Hadronic light-by-light scattering contribution to the anomalous moment of the muon from lattice QCD

Antoine Gérardin

Institut für Kernphysik, Johannes Gutenberg-Universität Mainz

December 6, 2017 - GDR QCD

Anomalous magnetic moment of the muon : $(g-2)_\mu$

Magnetic moment $\vec{\mu} = g\left(\frac{Qe}{2m}\right)\vec{s}$

Anomalous magnetic moment of the muon : $(g-2)_\mu$

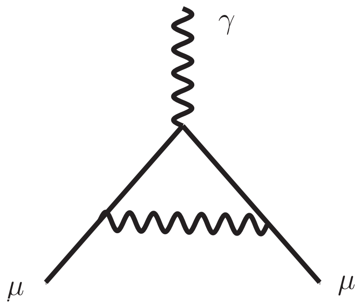
Magnetic moment $\vec{\mu} = g\left(\frac{Qe}{2m}\right)\vec{s}$

- Classical result : $g = 2$ - Quantum field theory : $a_\mu = \frac{g-2}{2} \neq 0$

Anomalous magnetic moment of the muon : $(g-2)_\mu$

Magnetic moment $\vec{\mu} = g \left(\frac{Qe}{2m} \right) \vec{s}$

- Classical result : $g = 2$ - Quantum field theory : $a_\mu = \frac{g-2}{2} \neq 0$



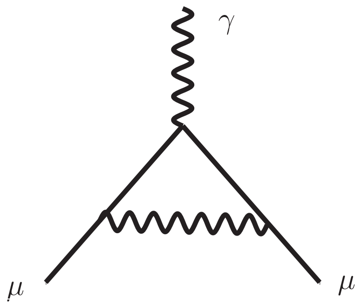
$$a_\mu^{(1)} = \frac{\alpha}{2\pi}$$

[Schwinger '48]

Anomalous magnetic moment of the muon : $(g-2)_\mu$

Magnetic moment $\vec{\mu} = g\left(\frac{Qe}{2m}\right)\vec{s}$

- Classical result : $g = 2$ - Quantum field theory : $a_\mu = \frac{g-2}{2} \neq 0$



$$a_\mu^{(1)} = \frac{\alpha}{2\pi}$$ [Schwinger '48]

- Experimental value : $a_\mu = (116\ 592\ 089 \pm 63) \times 10^{-11}$ [Bennett et al. '06]

$(g-2)_\mu$: present status

- Experimental value : $a_\mu = (116\ 592\ 089 \pm 63) \times 10^{-11}$ [Bennett et al. '06]
- Theory :

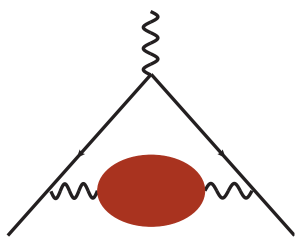
Contribution	$a_\mu \times 10^{11}$	
- QED (leptons, 5^{th} order)	$116\ 584\ 718.846 \pm 0.037$	[Aoyama et al. '12]
- Electroweak	153.6 ± 1.0	[Gnendiger et al. '13]
- Strong contributions		
HVP (LO)	$6\ 869.9 \pm 42.1$	[Jegerlehner '15]
HVP (NLO)	-98 ± 1	[Hagiwara et al. 11]
HVP (NNLO)	12.4 ± 0.1	[Kurtz et al. '14]
HLbL	102 ± 39	[Jegerlehner '15, Nyffeler '09]
Total (theory)	$116\ 591\ 811 \pm 62$	

$(g-2)_\mu$: present status

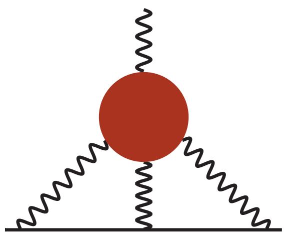
- Experimental value : $a_\mu = (116\ 592\ 089 \pm 63) \times 10^{-11}$ [Bennett et al. '06]
- Theory :

Contribution	$a_\mu \times 10^{11}$	
- QED (leptons, 5^{th} order)	$116\ 584\ 718.846 \pm 0.037$	[Aoyama et al. '12]
- Electroweak	153.6 ± 1.0	[Gnendiger et al. '13]
- Strong contributions		
HVP (LO)	$6\ 869.9 \pm 42.1$	[Jegerlehner '15]
HVP (NLO)	-98 ± 1	[Hagiwara et al. 11]
HVP (NNLO)	12.4 ± 0.1	[Kurtz et al. '14]
HLbL	102 ± 39	[Jegerlehner '15, Nyffeler '09]
Total (theory)	$116\ 591\ 811 \pm 62$	

Hadronic Vacuum Polarisation (HVP, α^2) :



Hadronic Light-by-Light (HLbL, α^3) :



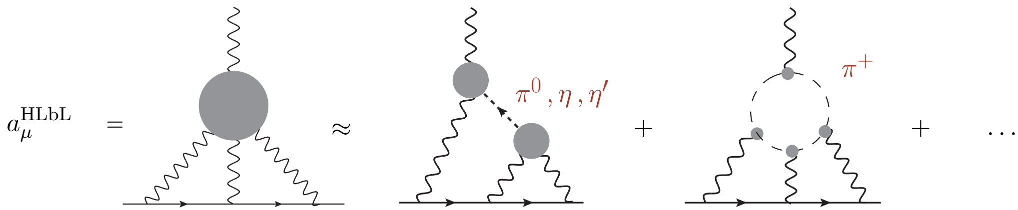
$(g-2)_\mu$: present status

- Experimental value : $a_\mu = (116\ 592\ 089 \pm 63) \times 10^{-11}$ [Bennett et al. '06]
- Theory :

Contribution	$a_\mu \times 10^{11}$	
- QED (leptons, 5^{th} order)	$116\ 584\ 718.846 \pm 0.037$	[Aoyama et al. '12]
- Electroweak	153.6 ± 1.0	[Gnendiger et al. '13]
- Strong contributions		
HVP (LO)	$6\ 869.9 \pm 42.1$	[Jegerlehner '15]
HVP (NLO)	-98 ± 1	[Hagiwara et al. 11]
HVP (NNLO)	12.4 ± 0.1	[Kurtz et al. '14]
HLbL	102 ± 39	[Jegerlehner '15, Nyffeler '09]
Total (theory)	$116\ 591\ 811 \pm 62$	

- $\Delta a_\mu = a_\mu^{\mathrm{exp}} - a_\mu^{\mathrm{th}} = 278 \times 10^{11}$ $\rightarrow$ $\sim 3-4\ \sigma$ discrepancy between experiment and theory
- Future experiments at Fermilab and J-PARC : reduction of the error by a factor of 4 $\rightarrow$ $\delta a_\mu = 16 \times 10^{-11}$
- Theory error is dominated by hadronic contributions

Hadronic Light-by-Light scattering (HLbL) : model calculations



[extracted from A. Nyffeler's slide]

Contribution	BPP	HKS, HK	KN	MV	BP, MdRR	PdRV	N, JN
π^0, η, η'	85±13	82.7±6.4	83±12	114±10	–	114±13	99 ± 16
axial vectors	2.5±1.0	1.7±1.7	–	22±5	–	15 ± 10	22 ± 5
scalars	−6.8±2.0	–	–	–	–	−7±7	−7±2
π, K loops	−19±13	−4.5±8.1	–	–	–	−19±19	−19±13
π,K loops +subl. N_C	–	–	–	0±10	–	–	–
quark loops	21±3	9.7±11.1	–	–	–	2.3 (c-quark)	21±3
Total	83±32	89.6±15.4	80±40	136±25	110±40	105 ± 26	116 ± 39

