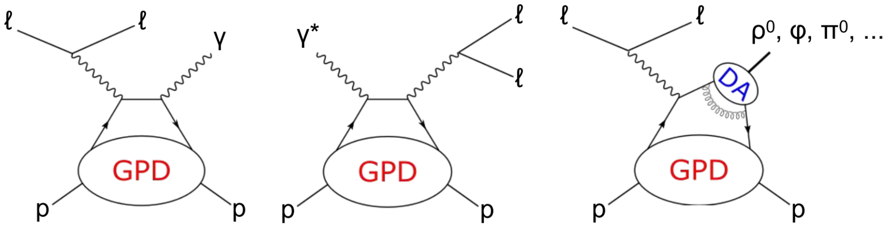

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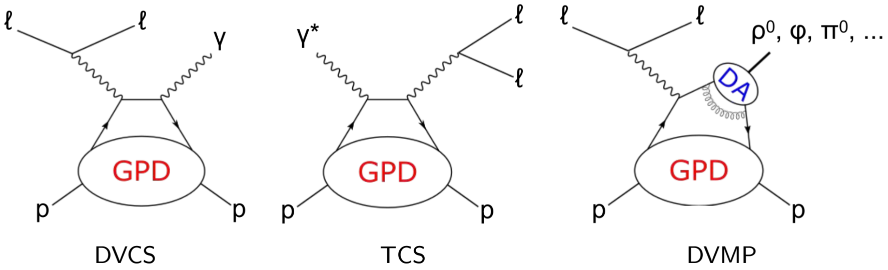

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- Observables are convolutions of:
  - a hard-scattering kernel, calculated with perturbative QCD (short distance interactions).
  - a soft part, i.e. the GPD, with long distance interactions encoded (non-perturbative QCD).

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- A given *hadronic state* is decomposed in a **Fock basis**: (Brodsky et al., 1981)

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- DD f is defined on the support $\Omega = \{|\beta| + |\alpha| \leq 1\}$.
- **Polynomial** in ξ:

$$\begin{aligned} \int_{-1}^{1} \mathrm{d}x\, x^m\, H(x,\xi,t) &\propto \int \mathrm{d}x\, x^m \int_\Omega \mathrm{d}\beta\,\mathrm{d}\alpha\, f(\beta,\alpha,t)\,\delta(x-\beta-\alpha\xi) \\ &\propto \int_\Omega \mathrm{d}\beta\,\mathrm{d}\alpha\,(\beta+\xi\alpha)^m\, f(\beta,\alpha,t)\ . \end{aligned} \tag{14}$$

- Positivity not manifest...

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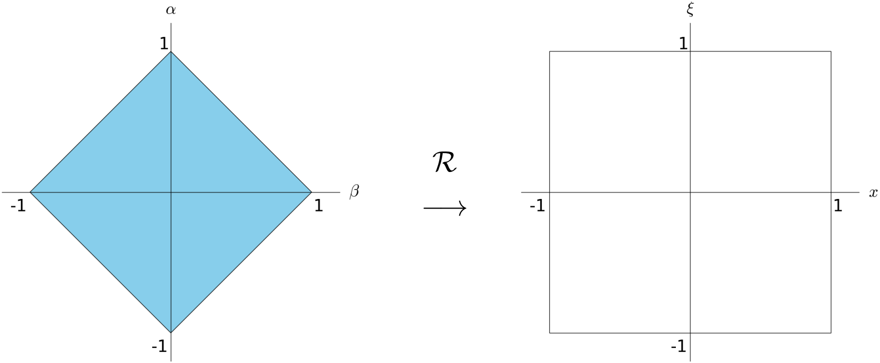

