Covariant extension of Generalized Parton Distributions at low Fock states

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GDR QCD, Orme des Merisiers, 5 Décembre, 2017

Outline

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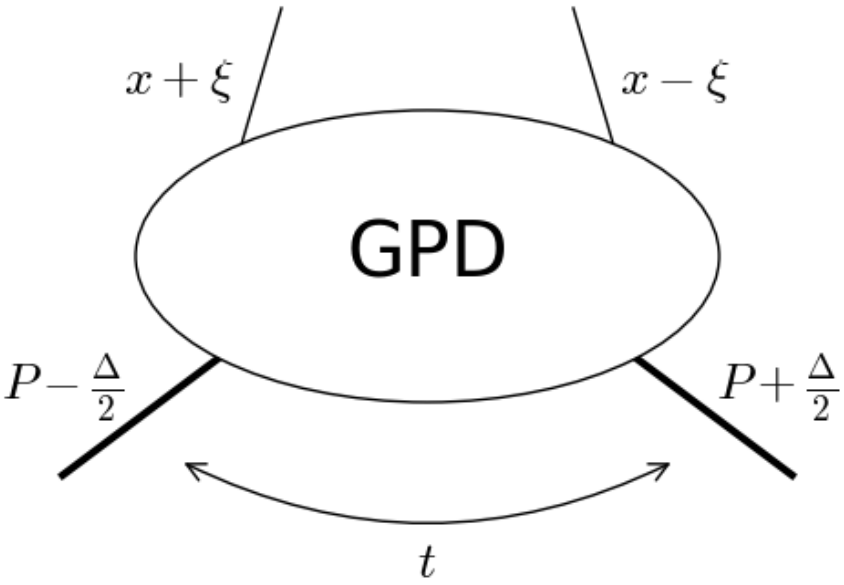
Definition of GPDs

- Quark GPD (twist-2, spin-0 hadron): (Müller et al., 1994; Radyushkin, 1996; Ji, 1997)

$$H^q(x,\xi,t) = \frac{1}{2}\int \frac{\mathrm{d}z^-}{2\pi}\, e^{i\,x\,P^+ z^-} \left\langle P + \frac{\Delta}{2} \left| \bar{q}(-z)\,\gamma^+ q(z) \right| P - \frac{\Delta}{2} \right\rangle \Big|_{z^+=0,\, z_\perp=0} . \tag{1}$$

with:

$$t = \Delta^2 \ , \ \xi = -\frac{\Delta^+}{2\,P^+} .$$



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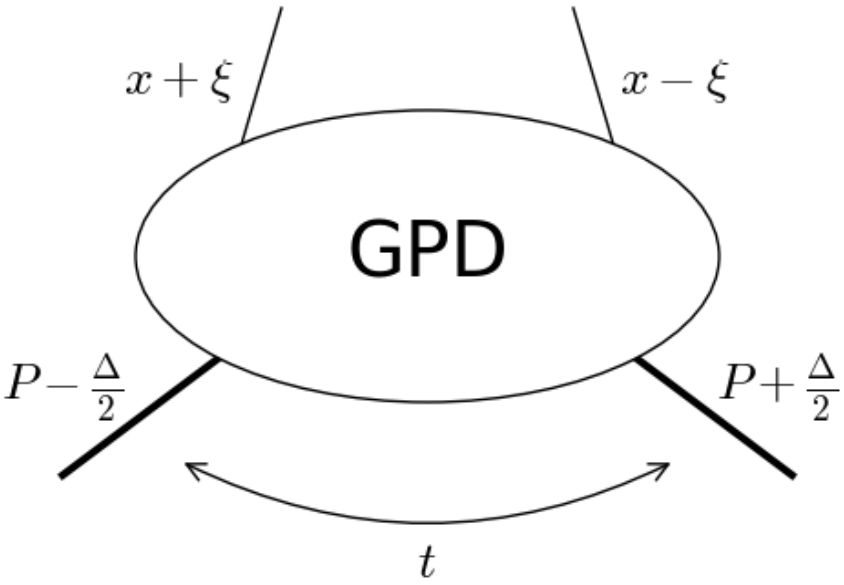

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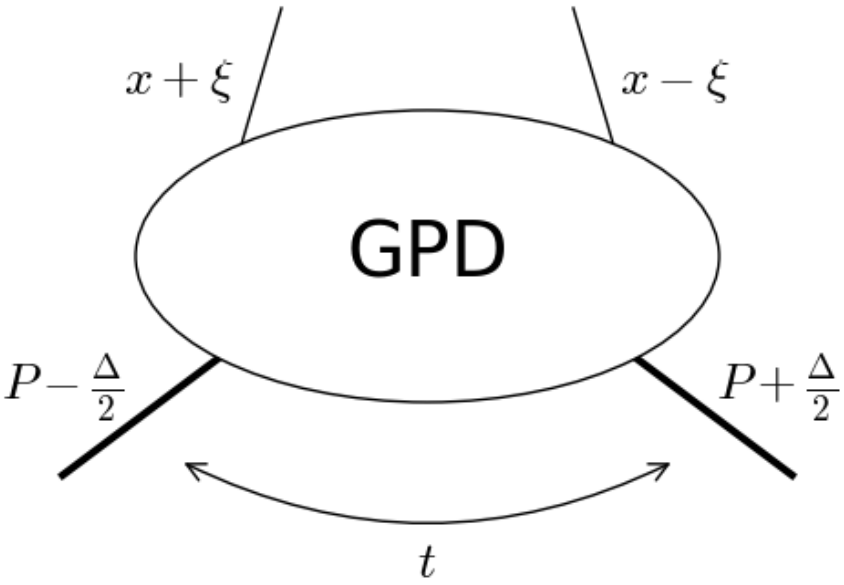

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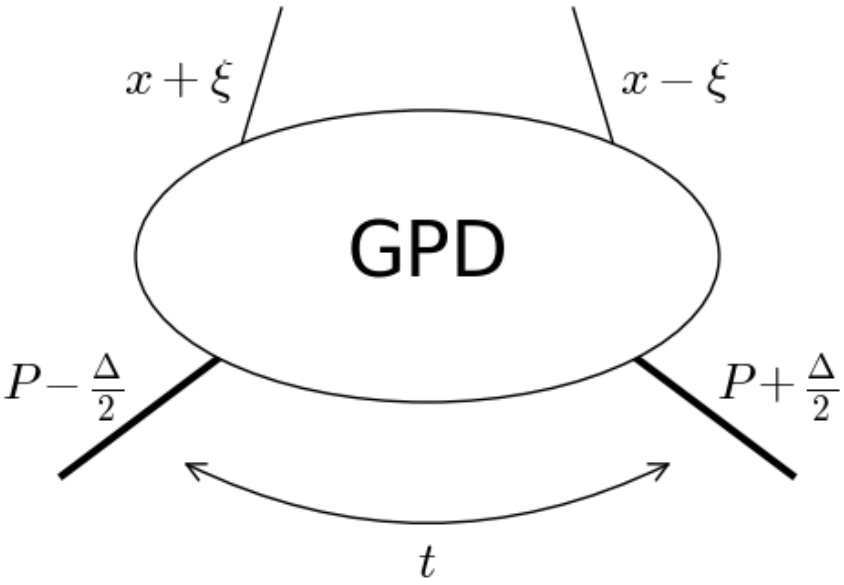

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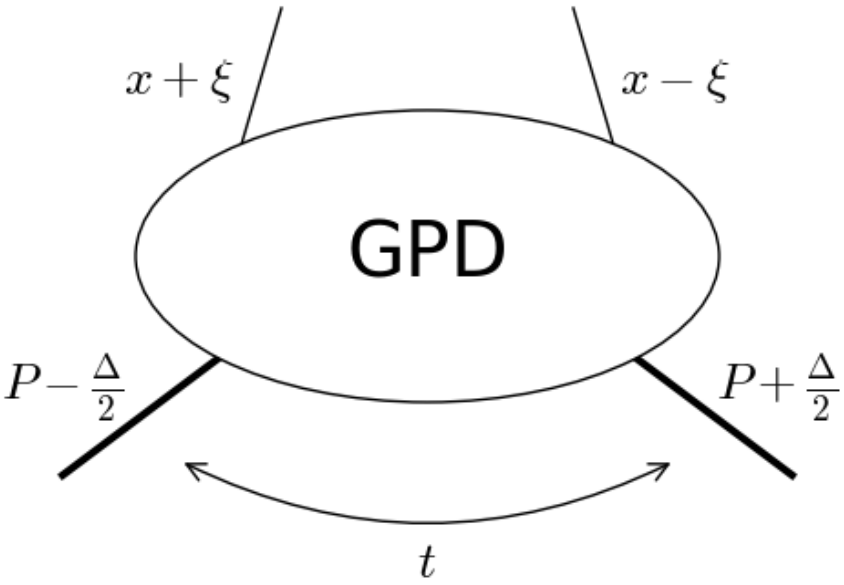

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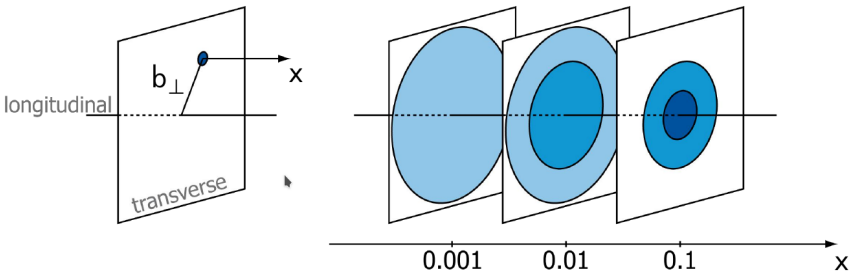

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  - Cauchy-Schwarz theorem in Hilbert space.

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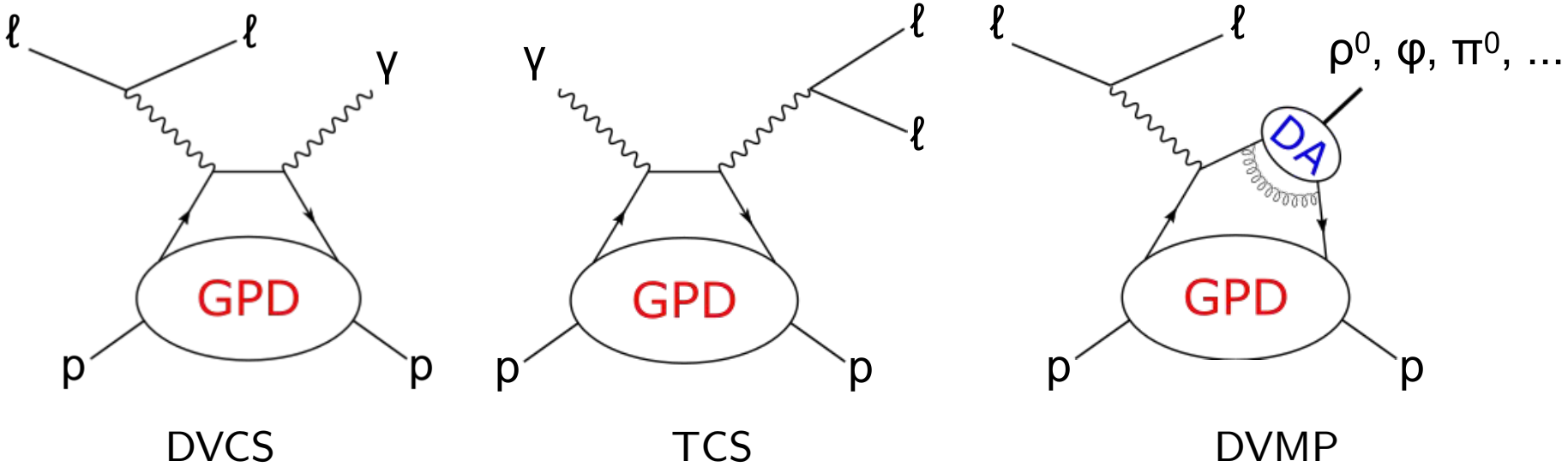

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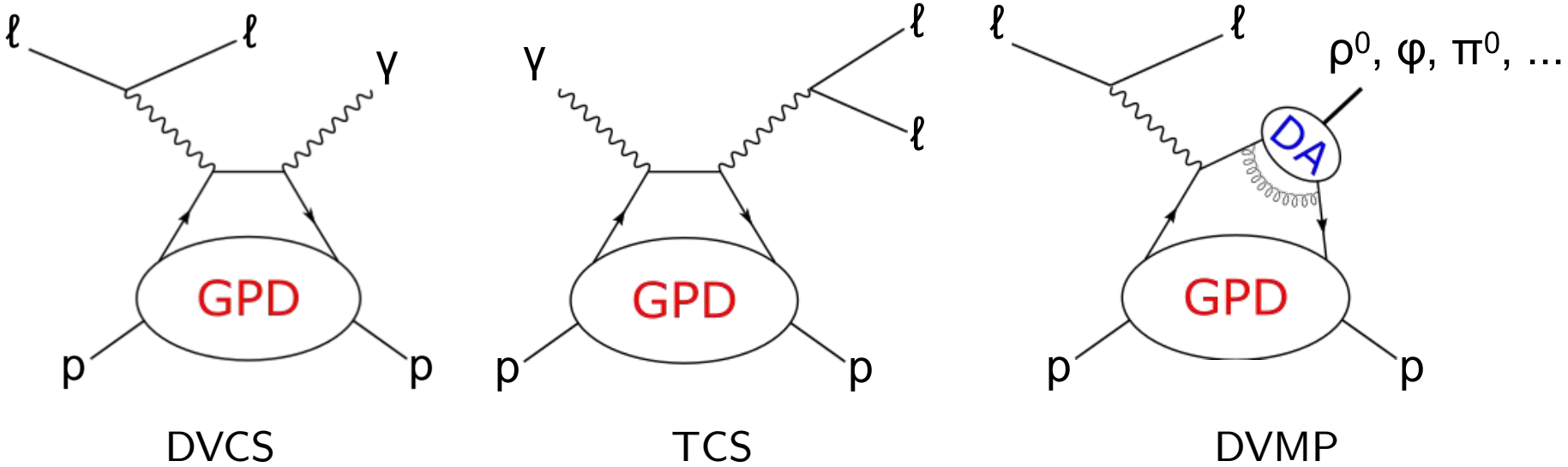

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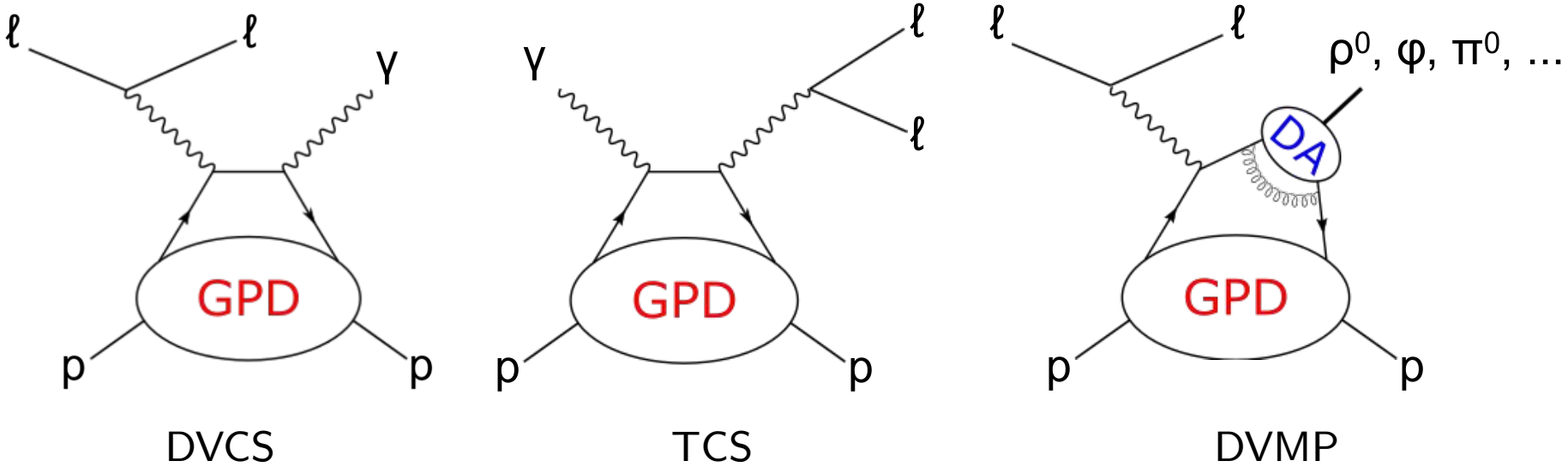