BPP = Bijnens, Pallante, Prades '95, '96, '02; HKS = Hayakawa, Kinoshita, Sanda '95, '96; HK = Hayakawa, Kinoshita '98, '02; KN = Knecht, AN '02; MV = Melnikov, Vainshtein '04; BP = Bijnens, Prades '07; MdRR = Miller, de Rafael, Roberts '07; PdRV = Prades, de Rafael, Vainshtein '09; N = AN '09, JN = Jegerlehner, AN '09

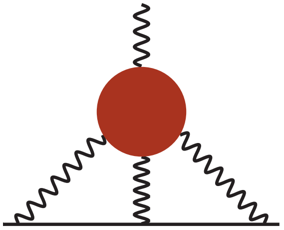
- Pseudoscalar pole contributions most important, but other contributions are not negligible
- Error are hard to estimate (model calculations)
- Need transition form factors as input parameters

HLbL from Lattice QCD

- **Long term project :** direct lattice calculation

  ↪ only one collaboration has published results so far [Blum et. al 14', 16']

  ↪ difficult calculation (4-pt correlation function), work in progress in Mainz

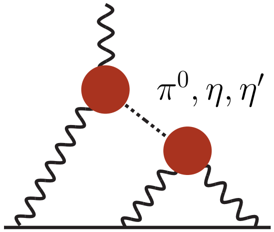


- **Pion-pole contribution**

  ↪ Dominant contribution to the HLbL scattering in $(g-2)_\mu$

  ↪ First-principle estimate

  ↪ Other pseudoscalars (η and η') can be included in a similar way



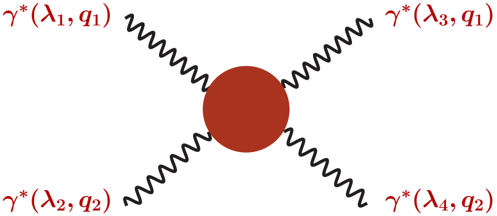
- **Hadronic Light-by-Light forward scattering amplitudes**

  ↪ The previous method works only for stable particles

  ↪ Full HLbL amplitudes contain more info than just a_μ

  ↪ Can be used to test the model (saturation)

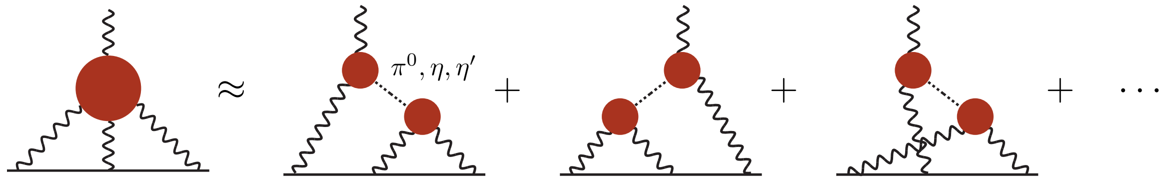
  ↪ Extract information about single-meson transition form factor



The pion-pole contribution

The pion-pole contribution

[Jegerlehner & Nyffeler '09]



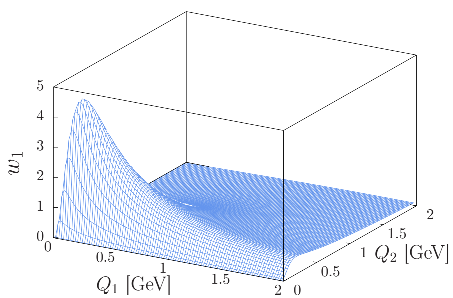
$$a_\mu^{\mathrm{HLbL};\pi^0} = \int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^1 d\tau \; w_1(Q_1,Q_2,\tau)\; \mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q_1^2, -(Q_1+Q_2)^2)\; \mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q_2^2, 0) \; + $$
$$w_2(Q_1,Q_2,\tau)\; \mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q_1^2, -Q_2^2)\; \mathcal{F}_{\pi^0\gamma^*\gamma^*}(-(Q_1+Q_2)^2, 0)$$

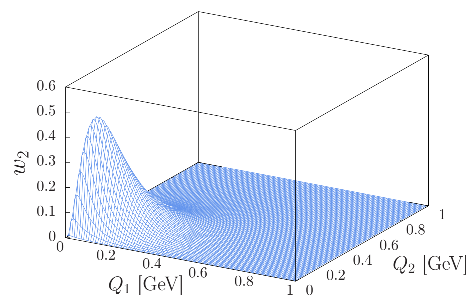
$\rightarrow$ $\tau = \cos(\theta)$ with $Q_1 \cdot Q_2 = Q_1 Q_2 \cos(\theta)$

$\rightarrow$ Product of one single-virtual and one double-virtual transition form factors

$\rightarrow$ $w_{1,2}(Q_1,Q_2,\tau)$ are known model-independent weight functions

$\rightarrow$ Weight functions are concentrated at small momenta below 1 GeV (here for $\tau = -0.5$)





The pion-pole contribution

To estimate the pion-pole contribution we need :

- The single and double virtual transition form factor for arbitrary space-like virtualities
- In the kinematical range $Q^2 \in [0-2]~\mathrm{GeV}^2$

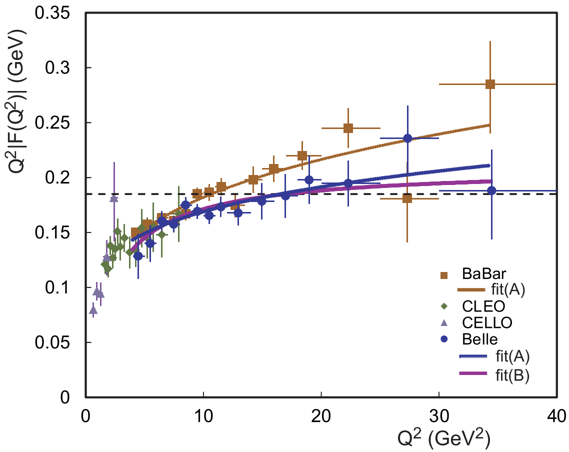
Present status :

- Experimental results available for the single-virtual form factor
- And only for relatively large virtualities $Q^2 > 0.6~\mathrm{GeV}^2$
- The theory imposes strong constraints for the normalisation and the asymptotic behavior of the TFF
  - $\hookrightarrow$ Anomaly constraint $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(0,0)$
  - $\hookrightarrow$ Brodsky-Lepage, OPE for large virtualities

$\hookrightarrow$ Most evaluations of the pion-pole contribution are therefore based on phenomenological models

$\hookrightarrow$ Systematic errors are difficult to estimate

Lattice QCD is particularly well suited to compute the form factor in the energy range relevant to $g-2$!