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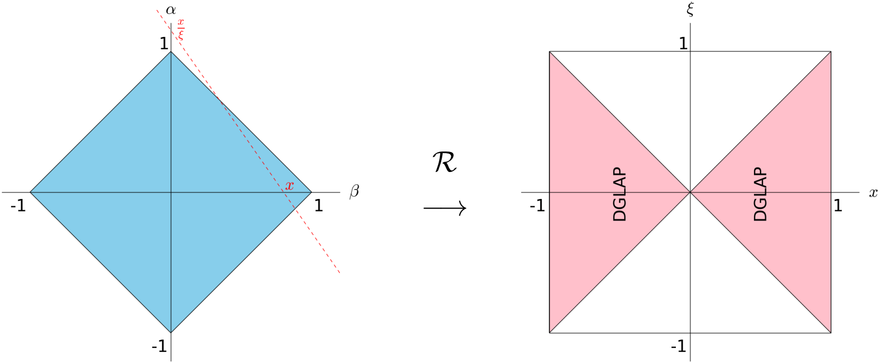

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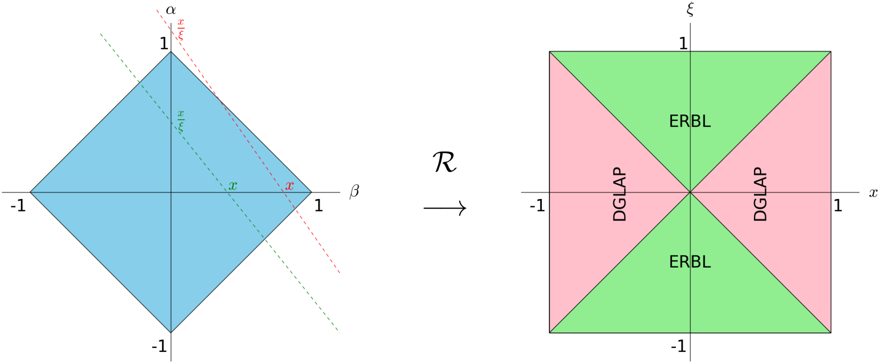

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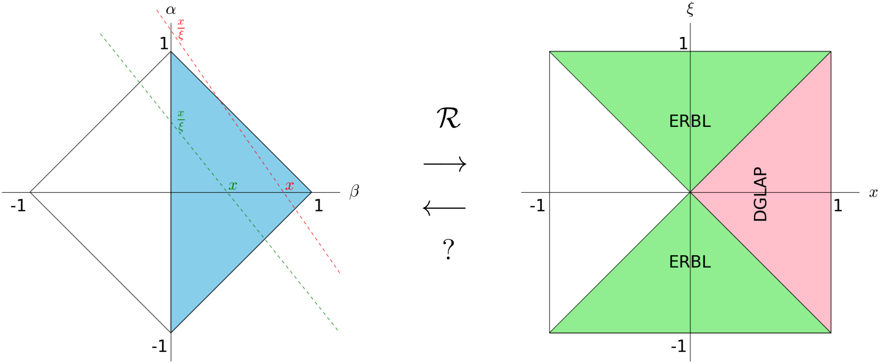
  - We can reconstruct the GPD everywhere.

Numerical Inversion



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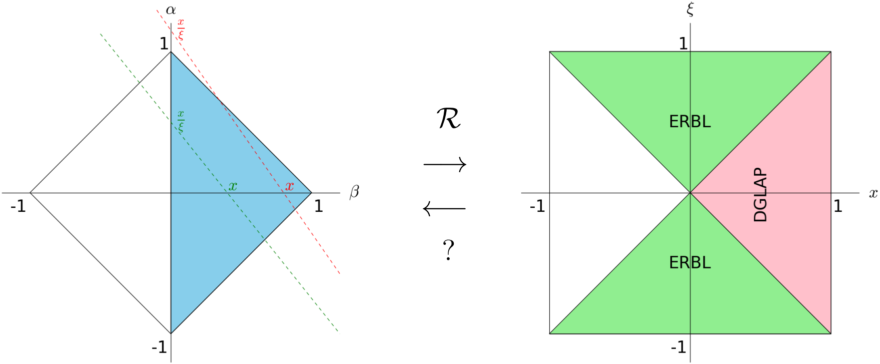