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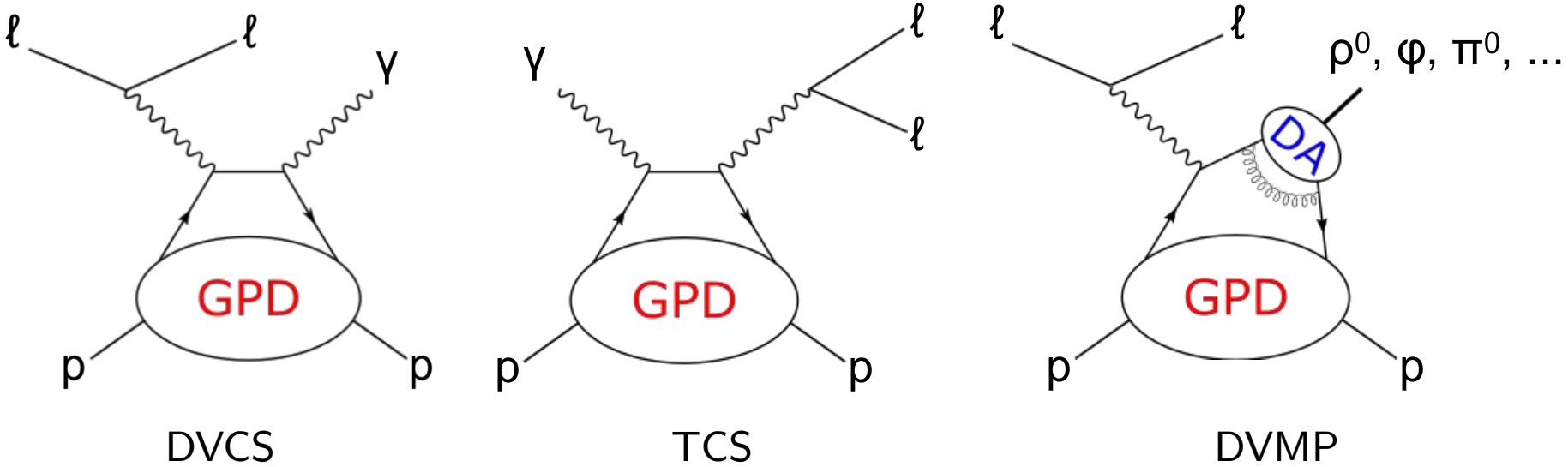

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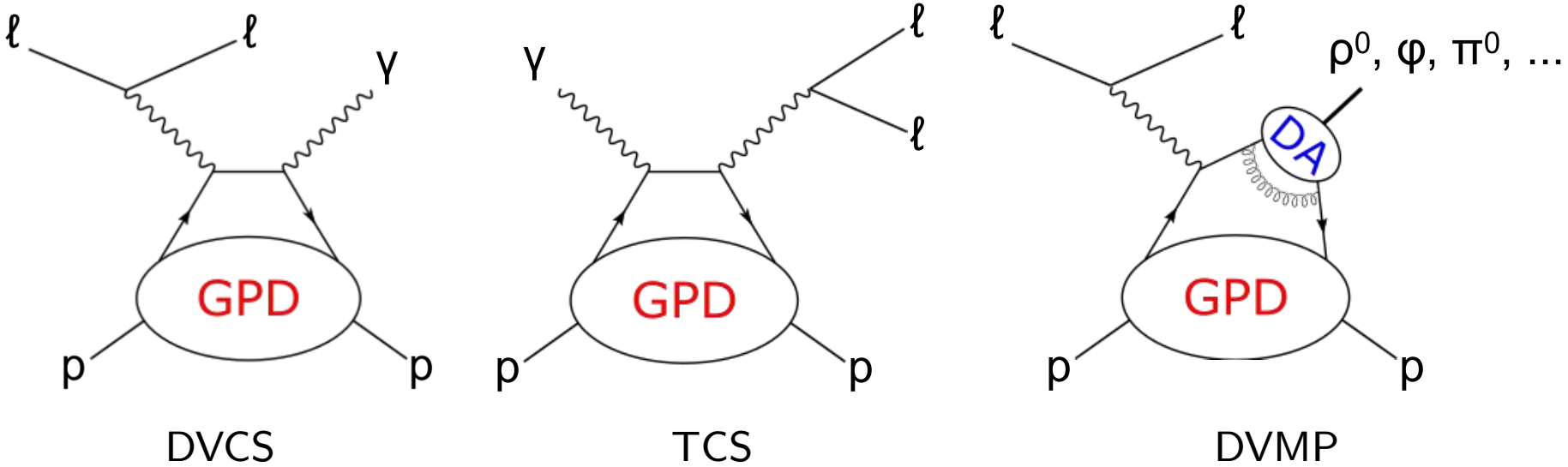

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  - a hard-scattering kernel, calculated with perturbative QCD (short distance interactions).

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Overlap of Light-cone wave functions

- A given *hadronic state* is decomposed in a **Fock basis**: (Brodsky et al., 1981)

$$|H; P, \lambda\rangle = \sum_{N,\beta} \int [\mathrm{d}x]_N \left[\mathrm{d}^2\mathbf{k}_\perp\right]_N \Psi^\lambda_{N,\beta}\left(x_1, \mathbf{k}_{\perp 1}, ...\right) |N, \beta; k_1, ..., k_N\rangle \ , \tag{10}$$

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Double Distributions (DDs)

- DD representation of GPDs:

$$H(x,\xi,t) \propto \int_{\Omega} \mathrm{d}\beta\, \mathrm{d}\alpha\, h(\beta,\alpha,t)\, \delta(x-\beta-\alpha\xi)\ . \tag{13}$$

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- Positivity not manifest...

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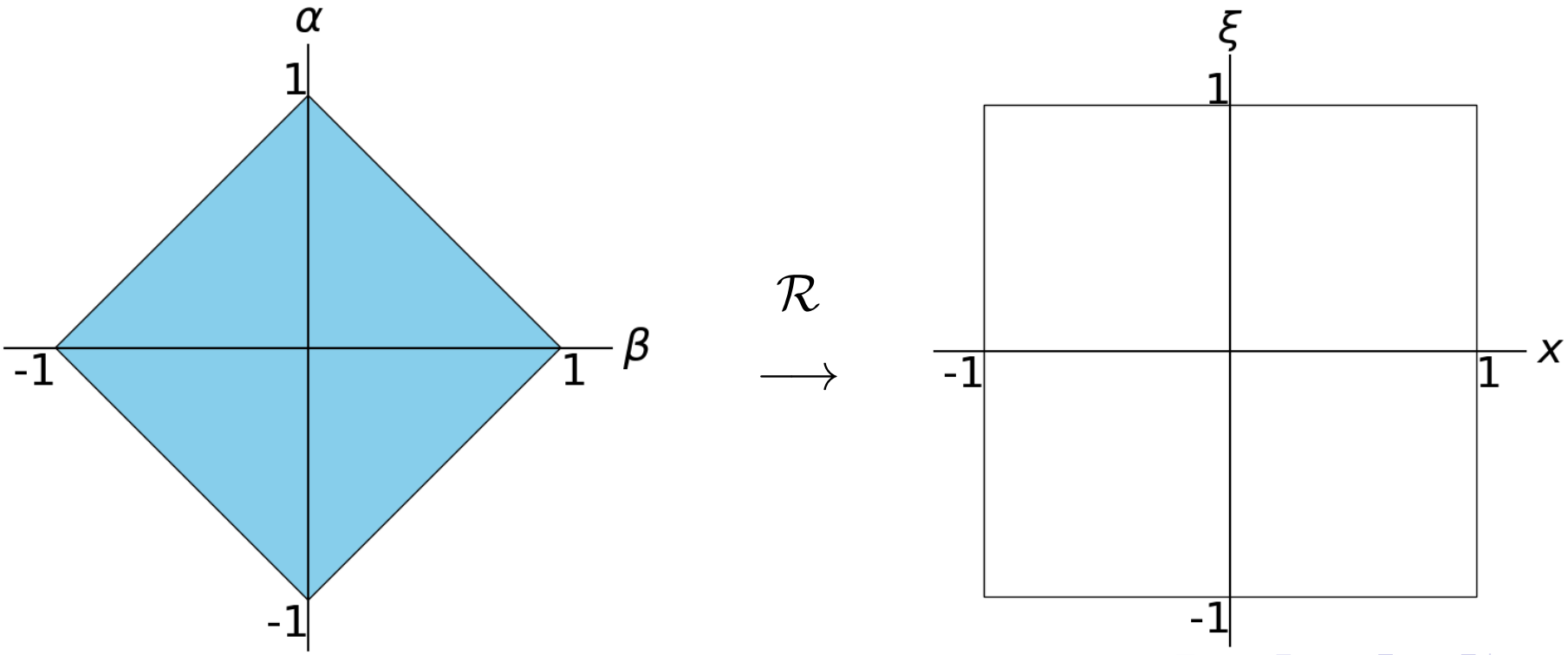

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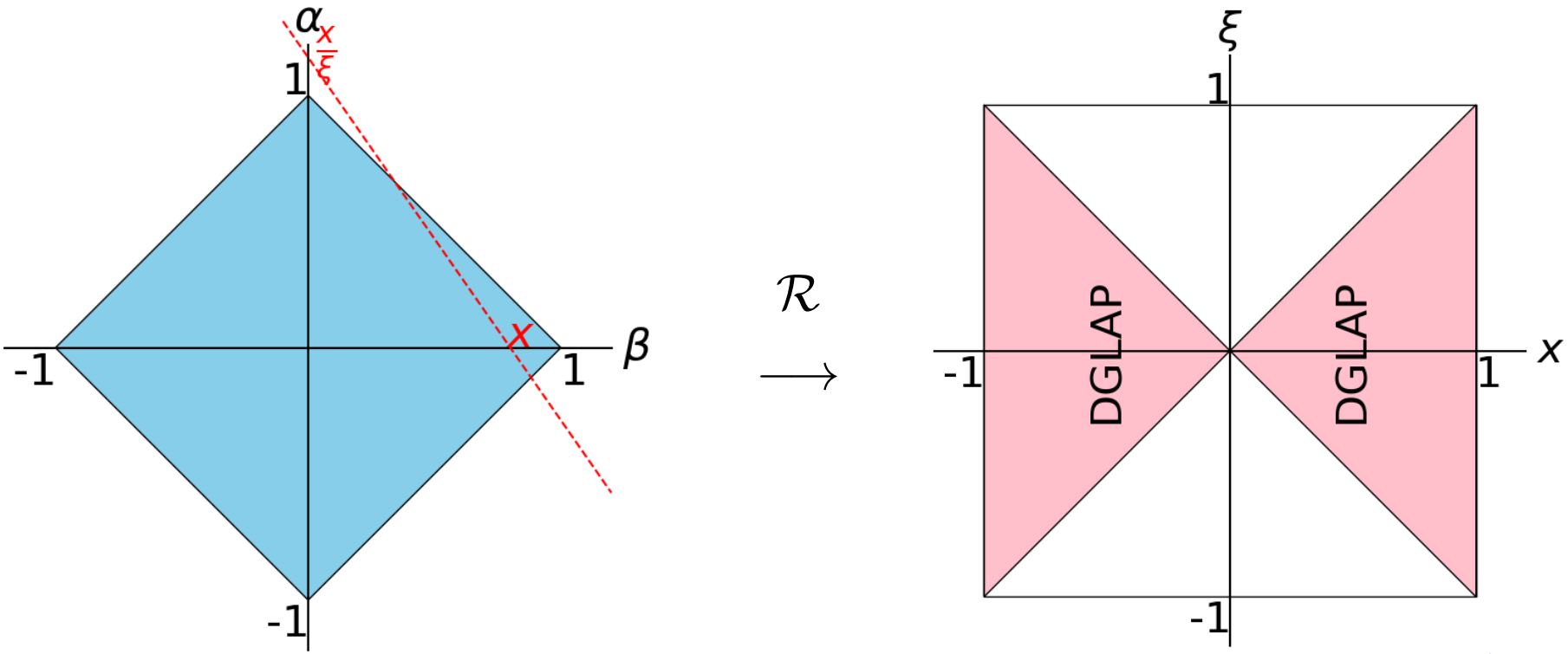

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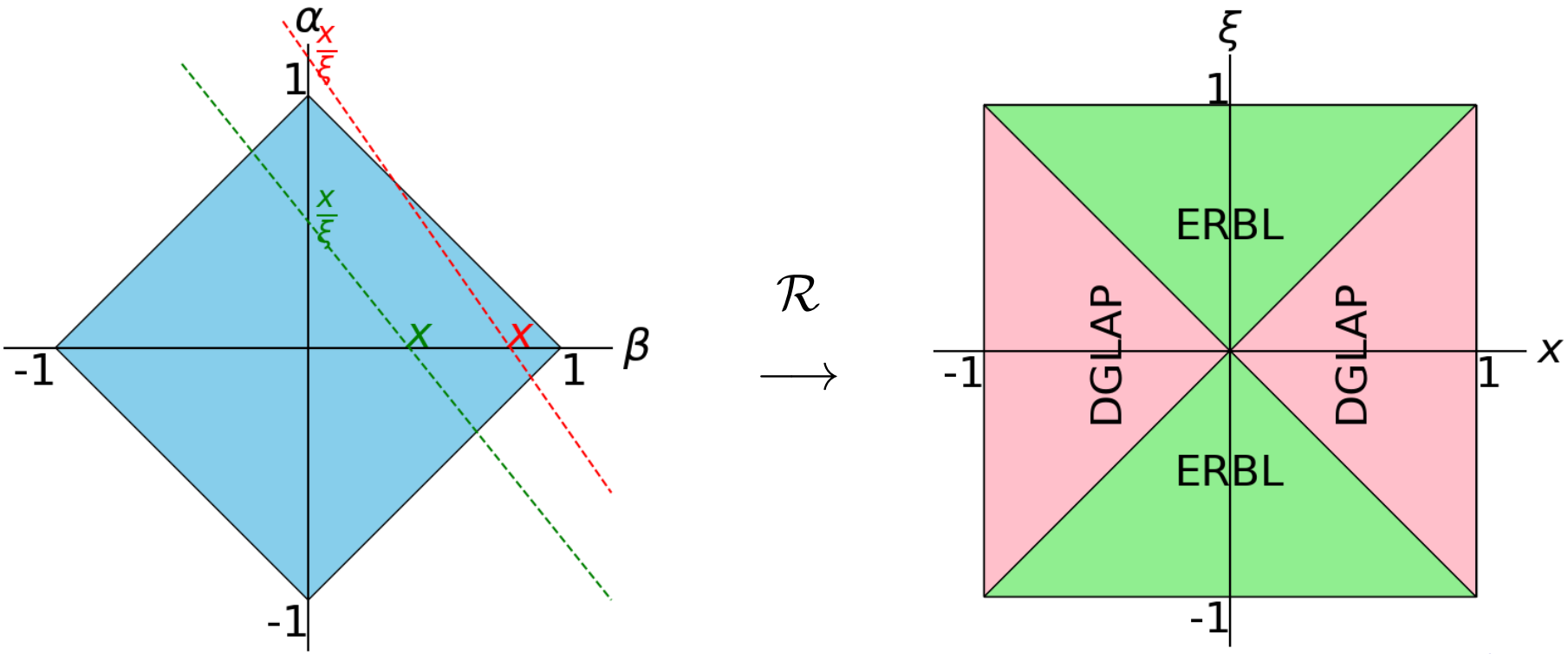
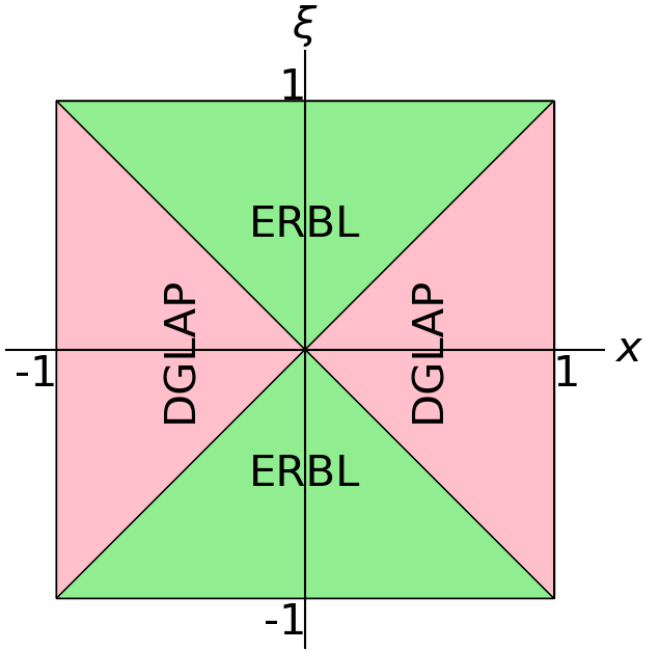