Lattice QCD : sources of systematic error

- Lattice QCD is not a model : specific regularisation of the theory adapted to numerical simulations
- However there are systematic errors that we need to understand :

1) We used $N_f = 2$ simulations (Only u and d quarks are dynamical)
2) Finite lattice spacing : discretisation errors

   $\rightarrow$ 3 lattice spacings ($a = 0.075, 0.065, 0.048$ fm) : extrapolation to the continuum limit $a = 0$
3) Unphysical quark masses

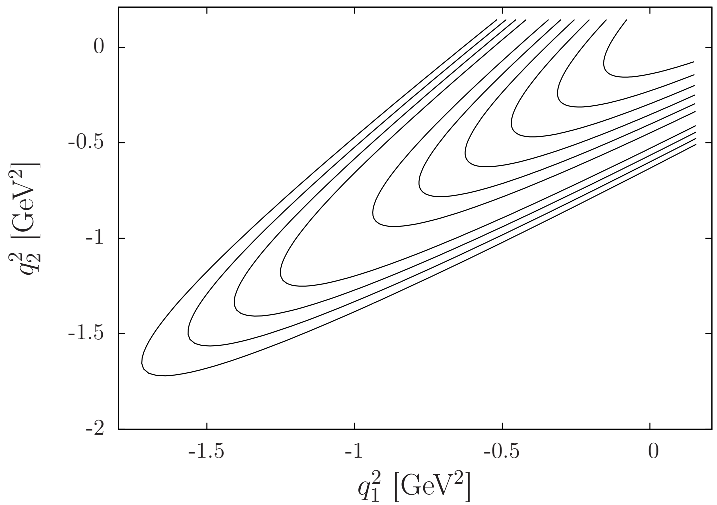
   $\rightarrow$ Different simulations with pion mass in the range [190-440] MeV : extrapolation to $m_\pi = m_\pi^{\text{exp}}$
4) Finite volume

   $\rightarrow$ Discrete spatial momenta $\vec{q} = 2\pi/L\vec{n}$

pion at rest : $q_1 = (\omega_1, \vec{q}_1)$

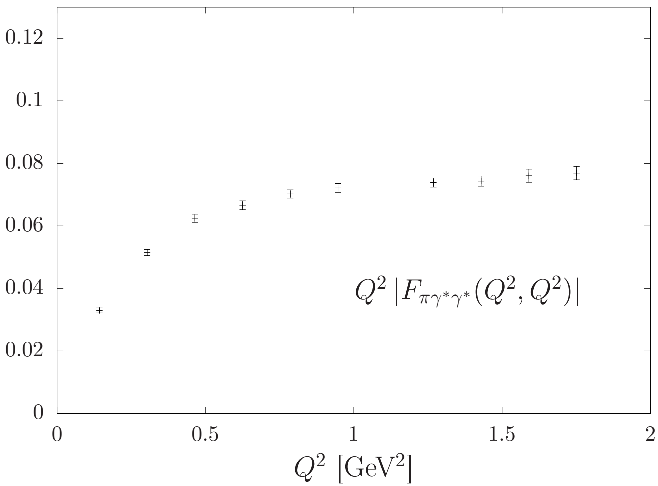
$$q_2 = (m_\pi - \omega_1, \vec{q}_2)$$

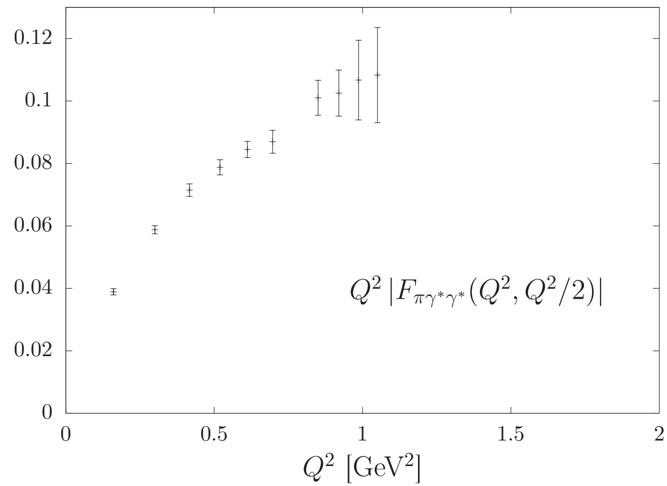
$$q_1^2 = \omega_1^2 - |\vec{q}_1|^2$$
$$q_2^2 = (m_\pi - \omega_1)^2 - |\vec{q}_1|^2$$



Transition form factor : results

- Results for one of the eight ensembles with $a = 0.048$ fm and $m_\pi = 270$ MeV



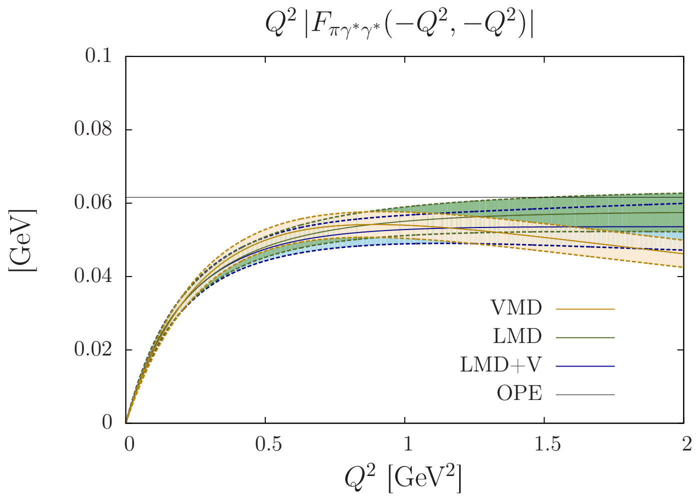


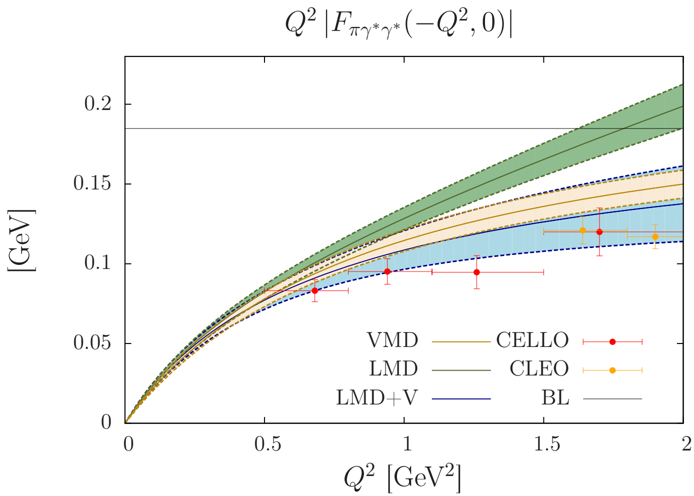
- Next step : extrapolate the results to the physical point
  - Use phenomenological models to describe the lattice data
  - Extrapolate the model parameters to the continuum and chiral limit

- **LMD model** (Lowest Meson Dominance) [Moussallam '94] [Knecht et al. '99]

$$\mathcal{F}^{\mathrm{LMD}}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = \frac{\alpha M_V^4 + \beta(q_1^2 + q_2^2)}{(M_V^2 - q_1^2)(M_V^2 - q_2^2)}$$

Final results





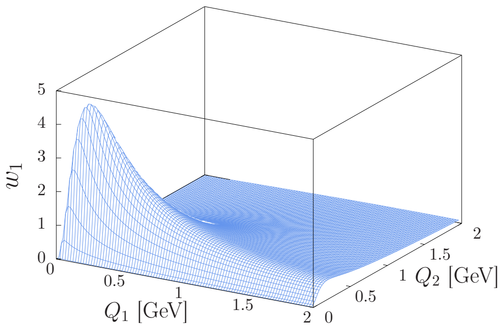
Final results : blue curves (the model satisfies all the theoretical constraint)

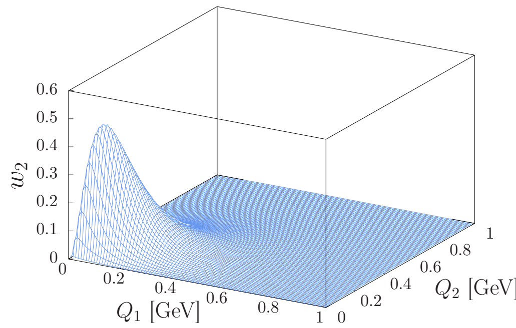
Back to phenomenology : the pion-pole contribution

