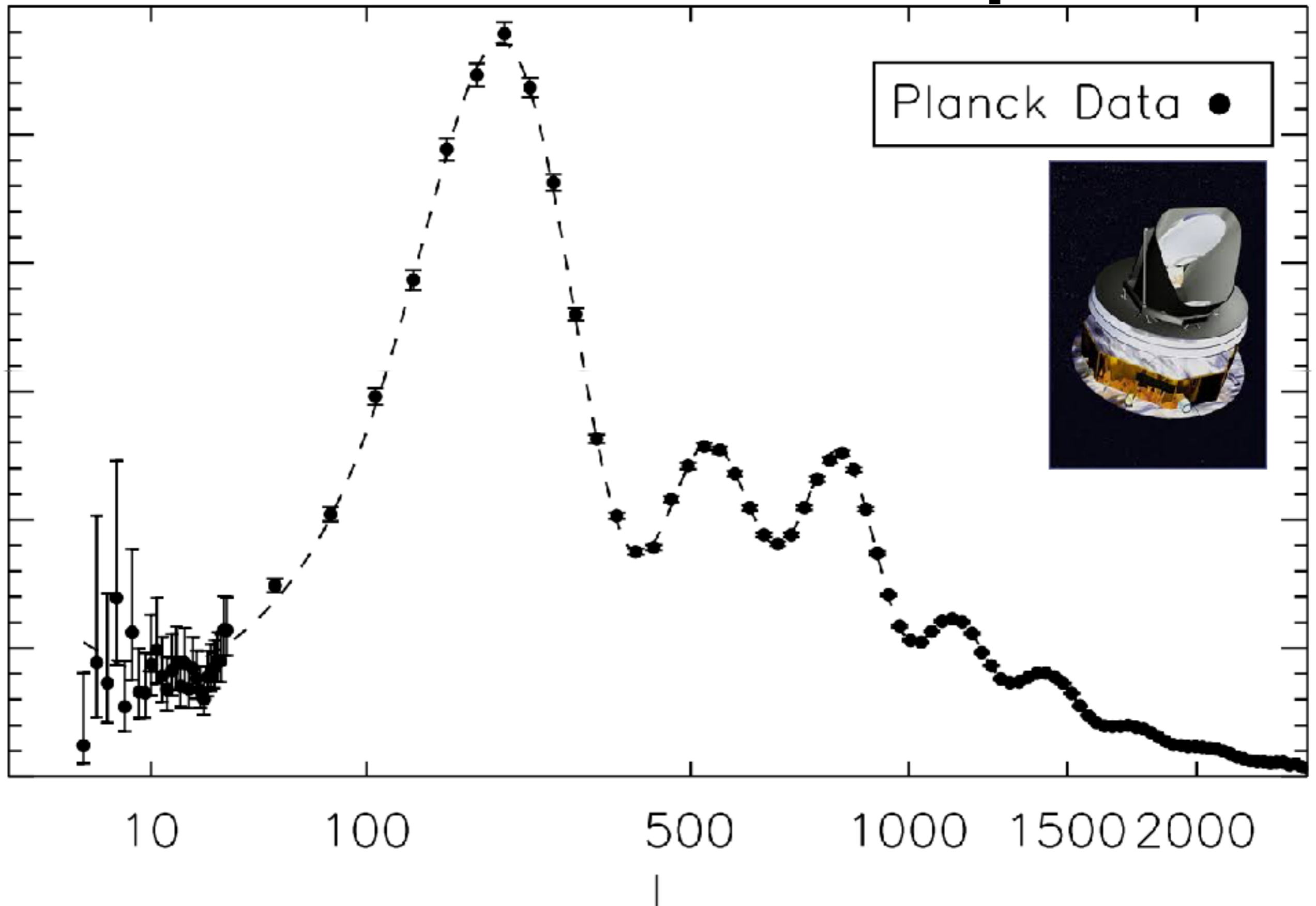


Lecture 2

- Temperature anisotropy from sound waves

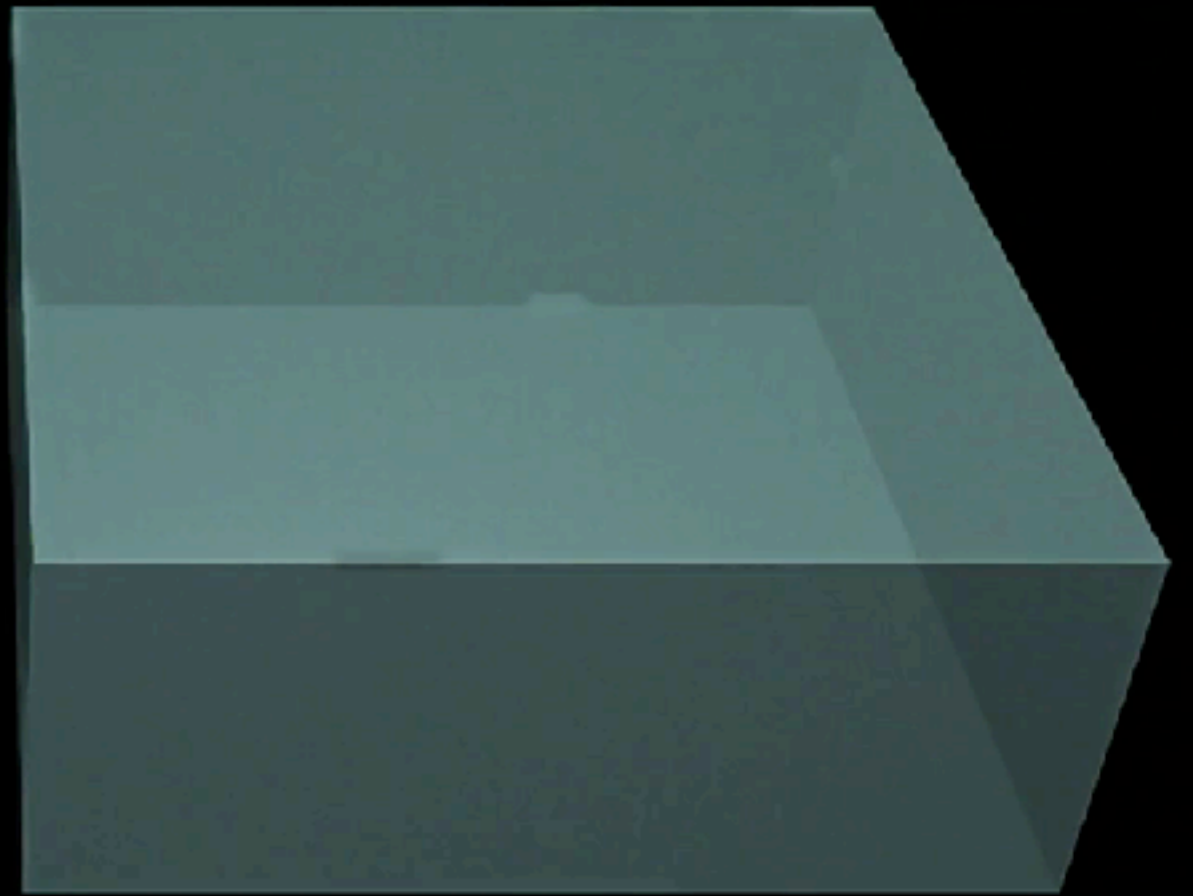
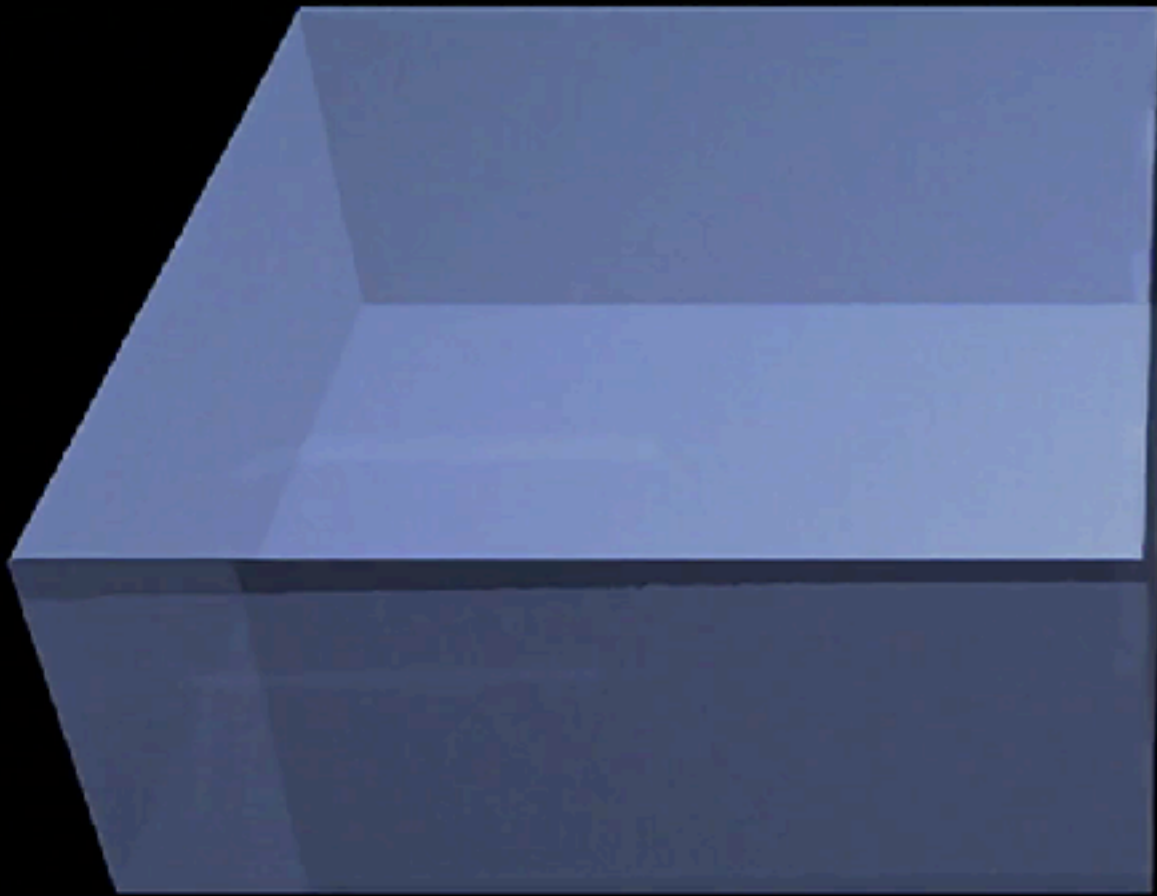
Planck 29-mo Power Spectrum

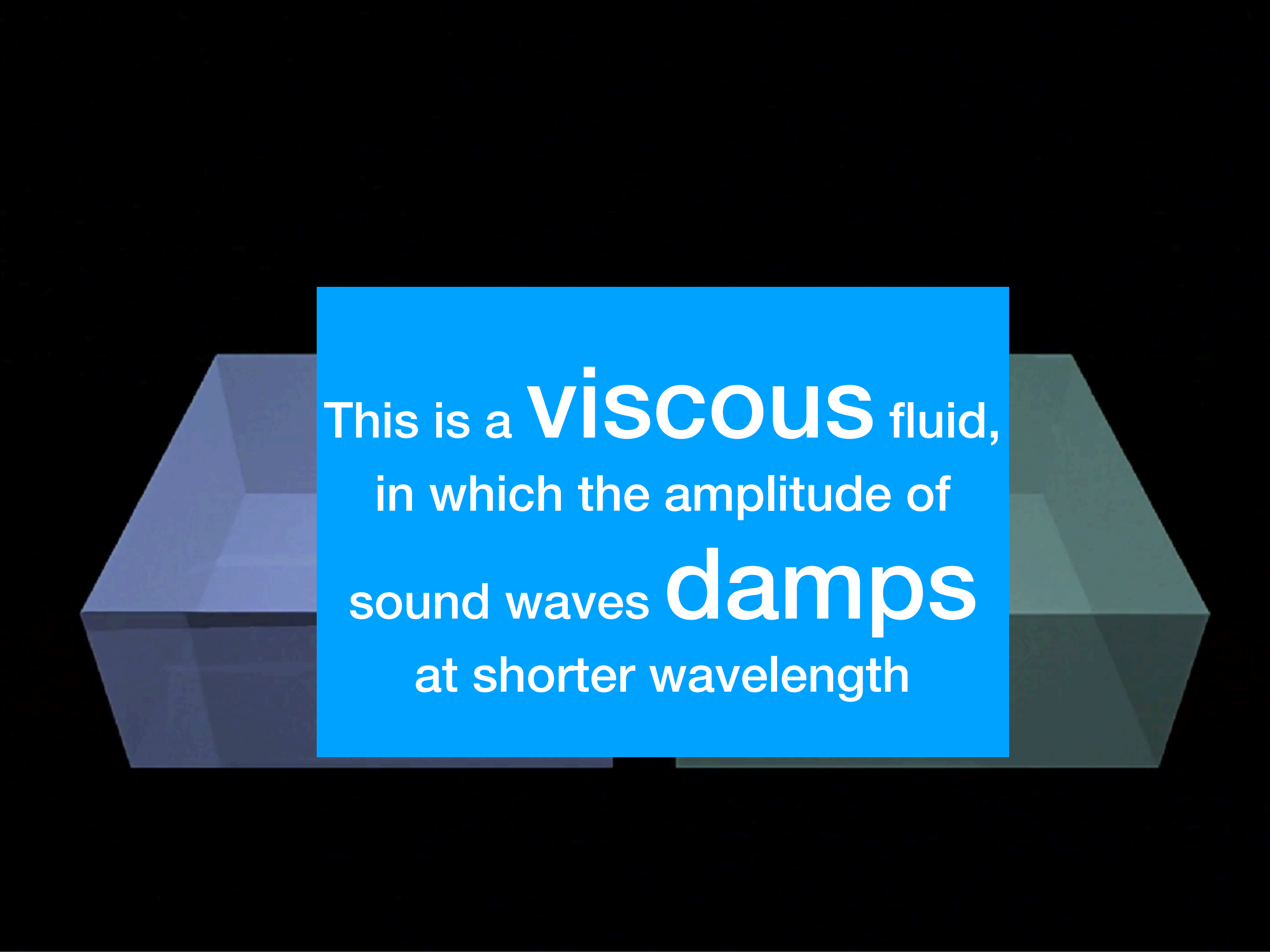




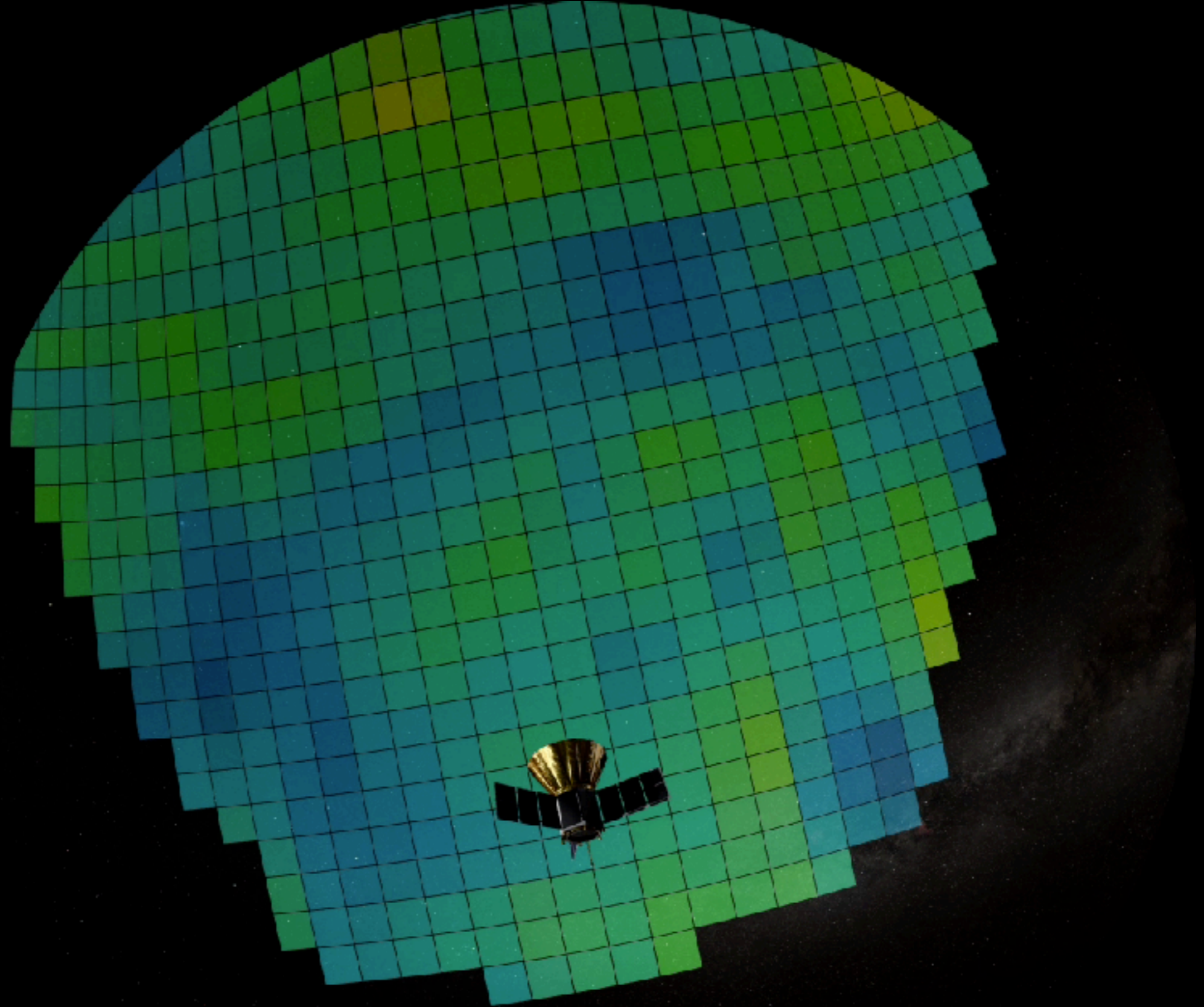
Cosmic Miso Soup

- When matter and radiation were hotter than 3000 K, matter was completely ionised. The Universe was filled with plasma, which behaves just like a soup
- Think about a Miso soup (if you know what it is). Imagine throwing Tofus into a Miso soup, while changing the density of Miso
- And imagine watching how ripples are created and propagate throughout the soup





This is a **viscous** fluid,
in which the amplitude of
sound waves **damps**
at shorter wavelength



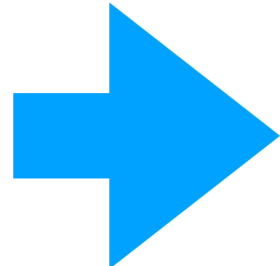
When do sound waves become important?