Outline

Covariant extension

- For the spatial tomography of hadrons, we need:



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- For the spatial tomography of hadrons, we need:
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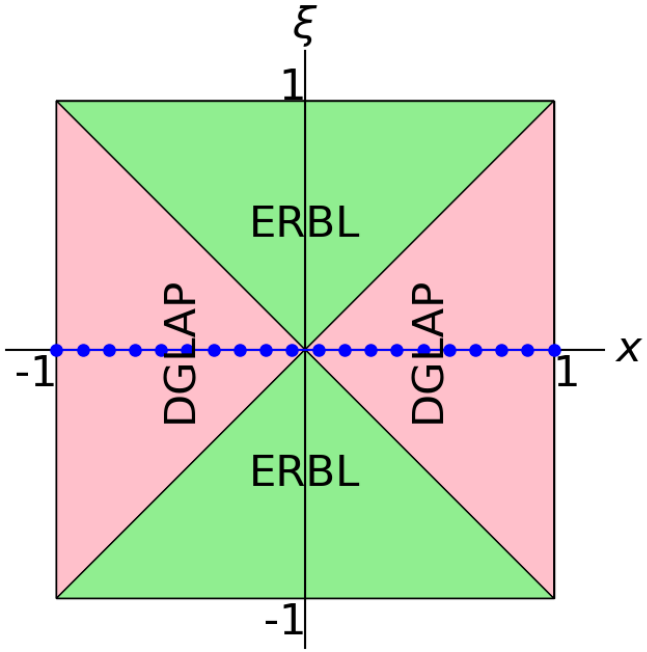

Covariant extension

- For the spatial tomography of hadrons, we need:
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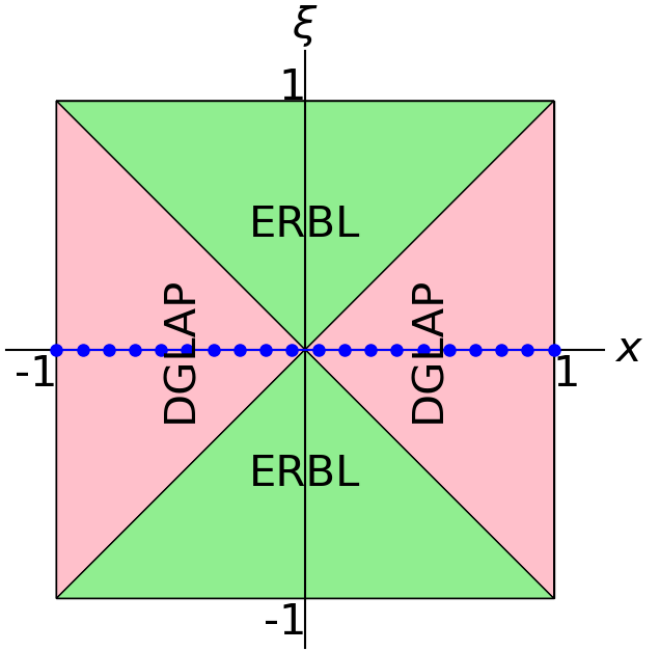
- Experimental access through exclusive processes:



Covariant extension

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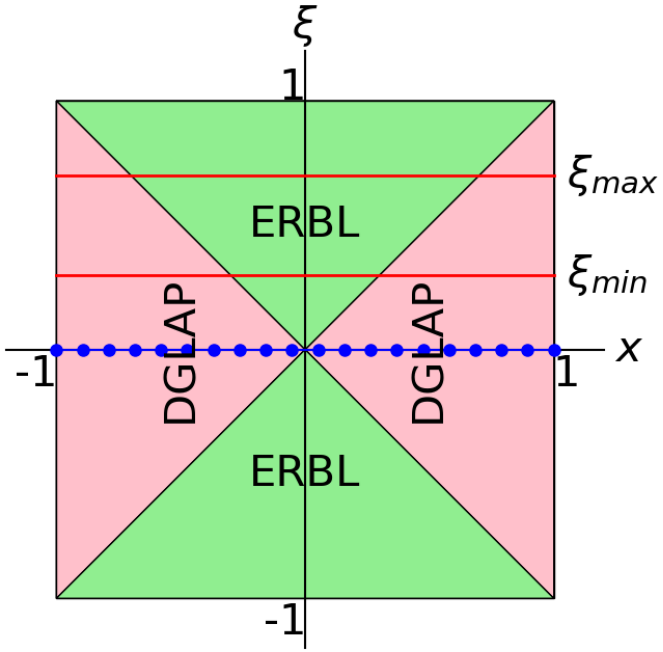

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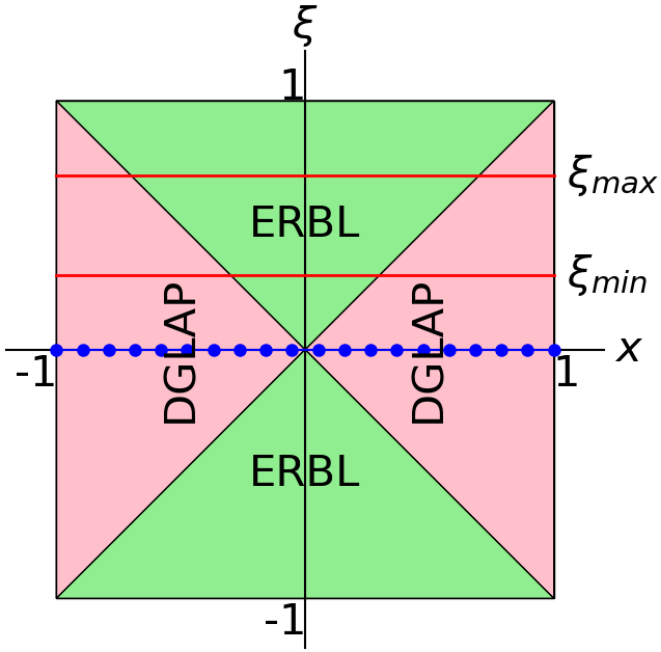

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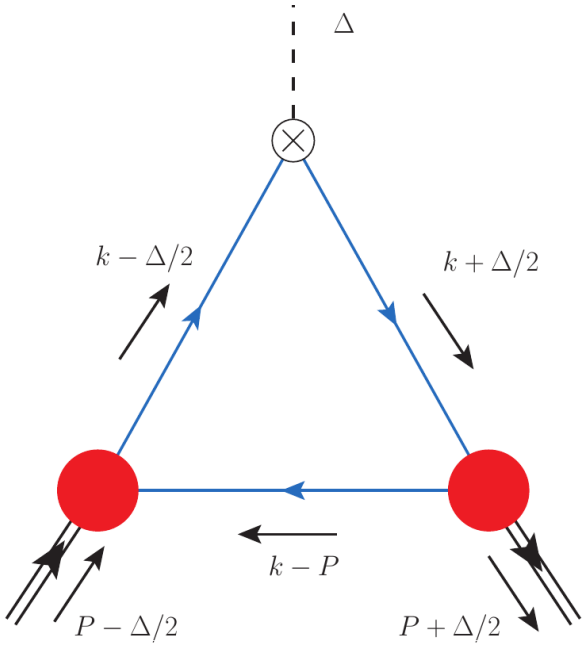

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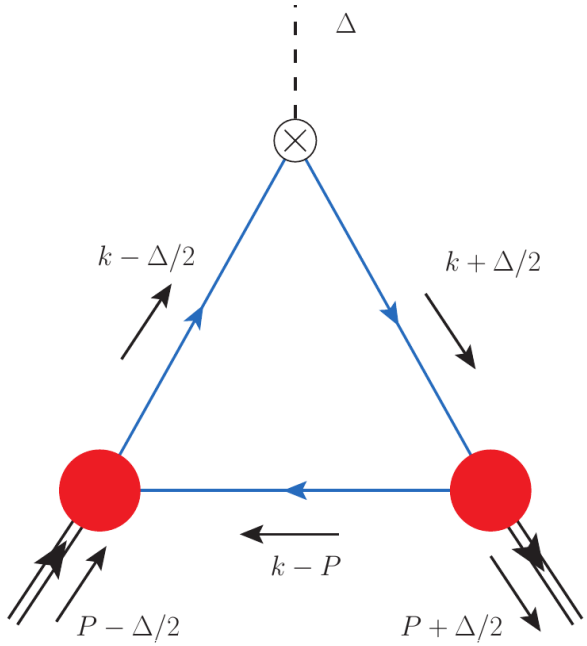

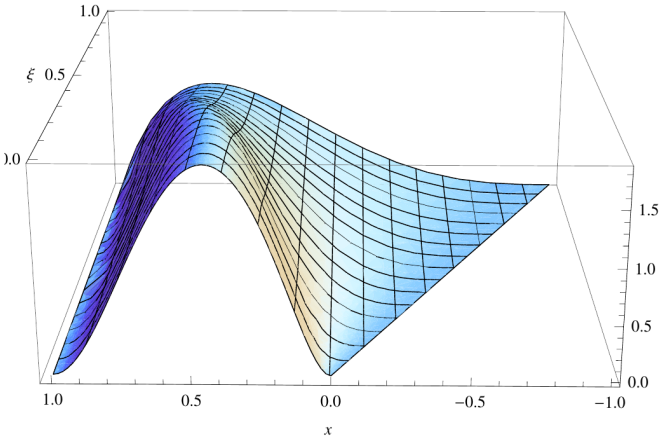

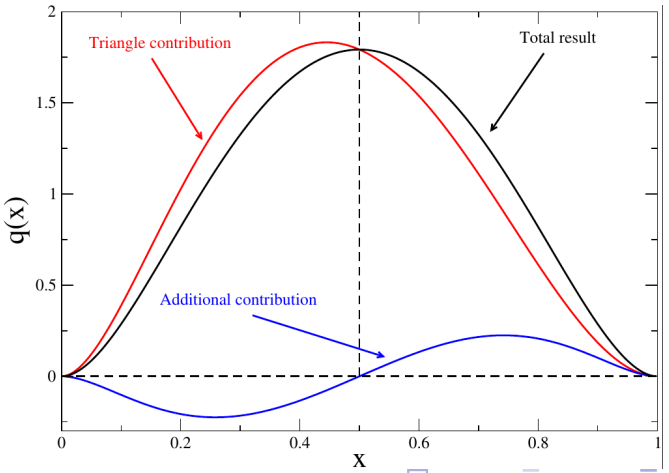

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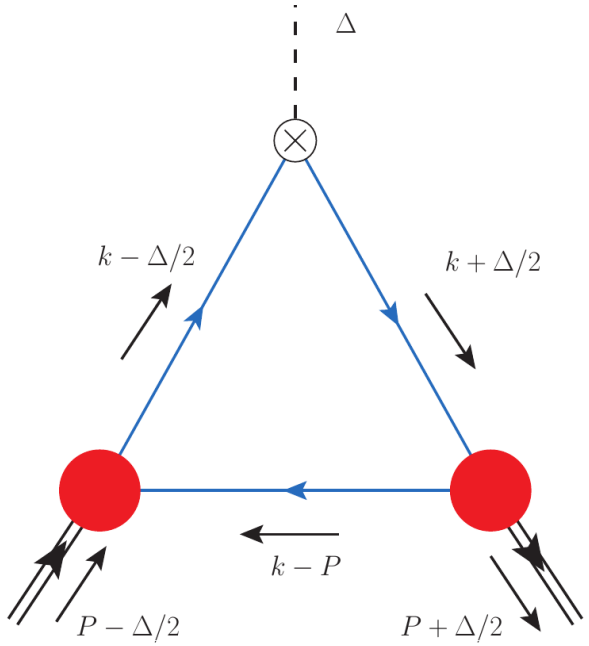
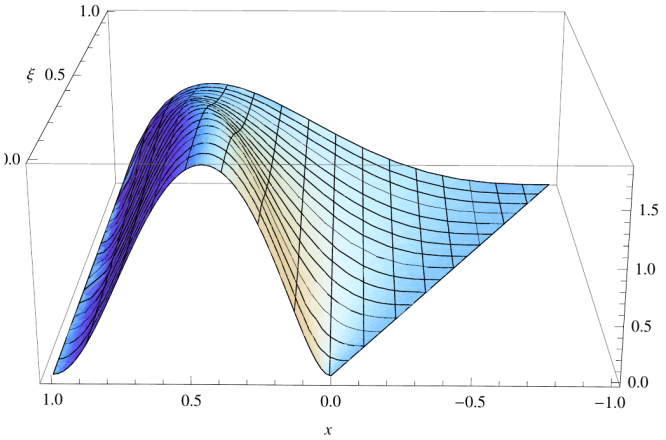
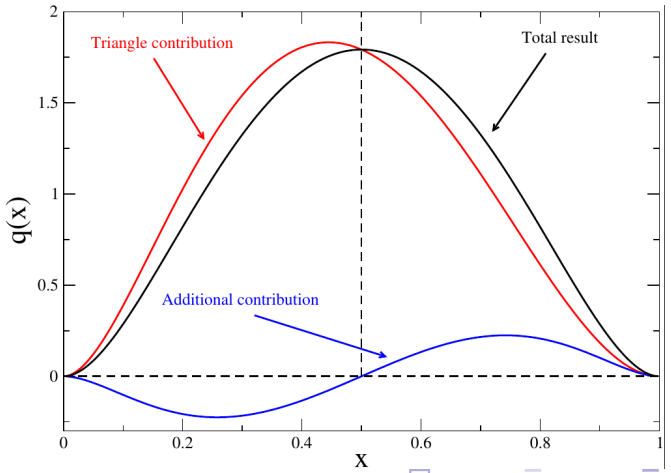
  - Loss of symmetries...