[Jegerlehner & Nyffeler '09]

$$a_\mu^{\mathrm{HLbL};\pi^0} = \int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^1 d\tau \; w_1(Q_1,Q_2,\tau)\; \mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q_1^2, -(Q_1+Q_2)^2)\; \mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q_2^2, 0) \; + \\ w_2(Q_1,Q_2,\tau)\; \mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q_1^2, -Q_2^2)\; \mathcal{F}_{\pi^0\gamma^*\gamma^*}(-(Q_1+Q_2)^2, 0)$$

$\rightarrow$ $w_{1,2}(Q_1,Q_2,\tau)$ are some model-independent weight functions (concentrated at small momenta below 1 GeV)





Back to phenomenology : the pion-pole contribution

[Jegerlehner & Nyffeler '09]

$$a_\mu^{\mathrm{HLbL};\pi^0} = \int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^1 d\tau \; w_1(Q_1,Q_2,\tau)\; \mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q_1^2, -(Q_1+Q_2)^2)\; \mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q_2^2,0) \; + $$
$$w_2(Q_1,Q_2,\tau)\; \mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q_1^2,-Q_2^2)\; \mathcal{F}_{\pi^0\gamma^*\gamma^*}(-(Q_1+Q_2)^2,0)$$

$\rightarrow$ $w_{1,2}(Q_1,Q_2,\tau)$ are some model-independent weight functions (concentrated at small momenta below 1 GeV)

$$a^{\mathrm{HLbL};\pi^0}_{\mu;\mathrm{LMD+V}} = (65.0 \pm 8.3)\times 10^{-11}$$

Model	$a_\mu^{\mathrm{HLbL};\pi^0} \times 10^{11}$
LMD (this work)	68.2(7.4)
LMD+V (this work)	65.0(8.3)
VMD (theory)	57.0
LMD (theory)	73.7
LMD+V (theory + phenomenology)	62.9

$\rightarrow$ Most model calculations yield results in the range $a_\mu^{\mathrm{HLbL};\pi^0} = (50-80)\times 10^{-11}$

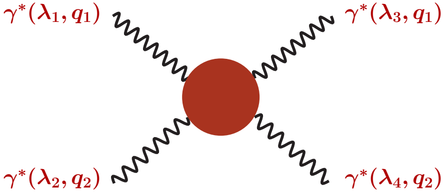
$\rightarrow$ We are currently working on the $N_f = 2+1$ result ?

$\rightarrow$ Contributions from other particles/resonances ?

Light-by-light forward scattering amplitudes

Light-by-light forward scattering amplitudes

- Forward scattering amplitudes $M_{\lambda_3\lambda_4\lambda_1\lambda_2}$: $\gamma^*(\lambda_1, q_1)\ \gamma^*(\lambda_2, q_2)\ \rightarrow\ \gamma^*(\lambda_3, q_1)\ \gamma^*(\lambda_4, q_2)$



  - 81 helicity amplitudes $(\lambda_i = 0, \pm 1)$
    $$\mathcal{M}_{\lambda_1'\lambda_2'\lambda_1\lambda_2} = \mathcal{M}_{\mu\nu\rho\sigma}\ \epsilon^{*\mu}(\lambda_1')\ \epsilon^{*\nu}(\lambda_2')\ \epsilon^{\rho}(\lambda_1)\ \epsilon^{\sigma}(\lambda_2)$$
  - Photons virtualities : $Q_1^2 = -q_1^2 > 0$ and $Q_2^2 = -q_2^2 > 0$
  - Crossing-symmetric variable : $\nu = q_1 \cdot q_2$

- Using parity and time invariance : only 8 independent amplitudes

$$(\mathcal{M}_{++,++} + \mathcal{M}_{+-,+-}),\ \mathcal{M}_{++,--},\ \mathcal{M}_{00,00},\ \mathcal{M}_{+0,+0},\ \mathcal{M}_{0+,0+},\ (\mathcal{M}_{++,00} + \mathcal{M}_{0+,-0}),$$
$$(\mathcal{M}_{++,++} - \mathcal{M}_{+-,+-}),\ (\mathcal{M}_{++,00} - \mathcal{M}_{0+,-0})$$

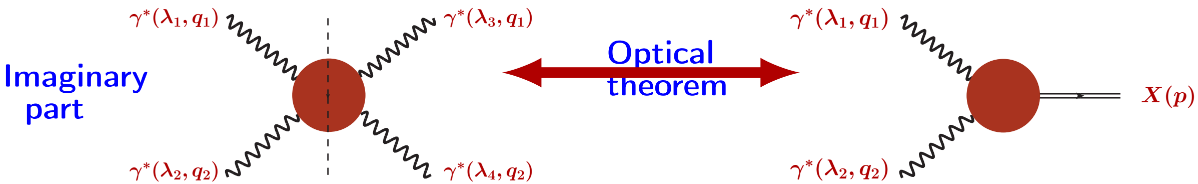
$\hookrightarrow$ Either even or odd with respect to ν
$\hookrightarrow$ The eight amplitudes have been computed on the lattice for different values of ν, Q_1^2, Q_2^2

- Relate the forward amplitudes to two-photon fusion cross sections using the optical theorem [Pascalutsa et. al '12]

$\hookrightarrow$ Eight independent dispersion relations for $\mathcal{M}_{TT}$, $\mathcal{M}_{TT}^t$, $\mathcal{M}_{TT}^a$, $\mathcal{M}_{TL}$, $\mathcal{M}_{LT}$, $\mathcal{M}_{TL}^a$, $\mathcal{M}_{TL}^t$ and $\mathcal{M}_{LL}$

Dispersion relations

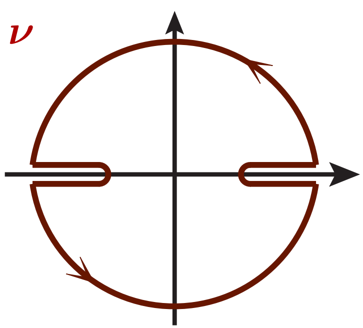
1) Optical theorem



$\hookrightarrow$ X can be any C-even final state

$$W_{\lambda_3\lambda_4,\lambda_1\lambda_2} = \mathrm{Im}\, M_{\lambda_3\lambda_4,\lambda_1\lambda_2} = \frac{1}{2}\int \mathrm{d}\Gamma_X (2\pi)^4 \delta(q_1 + 1_2 - p_X) \mathcal{M}_{\lambda_1\lambda_2}(q_1, q_2, p_X)\, \mathcal{M}^*_{\lambda_3\lambda_4}(q_1, q_2, p_X)$$

2) Dispersion relations [Pascalutsa et. al '12]



Once-subtracted sum rules : crossing-symmetric variable $\nu = q_1 \cdot q_2$

$$\mathcal{M}_{\mathrm{even}}(\nu) = \mathcal{M}_{\mathrm{even}}(0) + \frac{2\nu^2}{\pi}\int_{\nu_0}^{\infty} d\nu' \frac{1}{\nu'(\nu'^2 - \nu^2 - i\epsilon)} W_{\mathrm{even}}(\nu')$$

$$\mathcal{M}_{\mathrm{odd}}(\nu) = \nu\mathcal{M}_{\mathrm{odd}}(\nu) + \frac{2\nu^3}{\pi}\int_{\nu_0}^{\infty} d\nu' \frac{1}{\nu'(\nu'^2 - \nu^2 - i\epsilon)} W_{\mathrm{odd}}(\nu')$$

3) Higher mass singularities are suppressed with ν^2 :

$\hookrightarrow$ Only a few states X are necessary to saturate the sum rules and reproduce the lattice data