- In other words, when would the Sachs-Wolfe approximation (purely gravitational effects) become invalid?
- The key to the answer: **Sound-crossing Time**
- Sound waves cannot alter temperature anisotropy at a given angular scale if there was not enough time for sound waves to propagate to the corresponding distance at the last-scattering surface
 - The distance traveled by sound waves within a given time = **The Sound Horizon**

Comoving Photon Horizon

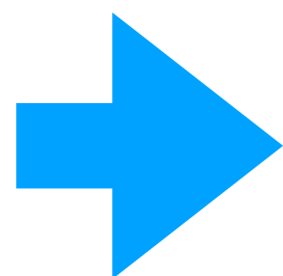
- First, the comoving distance traveled by photons is given by setting the space-time distance to be null:

$$ds^2 = -c^2 dt^2 + a^2(t) dr^2 = 0$$


$$r_{\text{photon}} = c \int_0^t \frac{dt'}{a(t')}$$

Comoving Sound Horizon

- Then, we replace the speed of light with a time-dependent speed of sound:


$$r_s = \int_0^t \frac{dt'}{a(t')} c_s(t')$$

- We cannot ignore the effects of sound waves if $qr_s > 1$

Sound Speed

- Sound speed of an adiabatic fluid is given by

$$c_s^2 = \delta P / \delta \rho$$

- δP : pressure perturbation
- $\delta \rho$: density perturbation

- For a baryon-photon system:

$$c_s^2 = \delta P_\gamma / (\delta \rho_\gamma + \delta \rho_B)$$

We can ignore the baryon pressure because it is much smaller than the photon pressure

Sound Speed

- Using the adiabatic relationship between photons and baryons:

$$\delta\rho_B/\bar{\rho}_B = \delta\rho_\gamma/(\bar{\rho}_\gamma + \bar{P}_\gamma) = 3\delta\rho_\gamma/4\bar{\rho}_\gamma$$

[i.e., the ratio of the number densities of baryons and photons is equal everywhere]

- and pressure-density relation of a relativistic fluid, $\delta P_\gamma = \delta\rho_\gamma/3$,
We obtain

$$c_s^2 = \delta P_\gamma/(\delta\rho_\gamma + \delta\rho_B) = 1/3(1 + 3\bar{\rho}_B/4\bar{\rho}_\gamma)$$

- Or equivalently

$$c_s = \frac{1}{\sqrt{3(1 + R)}}$$

sound speed is reduced!

where

$$R \equiv 3\bar{\rho}_B/4\bar{\rho}_\gamma$$

Value of R?

- The baryon mass density goes like a^{-3} , whereas the photon energy density goes like a^{-4} . Thus, the ratio of the two, R , goes like a .
- The proportionality constant is:

$$R = \frac{3\Omega_B}{4\Omega_\gamma} \frac{a}{a_0} = 0.6120 \left(\frac{\Omega_B h^2}{0.022} \right) \frac{1091}{1+z}$$

where we used

$$\Omega_\gamma \equiv \frac{8\pi G \rho_{\gamma 0}}{3H_0^2} = 2.471 \times 10^{-5} h^{-2} \quad \text{for } T_0 = 2.725 \text{ K}$$

For the last-scattering redshift of $z_L=1090$
(or last-scattering temperature of $T_L=2974$ K),

$$r_s = 145.3 \text{ Mpc}$$

We cannot ignore the effects of sound waves
if $qr_s > 1$. Since $l \sim qr_L$, this means

$$l > r_L/r_s = 96$$

where we used $r_L=13.95$ Gpc

Creation of Sound Waves: Basic Equations

1. Conservation equations (energy and momentum)
2. Equation of state, relating pressure to energy density

$$P = P(\rho)$$

3. General relativistic version of the “Poisson equation”, relating gravitational potential to energy density

$$\nabla^2 \Phi(t, \boldsymbol{x}) = 4\pi G a^2(t) \delta\rho_M(t, \boldsymbol{x})$$

4. Evolution of the “anisotropic stress” (viscosity)

Energy Conservation

- Total energy conservation:

$$\sum_{\alpha} \left\{ \delta \dot{\rho}_{\alpha} + \frac{\dot{a}}{a} (3\delta \rho_{\alpha} + 3\delta P_{\alpha} + \nabla^2 \pi_{\alpha}) - 3(\bar{\rho}_{\alpha} + \bar{P}_{\alpha}) \dot{\Psi} + \frac{1}{a^2} (\bar{\rho}_{\alpha} + \bar{P}_{\alpha}) \nabla^2 \delta u_{\alpha} \right\} = 0,$$

anisotropic stress:
[or, viscosity]

$$\Delta T_{ij} = a^2 \partial_i \partial_j \pi$$

velocity potential

$$\mathbf{v}_{\alpha} = \frac{1}{a} \nabla \delta u_{\alpha}$$

- C.f., Total energy conservation [unperturbed]

$$\sum_{\alpha} \left[\dot{\bar{\rho}}_{\alpha} + \frac{3\dot{a}}{a} (\bar{\rho}_{\alpha} + \bar{P}_{\alpha}) \right] = 0$$

Energy Conservation

- **Total energy conservation:**

$$\sum_{\alpha} \left\{ \delta \dot{\rho}_{\alpha} + \frac{\dot{a}}{a} (3\delta \rho_{\alpha} + 3\delta P_{\alpha} + \nabla^2 \pi_{\alpha}) - 3(\bar{\rho}_{\alpha} + \bar{P}_{\alpha}) \dot{\Psi} + \frac{1}{a^2} (\bar{\rho}_{\alpha} + \bar{P}_{\alpha}) \nabla^2 \delta u_{\alpha} \right\} = 0,$$

- **Again, this is the effect of locally-defined inhomogeneous scale factor, i.e.,**

- The spatial metric is given by $ds^2 = a^2(t) \exp(-2\Psi) d\mathbf{x}^2$

- Thus, locally we can define a new scale factor:

$$\tilde{a}(t, \mathbf{x}) = a(t) \exp(-\Psi)$$

Energy Conservation

- **Total energy conservation:**

$$\sum_{\alpha} \left\{ \delta \dot{\rho}_{\alpha} + \frac{\dot{a}}{a} (3\delta \rho_{\alpha} + 3\delta P_{\alpha} + \nabla^2 \pi_{\alpha}) - 3(\bar{\rho}_{\alpha} + \bar{P}_{\alpha}) \dot{\Psi} + \frac{1}{a^2} (\bar{\rho}_{\alpha} + \bar{P}_{\alpha}) \nabla^2 \delta u_{\alpha} \right\} = 0,$$

- **Momentum flux going outward (inward) -> reduction (increase) in the energy density**

$$\left(\begin{array}{l} \text{C.f., for a non-expanding medium:} \\ \dot{\rho} + \nabla \cdot (\rho \mathbf{v}) = 0 \end{array} \right)$$

Momentum Conservation

- **Total momentum conservation** $\mathbf{v}_\alpha = \frac{1}{a} \nabla \delta u_\alpha$

$$\sum_\alpha \left\{ \frac{\partial}{\partial t} [(\bar{\rho}_\alpha + \bar{P}_\alpha) \delta u_\alpha] + \frac{3\dot{a}}{a} (\bar{\rho}_\alpha + \bar{P}_\alpha) \delta u_\alpha \right. \\ \left. + (\bar{\rho}_\alpha + \bar{P}_\alpha) \Phi + \delta P_\alpha + \nabla^2 \pi_\alpha \right\} = 0,$$

- **Cosmological redshift of the momentum**
- **Gravitational force given by potential gradient**
- **Force given by pressure gradient**
- **Force given by gradient of anisotropic stress**

Equation of State

- Pressure of non-relativistic species (i.e., baryons and cold dark matter) can be ignored relative to the energy density. Thus, we set them to zero: $\mathbf{P}_B=0=\mathbf{P}_D$ and $\delta\mathbf{P}_B=0=\delta\mathbf{P}_D$

- Unperturbed pressure of relativistic species (i.e., photons and relativistic neutrinos) is given by the third of the energy density, i.e., $\mathbf{P}_\gamma=\rho_\gamma/3$ and $\mathbf{P}_\nu=\rho_\nu/3$

- Perturbed pressure involves contributions from the **bulk**

viscosity:
$$\delta P_\gamma = (\delta\rho_\gamma - \nabla^2\pi_\gamma)/3$$

$$\delta P_\nu = (\delta\rho_\nu - \nabla^2\pi_\nu)/3$$

The reason for this is that
trace of the stress-energy
of relativistic species

vanishes: $\sum_{\mu=0,1,2,3} T_{\mu}^{\mu} = 0$

$$T_0^0 + \sum_{i=1}^3 T_i^i = -\rho + 3P + \nabla^2 \pi = 0$$

- Perturbed pressure involves contributions from the **bulk**

viscosity: $\delta P_{\gamma} = (\delta \rho_{\gamma} - \nabla^2 \pi_{\gamma})/3$

$$\delta P_{\nu} = (\delta \rho_{\nu} - \nabla^2 \pi_{\nu})/3$$

Two Remarks

- In the standard scenario:
 - Energy densities are conserved separately; thus we do not need to sum over all species
 - Momentum densities of photons and baryons are NOT conserved separately but they are coupled via Thomson scattering. This must be taken into account when writing down separate conservation equations

Conservation Equations for Photons and Baryons

- Fourier transformation replaces $\nabla^2 \rightarrow -q^2$

$$X(t, \mathbf{x}) = (2\pi)^{-3} \int d^3q X_{\mathbf{q}}(t) \exp(i\mathbf{q} \cdot \mathbf{x})$$

$$\frac{\partial}{\partial t} (\delta\rho_{\gamma}/\bar{\rho}_{\gamma}) - \frac{4q^2}{3a^2} \delta u_{\gamma} = 4\dot{\Psi}$$

$$\frac{\partial}{\partial t} (\delta\rho_B/\bar{\rho}_B) - \frac{q^2}{a^2} \delta u_B = 3\dot{\Psi}$$

$$a \frac{\partial}{\partial t} (\delta u_{\gamma}/a) + \Phi + \frac{\delta\rho_{\gamma}}{4\bar{\rho}_{\gamma}} - \frac{q^2 \pi_{\gamma}}{2\bar{\rho}_{\gamma}} = \sigma_T \bar{n}_e (\delta u_B - \delta u_{\gamma})$$

$$\delta \dot{u}_B + \Phi = -\frac{\sigma_T \bar{n}_e}{R} (\delta u_B - \delta u_{\gamma})$$

$$R \equiv 3\bar{\rho}_B/4\bar{\rho}_{\gamma}$$

momentum transfer via scattering

Conservation Equations for Photons and Baryons

- Fourier transformation replaces $\nabla^2 \rightarrow -q^2$

$$X(t, \mathbf{x}) = (2\pi)^{-3} \int d^3q X_{\mathbf{q}}(t) \exp(i\mathbf{q} \cdot \mathbf{x})$$

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$$a \frac{\partial}{\partial t} (\delta u_{\gamma}/a) + \Phi + \frac{\delta\rho_{\gamma}}{4\bar{\rho}_{\gamma}} - \frac{q^2 \pi_{\gamma}}{2\bar{\rho}_{\gamma}} = \sigma_T \bar{n}_e (\delta u_B - \delta u_{\gamma})$$

what about photon's viscosity?

$$\delta\dot{u}_B + \Phi = -\frac{\sigma_T \bar{n}_e}{R} (\delta u_B - \delta u_{\gamma})$$

$$R \equiv 3\bar{\rho}_B/4\bar{\rho}_{\gamma}$$

Formation of a Photon-baryon Fluid

- **Photons are not a fluid.** Photons free-stream at the speed of light
 - The conservation equations are not enough because we need to specify the evolution of viscosity
 - Solving for viscosity requires information of the phase-space distribution function of photons: **Boltzmann equation**
- However, frequent scattering of photons with baryons* can make photons behave as a fluid: **Photon-baryon fluid**

**Photons scatter with electrons via Thomson scattering. Protons scatter with electrons via Coulomb scattering. Thus we can say, effectively, photons scatter with baryons*

Let's solve them!

- Fourier transformation replaces $\nabla^2 \rightarrow -q^2$

$$X(t, \mathbf{x}) = (2\pi)^{-3} \int d^3q X_{\mathbf{q}}(t) \exp(i\mathbf{q} \cdot \mathbf{x})$$

$$\frac{\partial}{\partial t} (\delta\rho_{\gamma}/\bar{\rho}_{\gamma}) - \frac{4q^2}{3a^2} \delta u_{\gamma} = 4\dot{\Psi}$$

$$\frac{\partial}{\partial t} (\delta\rho_B/\bar{\rho}_B) - \frac{q^2}{a^2} \delta u_B = 3\dot{\Psi}$$

$$a \frac{\partial}{\partial t} (\delta u_{\gamma}/a) + \Phi + \frac{\delta\rho_{\gamma}}{4\bar{\rho}_{\gamma}} - \frac{q^2 \pi_{\gamma}}{2\bar{\rho}_{\gamma}} = \sigma_T \bar{n}_e (\delta u_B - \delta u_{\gamma})$$

$$\delta \dot{u}_B + \Phi = -\frac{\sigma_T \bar{n}_e}{R} (\delta u_B - \delta u_{\gamma})$$

$$R \equiv 3\bar{\rho}_B/4\bar{\rho}_{\gamma}$$

Tight-coupling Approximation

- When Thomson scattering is efficient, the relative velocity between photons and baryons is small. We write

$$\delta u_B - \delta u_\gamma = d / \sigma_T \bar{n}_e$$

[d is an arbitrary dimensionless variable]

- And take $\sigma_T \bar{n}_e \rightarrow \infty$ *. We obtain

$$a \frac{\partial}{\partial t} (\delta u_\gamma / a) + \Phi + \frac{\delta \rho_\gamma}{4\bar{\rho}_\gamma} = d, \quad \delta \dot{u}_\gamma + \Phi = -\frac{d}{R}$$

**In this limit, viscosity π_γ is exponentially suppressed. This result comes from the Boltzmann equation but we do not derive it here. It makes sense physically.*

Tight-coupling Approximation

- Eliminating d and using the fact that R is proportional to the scale factor, we obtain

$$a \frac{\partial}{\partial t} [(1 + R) \delta u_\gamma / a] + (1 + R) \Phi + \frac{\delta \rho_\gamma}{4 \bar{\rho}_\gamma} = 0$$

- Using the energy conservation to replace δu_γ with $\delta \rho_\gamma / \rho_\gamma$, we obtain

$$\frac{1}{a(1 + R)} \frac{\partial}{\partial t} \left[a(1 + R) \frac{\partial}{\partial t} (\delta \rho_\gamma / \bar{\rho}_\gamma - 4\Psi) \right] + \frac{4q^2}{3a^2} \Phi + \frac{q^2}{a^2} \boxed{3(1 + R)} = 0$$

Wave Equation, with the speed of sound of $c_s^2 = 1/3(1+R)$!

Sound Wave!