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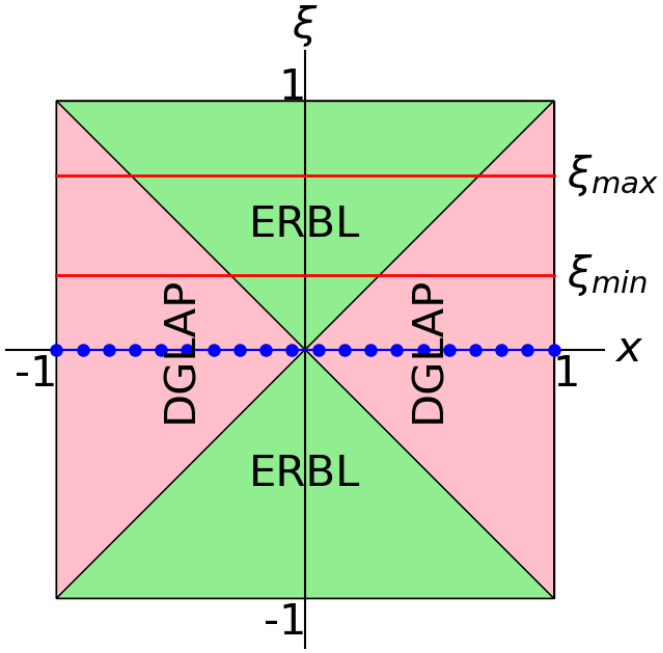

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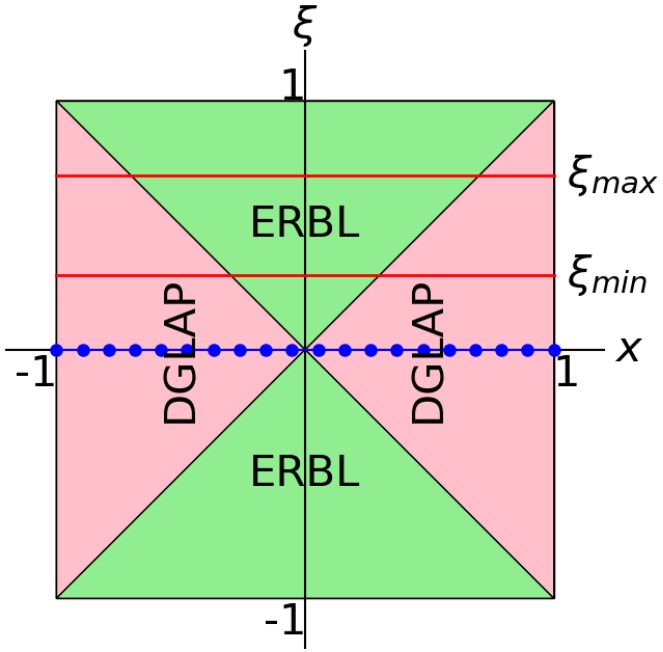

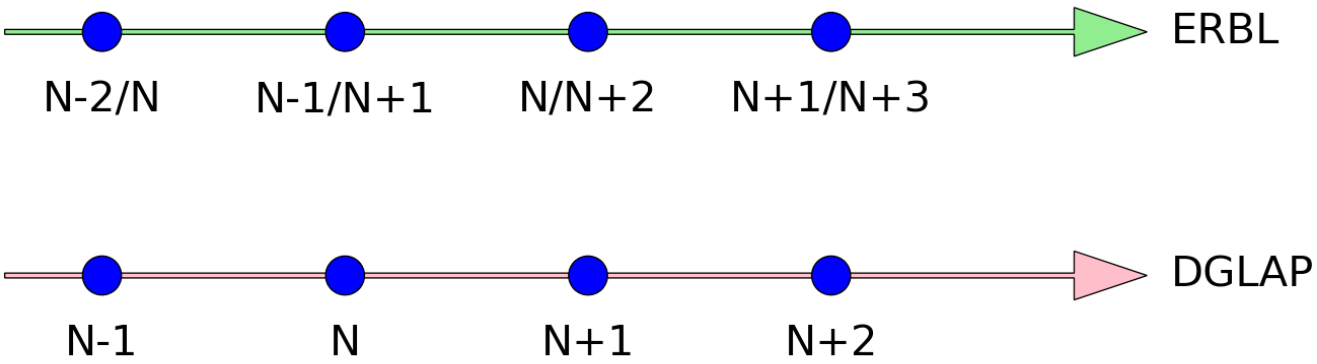

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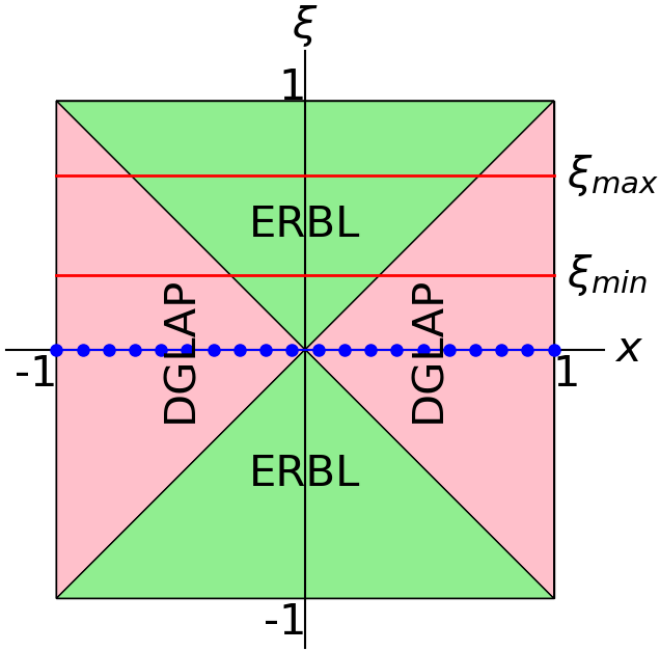

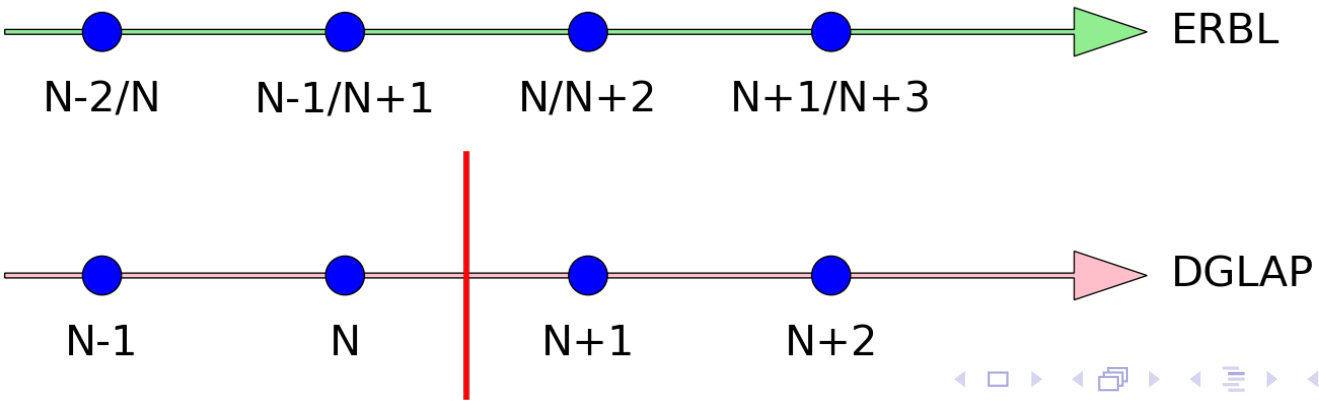

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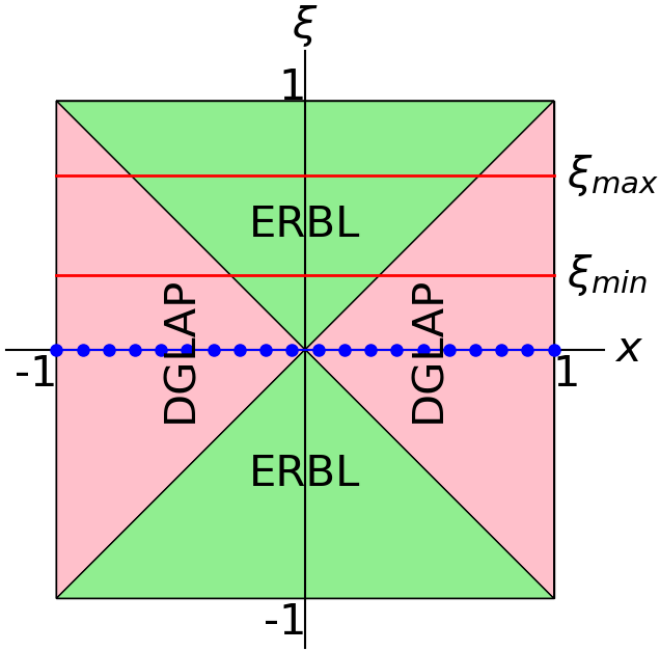

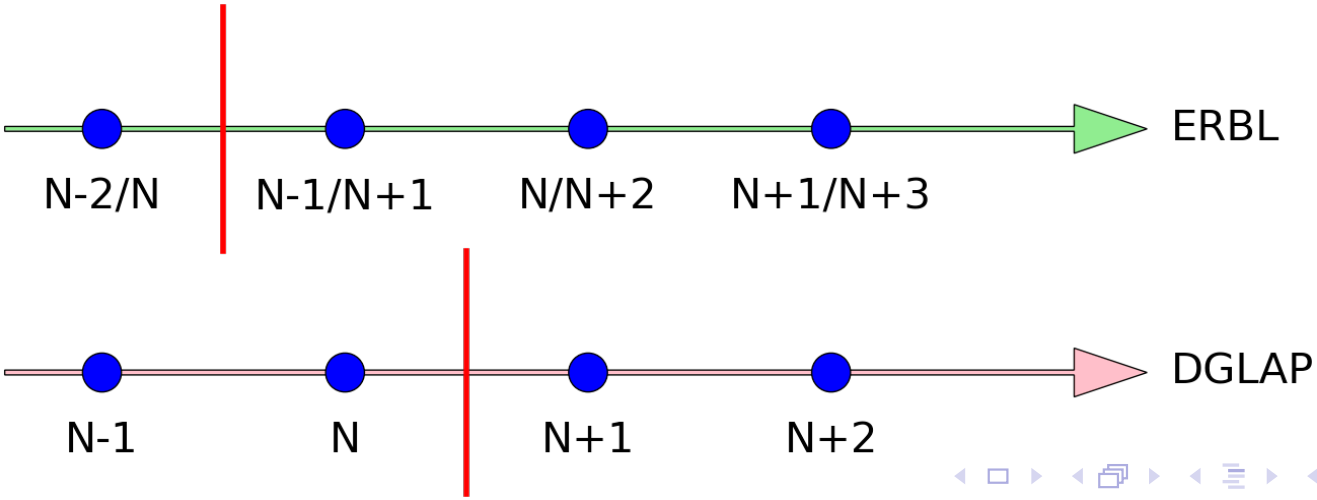

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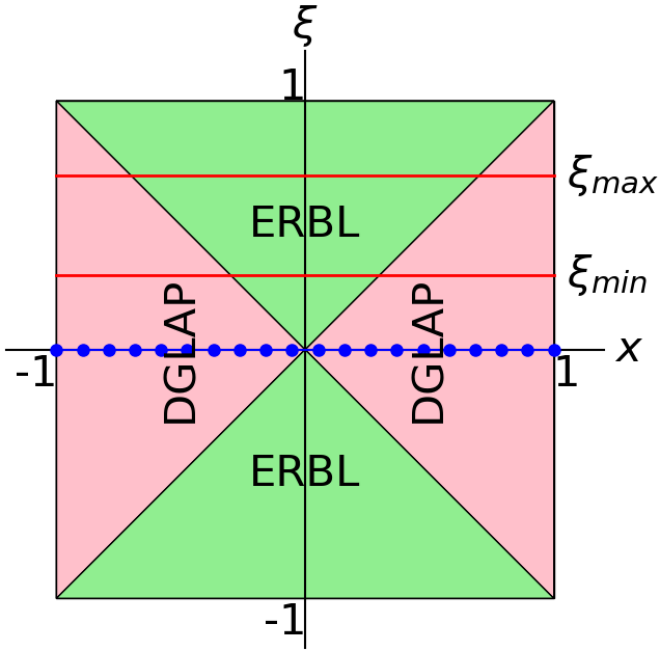

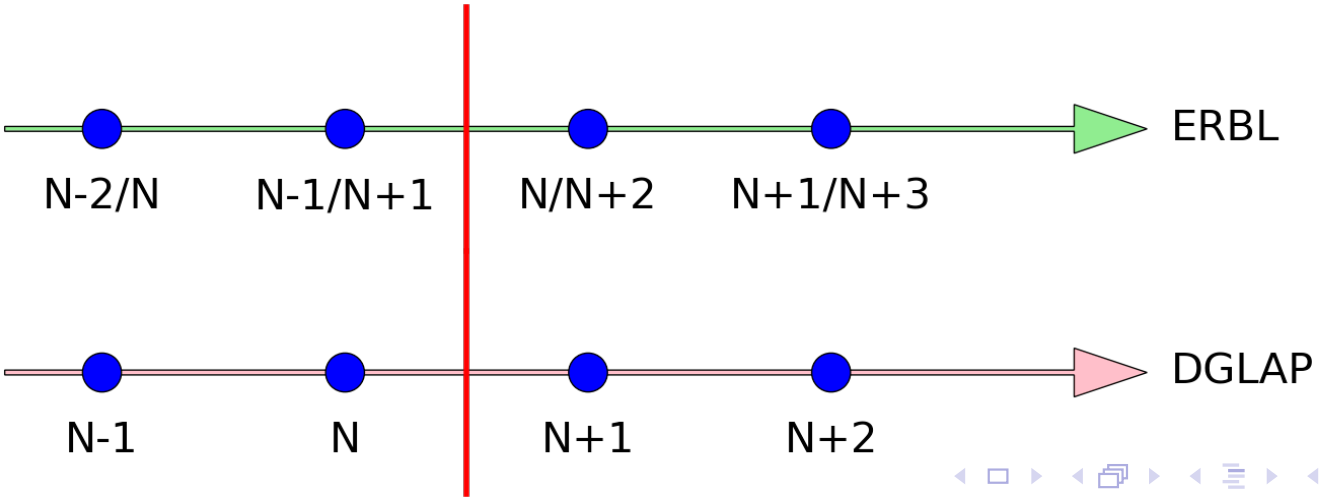

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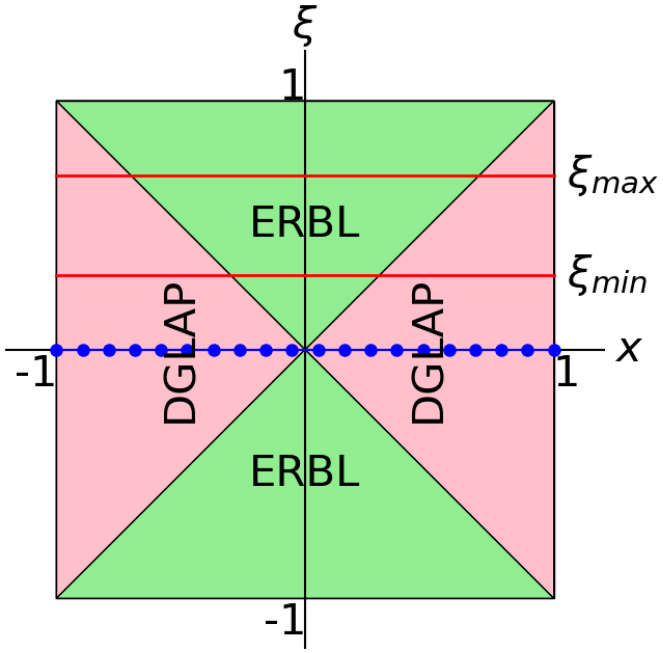