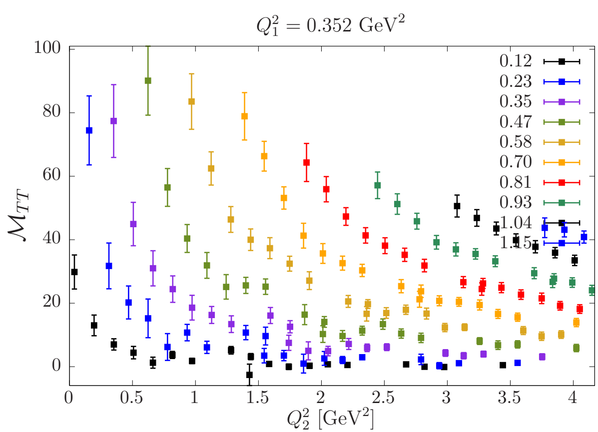
Description of the lattice data using phenomenology

→ For each of the eight amplitudes, we have a dispersion relation :

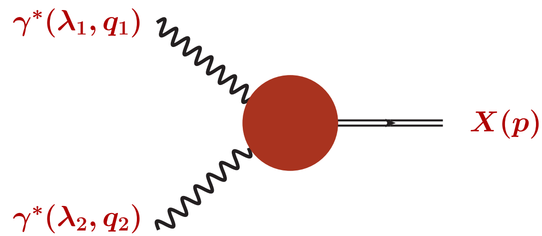
$$\overline{\mathcal{M}}_\alpha(\nu) = \frac{4\nu^2}{\pi} \int_{\nu_0}^{\infty} d\nu' \, \frac{\sqrt{X'}\,\sigma_\alpha/\tau_\alpha(\nu')}{\nu'(\nu'^2 - \nu^2 - i\epsilon)}$$

Lattice calculation

↪ 4-pt correlation function



$\gamma^*(\lambda_1, q_1) + \gamma^*(\lambda_2, q_2) \to \mathrm{X}(p_\mathrm{X})$ fusion cross sections



↪ Main contribution is expected from mesons :

Pseudoscalars (0^{-+}) Axial-vectors (1^{++})
Scalar (0^{++}) Tensors (2^{++})

↪ Input : transition form factors

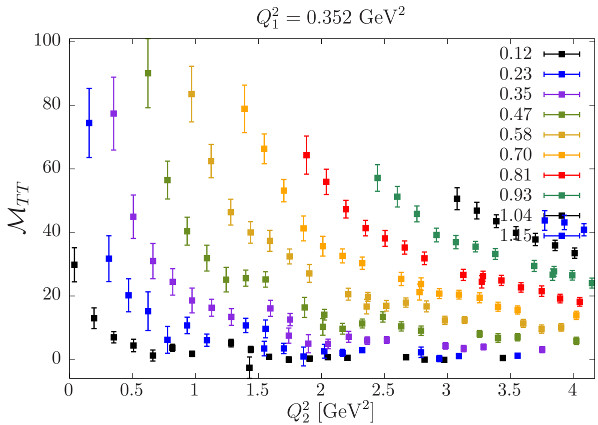
Description of the lattice data using phenomenology

→ For each of the eight amplitudes, we have a dispersion relation :

$$\overline{\mathcal{M}}_\alpha(\nu) = \frac{4\nu^2}{\pi} \int_{\nu_0}^{\infty} d\nu' \, \frac{\sqrt{X'}\, \sigma_\alpha/\tau_\alpha(\nu')}{\nu'(\nu'^2 - \nu^2 - i\epsilon)}$$

Lattice calculation

↪ 4-pt correlation function



$\gamma^*(\lambda_1, q_1) + \gamma^*(\lambda_2, q_2) \to \mathrm{X}(p_\mathrm{X})$ fusion cross sections

↪ Consider only one particle in each channel

↪ $N_f = 2$: no η meson

↪ Isospin symmetry + large-N_c approximation : isovector only with an overall factor $34/9$

	Isovector	Isoscalar	Isoscalar
0^{-+}	π	η'	η
0^{++}	$a_0(980)$	$f_0(980)$	$f_0(600)$
1^{++}	$a_1(1260)$	$f_1(1285)$	$f_1(1420)$
2^{++}	$a_2(1320)$	$f_2(1270)$	$f_2'(1525)$

Assumptions on form factors

- Pseudoscalar meson : Lattice QCD (Previous calculation)
- Other particles (resonances) : Only measurement for isoscalar, single virtual TFF so far. And at rather large virtualities.

1) Scalar mesons

$$\boxed{\frac{F^T_{\mathcal{S}\gamma^*\gamma^*}(Q_1^2,Q_2^2)}{F^T_{\mathcal{S}\gamma^*\gamma^*}(0,0)} = \frac{1}{(1+Q_1^2/M_S^2)(1+Q_2^2/M_S^2)}}$$

2) Tensor mesons

$\hookrightarrow$ Amplitudes are described by four form factors $F^{(\Lambda)}_{\mathcal{T}\gamma^*\gamma^*}(Q_1^2,Q_2^2)$ with $\Lambda = (0,T),\ (0,L),\ 1,\ 2$

$\hookrightarrow$ data are compatible with a dipole form factor [Danilkin '16]

$$\boxed{\frac{F^{(\Lambda)}_{\mathcal{T}\gamma^*\gamma^*}(Q_1^2,Q_2^2)}{F^{(\Lambda)}_{\mathcal{T}\gamma^*\gamma^*}(0,0)} = \frac{1}{(1+Q_1^2/M_{T,(\Lambda)}^2)^2(1+Q_2^2/M_{T,(\Lambda)}^2)^2}}$$

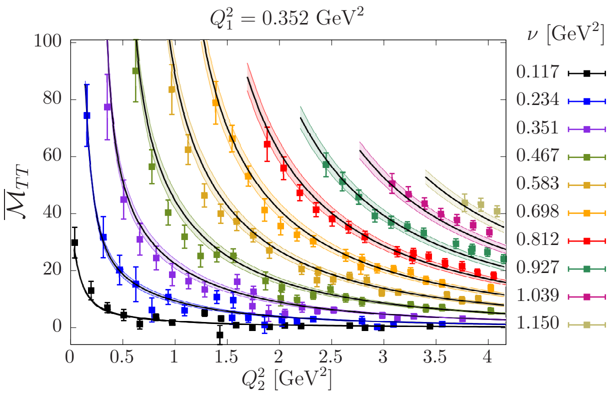
3) Axial mesons : Dipole form factor based on a quark model parametrisation : M_A

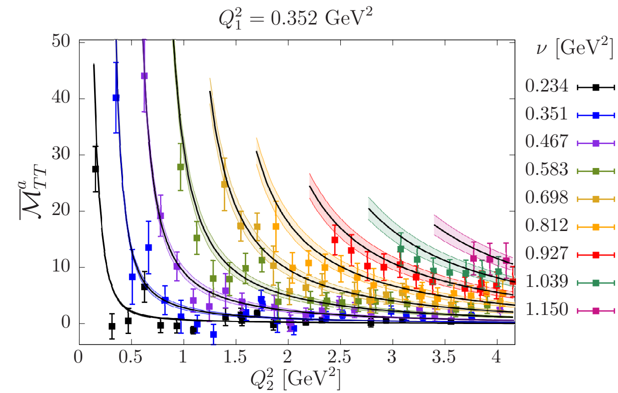
- Scalar QED

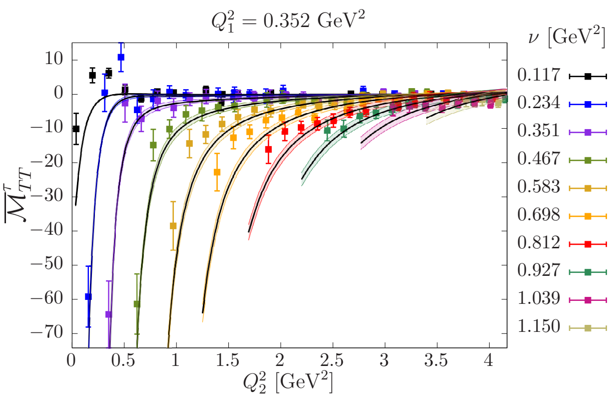
$\gamma^*\gamma^* \to \pi^+\pi^-$ evaluated using scalar QED dressed with monopole form factors ($M =$ lattice tho mass)