- To simplify the equation, let's first look at the high-frequency solution
- Specifically, we take $q \gg aH$ (the wavelength of fluctuations is much shorter than the Hubble length). Then we can ignore time derivatives of R and Ψ because they evolve in the Hubble time scale:

$$\frac{1}{a} \frac{\partial}{\partial t} \left[a \frac{\partial}{\partial t} (\delta\rho_\gamma / \bar{\rho}_\gamma) \right] + \frac{q^2 c_s^2}{a^2} [\delta\rho_\gamma / \bar{\rho}_\gamma + 4(1 + R)\Phi] = 0$$

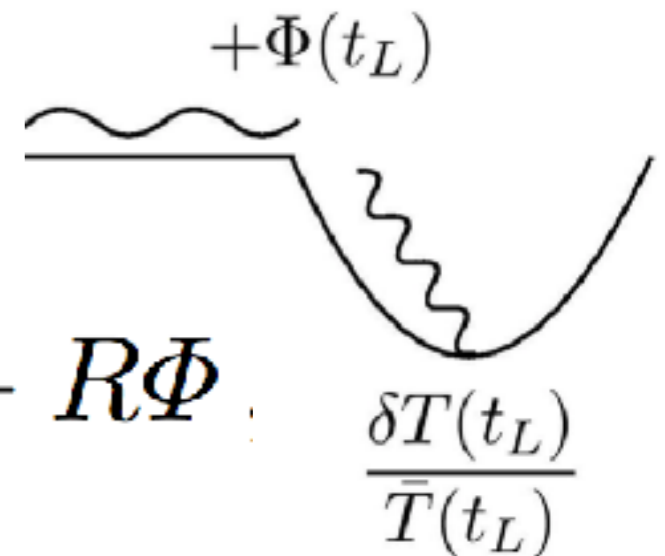
Solution: SOUND WAVE!

$$\frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma} + \Phi = A \cos(qr_s) + B \sin(qr_s) - R\Phi$$

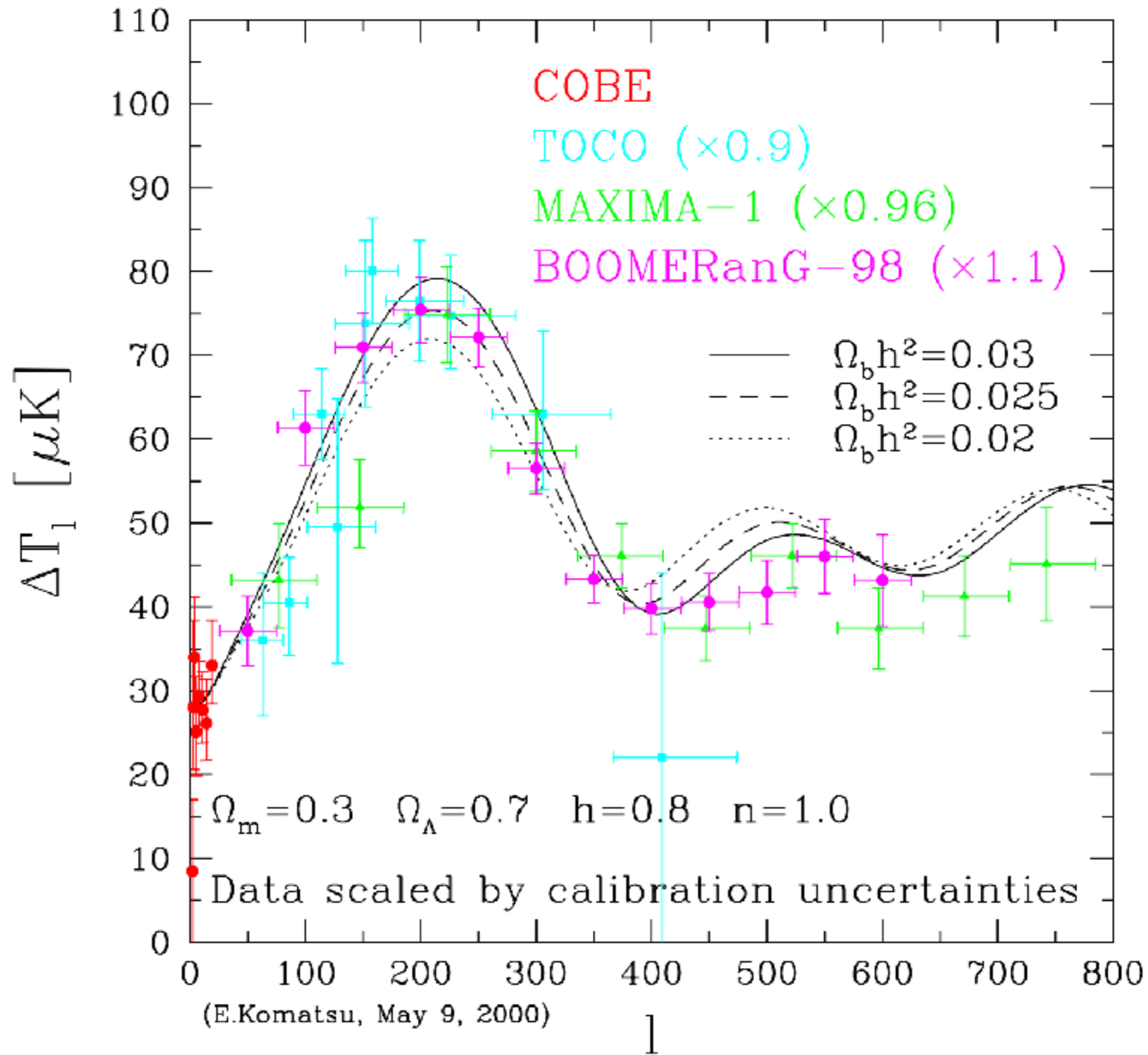
Recap

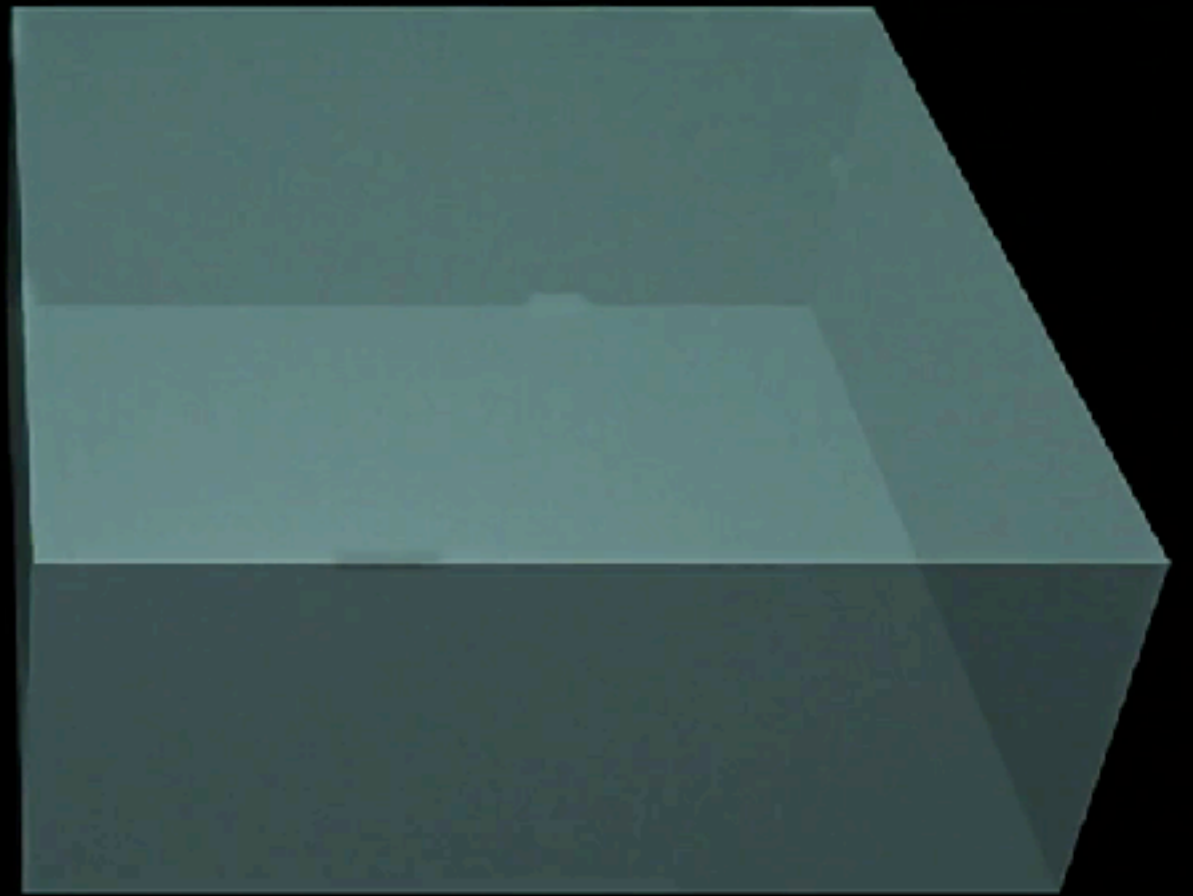
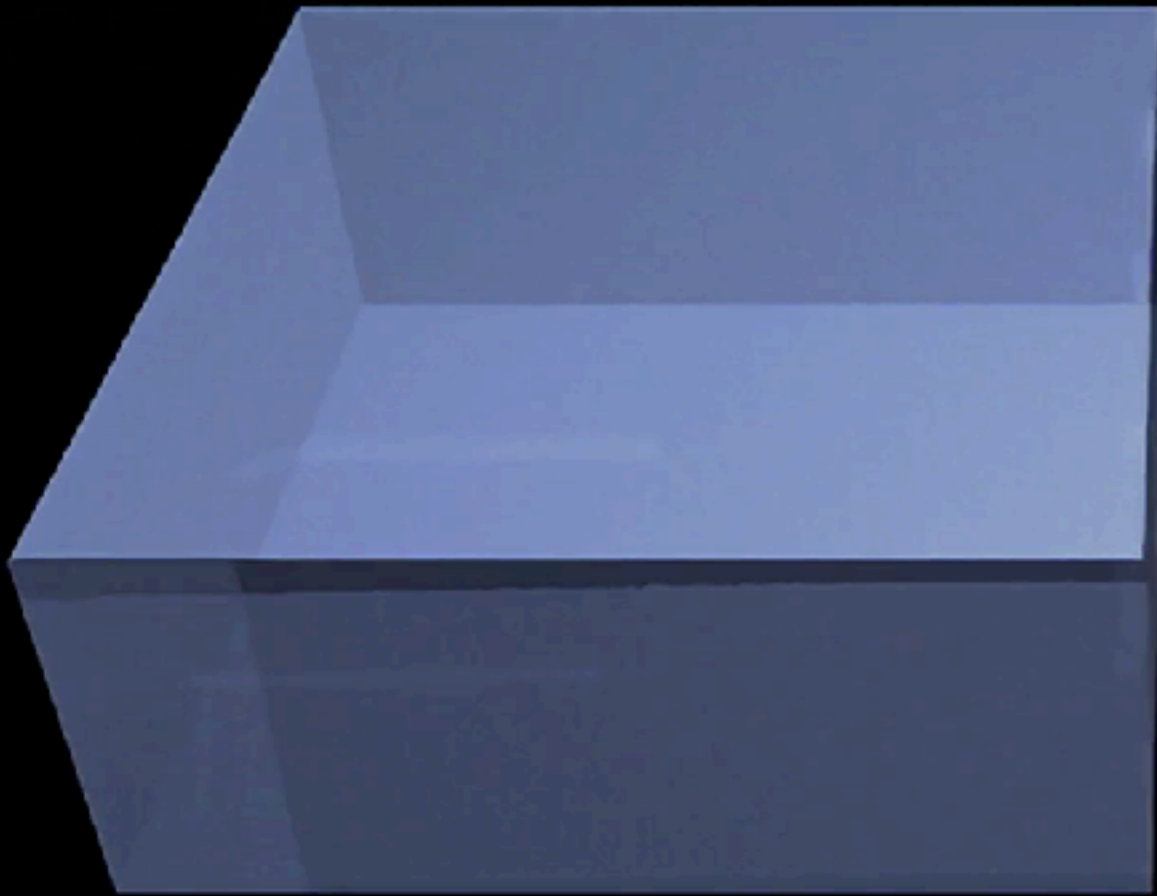
- Photons are not a fluid; but Thomson scattering couples photons to baryons, forming a **photon-baryon fluid**
- The reduced sound speed, $c_s^2=1/3(1+R)$, emerges automatically
- $\delta\rho_\gamma/4\rho_\gamma$ is the temperature anisotropy at the bottom of the potential well. Adding gravitational redshift, the observed temperature anisotropy is $\delta\rho_\gamma/4\rho_\gamma + \Phi$, which is given by

$$\frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma} + \Phi = A \cos(qr_s) + B \sin(qr_s) - R\Phi \quad ; \quad \frac{\delta T(t_L)}{\bar{T}(t_L)}$$



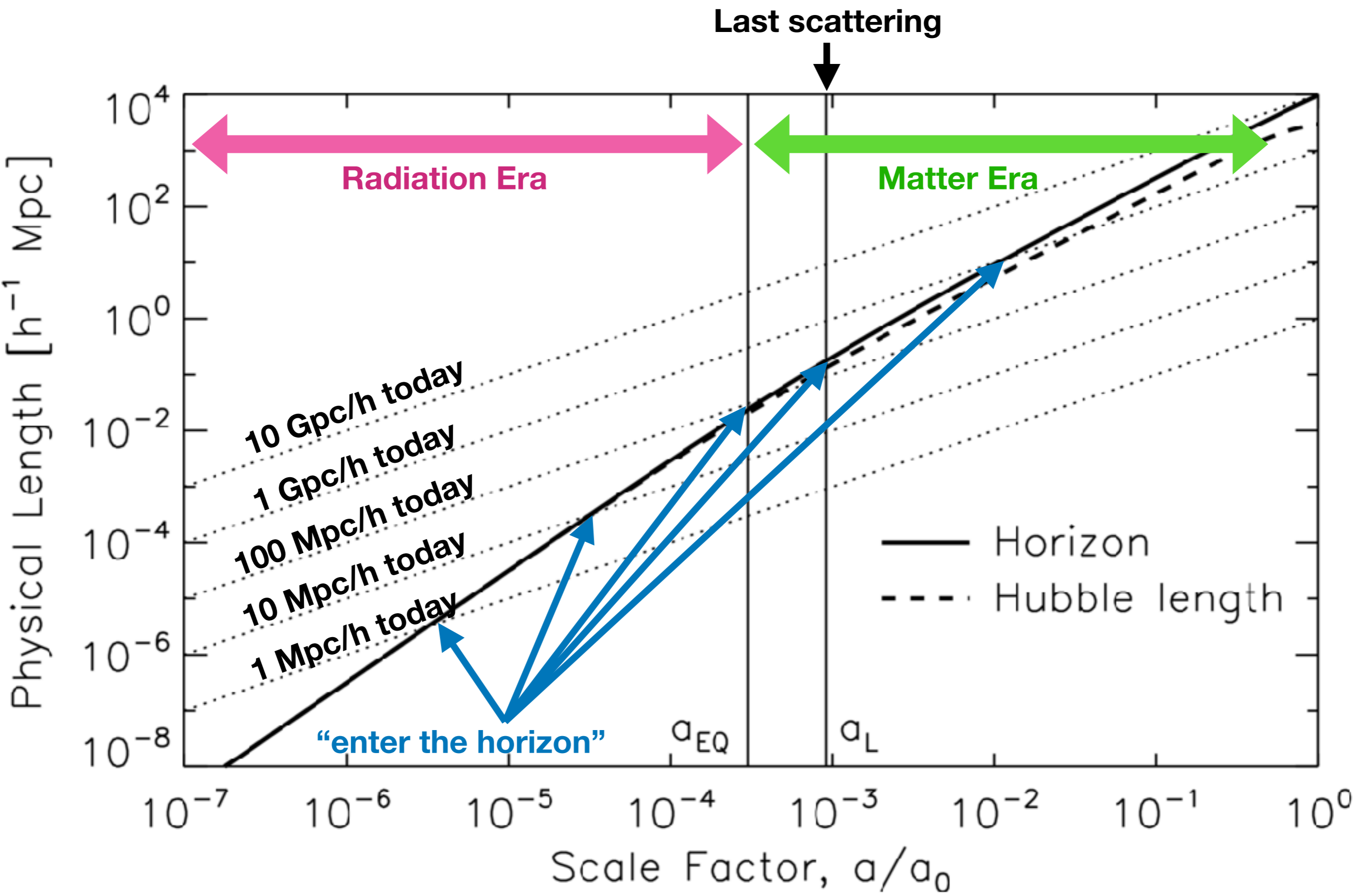
Effect of Baryon-Density





Stone: Fluctuations “entering the horizon”

- This is a tricky concept, but it is important
- Suppose that there are fluctuations at all wavelengths, including the ones that exceed the Hubble length (which we loosely call our “horizon”)
 - Let’s not ask the origin of these “super-horizon fluctuations”, but just assume their existence
- As the Universe expands, our horizon grows and we can see longer and longer wavelengths
 - **Fluctuations “entering the horizon”**



Three Regimes

- **Super-horizon scales [$q < aH$]**
 - Only gravity is important
 - Evolution differs from Newtonian
- **Sub-horizon but super-sound-horizon [$aH < q < aH/c_s$]**
 - Only gravity is important
 - Evolution similar to Newtonian
- **Sub-sound-horizon scales [$q > aH/c_s$]**
 - Hydrodynamics important -> Sound waves

ζ :

Conserved on large scales

- For the adiabatic initial condition, there exists a useful quantity, ζ , which **remains constant on large scales** (*super-horizon scales, $q \ll aH$*) regardless of the contents of the Universe
 - ζ is conserved regardless of whether the Universe is radiation-dominated, matter-dominated, or whatever
- Energy conservation for $q \ll aH$:

$$\delta \dot{\rho}_\alpha + \frac{3\dot{a}}{a} (\delta \rho_\alpha + \delta P_\alpha) - 3(\bar{\rho}_\alpha + \bar{P}_\alpha) \dot{\Psi} = 0$$

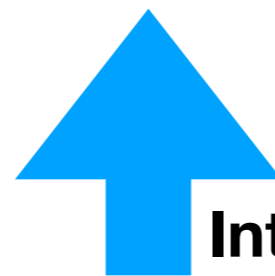
ζ :

Conserved on large scales

- If pressure is a function of the energy density only, i.e., $P_\alpha = P_\alpha(\rho_\alpha)$, then

$$\frac{1}{3} \frac{\delta \rho_\alpha(t, \mathbf{x})}{\bar{\rho}_\alpha(t) + \bar{P}_\alpha(t)} - \Psi(t, \mathbf{x}) = \zeta_\alpha(\mathbf{x})$$

integration constant



Integrate

$$\delta \dot{\rho}_\alpha + \frac{3\dot{a}}{a} (\delta \rho_\alpha + \delta P_\alpha) - 3(\bar{\rho}_\alpha + \bar{P}_\alpha) \dot{\Psi} = 0$$

ζ :

Conserved on large scales

- If pressure is a function of the energy density only, i.e., $P_\alpha = P_\alpha(\rho_\alpha)$, then

$$\frac{1}{3} \frac{\delta \rho_\alpha(t, \mathbf{x})}{\bar{\rho}_\alpha(t) + \bar{P}_\alpha(t)} - \Psi(t, \mathbf{x}) = \zeta_\alpha(\mathbf{x})$$