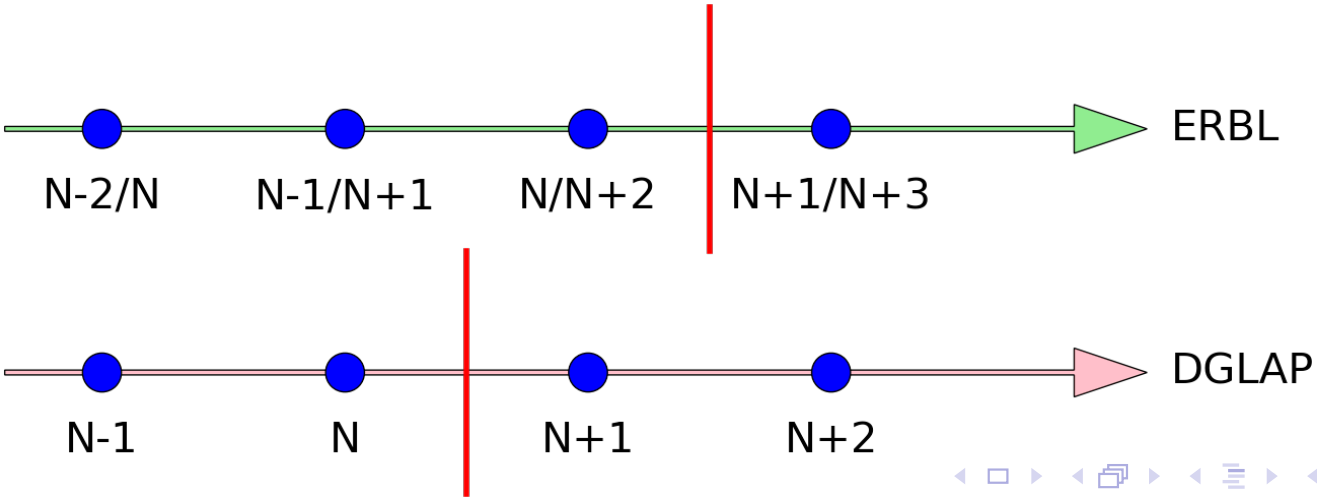

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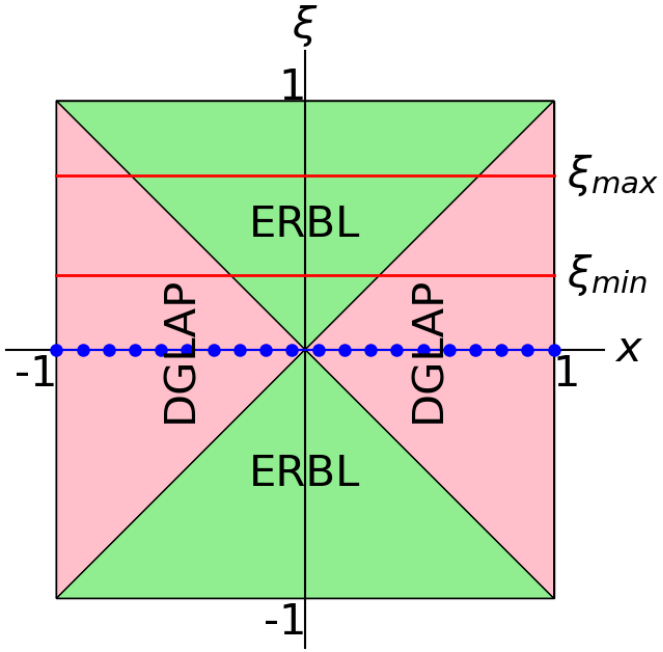

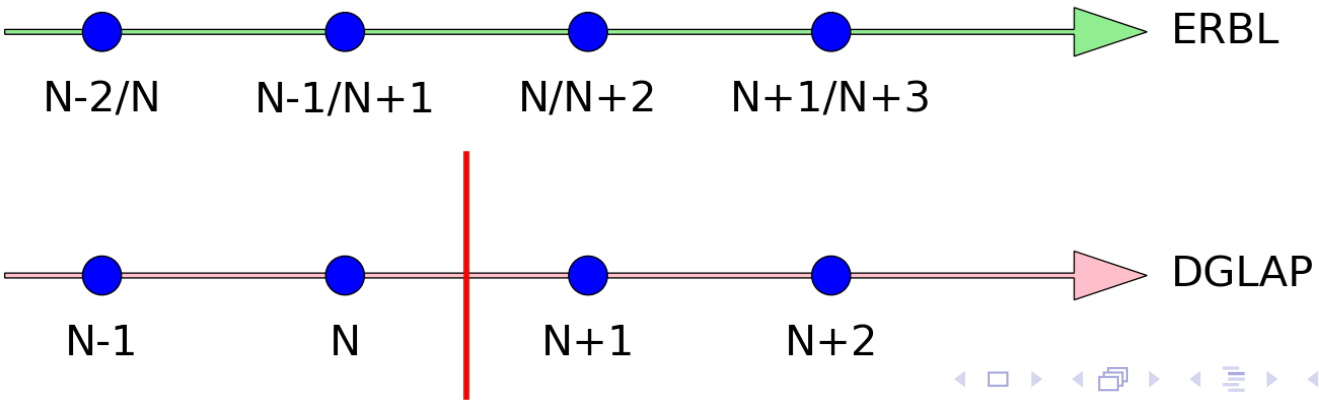

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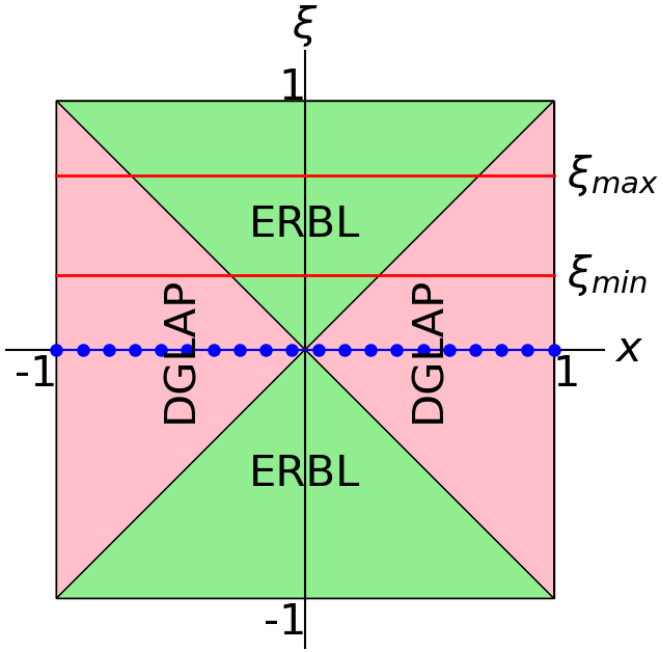

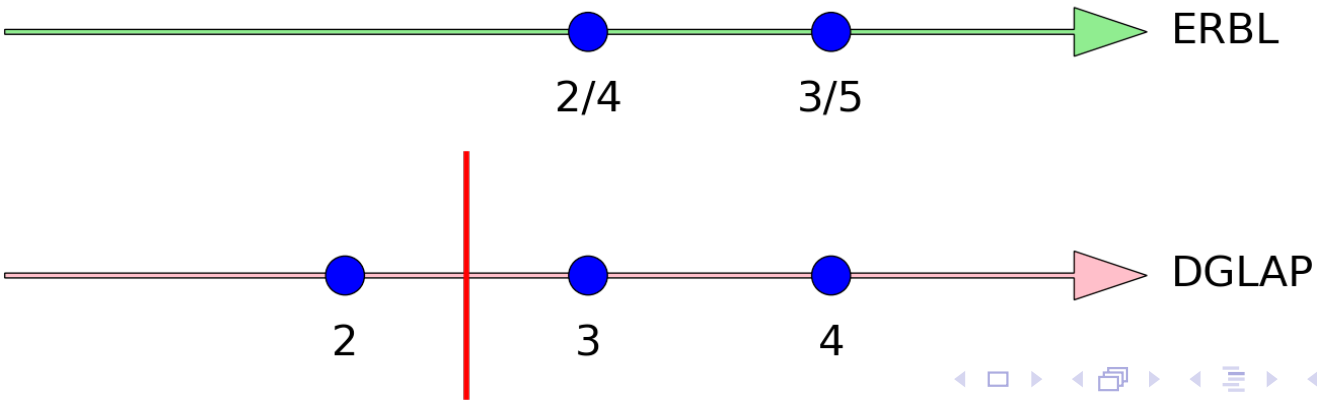

Inversion

Problem

Find $h(\beta, \alpha)$ on square $\{|\alpha| + |\beta| \leq 1\}$ such that

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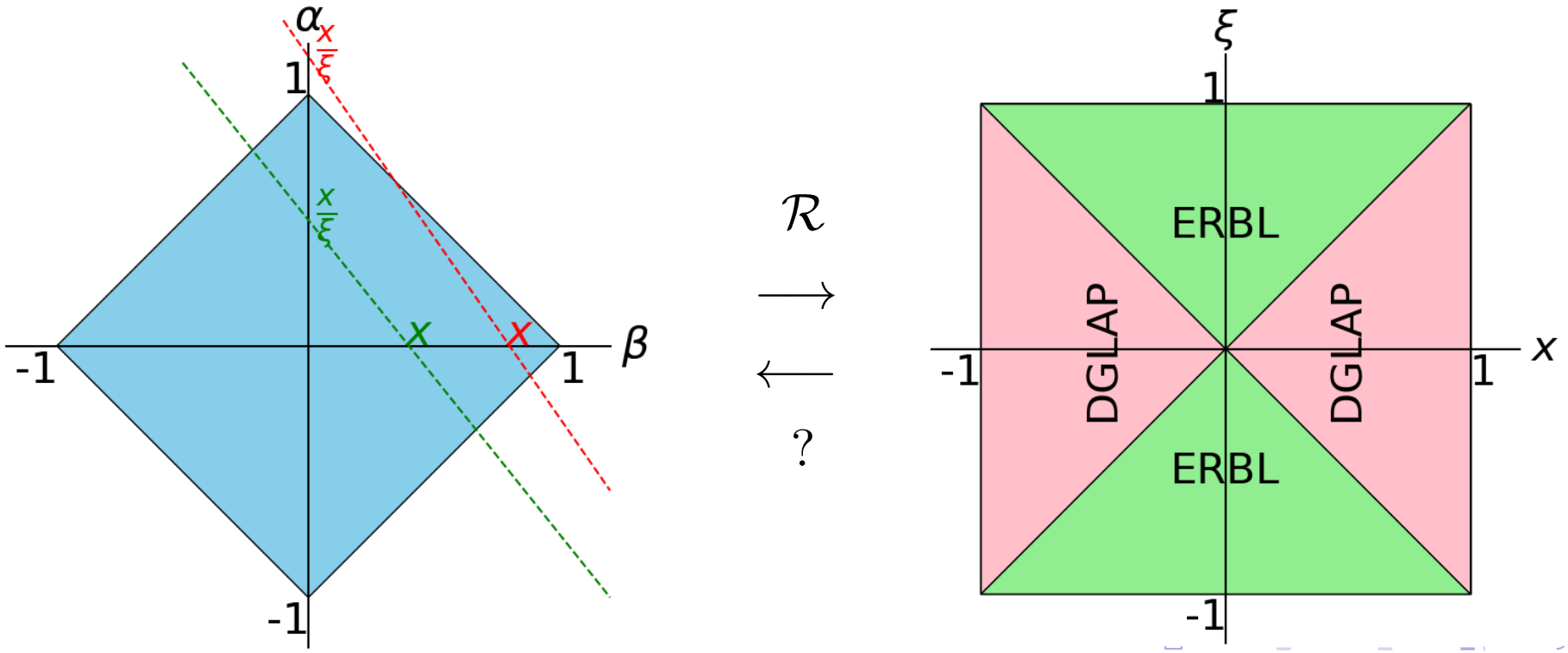

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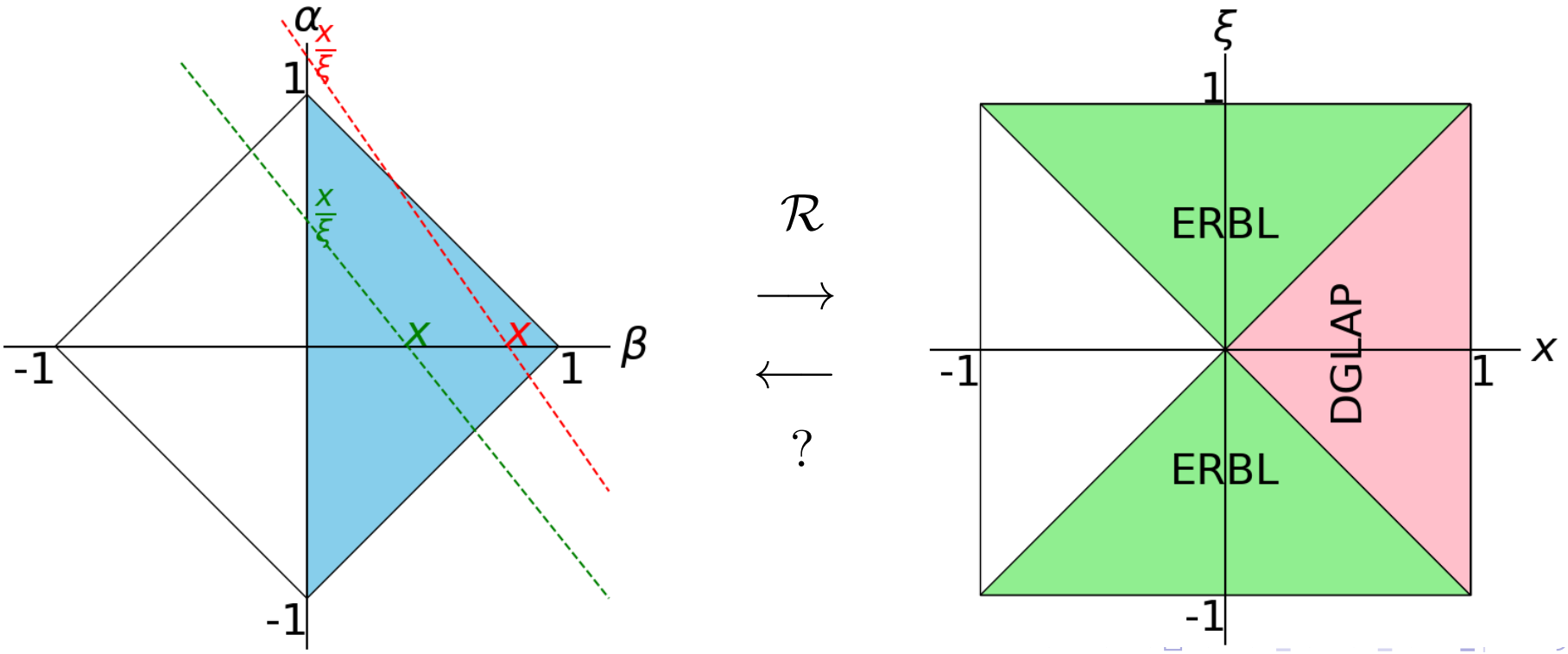