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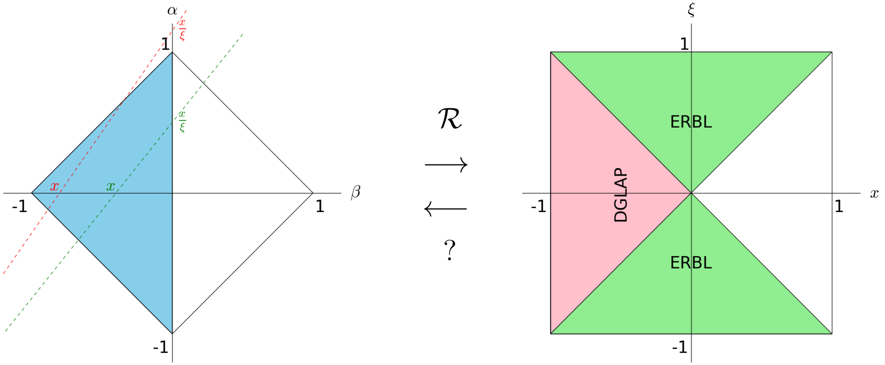

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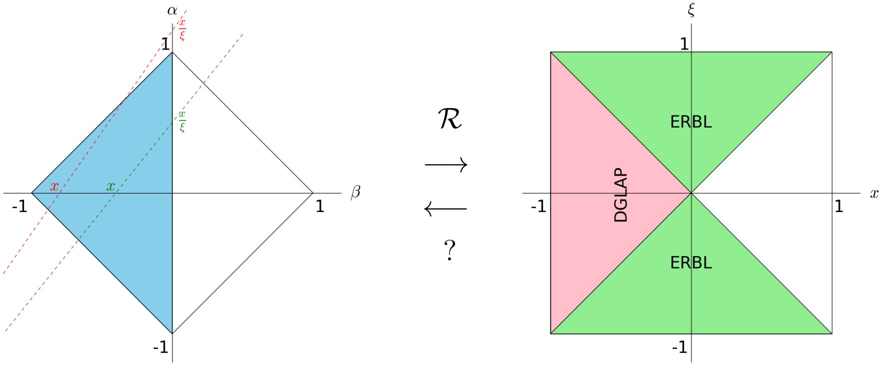

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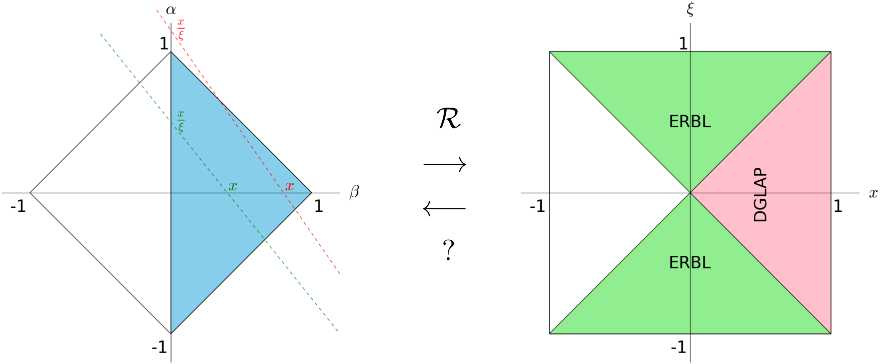