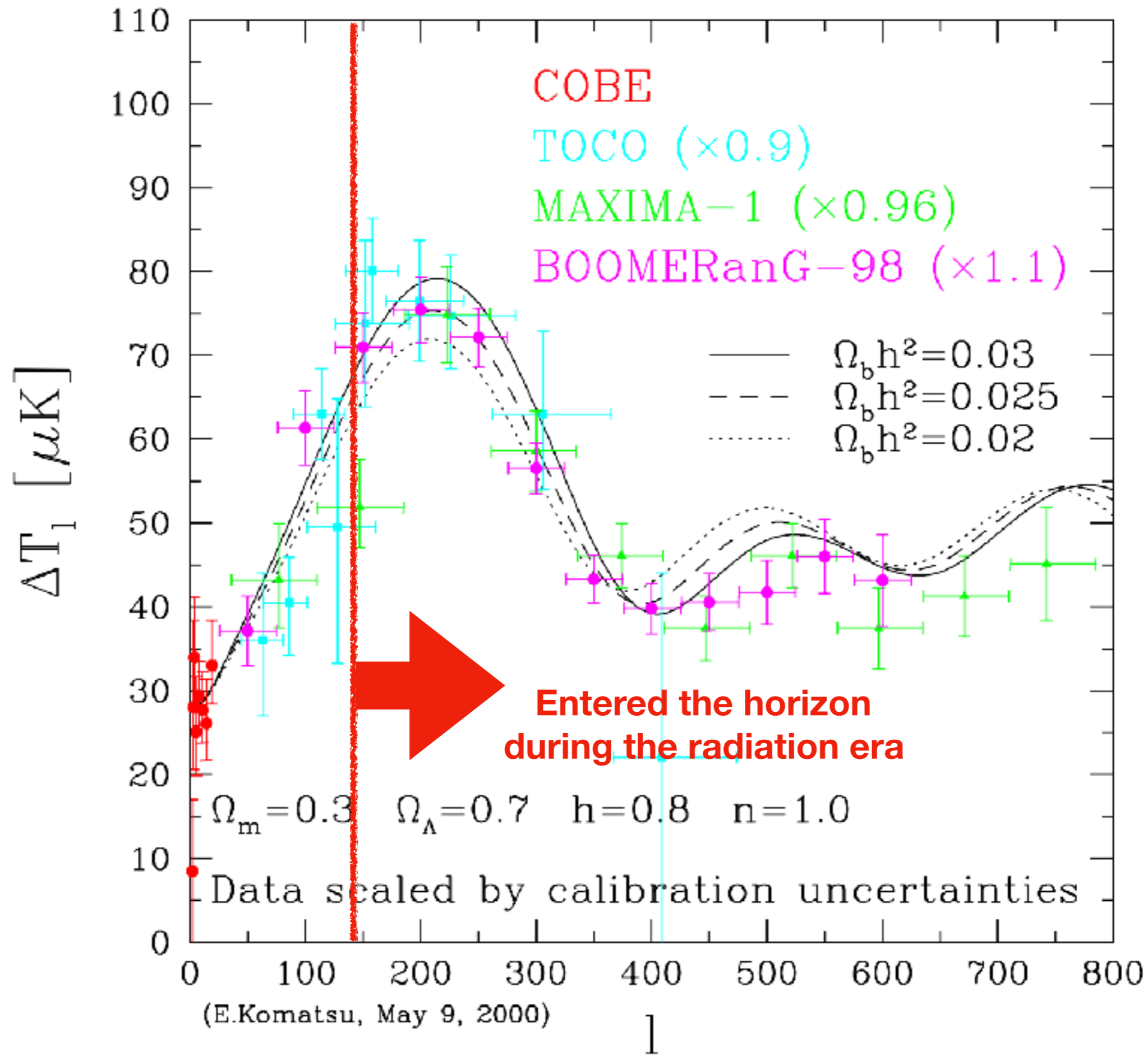
integration constant

For the adiabatic initial condition, all species share the same value of ζ_α , i.e., $\zeta_\alpha = \zeta$

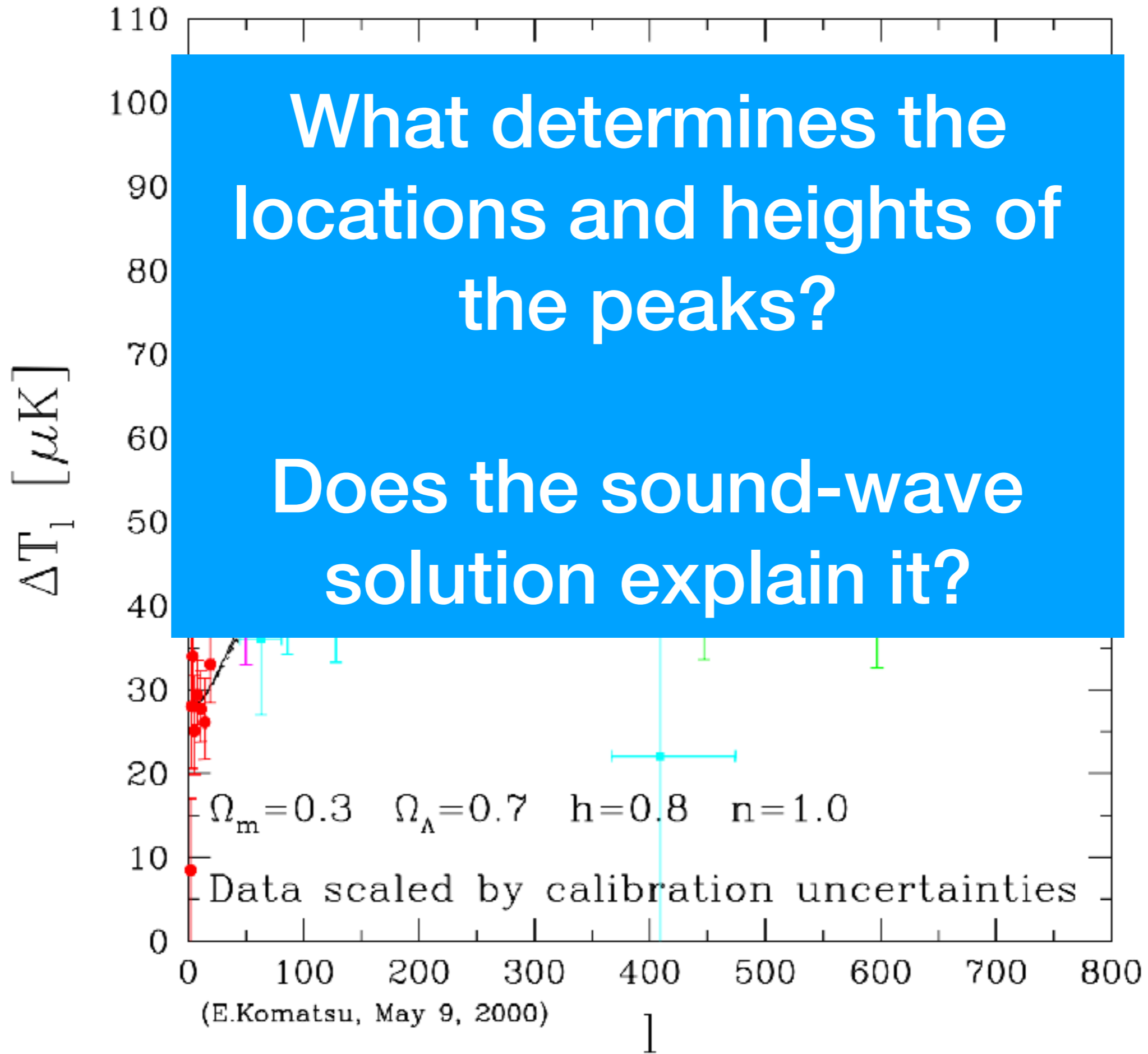
q_{EQ}

- Which fluctuation entered the horizon before the matter-radiation equality?
- $q_{EQ} = a_{EQ}H_{EQ} \sim 0.01 (\Omega_M h^2 / 0.14) \text{ Mpc}^{-1}$
- At the last scattering surface, this subtends the multipole of $l_{EQ} = q_{EQ}r_L \sim 140$

Effect of Baryon-Density



Effect of Baryon-Density



Peak Locations?

High-frequency solution, for $q \gg aH$

$$\frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma} + \Phi = A \cos(qr_s) + B \sin(qr_s) - R\Phi$$

- VERY roughly speaking, the angular power spectrum C_l is given by $\left[\frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma} + \Phi \right]^2$ with $q \rightarrow l/r_L$
 - Question: What are the integration constants, **A** and **B**?
 - Answer: They depend on the initial conditions; namely, adiabatic or not?
 - For adiabatic initial condition, **$A \gg B$ when q is large**
[We will show it later.]

Peak Locations?

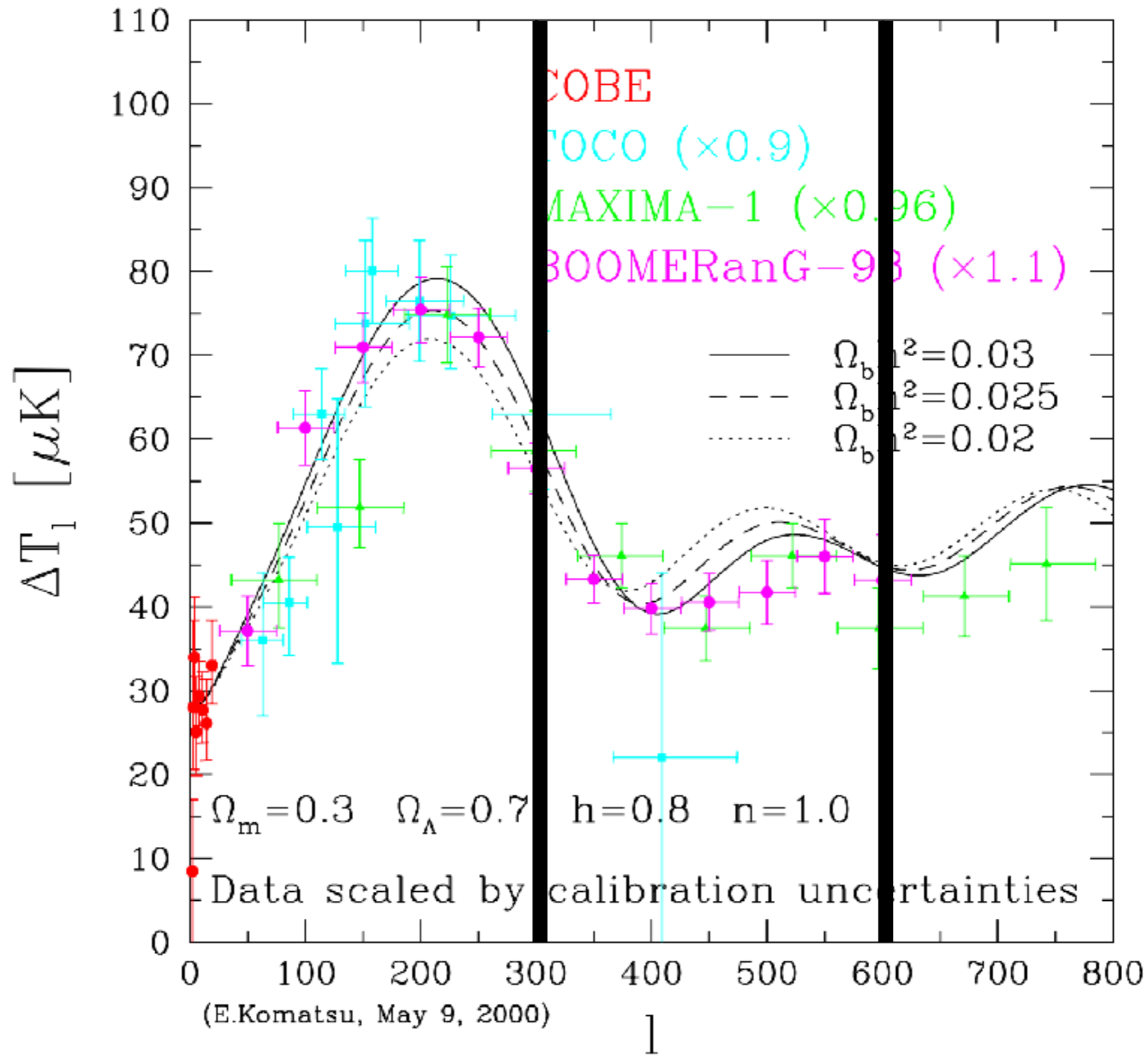
High-frequency solution, for $q \gg aH$

$$\frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma} + \Phi = A \cos(qr_s) + B \sin(qr_s) - R\Phi$$

- VERY roughly speaking, the angular power spectrum C_l is given by $\left[\frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma} + \Phi \right]^2$ with $q \rightarrow l/r_L$
- If $A \gg B$, the locations of peaks are

$$l = (1, 2, \dots)\pi r_L / r_s(t_L) = (1, 2, \dots) \times 302$$

Effect of Baryon-Density



ΔT_1 [μK]110
100
90
80
70
60
50
40
30
20
10
0

The simple estimates do not match!

This is simply because these angular scales do not satisfy $q \gg aH$, i.e., the oscillations are not pure cosine even for the adiabatic initial condition.

We need a better solution!

10

Better Solution in Radiation-dominated Era

Going back to the original tight-coupling equation..

$$\frac{1}{a(1+R)} \frac{\partial}{\partial t} \left[a(1+R) \frac{\partial}{\partial t} (\delta\rho_\gamma/\bar{\rho}_\gamma - 4\Psi) \right] + \frac{4q^2}{3a^2} \Phi + \frac{q^2}{a^2} \frac{\delta\rho_\gamma/\bar{\rho}_\gamma}{3(1+R)} = 0$$

- In the radiation-dominated era, $R \ll 1$
- Change the independent variable from the time (t) to

$$\varphi \equiv qr_s = 2qt/\sqrt{3}a$$

Better Solution in Radiation-dominated Era

Then the equation simplifies to

$$\partial^2 X / \partial \varphi^2 + X + \Phi + \Psi = 0$$

where $X \equiv \delta \rho_\gamma / 4\bar{\rho}_\gamma - \Psi$

- In the radiation-dominated era, $R \ll 1$
- Change the independent variable from the time (t) to

$$\varphi \equiv qr_s = 2qt / \sqrt{3}a$$

Better Solution in Radiation-dominated Era

Then the equation simplifies to

$$\partial^2 X / \partial \varphi^2 + X + \Phi + \Psi = 0$$

where $X \equiv \delta\rho_\gamma / 4\bar{\rho}_\gamma - \Psi$

The solution is

$$X = \tilde{A} \cos \varphi + \tilde{B} \sin \varphi - \int_0^\varphi d\varphi' \sin(\varphi - \varphi') (\Phi + \Psi)(\varphi')$$

We rewrite this using the formula for trigonometry:

$$\sin(\varphi - \varphi') = \sin(\varphi) \cos(\varphi') - \cos(\varphi) \sin(\varphi')$$

Better Solution in Radiation-dominated Era

Then the equation simplifies to

$$\partial^2 X / \partial \varphi^2 + X + \Phi + \Psi = 0$$

where $X \equiv \delta \rho_\gamma / 4\bar{\rho}_\gamma - \Psi$

The solution is

$$X = (\tilde{A} + \Delta A) \cos \varphi + (\tilde{B} + \Delta B) \sin \varphi$$

where $\Delta A(\varphi) \equiv \int_0^\varphi d\varphi' \sin \varphi' (\Phi + \Psi)(\varphi'),$

$$\Delta B(\varphi) \equiv - \int_0^\varphi d\varphi' \cos \varphi' (\Phi + \Psi)(\varphi')$$

Einstein's Equations

- Now we need to know Newton's gravitational potential, ϕ , and the scalar curvature perturbation, ψ .
- Einstein's equations - let's look up any text books:

$$\nabla^2 \Psi = 4\pi G a^2 \sum_{\alpha} \left[\delta \rho_{\alpha} - \frac{3\dot{a}}{a} (\bar{\rho}_{\alpha} + \bar{P}_{\alpha}) \delta u_{\alpha} \right]$$

$$\dot{\Psi} + \frac{\dot{a}}{a} \Phi = -4\pi G \sum_{\alpha} (\bar{\rho}_{\alpha} + \bar{P}_{\alpha}) \delta u_{\alpha}$$

$$\partial_i \partial_j (\Phi - \Psi) = -8\pi G a^2 \partial_i \partial_j \sum_{\alpha} \pi_{\alpha}$$

Einstein's Equations

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- Einstein's equations - let's look up any text books:

$$\nabla^2 \Psi = 4\pi G a^2 \sum_{\alpha} \left[\delta \rho_{\alpha} - \frac{3\dot{a}}{a} (\bar{\rho}_{\alpha} + \bar{P}_{\alpha}) \delta u_{\alpha} \right]$$

$$\dot{\Psi} + \frac{\dot{a}}{a} \Phi = -4\pi G \sum_{\alpha} (\bar{\rho}_{\alpha} + \bar{P}_{\alpha}) \delta u_{\alpha}$$

$$\partial_i \partial_j (\Phi - \Psi) = -8\pi G a^2 \partial_i \partial_j \left[\sum_{\alpha} \pi_{\alpha} \right]$$

Einstein's Equations

- Now we need to know Newton's gravitational potential, ϕ , and the scalar curvature perturbation, ψ .
- Einstein's equations - let's look up any text books:

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Will come back to this later.
For now, let's ignore any viscosity.