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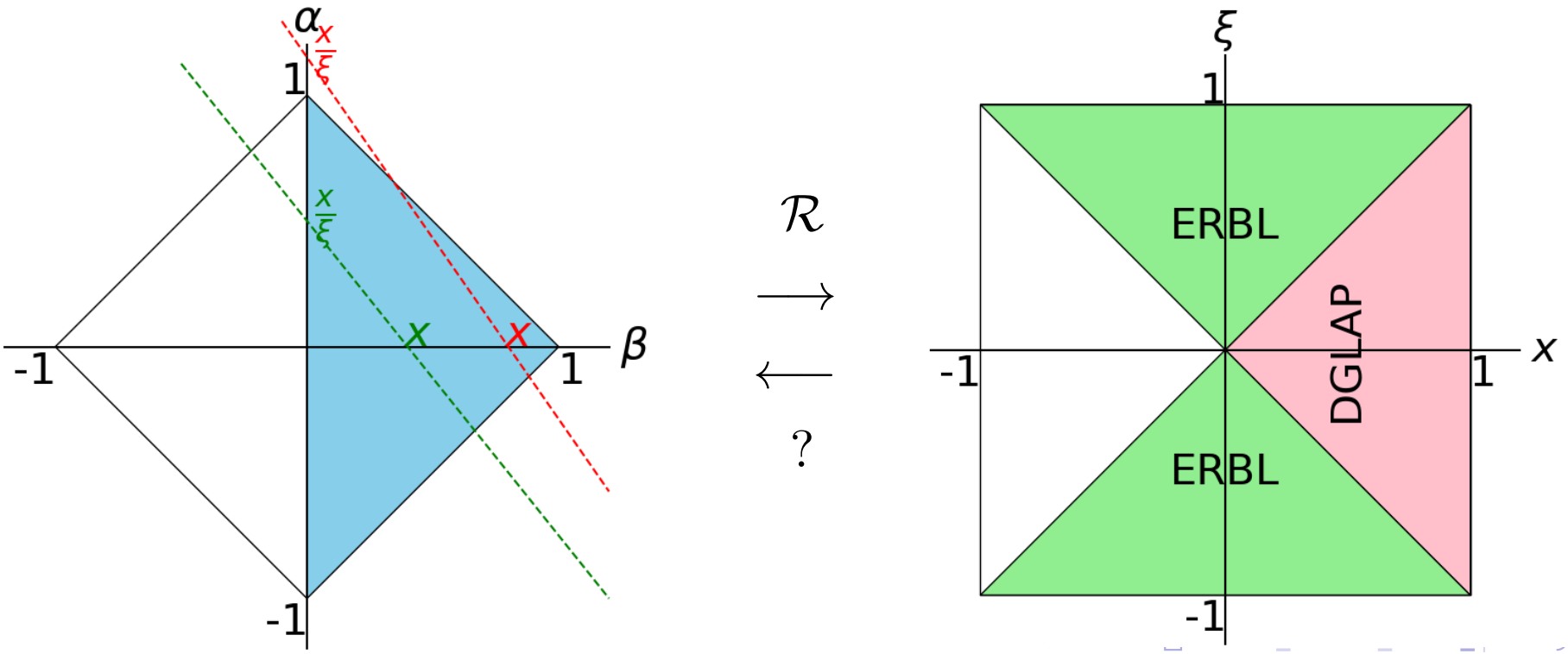

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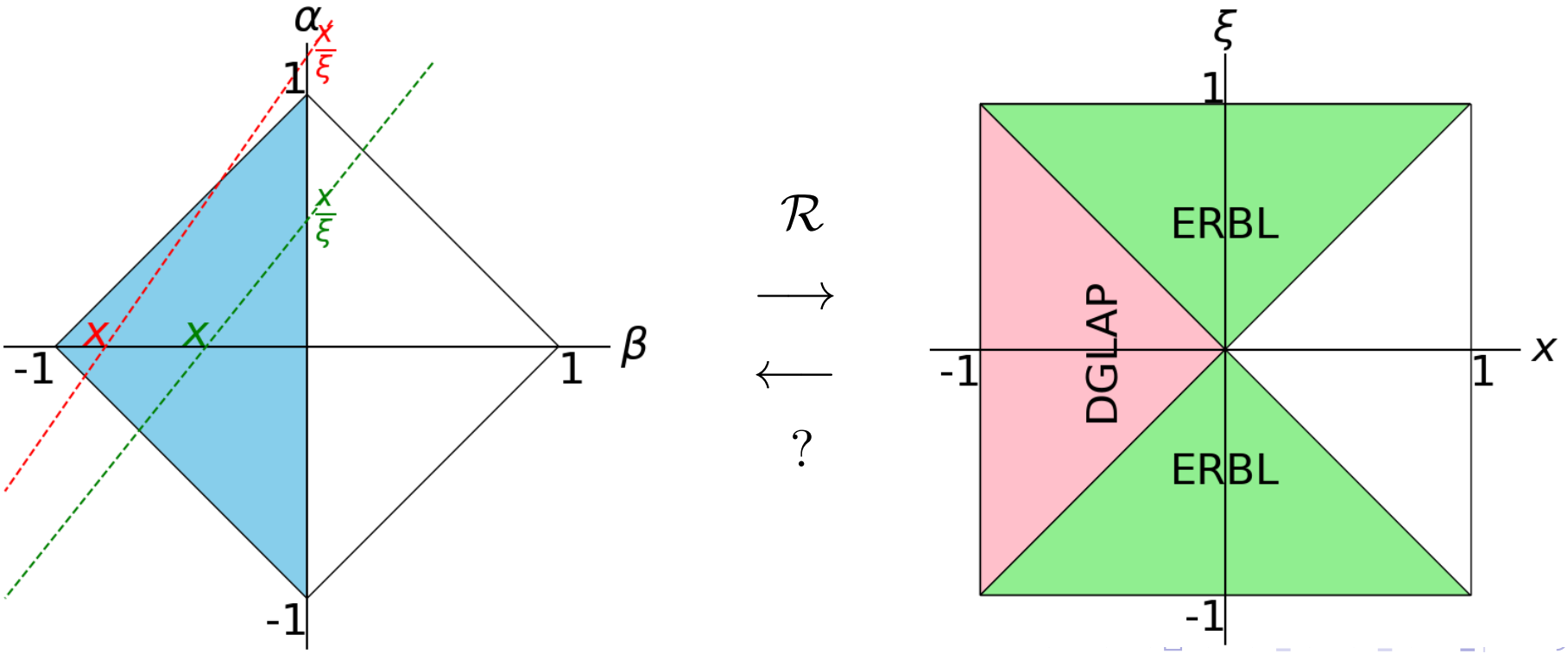

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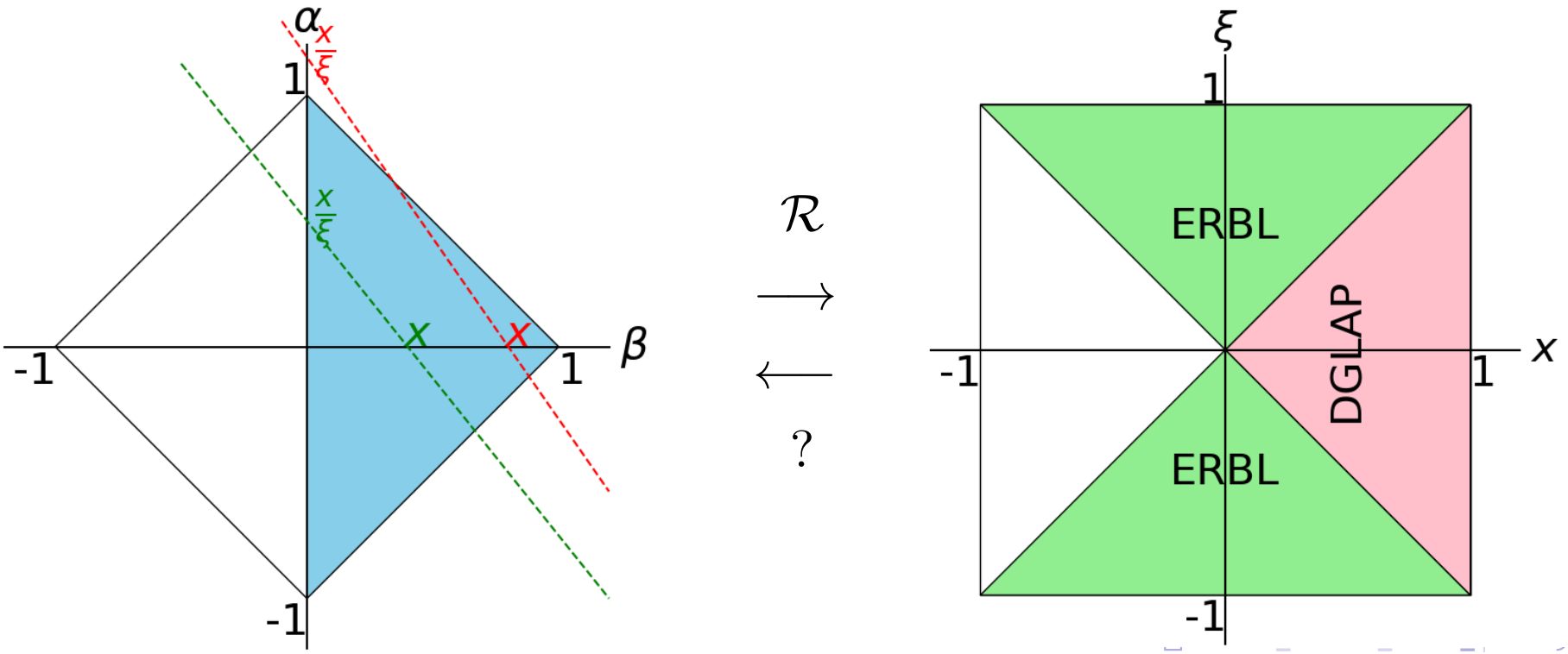
  - Better numerical stability.
  - Lesser complexity: $O(N^p + N^p) \ll O((N+N)^p)$.



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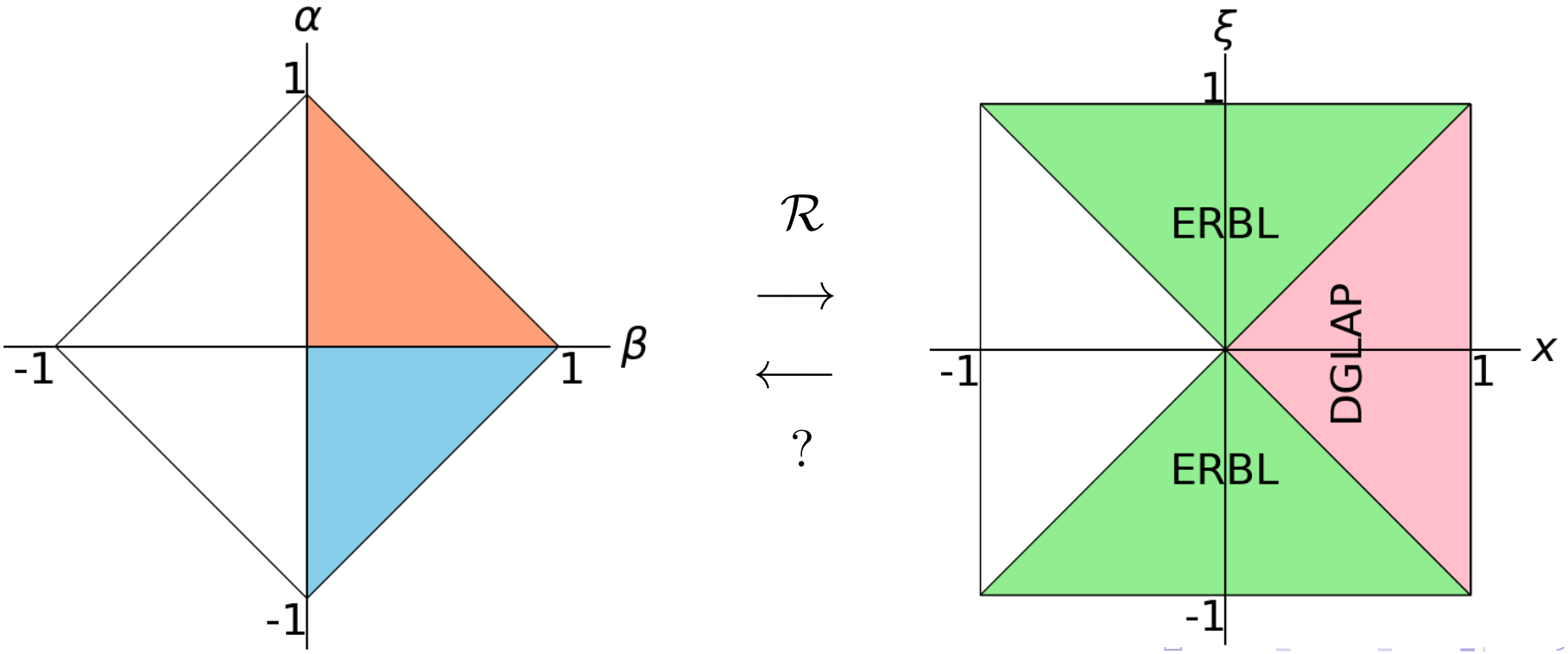
- α-parity of the DD: $h(\beta,-\alpha) = h(\beta,\alpha)$.



Discretization

- Expansion of the DD into basis functions $\{v_j\}$:

$$h(\beta, \alpha) = \sum_j h_j \, v_j(\beta, \alpha) \, , \tag{15}$$

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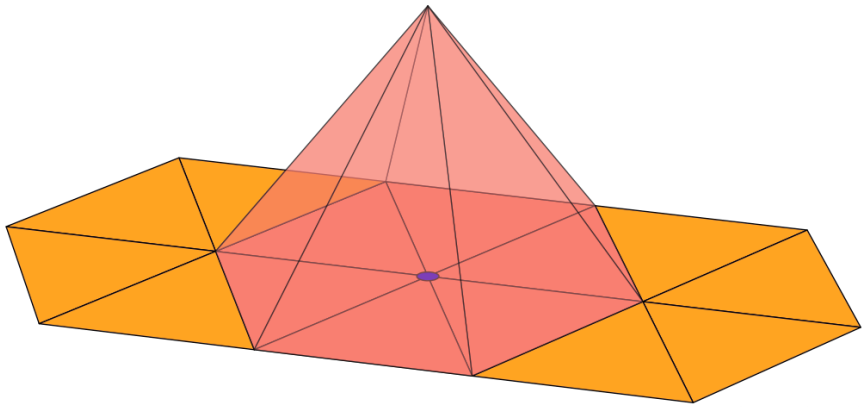

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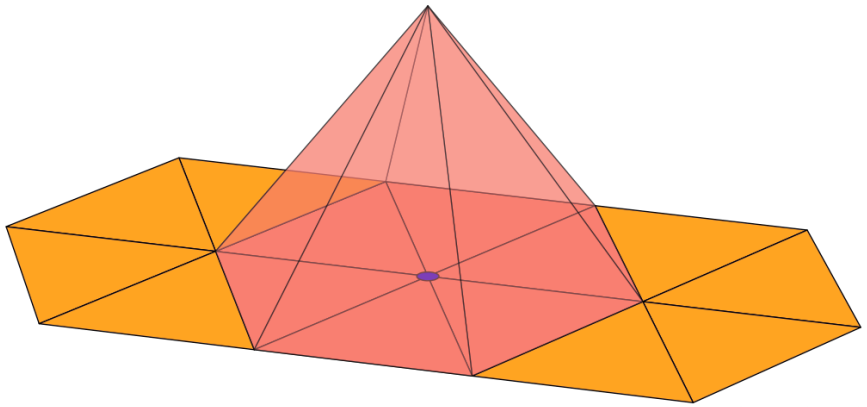

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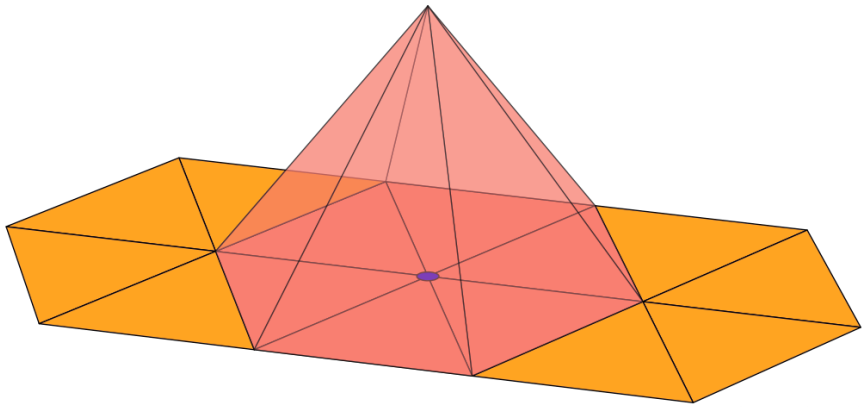

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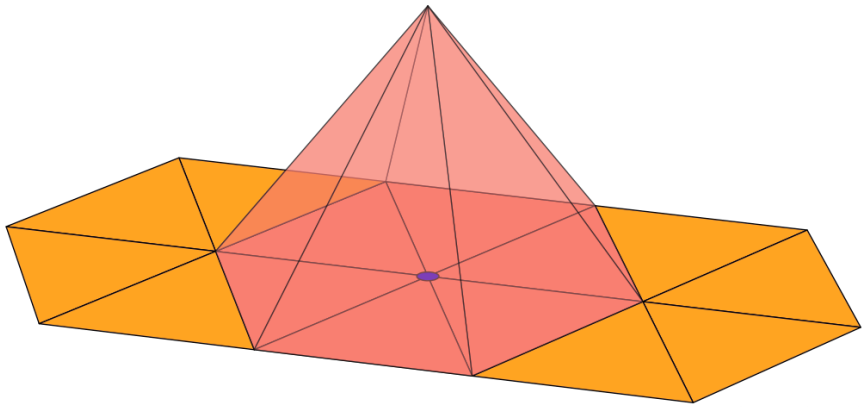
- Sampling:
  - Random couples $(x, \xi) \longrightarrow m \geq n$ lines of the matrix.