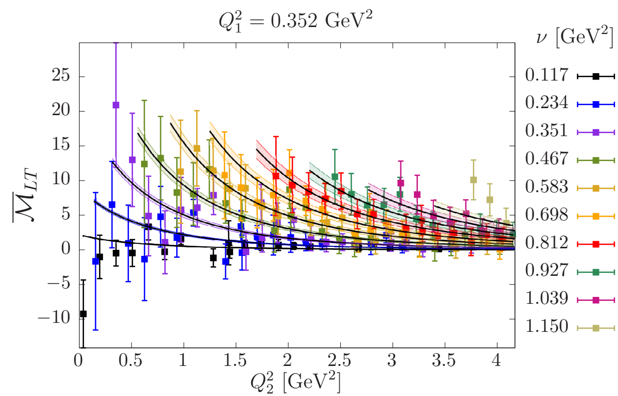
Preliminary results : F7 - dependance on ν and Q_2^2

- Each plot correspond to a fixed Q_1^2
- Different colours correspond to different values of $\nu = Q_1^2 \cdot Q_2^2$

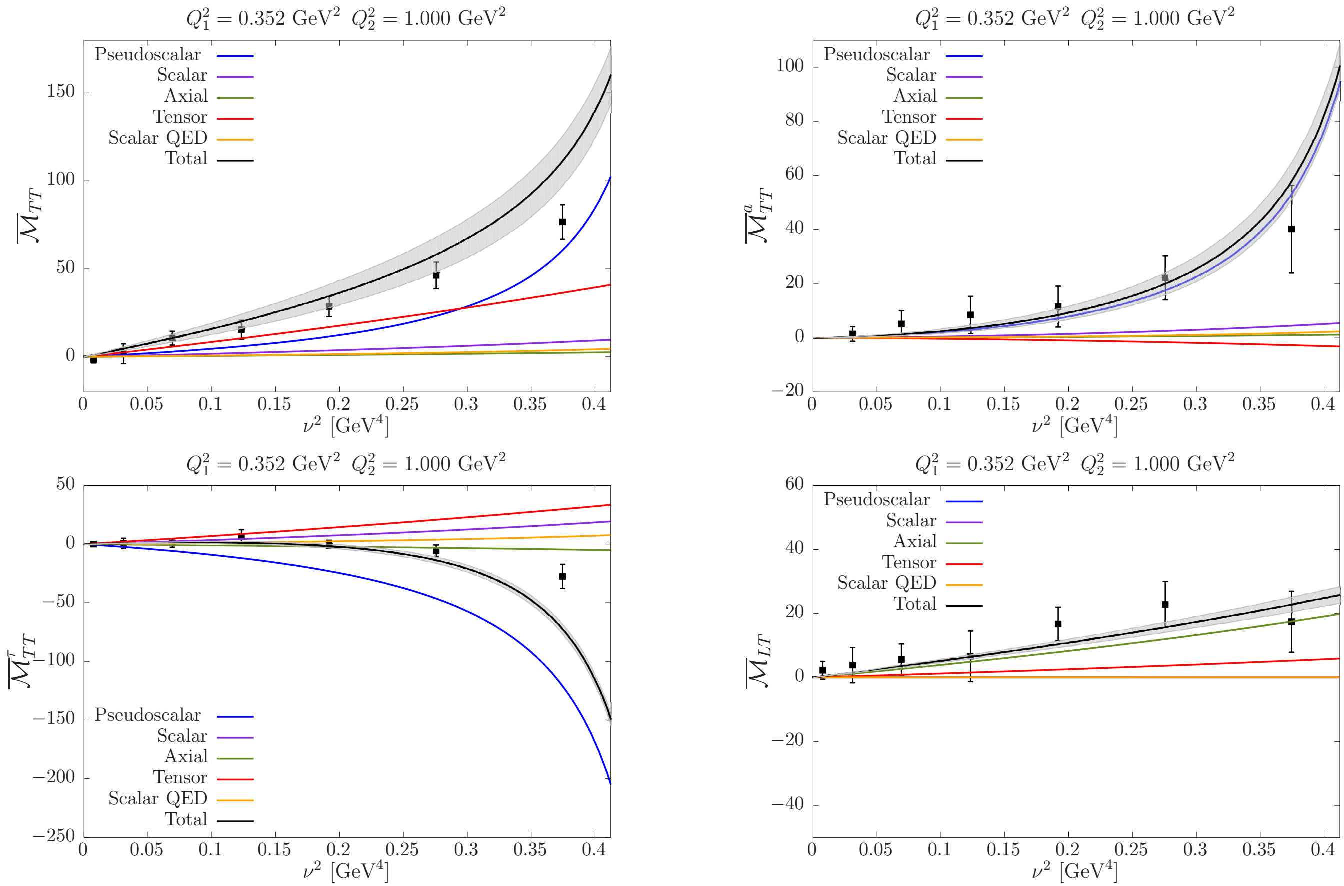








Preliminary results : F7 - contributions from different channels



Preliminary results : monopole and dipole masses

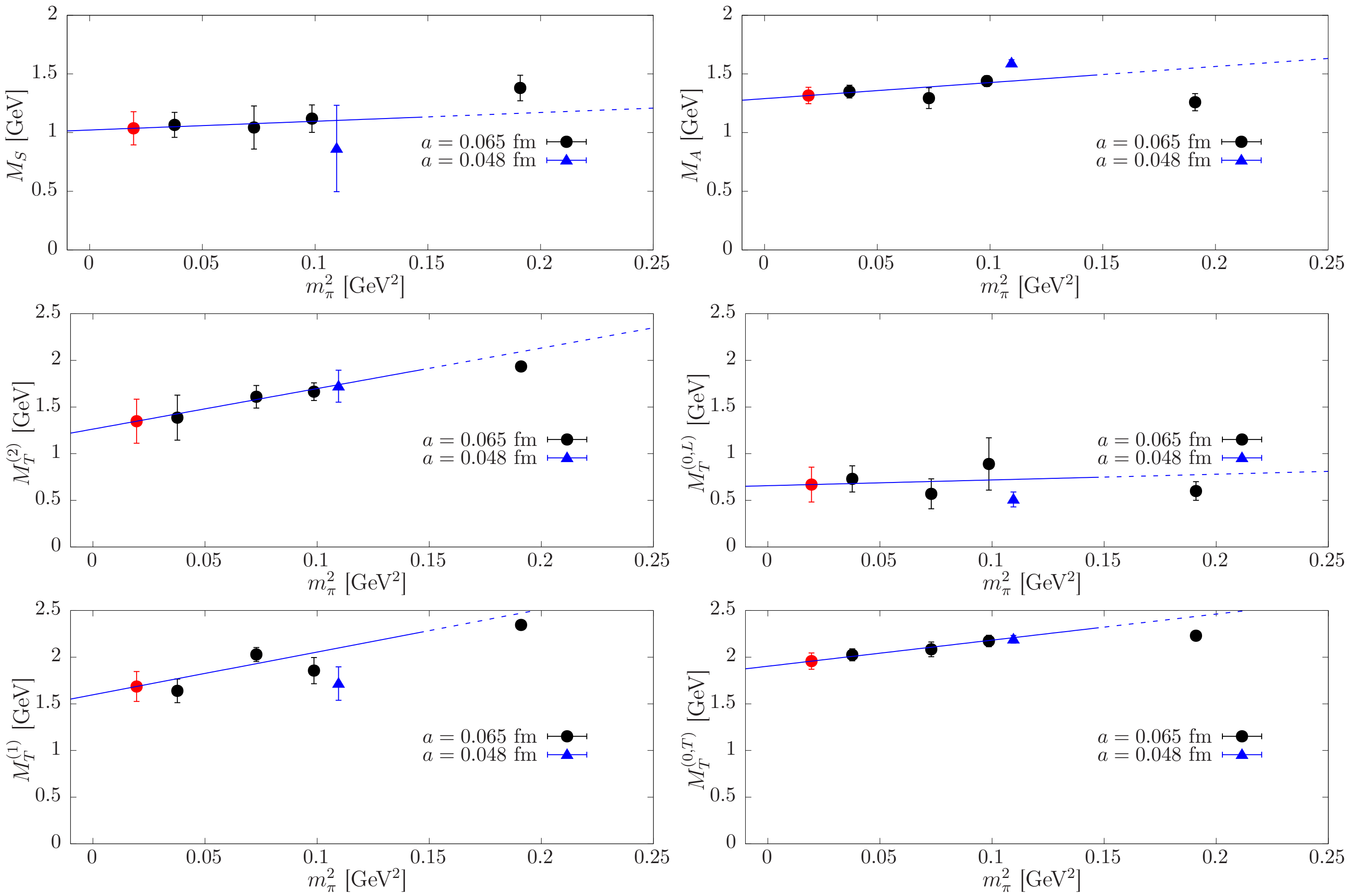
Monopole FF / **Dipole FF**

$$F_X(Q_1^2, Q_2^2) = \frac{F_X(0,0)}{(1 + Q_1^2/\Lambda_X^2)\,(1 + Q_2^2/\Lambda_X^2)} \quad , \quad F_X(Q_1^2, Q_2^2) = \frac{F_X(0,0)}{(1 + Q_1^2/\Lambda_X^2)^2\,(1 + Q_2^2/\Lambda_X^2)^2}$$

- Global fit of the eight amplitudes

	M_S [GeV]	M_A [GeV]	$M_T^{(2)}$ [GeV]	$M_T^{(0,T)}$ [GeV]	$M_T^{(1)}$ [GeV]	$M_T^{(0,L)}$ [GeV]	$\chi^2/\mathrm{d.o.f}$
E5	1.38(11)	1.26(10)	1.93(3)	2.24(5)	2.36(4)	0.60(10)	4.22
F6	1.12(14)	1.44(5)	1.66(9)	2.17(5)	1.85(14)	0.89(28)	1.15
F7	1.04(18)	1.29(8)	1.61(12)	2.08(7)	2.03(7)	0.57(16)	1.19
G8	1.07(10)	1.36(5)	1.37(24)	2.03(6)	1.63(13)	0.73(14)	1.13
N6	0.86(37)	1.59(3)	1.72(17)	2.19(4)	1.72(18)	0.51(8)	1.35

Monopole and dipole masses : chiral extrapolations (preliminary, stat error only)



Results

Chiral extrapolations :

- $M_S = 0.94(12)$ GeV : slightly above the experimental result from the Belle Collaboration ($M_S = 796(54)$ MeV for the isoscalar scalar meson [Masuda '15])
- $M_A = 1.40(7)$ GeV to be compared with the experimental value by the L3 Collaboration $M_A = 1040(80)$ MeV for the isoscalar meson $f_1(1285)$ [Achard '01 '07].
- $M_T^{(2)} = 1.39(12)$ GeV, $M_{0,T}^{(1)} = 1.67(10)$ GeV and $M_{(1)}^{(0,T)} = 2.01(7)$ GeV, above the experimental values for the isoscalar $f_2(1270)$ mesons obtained by fitting the single-virtual form factor [Masuda '15, Danilkin '16].

$\hookrightarrow$ Form factors : input for the contribution of resonances to HLbL in $(g-2)_\mu$

$\hookrightarrow$ We are able do describe the lattice data with one particle in each channel (confirmation of the model)

$\hookrightarrow$ We can also make a prediction on the contribution of disconnected diagrams in two limits :

- $m_s = \infty$, which corresponds to the two-flavor theory ;