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$$\Phi = \Psi$$

**Will come back to
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Einstein's Equations in Radiation-dominated Era

- Now we need to know Newton's gravitational potential, Φ , and the scalar curvature perturbation, ψ .
- Einstein's equations - let's look up any text books:

$$\frac{\partial^2 \Phi}{\partial \varphi^2} + \frac{4}{\varphi} \frac{\partial \Phi}{\partial \varphi} + \Phi = \frac{3}{2\varphi^2} \frac{\delta \mathcal{P}}{\bar{\rho}_R}$$

$$\sum_{\alpha} \delta P_{\alpha}(t, \mathbf{x}) = \frac{\sum_{\alpha} \dot{P}_{\alpha}(t)}{\sum_{\alpha} \dot{\bar{\rho}}_{\alpha}(t)} \sum_{\alpha} \delta \rho_{\alpha}(t, \mathbf{x}) + \delta \mathcal{P}(t, \mathbf{x})$$

“non-adiabatic” pressure

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We shall ignore this

$$\sum_{\alpha} \delta P_{\alpha}(t, \mathbf{x}) = \frac{\sum_{\alpha} \dot{P}_{\alpha}(t)}{\sum_{\alpha} \dot{\rho}_{\alpha}(t)} \sum_{\alpha} \delta \rho_{\alpha}(t, \mathbf{x}) + \delta \mathcal{P}(t, \mathbf{x})$$

“non-adiabatic” pressure

Solution (Adiabatic) in Radiation-dominated Era

$$\Phi_{\text{ADI}} = -2\zeta (\sin \varphi - \varphi \cos \varphi) / \varphi^3$$

where

$$\varphi \equiv qr_s = 2qt / \sqrt{3}a$$

- Low-frequency limit (*super-sound-horizon scales*, $qr_s \ll 1$)
 - $\Phi_{\text{ADI}} \rightarrow -2\zeta/3 = \text{constant}$
- High-frequency limit (*sub-sound-horizon scales*, $qr_s \gg 1$)
 - $\Phi_{\text{ADI}} \rightarrow 2\zeta \cos \varphi / \varphi^2 \propto a^{-2}$ **damp**

Solution (Adiabatic) in Radiation-dominated Era

$$\Phi_{\text{ADI}} = -2\zeta (\sin \varphi - \varphi \cos \varphi) / \varphi^3$$

where

Poisson Equation

$$-q^2 \Phi = 4\pi G a^2 \delta \rho$$

- Low-frequency (sub sound horizon scales, $qr_s \ll 1$)

& oscillation solution for radiation

- $\Phi_{\text{ADI}} \rightarrow -2\zeta/3$

$$\delta \rho_R / \bar{\rho}_R \propto \cos \varphi$$

- High-frequency in (sub sound horizon scales, $qr_s \gg 1$)

- $\Phi_{\text{ADI}} \rightarrow 2\zeta \cos \varphi / \varphi^2 \propto a^{-2}$ **damp**

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Sound Wave Solution in the Radiation-dominated Era

The solution is

$$X = (\tilde{A} + \Delta A) \cos \varphi + (\tilde{B} + \Delta B) \sin \varphi$$

where

$$X \equiv \delta\rho_\gamma / 4\bar{\rho}_\gamma - \Psi$$

$$\Delta A(\varphi) \equiv \int_0^\varphi d\varphi' \sin \varphi' (\Phi + \Psi)(\varphi'),$$

$$\Delta B(\varphi) \equiv - \int_0^\varphi d\varphi' \cos \varphi' (\Phi + \Psi)(\varphi')$$

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$$\Delta A(\varphi) \equiv \int_0^\varphi d\varphi' \sin \varphi' (\Phi + \Psi)(\varphi') = \underline{-2\zeta(1 - \sin^2 \varphi / \varphi^2)}$$

$$\begin{aligned} \Delta B(\varphi) &\equiv - \int_0^\varphi d\varphi' \cos \varphi' (\Phi + \Psi)(\varphi') \\ &= \underline{2\zeta(\varphi - \cos \varphi \sin \varphi) / \varphi^2} \end{aligned}$$

Sound Wave Solution in the Radiation-dominated Era

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$$X = (\tilde{A} + \Delta A) \cos \varphi + (\tilde{B} + \Delta B) \sin \varphi$$

where

$$X \equiv \delta\rho_\gamma/4\bar{\rho}_\gamma - \Psi \xrightarrow{\varphi \ll 1} \zeta \quad \text{i.e., } \underline{\tilde{A}_{\text{ADI}} = \zeta, \tilde{B}_{\text{ADI}} = 0}$$

$$\Delta A(\varphi) \equiv \int_0^\varphi d\varphi' \sin \varphi' (\Phi + \Psi)(\varphi') = \underline{-2\zeta(1 - \sin^2 \varphi/\varphi^2)}$$

$$\begin{aligned} \Delta B(\varphi) &\equiv -\int_0^\varphi d\varphi' \cos \varphi' (\Phi + \Psi)(\varphi') \\ &= \underline{2\zeta(\varphi - \cos \varphi \sin \varphi)/\varphi^2} \end{aligned}$$

Sound Wave Solution in the Radiation-dominated Era

The adiabatic solution is

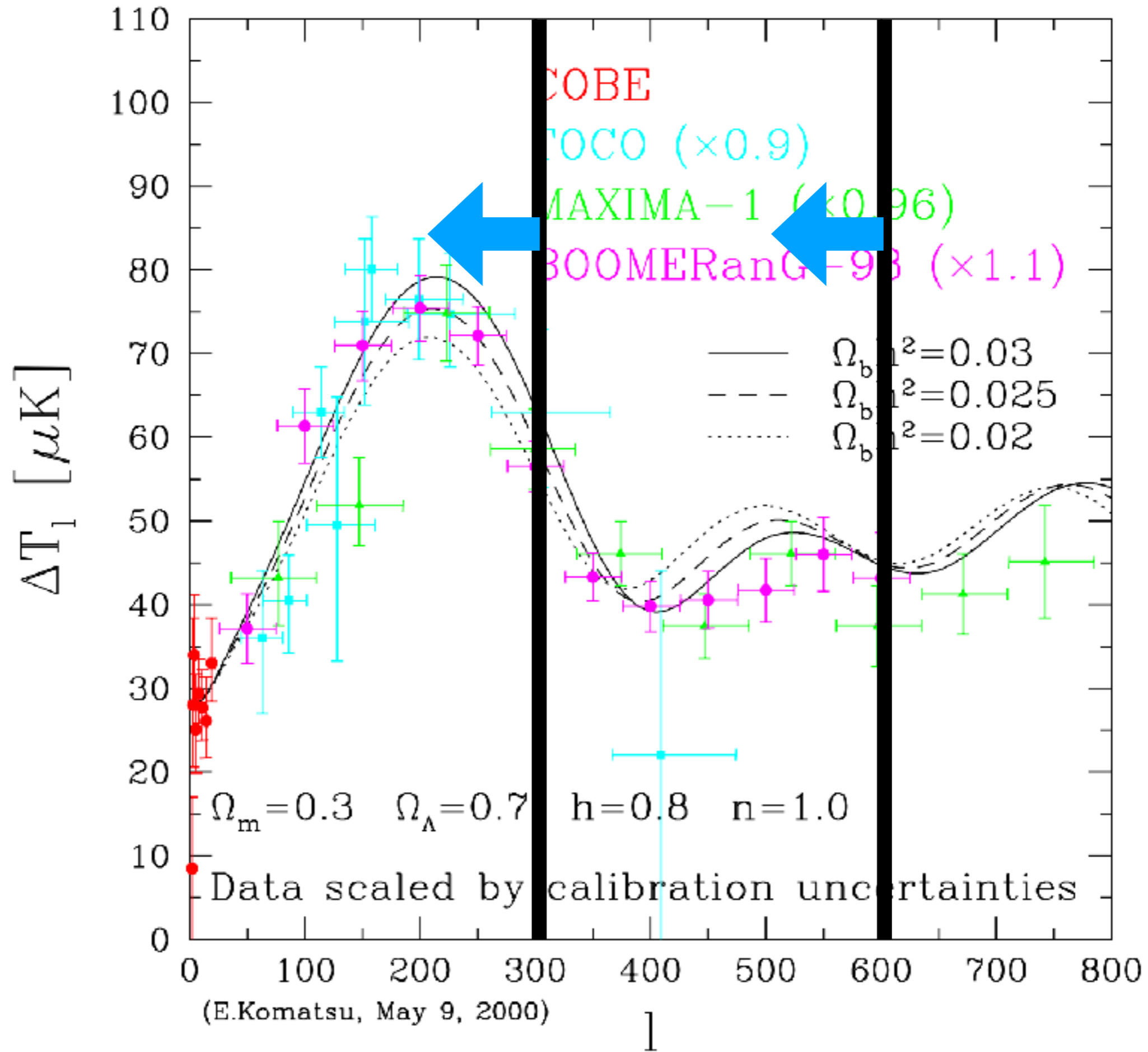
$$X = \frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma} - \Psi = \zeta \left(-\cos\varphi + \frac{2}{\varphi} \sin\varphi \right)$$

with

$$\Phi = \Psi = -2\zeta (\sin\varphi - \varphi \cos\varphi) / \varphi^3$$

Therefore, the solution is a **pure cosine**
only in the **high-frequency** limit!

Effect of Baryon-Density



Roles of viscosity

- **Neutrino viscosity**

- Modify potentials: $\partial_i \partial_j (\Phi - \Psi) = -8\pi G a^2 \partial_i \partial_j \pi_\nu$

- **Photon viscosity**

- Viscous photon-baryon fluid: **damping of sound waves**

Silk (1968) "Silk damping"

$$a \frac{\partial}{\partial t} (\delta u_\gamma / a) + \Phi + \frac{\delta \rho_\gamma}{4\bar{\rho}_\gamma} - \frac{q^2 \pi_\gamma}{2\bar{\rho}_\gamma} = \sigma_T \bar{n}_e (\delta u_B - \delta u_\gamma)$$

High-frequency solution without neutrino viscosity

The solution is

$$X = (\zeta + \Delta A) \cos \varphi + (\Delta B) \sin \varphi$$

where

$$X \equiv \delta\rho_\gamma / 4\bar{\rho}_\gamma - \Psi$$

$$\Delta A(\varphi) \equiv \int_0^\varphi d\varphi' \sin \varphi' (\Phi + \Psi)(\varphi') = -2\zeta (1 - \sin^2 \varphi / \varphi^2)$$

$$\Delta B(\varphi) \equiv - \int_0^\varphi d\varphi' \cos \varphi' (\Phi + \Psi)(\varphi') \xrightarrow{\varphi \gg 1} -2\zeta$$

$$= 2\zeta (\varphi - \cos \varphi \sin \varphi) / \varphi^2 \xrightarrow{\varphi \gg 1} 0$$

High-frequency solution with neutrino viscosity

The solution is

$$X = (-\zeta + \Delta A_\nu) \cos \varphi + \Delta B_\nu \sin \varphi$$

where

$$X \equiv \delta\rho_\gamma / 4\bar{\rho}_\gamma - \Psi$$

$$\Delta A_\nu \xrightarrow{\varphi \gg 1} 0.338 R_\nu \zeta$$

$$\Delta B_\nu \xrightarrow{\varphi \gg 1} 0.418 R_\nu \zeta$$

non-zero value!

$$R_\nu \equiv \bar{\rho}_\nu / (\bar{\rho}_\gamma + \bar{\rho}_\nu) \approx 0.409$$

High-frequency solution with neutrino viscosity

Using the formula for trigonometry, we write

$$X = -C \cos(\varphi + \theta)$$

where

$$C \equiv \sqrt{(-\zeta + \Delta A_\nu)^2 + \Delta B_\nu^2}$$

$$\approx \zeta (1 + 4R_\nu/15)^{-1} \quad \text{Hu \& Sugiyama (1996)}$$

$$\tan \theta = -\frac{\Delta B_\nu}{-\zeta + \Delta A_\nu} \approx 0.063\pi \quad \text{Phase shift!}$$

Bashinsky & Seljak (2004)

High-frequency solution

with neutrino viscosity

Thus, the neutrino viscosity will:

- (1) Reduce the amplitude of sound waves at large multipoles
- (2) Shift the peak positions of the temperature power spectrum

$$\tan \theta = -\frac{\Delta B_\nu}{-\zeta + \Delta A_\nu} \approx 0.063\pi \quad \text{Phase shift!}$$

Bashinsky & Seljak (2004)

Photon Viscosity

- In the tight-coupling approximation, the photon viscosity damps exponentially
- To take into account a non-zero photon viscosity, we go to a higher order in the tight-coupling approximation

Yesterday: Tight-coupling Approximation (1st-order)

- When Thomson scattering is efficient, the relative velocity between photons and baryons is small. We write

$$\delta u_B - \delta u_\gamma = d / \sigma_T \bar{n}_e$$

[d is an arbitrary dimensionless variable]

- And take $\sigma_T \bar{n}_e \rightarrow \infty$ *. We obtain

$$a \frac{\partial}{\partial t} (\delta u_\gamma / a) + \Phi + \frac{\delta \rho_\gamma}{4\bar{\rho}_\gamma} = d, \quad \delta \dot{u}_\gamma + \Phi = -\frac{d}{R}$$

**In this limit, viscosity π_γ is exponentially suppressed. This result comes from the Boltzmann equation but we do not derive it here. It makes sense physically.*

Today: Tight-coupling Approximation (2nd-order)

- When Thomson scattering is efficient, the relative velocity between photons and baryons is small. We write

$$\delta u_B - \delta u_\gamma = d_1 / \sigma_T \bar{n}_e + \underline{q d_2 / (\sigma_T \bar{n}_e)^2}$$

where

$$d_1 = -R(\delta \dot{u}_\gamma + \Phi) \quad [d_2 \text{ is an arbitrary dimensionless variables}]$$

- And take $\sigma_T \bar{n}_e \rightarrow \infty$. We obtain

$$a \frac{\partial}{\partial t} (\delta u_\gamma / a) + \Phi + \frac{\delta \rho_\gamma}{4\bar{\rho}_\gamma} - \frac{q^2 \pi_\gamma}{2\bar{\rho}_\gamma} = -R(\delta \dot{u}_\gamma + \Phi) + \frac{q}{\sigma_T \bar{n}_e} d_2$$

$$\frac{\partial}{\partial t} \left[\frac{R(\delta \dot{u}_\gamma + \Phi)}{\sigma_T \bar{n}_e} \right] = \frac{q}{R \sigma_T \bar{n}_e} d_2$$

Tight-coupling Approximation (2nd-order)

- Eliminating d_2 and using the fact that R is proportional to the scale factor, we obtain

$$a \frac{\partial}{\partial t} [(1 + R)\delta u_\gamma / a] + (1 + R)\Phi + \frac{\delta \rho_\gamma}{4\bar{\rho}_\gamma} - \frac{q^2 \pi_\gamma}{2\bar{\rho}_\gamma} + R \frac{\partial}{\partial t} \left[\frac{R(\delta \dot{u}_\gamma + \Phi)}{\sigma_T \bar{n}_e} \right] = 0$$

- Getting π_γ requires an approximate solution of the Boltzmann equation in the 2nd-order tight coupling. We do not derive it here. The answer is

$$\Delta T_{ij} = a^2 \partial_i \partial_j \pi_\gamma = - \frac{32}{45} \frac{\bar{\rho}_\gamma}{\sigma_T \bar{n}_e} \partial_i \partial_j \delta u_\gamma$$

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- Eliminating d_2 and using the fact that R is proportional to the scale factor, we obtain

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- Getting π_γ requires an approximation equation in the 2nd-order tight coupling here. The answer is

Given by the **spatial gradient of the velocity field**

- a well-known result in fluid dynamics

$$\Delta T_{ij} = a^2 \partial_i \partial_j \pi_\gamma = - \frac{32}{45} \frac{\bar{\rho}_\gamma}{\sigma_T \bar{n}_e} \partial_i \partial_j \delta u_\gamma$$

Damped Oscillator

- Using the energy conservation to replace δu_γ with $\delta\rho_\gamma/\rho_\gamma$, we obtain, for $q \gg aH$,

$$\frac{1}{a} \frac{\partial}{\partial t} \left[a \frac{\partial}{\partial t} (\delta\rho_\gamma/\bar{\rho}_\gamma) \right] + 2\Gamma \frac{\partial}{\partial t} (\delta\rho_\gamma/\bar{\rho}_\gamma) + \frac{q^2 c_s^2}{a^2} [\delta\rho_\gamma/\bar{\rho}_\gamma + 4(1+R)\Phi] = 0$$

New term, giving damping!

where

$$\Gamma(q, t) \equiv \frac{q^2}{6a^2 \sigma_T \bar{n}_e} \left[\frac{16}{15(1+R)} + \frac{R^2}{(1+R)^2} \right]$$

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Important for high frequencies
(large multipoles)

$$\Gamma(q, t) \equiv \frac{q^2}{6a^2 \sigma_T \bar{n}_e} \left[\frac{16}{15(1 + R)} + \frac{R^2}{(1 + R)^2} \right]$$

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New term, giving damping!

SOLUTION:

$$\frac{\delta \rho_\gamma}{4\bar{\rho}_\gamma} + \Phi = [A \cos(qr_s) + B \sin(qr_s)] \exp \left[- \int_0^t dt' \Gamma(q, t') \right] - R\Phi$$

Exponential damping!

$$\approx \exp \left(-q^2 / \sigma \tau \bar{n}_e H \right)$$

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New term, giving damping!

SOLUTION:

$$\frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma} + \Phi = [A \cos(qr_s) + B \sin(qr_s)] \exp\left(-q^2 / q_{\text{Silk}}^2\right) - R\Phi$$

Exponential damping!