Discretization

- Expansion of the DD into basis functions $\{v_j\}$:

$$h(\beta, \alpha) = \sum_j h_j \, v_j(\beta, \alpha) \ , \tag{15}$$

  - Piece-wise constant, piece-wise linear, etc.
  - n columns of the matrix.
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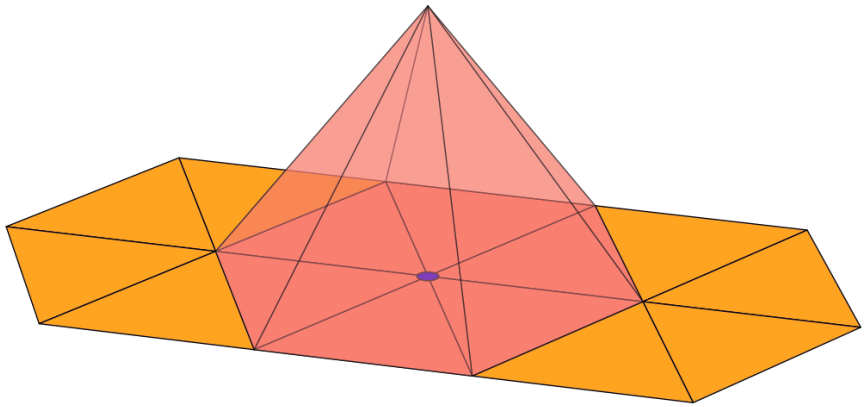

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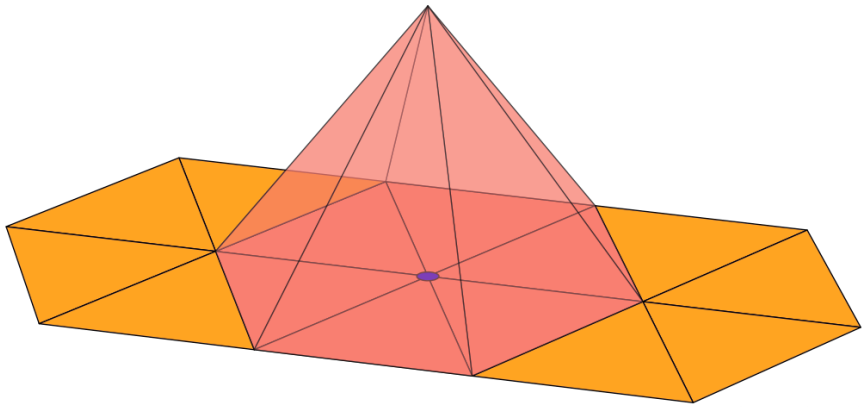

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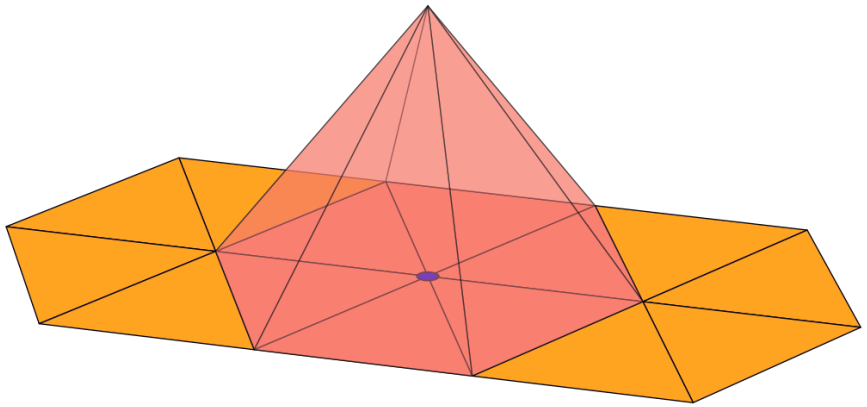

Some examples (Dyson-Schwinger model)

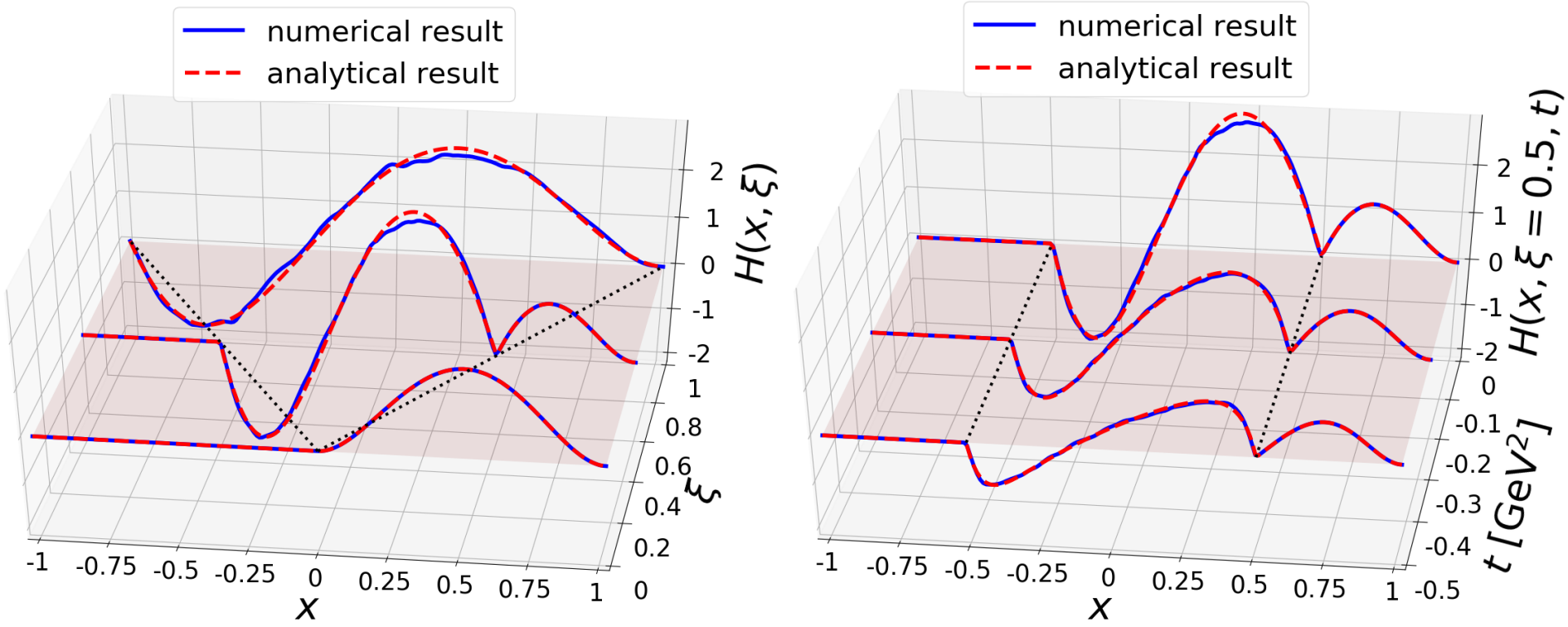

Figure: Extension of GPDs for the pion DSE model of Refs. **(Mezrag, 2015; Mezrag et al., 2016)**. Comparison to the analytical result. Left: Plot for fixed ξ values 0, 0.5 and 1, at $t = 0$ GeV^2. Right: Plot for fixed t values 0, -0.25 and -0.5 GeV^2, at $\xi = 0.5$.

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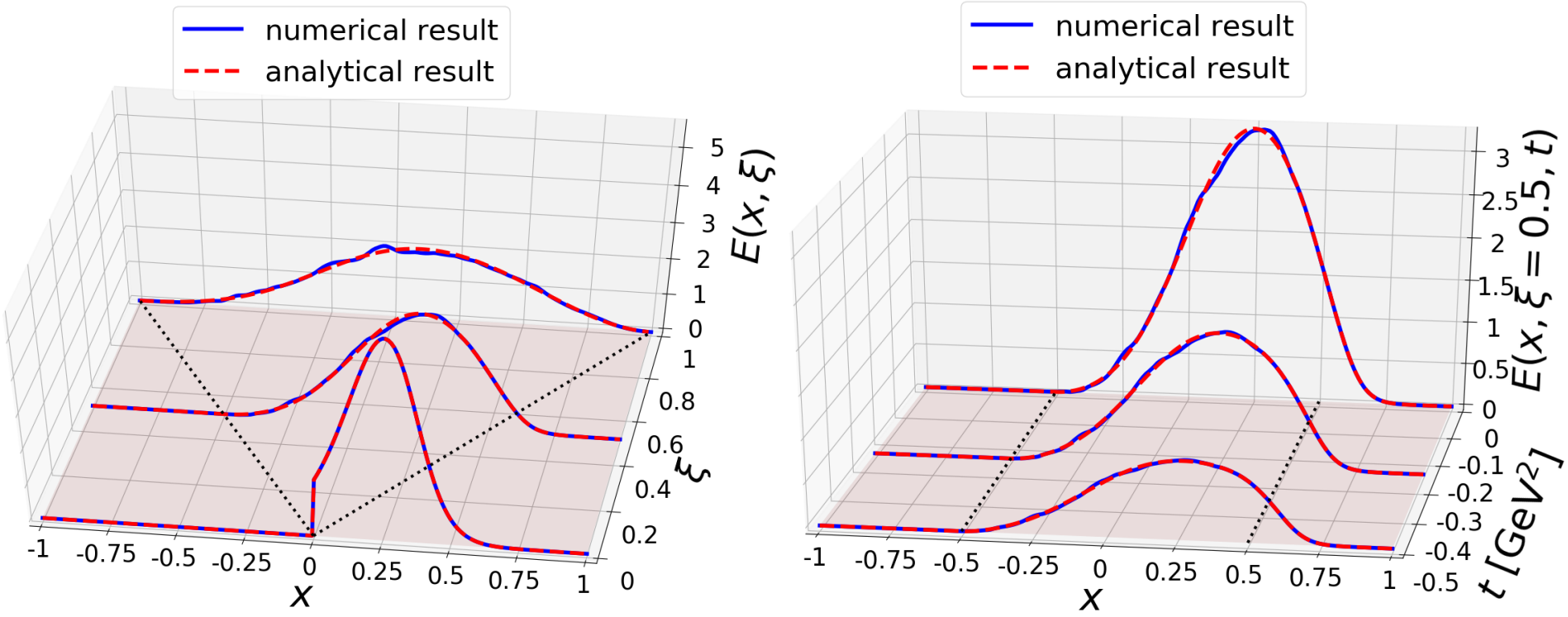

Figure: Extension of GPD E for the nucleon model of Ref. **(Hwang and Mueller, 2008)**. Comparison to the analytical result of the authors. Left: Plot for fixed ξ values 0, 0.5 and 1, at $t = 0$ GeV^2. Right: Plot for fixed t values 0, -0.25 and -0.5 GeV^2, at $\xi = 0.5$.

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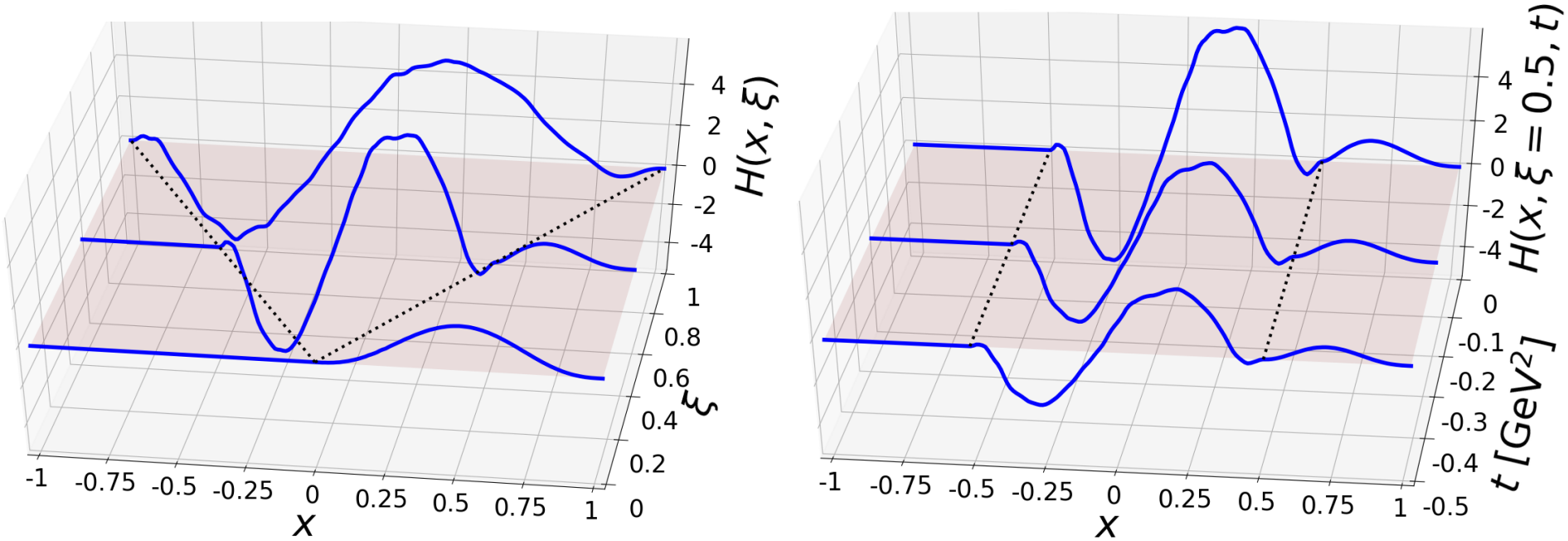

Figure: Extension of GPD for a gaussian pion model (in the vein of AdS/QCD). Left: Plot for fixed ξ values 0, 0.5 and 1, at $t = 0$ GeV^2. Right: Plot for fixed t values 0, -0.25 and -0.5 GeV^2, at $\xi = 0.5$.