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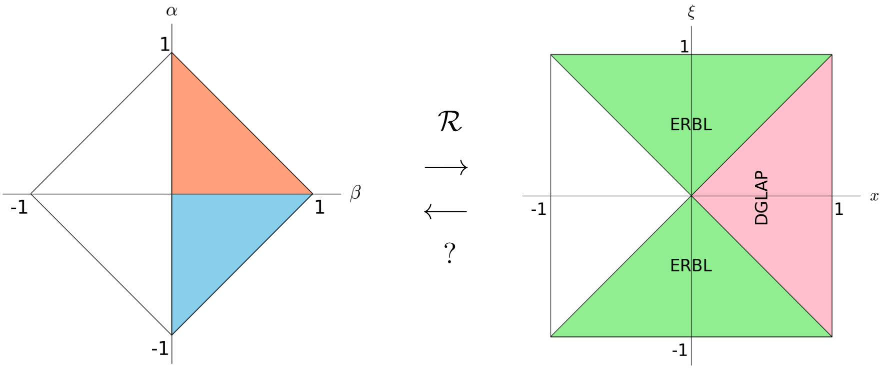

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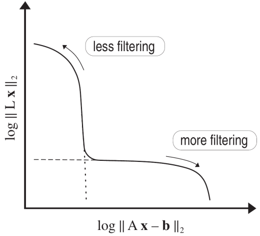
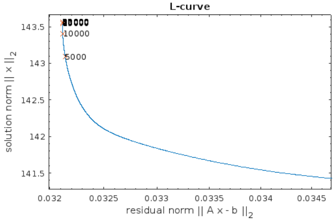


Theoretical "L-curve": curve parameterized by the regularization factor.

(fig. taken from Ref. **(Hansen, 2007)**)



L-curve with the iteration number as regularization factor.

Some examples (Dyson-Schwinger model)

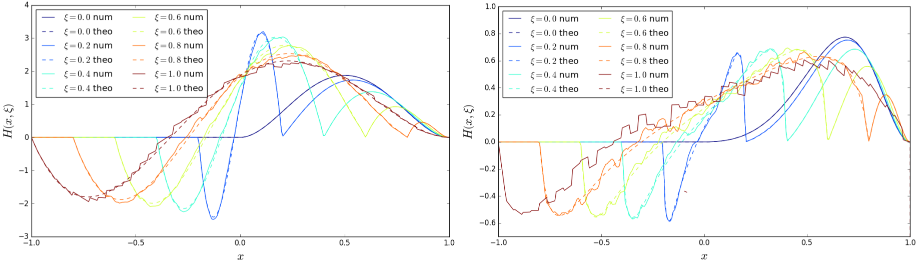


Figure: Extension of GPDs for the pion DSE model of Refs. **(Mezrag, 2015; Mezrag et al., 2016)**. Left: $t = 0$ GeV^2. Right: $t = 1$ GeV^2. Comparison to the analytical result.

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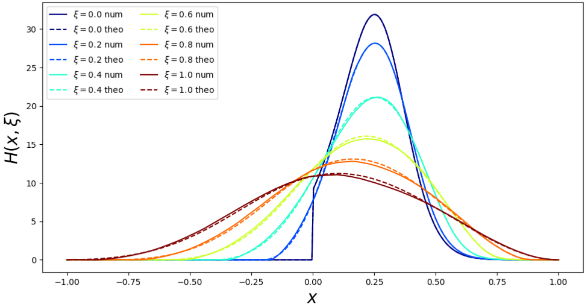

Figure: Extension of GPD E for the nucleon model of Ref. **(Hwang and Mueller, 2008)**. Comparison to the analytical result of the paper.

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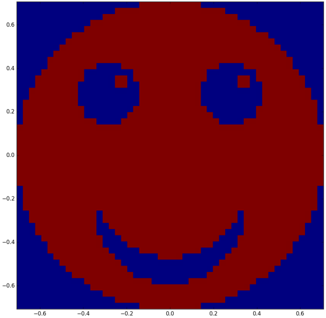

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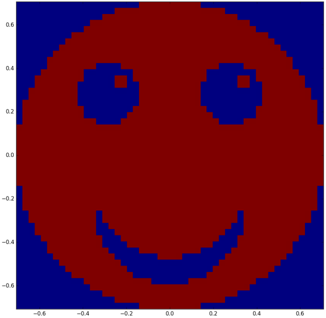

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