- $m_s = m_{ud}$, which corresponds to the SU(3)-flavor symmetric theory.

We expect the real world to lie between these two predictions :

$$a_\mu^{\mathrm{HLbL},(2+2)} \approx \begin{cases} -\frac{25}{9} a_\mu^{\mathrm{HLbL},\pi^0} + a_\mu^{\mathrm{HLbL},\eta'} = -(162 \pm 27) \cdot 10^{-11} & m_s = \infty, \\ -2(a_\mu^{\mathrm{HLbL},\pi^0} + a_\mu^{\mathrm{HLbL},\eta}) + a_\mu^{\mathrm{HLbL},\eta'} = -(142 \pm 19) \cdot 10^{-11} & m_s = m_{ud}. \end{cases}$$

Conclusion and perspectives

- We have performed a lattice calculation of the pion transition form factor (TFF) in the momentum region relevant for the $(g-2)_\mu$.

  $\rightarrow$ Provides a first lattice estimate of the pion-pole contribution to the hadronic light-by-light scattering in the $g-2$ of the muon

  $$a_{\mu;\mathrm{LMD+V}}^{\mathrm{HLbL};\pi^0} = (65.0 \pm 8.3) \times 10^{-11}$$

- The eight forward light-by-light amplitudes have been computed on the lattice

  $\rightarrow$ They are well described by the cross sections $\gamma^*\gamma^* \rightarrow$ a few resonances via dispersion relations.

  $\rightarrow$ Allows us to put constraints on form factors used to estimate the HLbL contribution to the $(g-2)_\mu$

- Perspectives

  $\rightarrow$ Pion TFF : $N_f = 2+1$ with full $\mathcal{O}(a)$-improvement to reduce discretization effects

  $\rightarrow$ Include a new kinematical configuration where the pion has one unit of momentum

  $\rightarrow$ Compute the HLbL directly on the lattice

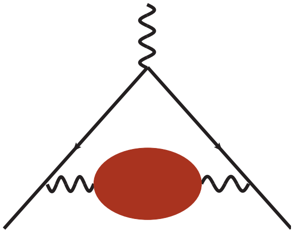
$(g-2)_\mu$: Mainz effort

- **Hadronic Vacuum Polarisation (LO)**

  ↪ Recent publication with $N_f = 2$ flavours [arXiv :1705.1775]

  ↪ Focus on methodology

  ↪ Now generating data with $N_f = 2+1$ flavours



- **Hadronic Light-by-Light**

  ↪ Direct lattice calculation

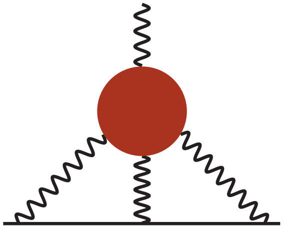
  ↪ Exact QED kernel in position space :

$$a_\mu^{\mathrm{HLbL}} = \frac{me^3}{3} \int \mathrm{d}^4 y \int \mathrm{d}^4 x \; \overline{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y) \; i\Pi_{\rho,\mu\nu\lambda\sigma}(x,y)$$

$$i\Pi_{\rho,\mu\nu\lambda\sigma}(x,y) = -\int \mathrm{d}^4 z \, z_\rho \, \langle J_\mu(x) J_\nu(y) J_\sigma(z) J_\lambda(0) \rangle$$