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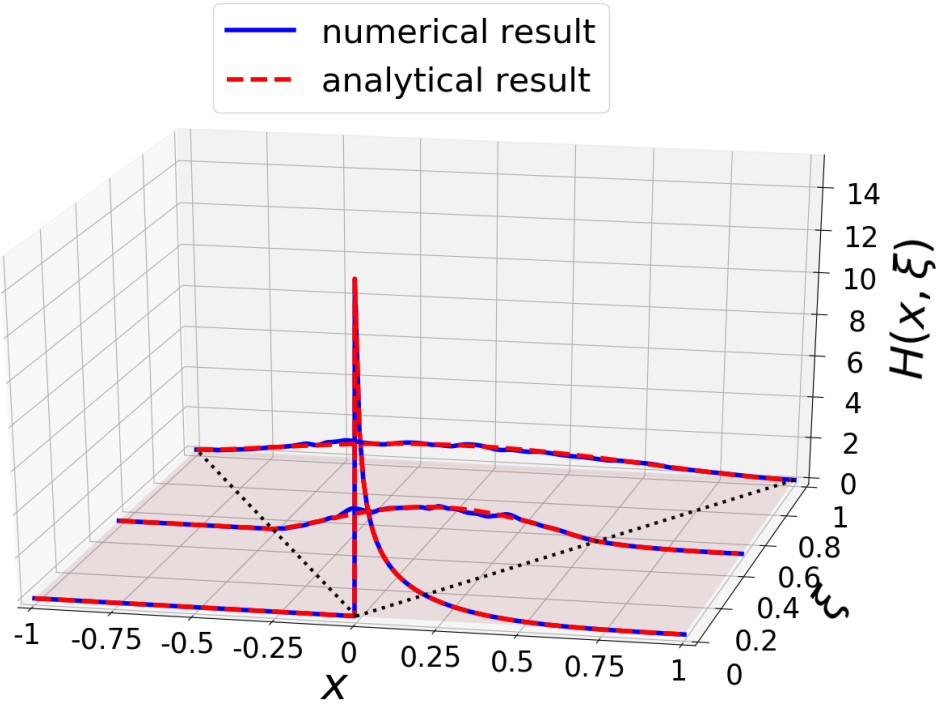

Figure: Extension of GPD for a nucleon toy model with Regge behavior. Plot for fixed ξ values 0, 0.5 and 1.

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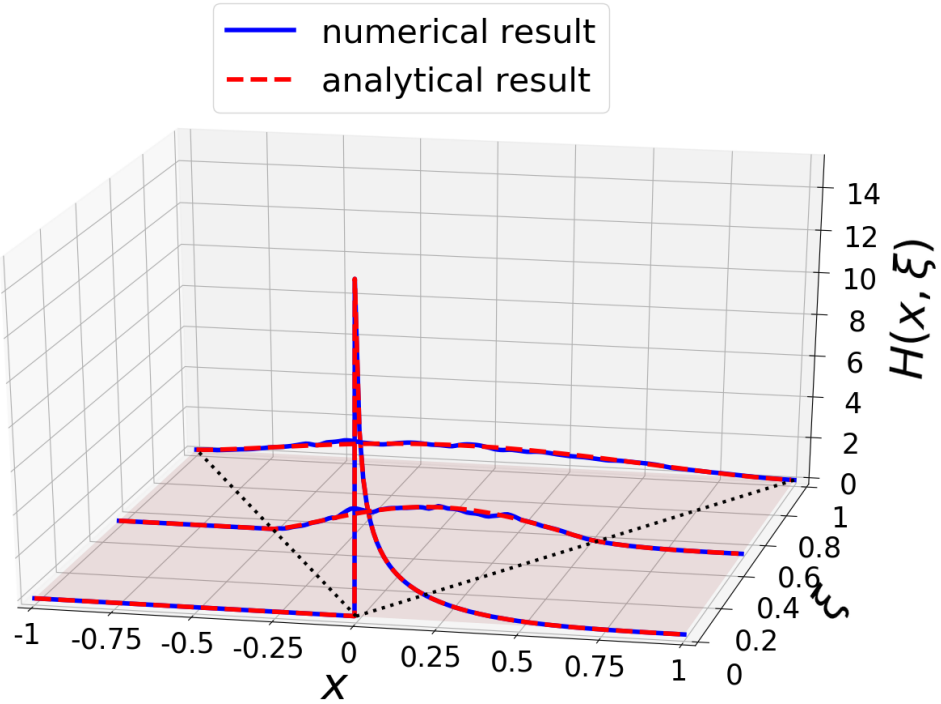

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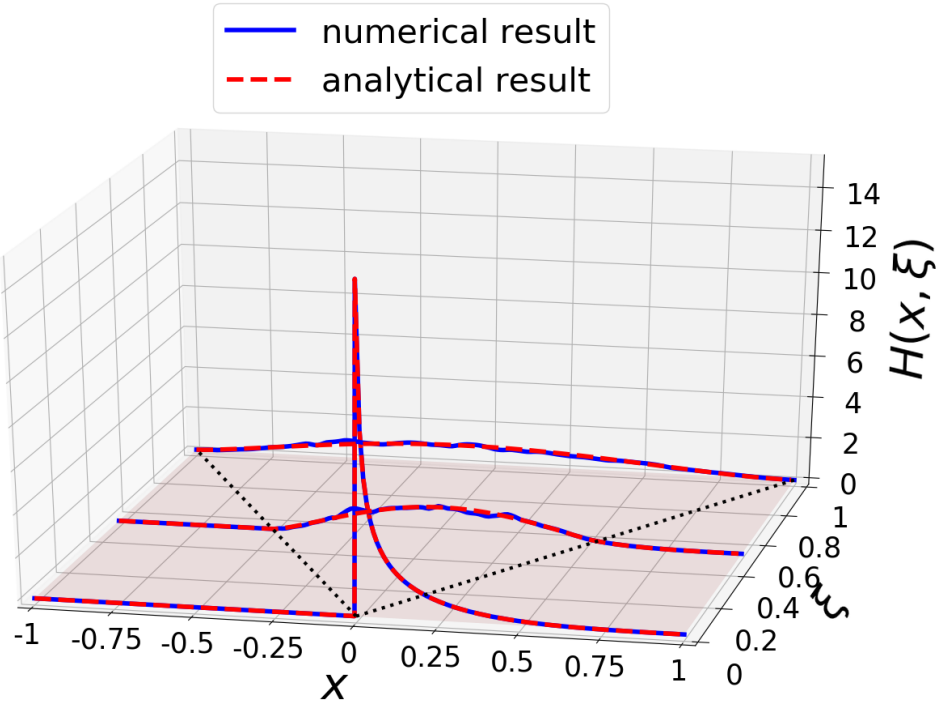

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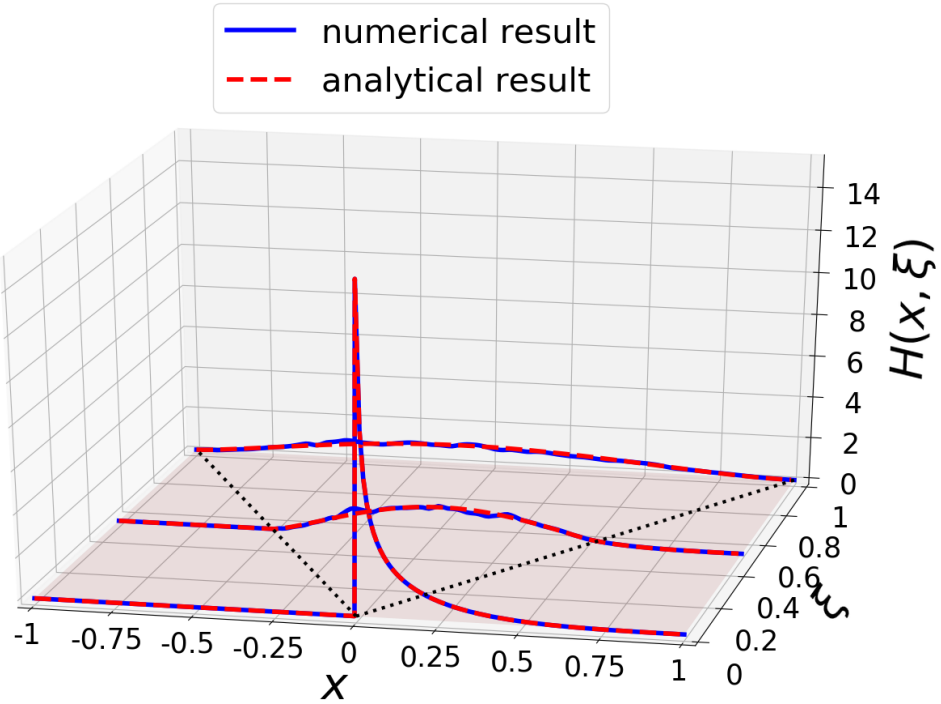

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  - We solve for $\sqrt{\beta}\, h(\beta,\alpha)$ instead of $h(\beta,\alpha)$!

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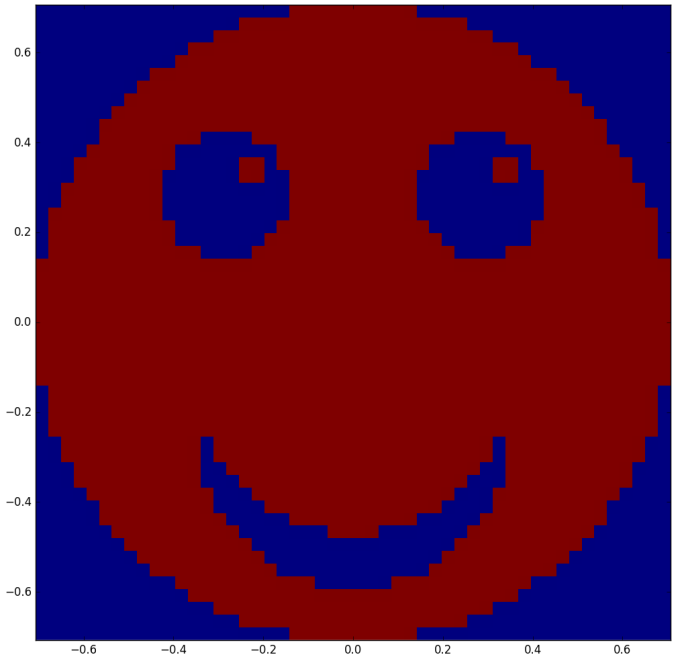

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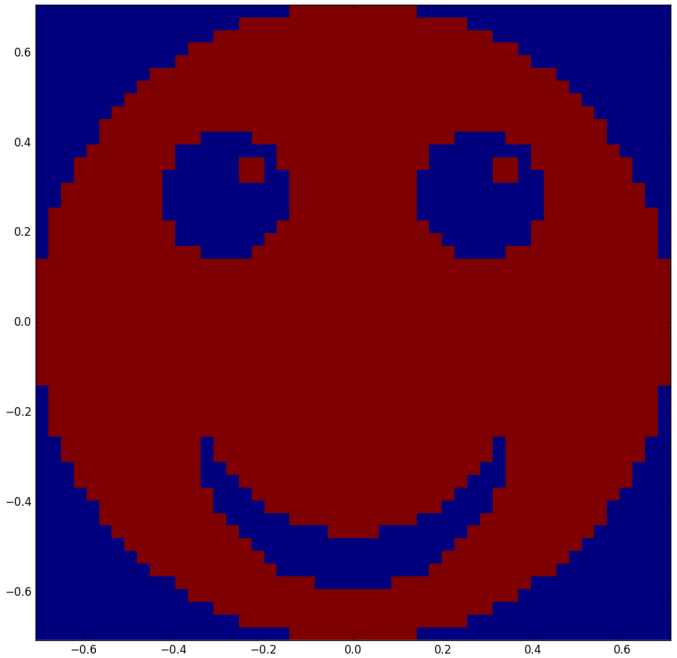
  - Any questions?

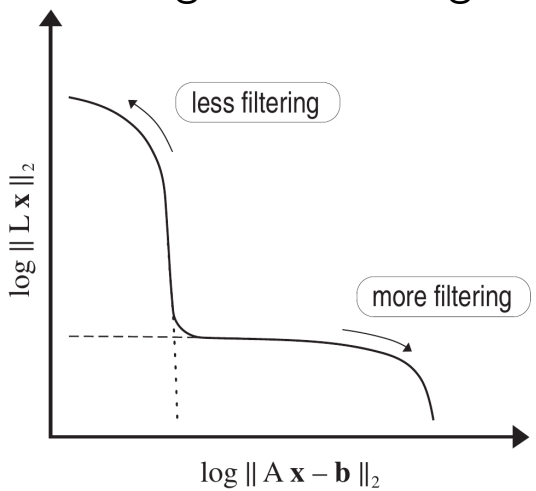


Ill-posed problems and Regularization

- Ill-posed problems?
  - For example the inversion of a Fredholm equation of the first kind:

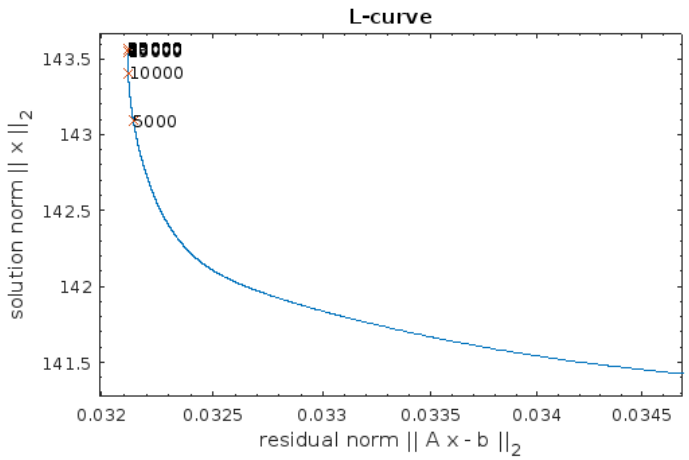
$$\int K(x,y)\, f(y)\, \mathrm{d}y = g(x)\,. \tag{16}$$

  - The inverse is not continuous: an arbitrarily small variation Δg of the rhs can lead to an arbitrarily large variation Δf of the solution.
- The corresponding discrete problem needs to be regularized.
  - E.g Tikhonov regularization: $\min \left\{ \|AX - B\|^2 + \epsilon \|X\|^2 \right\}$.



Theoretical "L-curve": curve parameterized by the regularization factor.

(fig. taken from Ref. (Hansen, 2007))



L-curve with the iteration number as regularization factor.

D-term considerations

- Polynomiality property:

$$\int_{-1}^{1} \mathrm{d}x\, x^m H(x,\xi,t) = \sum_{\substack{k=0 \\ k \text{ even}}}^{m+1} c_k^{(m)}(t)\xi^k \,. \tag{17}$$

- Recast polynomiality property for $H - D$:

$$\int_{-1}^{1} \mathrm{d}x\, x^m \left(H(x,\xi,t) - D\left(\frac{x}{\xi},t\right)\right) = \sum_{\substack{k=0 \\ k \text{ even}}}^{m} c_k^{(m)}(t)\xi^k \,, \tag{18}$$

  where $D\left(\frac{x}{\xi},t\right)$ is the so-called D-term with support on $-\xi < x < \xi$.
- $H - D$ is a Radon Transform:

$$H(x,\xi,t) - D\left(\frac{x}{\xi},t\right) = \int_{\Omega} \mathrm{d}\beta\, \mathrm{d}\alpha\, h_{\mathrm{PW}}(\beta,\alpha)\, \delta(x-\beta-\alpha\xi) \,. \tag{19}$$

  - **The DGLAP region gives no information on the D-term.**
- With other DD representations, we can generate intrinsic D-terms, e.g. Pobylitsa representation:

$$H(x,\xi,t) = (1-x) \int_{\Omega} \mathrm{d}\beta\, \mathrm{d}\alpha\, h_{\mathrm{P}}(\beta,\alpha)\, \delta(x-\beta-\alpha\xi) \,. \tag{20}$$

  - **Still freedom of extra D-term.**

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