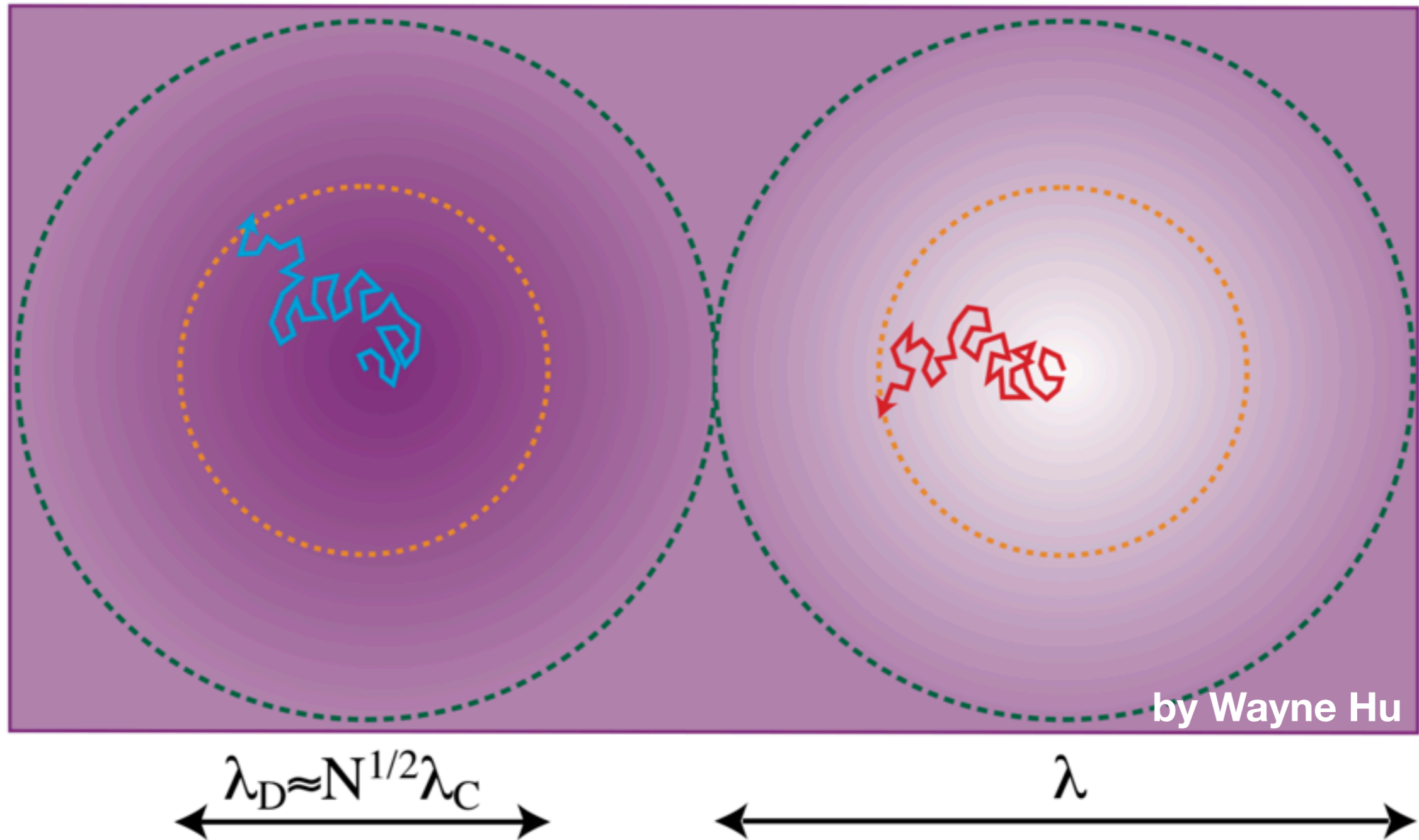
$$a/q_{\text{Silk}} \approx (\sigma_T \bar{n}_e H)^{-1/2} \quad \text{“diffusion length”}$$

= length traveled by photon's random walks

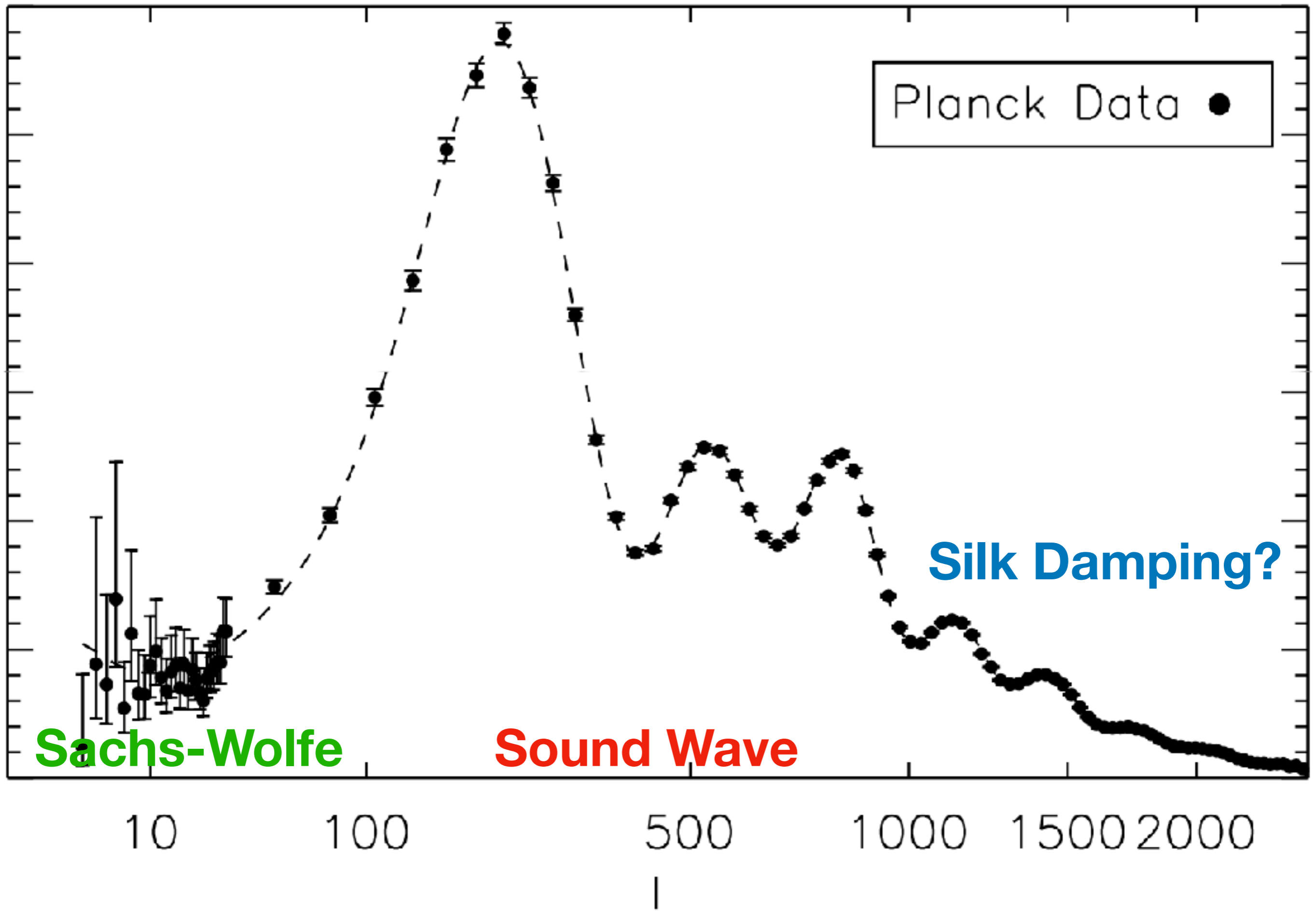
Diffusion Length

- The mean free path of the photon between scatterings is $(\sigma_T n_e)^{-1}$
 - Below this scale, you do not have a photon-baryon fluid
- The number of scatterings per Hubble time is $N_{\text{scattering}} = \sigma_T n_e / H$
- Then, the length traveled by photons by random walks within the Hubble time is $(\sigma_T n_e)^{-1}$ times $\sqrt{N_{\text{scatterings}}}$
 - The diffusion length is thus $(\sigma_T n_e)^{-1}$ times $\sqrt{N_{\text{scatterings}}} = (\sigma_T n_e H)^{-1/2}$

Diffusion Damping



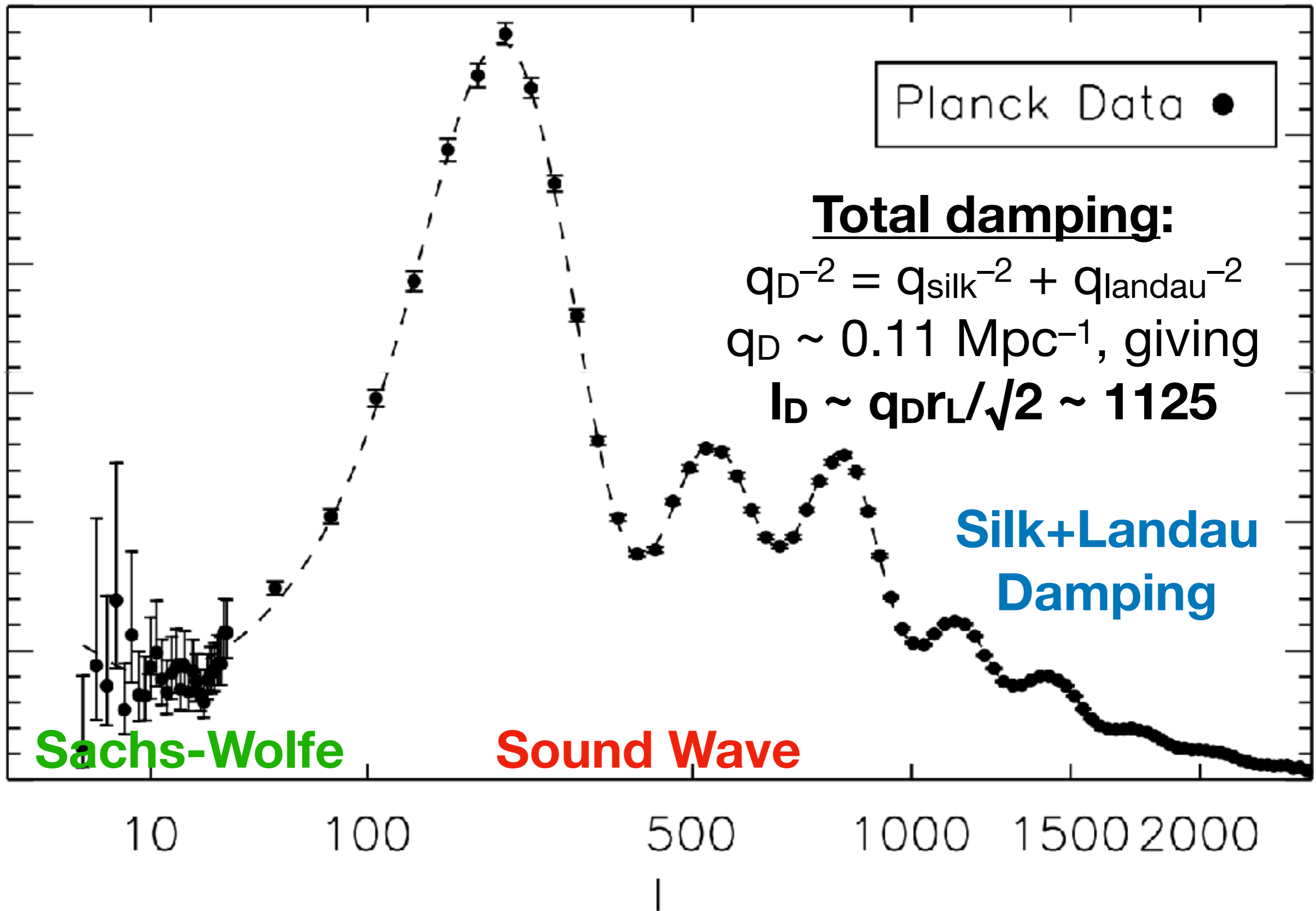
- Diffusion mixes hot and cold photons -> Damping of anisotropies



Additional Damping

- The power spectrum is $\left[\frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma} + \Phi\right]^2$ with $q \rightarrow 1/r_L$. The damping factor is thus $\exp(-2q^2/q_{\text{silk}}^2)$
- $q_{\text{silk}}(t_L) = 0.139 \text{ Mpc}^{-1}$. This corresponds to a multipole of $l_{\text{silk}} \sim q_{\text{silk}} r_L/\sqrt{2} = 1370$. Seems too large, compared to the exact calculation
- There is an additional damping due to a finite width of the last scattering surface, $\sigma \sim 250 \text{ K}$
 - “Fuzziness damping” – Bond (1996)
 - “Landau damping” - Weinberg (2001)

$$q_{\text{Landau}}^{-2} = \frac{3\sigma^2 t_L^2}{8a_0^2 T_0^2 (1 + R_L)} \approx \left(0.20 \text{ Mpc}^{-1}\right)^{-2}$$



Recap

- The basic structure of the temperature power spectrum is
 - The Sachs-Wolfe “plateau” at low multipoles
 - Sound waves at intermediate multipoles
 - 1st-order tight-coupling
 - Silk damping and Landau damping at high multipoles
 - 2nd-order tight-coupling

Matching Solutions

- We have a very good analytical solution valid at low and high frequencies during the radiation era: $\varphi \equiv qr_s$

$$\frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma} - \Psi = \zeta \left(-\cos\varphi + \frac{2}{\varphi} \sin\varphi \right)$$

- Now, match this to a high-frequency solution valid at the last-scattering surface (when R is no longer small)

$$\frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma} + \Phi = A \cos(qr_s) + B \sin(qr_s) - R\Phi$$

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Slightly improved solution, with a weak time dependence of R using the WKB method
[Peebles & Yu (1970)]

$$\frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma} + \Phi = (1 + R)^{-1/4} [A \cos(qr_s) + B \sin(qr_s)] - R\Phi$$

High-frequency Solution(*) at the Last Scattering Surface

$$\frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma} + \Phi = \frac{\zeta}{5} \left\{ 3RT(q) - (1+R)^{-1/4} \mathcal{S}(q) \cos[qr_s + \theta(q)] \right\}$$

where $T(q)$, $\mathcal{S}(q)$, $\theta(q)$ are "transfer functions" that smoothly interpolate two limits as

$$\mathbf{q} \ll \mathbf{q}_{\text{EQ}}: \quad \mathcal{S} \rightarrow 1, \quad \mathcal{T} \rightarrow 1, \quad \theta \rightarrow 0$$

$$\mathbf{q} \gg \mathbf{q}_{\text{EQ}}: \quad \mathcal{S} \rightarrow 5, \quad \mathcal{T} \propto \ln q / q^2, \quad \theta \rightarrow 0.062\pi$$

"EQ" for "matter-radiation Equality epoch"

with $q_{\text{EQ}} = a_{\text{EQ}} H_{\text{EQ}} \sim 0.01 \text{ Mpc}^{-1}$, giving $l_{\text{EQ}} = q_{\text{EQ}} r_L \sim 140$

- (*) To a good approximation, the low-frequency solution is given by setting $R=0$ because sound waves are not important at large scales

High-frequency Solution(*) at the Last Scattering Surface

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"EQ" for "matter-radiation Equality epoch"

with q_{EQ}

$q_{\text{L}} \sim 140$

- (*) To a given l , the most important l modes are those that were produced at large scales. The solution is not

Due to the decay of gravitational potential during the radiation dominated era

High-frequency Solution(*) at the Last Scattering Surface

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"EQ" for "matter-radiation Equality epoch"

with $q_{\text{EQ}} = a_{\text{EQ}} H_{\text{EQ}} \sim 0.01 \text{ Mpc}^{-1}$, given

- (*) To a good approximation, the low- q limit is given by setting $R=0$ because sound speed is important at large scales

Due to the neutrino
anisotropic stress

High-frequency Solution(*) at the Last Scattering Surface

$$\frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma} + \Phi = \frac{\zeta}{5} \left\{ 3RT(q) - (1+R)^{-1/4} \mathcal{S}(q) \cos[qr_s + \theta(q)] \right\}$$

$$\xrightarrow{q \rightarrow 0(*)} -\frac{\zeta}{5}$$

This should agree with the Sachs-Wolfe result: $\Phi/3$; thus,

$$\Phi = -3\zeta/5 \quad \text{in the matter-dominated era}$$

- (*) To a good approximation, the low-frequency solution is given by setting $R=0$ because sound waves are not important at large scales

High-frequency Solution(*) at the Last Scattering Surface

$$\frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma} + \Phi = \frac{\zeta}{5} \left\{ 3RT(q) - (1+R)^{-1/4} \mathcal{S}(q) \cos[qr_s + \theta(q)] \right\}$$

$$\xrightarrow{q/q_{\text{EQ}} \gg 1} - (1+R)^{-1/4} \zeta \cos[qr_s + \theta(q)]$$

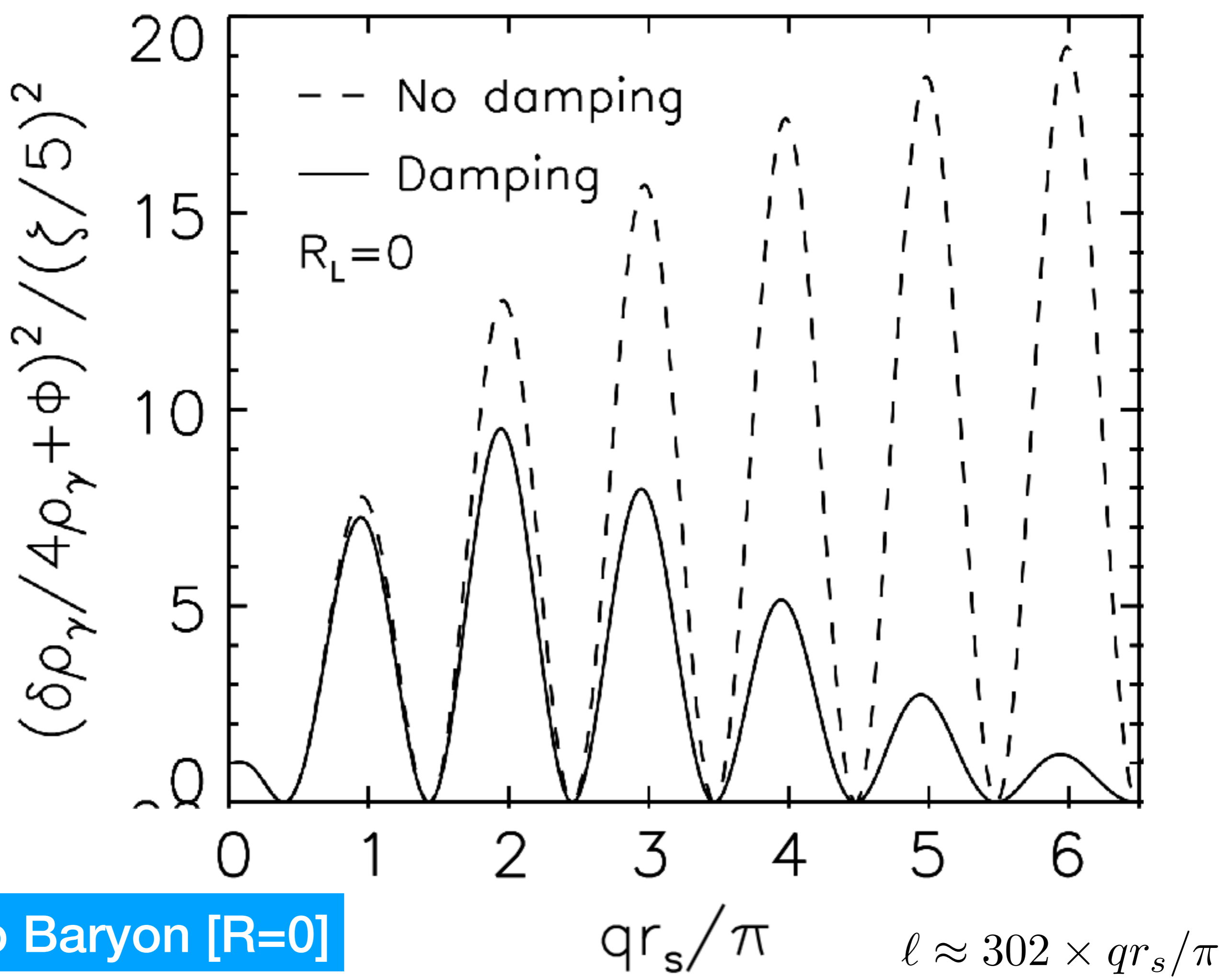
- The amplitude of the oscillation on small scales is a factor of $5(1+R)^{-1/4}$ times the Sachs-Wolfe plateau!