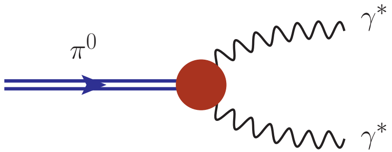
  ↪ Four-point correlation function computed on the lattice

  ↪ Only one collaboration has published results so far [RBC/UKQCD]



- **Pseudoscalar transition Form Factor**

  ↪ Gives the dominant contribution to HLbL from first principles



Experimental status

- Decay width : $\Gamma_{\pi^0\gamma\gamma} = 7.82(22)\ \mathrm{eV} \quad \sim 3\%$ [PrimEx '10]

$$\Gamma_{\pi^0\gamma\gamma} = \frac{\pi\alpha_e^2 m_\pi^3}{4}\,\mathcal{F}^2_{\pi^0\gamma^*\gamma^*}(0,0)$$

$\rightarrow$ Consistent with current theoretical predictions

$\rightarrow$ Experimental test of the chiral anomaly

$\rightarrow$ A further reduction of the error by a factor of two is expected soon

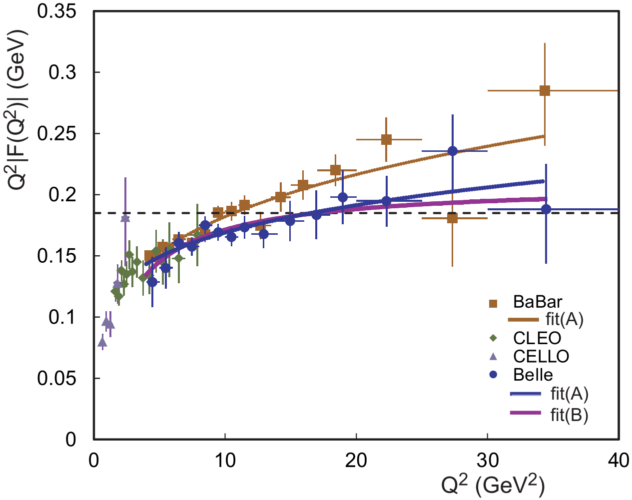
- The single-virtual form factor has been measured (CELLO, CLEO, BaBar, Belle)

[Belle '12]

$\rightarrow$ Belle data seem to confirm the Brodsky-Lepage behavior $\sim 1/Q^2$.

$\rightarrow$ Belle and Babar results are quite different

$\rightarrow$ No measurement at low $Q < 0.8$ GeV (dominant contribution)



- No result yet for the double-virtual form factor

$\hookrightarrow$ measurement planned at BESIII

$\hookrightarrow$ challenging (small cross section)

Continuum and chiral extrapolation

- **VMD model** (Vector Meson Dominance)

$$\mathcal{F}^{\mathrm{VMD}}_{\pi^0\gamma^*\gamma^*}(q_1^2,q_2^2) = \frac{\alpha M_V^4}{(M_V^2-q_1^2)(M_V^2-q_2^2)}$$

- **LMD model** (Lowest Meson Dominance) [Moussallam '94] [Knecht et al. '99]

$$\mathcal{F}^{\mathrm{LMD}}_{\pi^0\gamma^*\gamma^*}(q_1^2,q_2^2) = \frac{\alpha M_V^4 + \beta(q_1^2+q_2^2)}{(M_V^2-q_1^2)(M_V^2-q_2^2)}$$

- **LMD+V model** [Knecht & Nyffeler '01]

$$\mathcal{F}^{\mathrm{LMD+V}}_{\pi^0\gamma^*\gamma^*}(q_1^2,q_2^2) = \frac{\widetilde{h}_0\, q_1^2 q_2^2 (q_1^2+q_2^2) + \widetilde{h}_1 (q_1^2+q_2^2)^2 + \widetilde{h}_2\, q_1^2 q_2^2 + \widetilde{h}_5\, M_{V_1}^2 M_{V_2}^2\, (q_1^2+q_2^2) + \alpha\, M_{V_1}^4 M_{V_2}^4}{(M_{V_1}^2-q_1^2)(M_{V_2}^2-q_1^2)(M_{V_1}^2-q_2^2)(M_{V_2}^2-q_2^2)}$$

- Anomaly constraint with $\alpha = 1/4\pi^2 F_\pi$

$$\mathcal{F}_{\pi^0\gamma^*\gamma^*}(0,0) = \alpha = 1/4\pi^2 F_\pi$$

- Short distances constraints :

$$\mathcal{F}^{\mathrm{OPE}}_{\pi^0\gamma^*\gamma^*}(-Q^2,-Q^2) \sim 2F_\pi/(3Q^2)$$

$$\mathcal{F}^{\mathrm{BL}}_{\pi^0\gamma^*\gamma^*}(-Q^2,0) \sim 1/Q^2$$

	Anomaly	OPE	BL
VMD	✔	✗	✔
LMD	✔	✔	✗
LMD+V	✔	✔	✔

Contributions to the eight independent amplitudes

$$\overline{\mathcal{M}}(\nu) = \frac{4\nu^2}{\pi} \int_{\nu_0}^{\infty} d\nu' \, \frac{\sqrt{X'}\, \sigma/\tau(\nu')}{\nu'(\nu'^2 - \nu^2 - i\epsilon)}$$

Amplitude	Pseudoscalar	Scalar	Axial	Tensor	Scalar QED
$\mathcal{M}_{TT}$	$\sigma_0/2$	$\sigma_0/2$	$\sigma_0/2$	$\frac{\sigma_0+\sigma_2}{2}$	σ_{TT}
$\mathcal{M}^t_{TT}$	$-\sigma_0$	σ_0	$-\sigma_0$	σ_0	τ_{TT}
$\mathcal{M}^a_{TT}$	$\sigma_0/2$	$\sigma_0/2$	$\sigma_0/2$	$\frac{\sigma_0-\sigma_2}{2}$	τ^a_{TT}
$\mathcal{M}_{TL}$	$\times$	$\times$	σ_{TL}	σ_{TL}	σ_{TL}
$\mathcal{M}_{LT}$	$\times$	$\times$	σ_{LT}	σ_{LT}	σ_{LT}
$\mathcal{M}^t_{TL}$	$\times$	τ_{TL}	τ_{TL}	τ_{TL}	τ_{TL}
$\mathcal{M}^a_{TL}$	$\times$	τ_{TL}	$-\tau_{TL}$	τ^a_{TL}	τ^a_{TL}
$\mathcal{M}_{LL}$	$\times$	σ_{LL}	$\times$	σ_{LL}	σ_{LL}

- Example : contribution of the pseudoscalar to the amplitude M_{TT}

$$\sigma_0 = 16\pi^2 \delta(s - m_P^2) \frac{2\sqrt{X}}{m_P^2} \times \frac{\Gamma_{\gamma\gamma}}{m_P} \times \left[\frac{F_{\mathcal{P}\gamma^*\gamma^*}(Q_1^2, Q_2^2)}{F_{\mathcal{P}\gamma^*\gamma^*}(0,0)} \right]^2$$

- We assume a constant mass shift in the spectrum (scalar, axial, tensor)

$$m_X = m_X^{\rm phys} + (m_\rho^{\rm lat} - m_\rho^{\rm phys})$$

- The two-photons decay widths $\Gamma_{\gamma\gamma} = \frac{\pi\alpha^2}{4} m_S \left[F^T_{\mathcal{S}\gamma^*\gamma^*}(0,0)\right]^2$ are taken from experiment