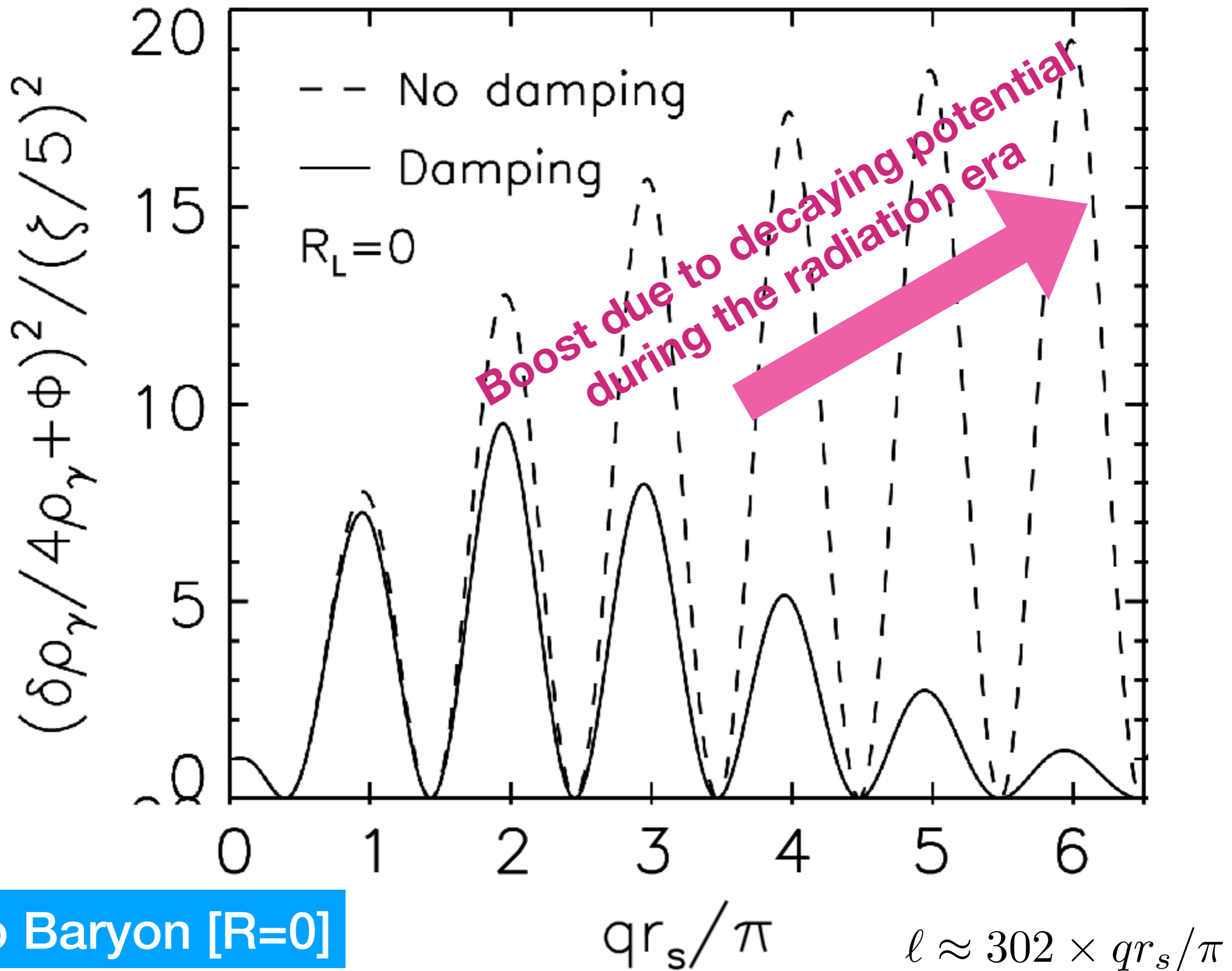
Effect of Baryons

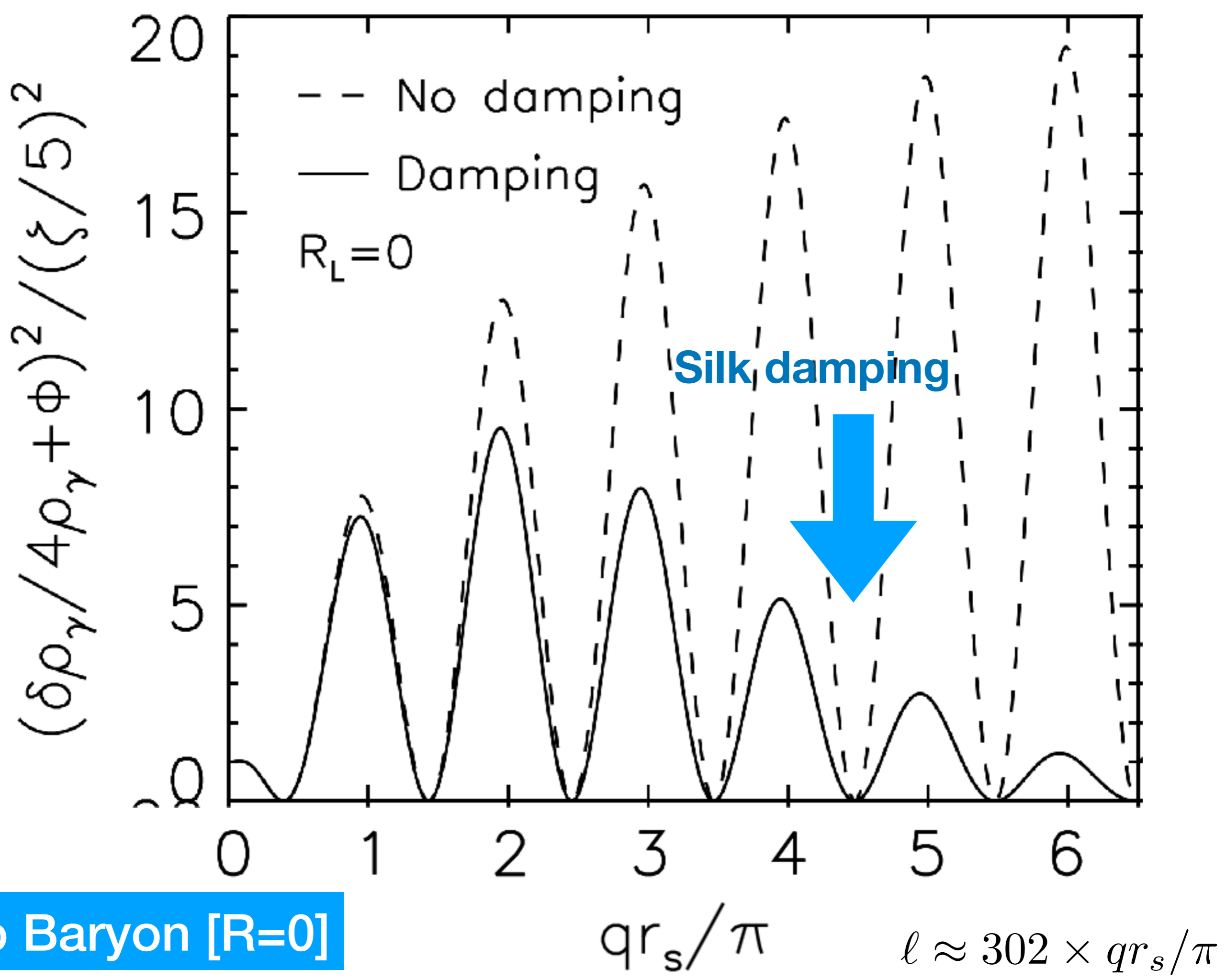
$$\frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma} + \Phi = \frac{\zeta}{5} \left\{ \boxed{3RT(q)} - \boxed{(1+R)^{-1/4}} \mathcal{S}(q) \cos[qr_s + \theta(q)] \right\}$$

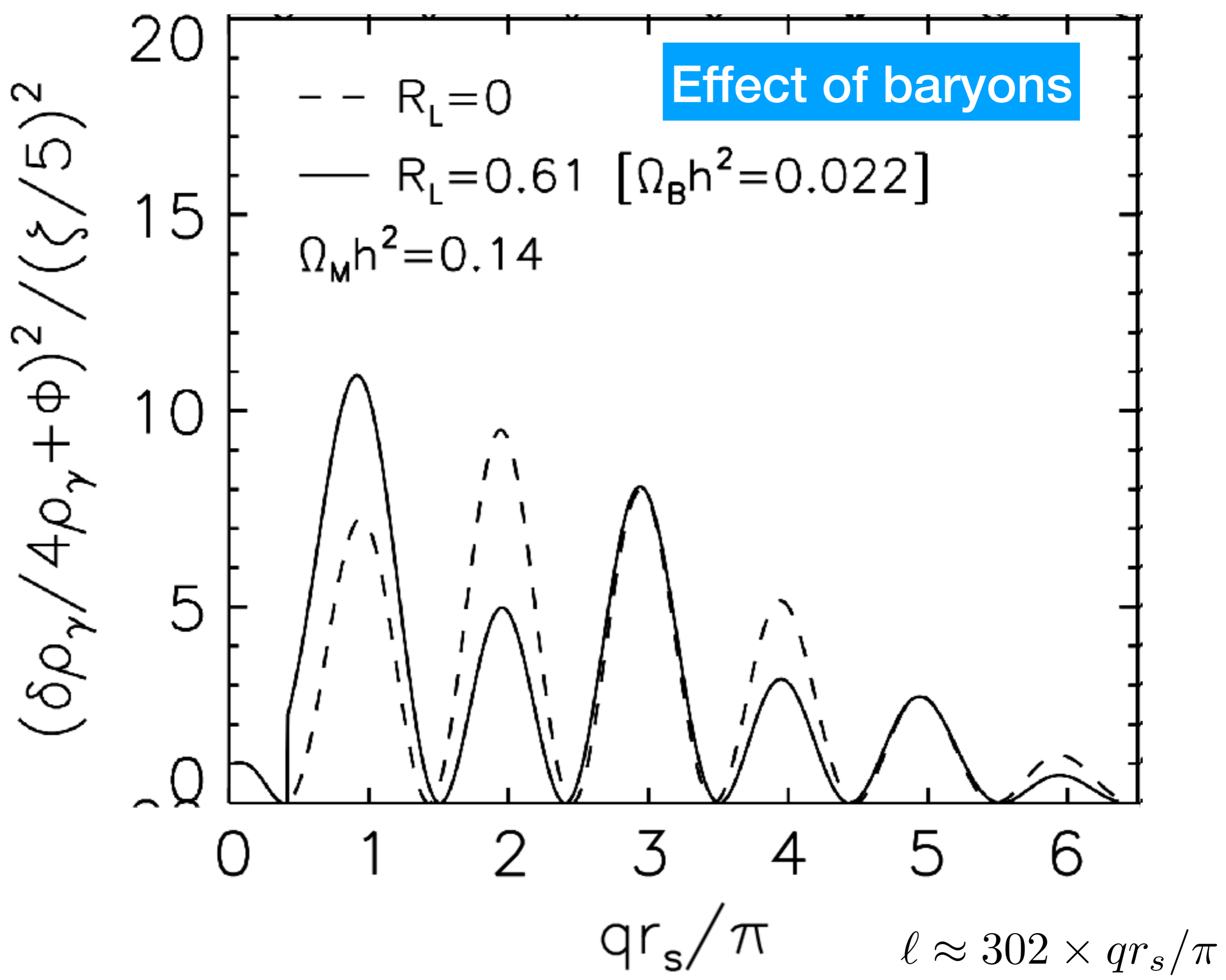
Shift the zero-point of oscillations
Reduce the amplitude of oscillations

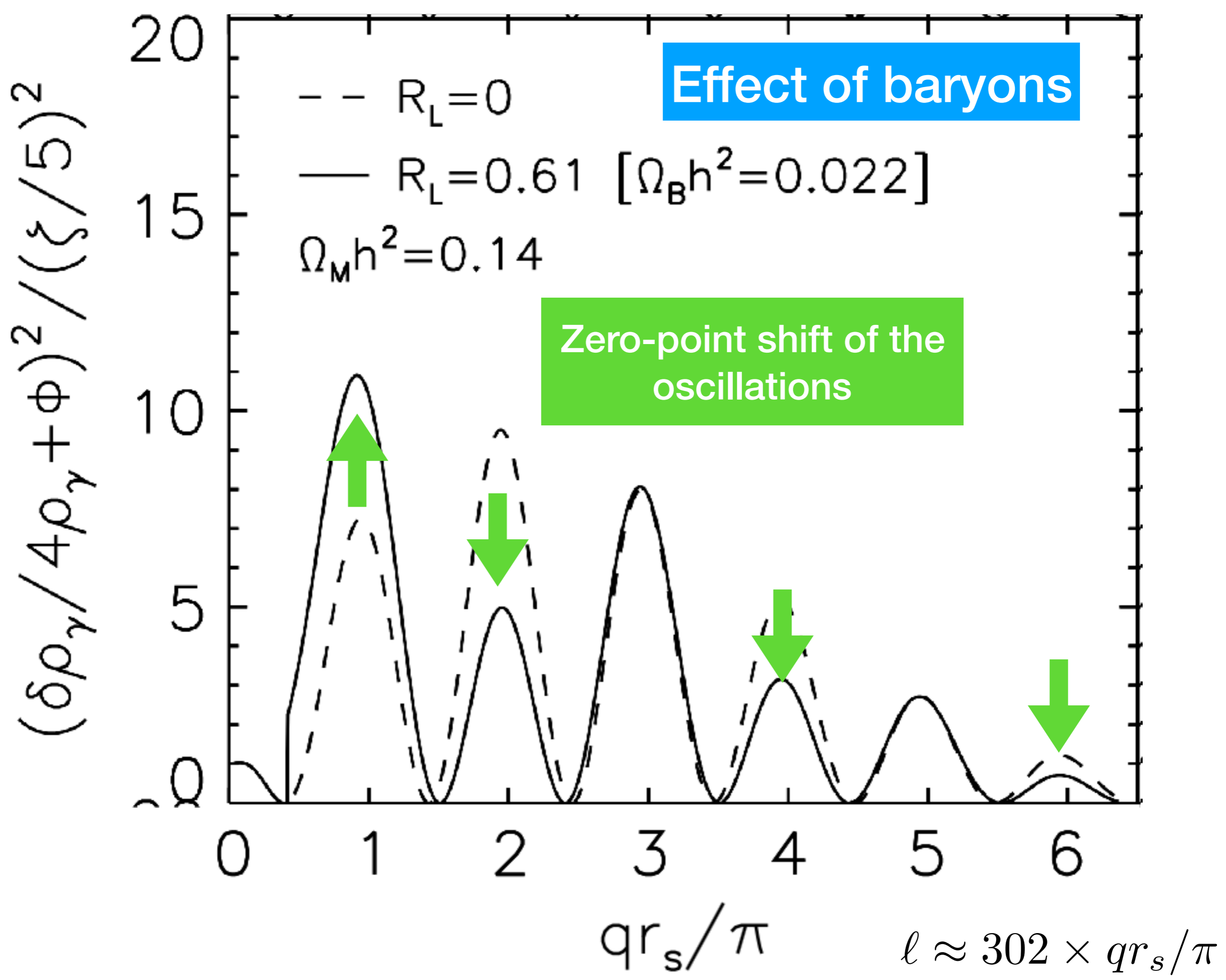
- (*) To a good approximation, the low-frequency solution is given by setting $R=0$ because sound waves are not important at large scales

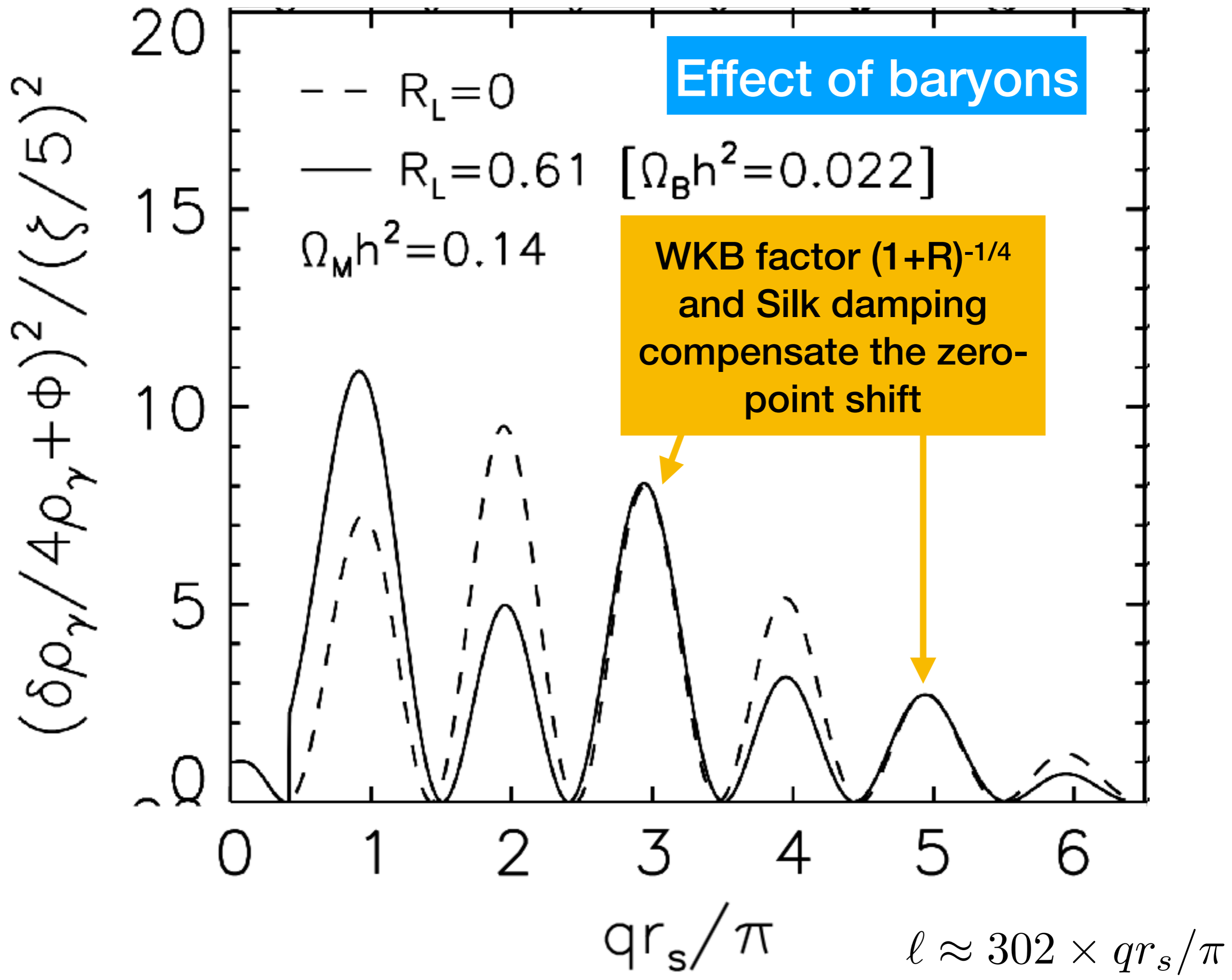












Effect of Total Matter

$$\frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma} + \Phi = \frac{\zeta}{5} \left\{ 3RT(q) - (1+R)^{-1/4} \mathcal{S}(q) \cos[qr_s + \theta(q)] \right\}$$

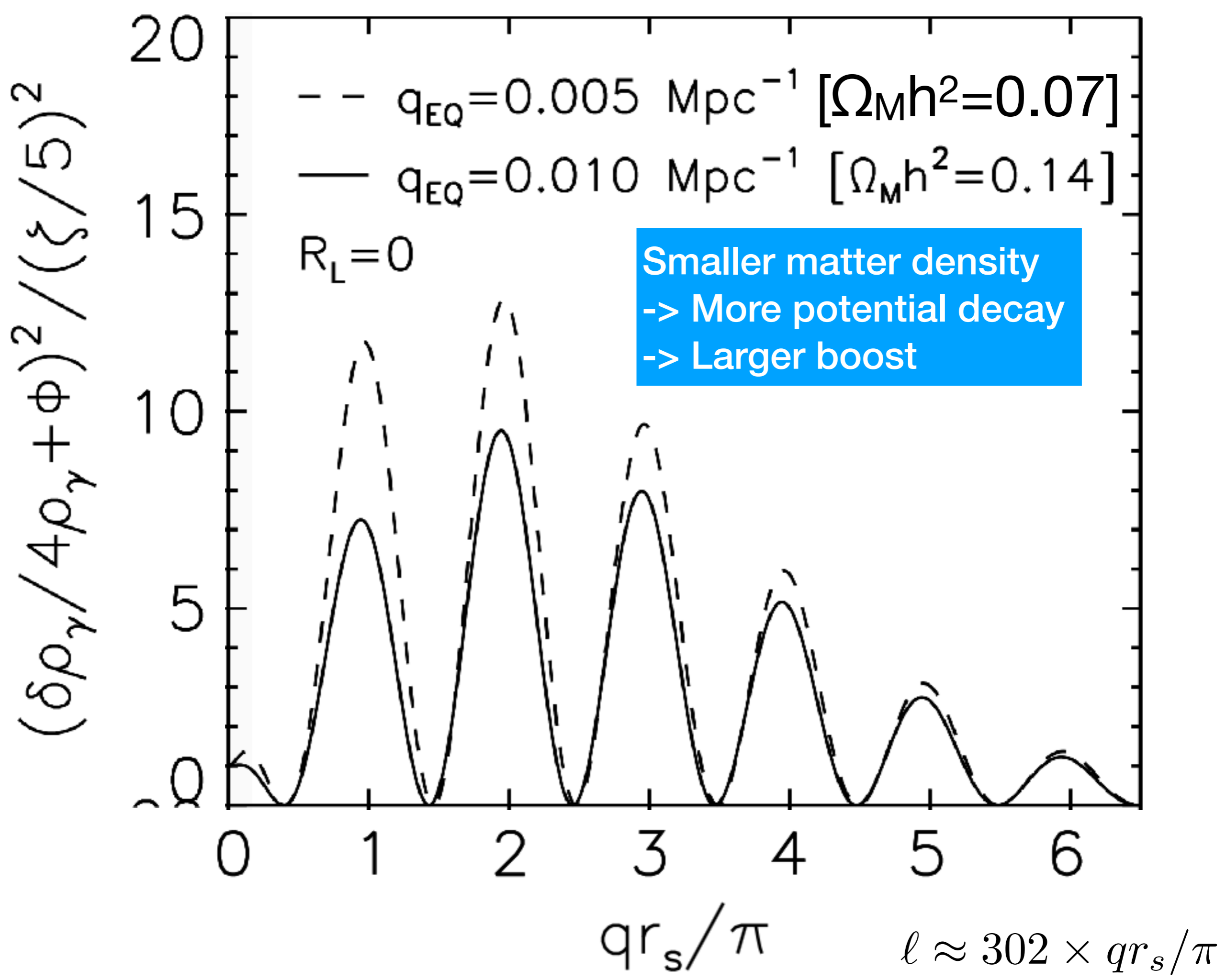
where $T(q)$, $\mathcal{S}(q)$, $\theta(q)$ are "transfer functions" that smoothly interpolate two limits as

$$\mathbf{q} \ll \mathbf{q}_{\text{EQ}}: \quad \mathcal{S} \rightarrow 1, \quad \mathcal{T} \rightarrow 1, \quad \theta \rightarrow 0$$

$$\mathbf{q} \gg \mathbf{q}_{\text{EQ}}: \quad \mathcal{S} \rightarrow 5, \quad \mathcal{T} \propto \ln q / q^2, \quad \theta \rightarrow 0.062\pi$$

"EQ" for "matter-radiation Equality epoch"

with $q_{\text{EQ}} = a_{\text{EQ}} H_{\text{EQ}} \sim 0.01 (\Omega_M h^2 / 0.14) \text{ Mpc}^{-1}$



Recap

- Decay of gravitational potentials boosts the temperature anisotropy dT/T at high multipoles by **$5(1+R)^{-1/4}$** **compared to the Sachs-Wolfe plateau**
 - Where this boost starts depends on the total matter density
- Baryon density shifts the zero-point of the oscillation, boosting the odd peaks relative to the even peaks
 - However, the WKB factor $(1+R)^{-1/4}$ and damping make the **boosting of the 3rd and 5th peaks not so obvious**

Not quite there yet...

- **The first peak is too low**
 - We need to include the “integrated Sachs-Wolfe effect”
- **How to fill zeros between the peaks?**
 - We need to include the Doppler shift of light

Doppler Shift of Light

$$\frac{\Delta T(\hat{n})}{T_0} = \frac{\delta\rho_\gamma(t_L, \hat{n}r_L)}{4\bar{\rho}_\gamma(t_L)} + \Phi(t_L, \hat{n}r_L) - \hat{n} \cdot \mathbf{v}_B(t_L, \hat{n}r_L)$$

\mathbf{v}_B is the bulk velocity of a baryon fluid

- Using the velocity potential, we write $-\hat{n} \cdot \nabla \delta u_B / a$

- In tight coupling, $\delta u_B = \delta u_\gamma$

- Using energy conservation,

$$\delta u_\gamma = (3a^2 / q^2) \partial(\delta\rho_\gamma / 4\bar{\rho}_\gamma) / \partial t$$

Line-of-sight direction

$$\hat{n}^i = -\gamma^i$$

Coming distance (r)

$$x^i = \hat{n}^i r$$

$$r(t) = \int_t^{t_0} \frac{dt'}{a(t')}$$

Doppler Shift of Light

$$\frac{\Delta T(\hat{n})}{T_0} = \frac{\delta\rho_\gamma(t_L, \hat{n}r_L)}{4\bar{\rho}_\gamma(t_L)} + \Phi(t_L, \hat{n}r_L) - \hat{n} \cdot \mathbf{v}_B(t_L, \hat{n}r_L)$$

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$$\delta u_\gamma = (3a^2 / q^2) \partial(\delta\rho_\gamma / 4\bar{\rho}_\gamma) / \partial t$$

Velocity potential is a **time-derivative** of the energy density:

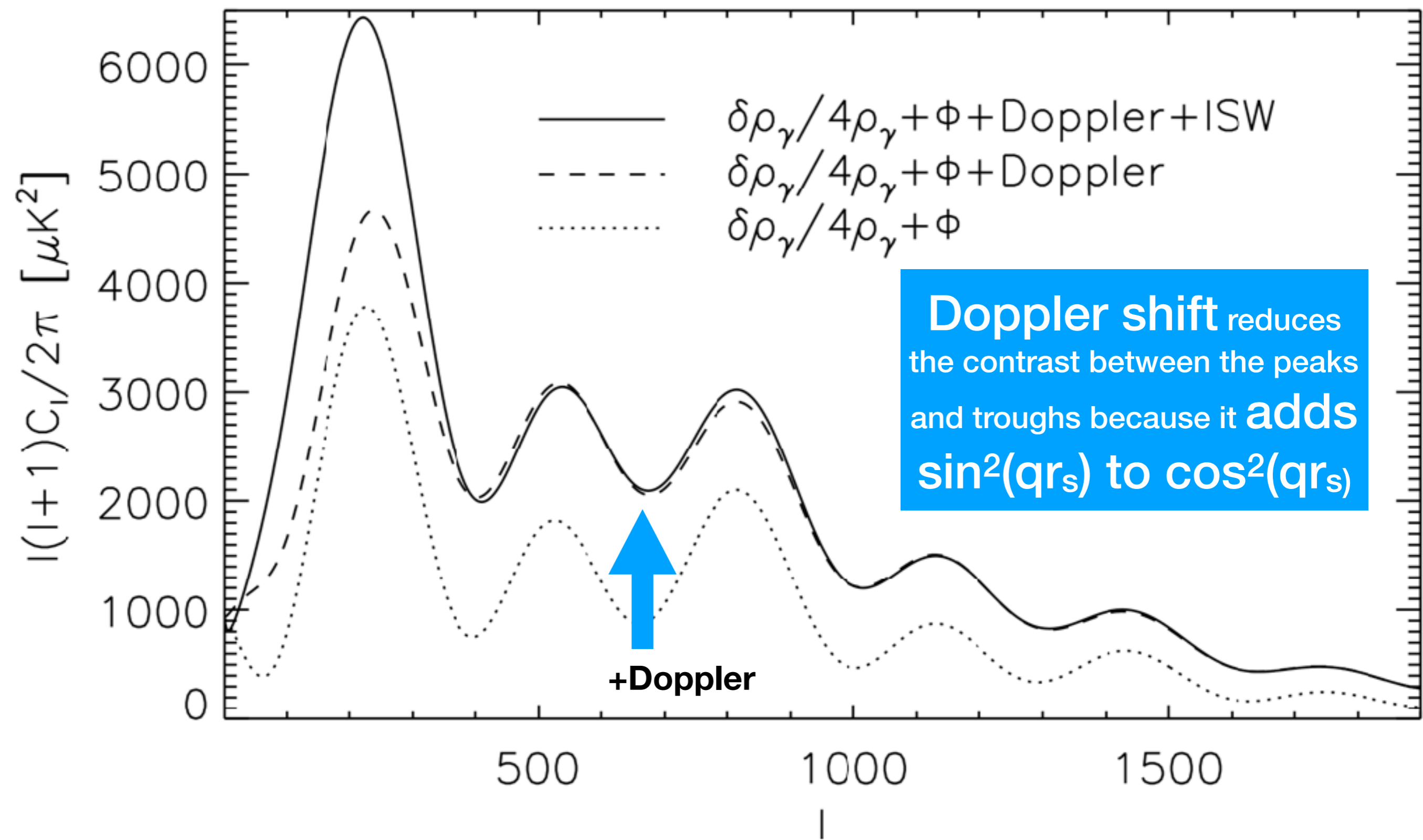
cos(qr_s) becomes sin(qr_s)!

Temperature Anisotropy from Doppler Shift

$$\frac{q}{a} \delta u_\gamma = \frac{\sqrt{3}\zeta}{5} (1 + R)^{-3/4} \mathcal{S}(\kappa) \sin[qr_s + \theta(\kappa)]$$

- To this, we should multiply the damping factor

$$\exp(-q^2 / q_{\text{Damp}}^2)$$



(Early) ISW

$$\frac{\Delta T(\hat{n})}{T_0} = \frac{\delta T(t_L, \hat{n}r_L)}{\bar{T}(t_L)} + \Phi(t_L, \hat{n}r_L) - \Phi(t_0, 0)$$

“integrated Sachs-Wolfe” (ISW) effect

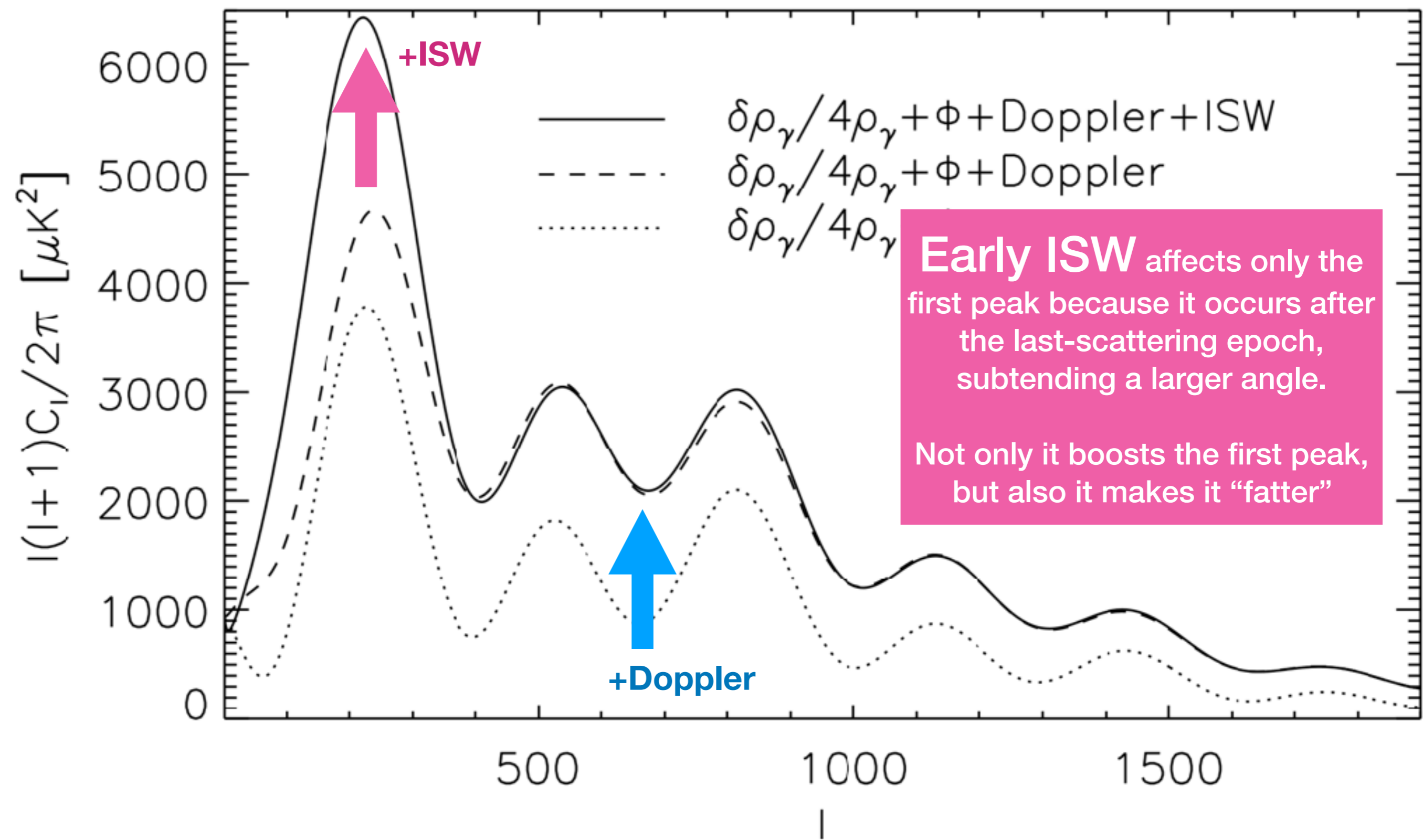
$$+ \int_{t_L}^{t_0} dt (\dot{\Phi} + \dot{\Psi})(t, \hat{n}r)$$

$$+ 2 \int dt \dot{\Phi}$$

$$+ \Phi(t_L)$$

$$\frac{\delta T(t_L)}{\bar{T}(t_L)}$$

Gravitational potentials still decay after last-scattering because the Universe then was not completely matter-dominated yet



We are ready!

$$\frac{\Delta T(\hat{n})}{T_0} = \frac{\delta\rho_\gamma(t_L, \hat{n}r_L)}{4\bar{\rho}_\gamma(t_L)} + \Phi(t_L, \hat{n}r_L) - \hat{n} \cdot \mathbf{v}_B(t_L, \hat{n}r_L)$$

$$\frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma} + \Phi = \frac{\zeta}{5} \left\{ 3R\mathcal{T}(\kappa) - (1+R)^{-1/4} \mathcal{S}(\kappa) \cos[qr_s + \theta(\kappa)] \right\}$$

$$\frac{q}{a} \delta u_\gamma = \frac{\sqrt{3}\zeta}{5} (1+R)^{-3/4} \mathcal{S}(\kappa) \sin[qr_s + \theta(\kappa)]$$

- We are ready to understand the effects of all the cosmological parameters.
- Let's do this tomorrow