

Intro to conformal bootstrap

BOOTSTAT 2021

Outline of the lectures

1 - Introduction

- Conformal transformations in $d \geq 2$ spacetime dimensions
- Irreps

2 - Correlation functions (2pt, 3pt, 4pt of scalars)

- Unitarity / Reflection positivity

3 - Operator Product Expansion (OPE)

- Conformal blocks
- Crossing symmetry
- Recasting crossing eq. to convex optimization problems

4 - Numerical Conformal Bootstrap (philosophy)

- Applications : Ising model in 2d and 3d

References:

- review on Conformal bootstrap: 1805.04405
- EPFL lectures on CFT in $D \geq 3$ dimensions: 1601.05000
- Tasi lectures on CFTs: 1602.07982
- Weizmann Lectures on the numerical conformal bootstrap: 1907.05147
- Book: "Conformal Field Theories"; Di Francesco, Mathieu, Sénéchal
- Book: "String theory"; Polchinski

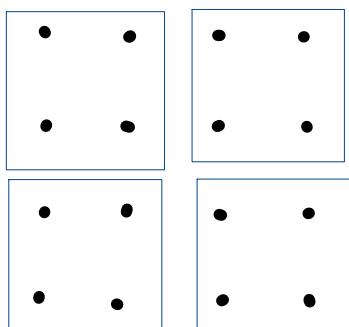
Lecture 1

Our favourite example : the Ising model in $d \geq 2$
 ($d =$ Euclidean space-time dimension)

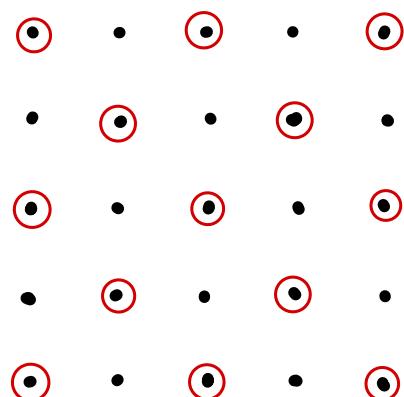
$$H = -J \sum_{\langle i,j \rangle} S_i S_j \quad Z = e^{-\beta H}, \quad \beta = \frac{1}{kT}$$

Fixed point of Renormalization Group at finite temperature for $2 \leq d < 4$

blocking



decimation



$$\{ J, K, \dots \}$$

↓ RG transformation

J (nearest neighbour coupling)

K (next to nearest neighbour c.)

:

$$\{ J', K', \dots \}$$

$$J' = K + 2J^2 + O(J^4)$$

$$K' = J^2 + O(J^4)$$

non trivial
fixed point

$$(J, K) = \left(\frac{1}{3}, \frac{1}{9}\right)$$

We learn:

- existence of a fixed point :
invariance under RG transformations \longleftrightarrow scale invariance
- divergence of correlation length \longleftrightarrow scale invariance
- \mathbb{Z}_2 symmetry present at f.p.
- Only one \mathbb{Z}_2 preserving unstable direction
- Eventually one \mathbb{Z}_2 breaking unstable direction

} useful in
lecture 4

Goal:

Framework to study quantitatively critical points

- works in any $d \geq 2$
- needs basic ingredients (symmetries, # unstable directions
...)
- non-perturbative
- rigorous

Our framework :

- Work in the realm of quantum field theories (QFT)

$$\mathcal{Z}_{\text{latt}} = \sum_{\{\sigma_i\}} e^{-\beta H} \longrightarrow \mathcal{Z}_{\text{QFT}} = \int \mathcal{D}[\phi] e^{-S[\phi]}$$

There exist particular lattice observables for which is well defined the scaling limit

$$\lim_{a \rightarrow 0} \prod_{i=1}^N a^{-\Delta_i} \underbrace{\langle \phi_1^{\text{latt}}(x_1) \dots \phi_N^{\text{latt}}(x_N) \rangle}_{\sum_{\{\sigma_i\}} \phi_1^{\text{latt}}(x_1) \dots \phi_N^{\text{latt}}(x_N) e^{-\beta H}}$$

→ $\langle \phi_1(x_1) \dots \phi_N(x_N) \rangle$ expectation value of local quantum operators weighted by a path integral measure..

- Restrict to unitary (reflection positive) QFTs

- scaling dimensions of operators $\Delta_i \geq \Delta_* > 0$ (except identity)

- central charge positive
- no logarithms

$\begin{cases} \text{Non Unitary theories} \\ \cdot \text{ non integer Q-Potts} \\ \cdot \text{ percolation} \\ \cdot \text{ Lee Yang minimal model} \\ \cdot \text{ Random Walk} \end{cases}$

Conformal symmetry ($d \geq 2$)

Special operator: stress tensor

$$SS = \int d^d x T^{\mu\nu} \partial_\mu \xi_\nu(x)$$

response of action under $\overset{\text{local}}{\downarrow}$ transformation $x_\mu \rightarrow x_\mu + \xi_\nu(x)$

- Invariance under translations:

$$\partial_\mu T^{\mu\nu} = 0$$

- Invariance under $SO(d)$ -rotations

$$T^{\mu\nu} = T^{\nu\mu}$$

- Invariance under scale transformations (dilatations)

$$x_\mu \rightarrow x_\mu + \lambda x_\mu \quad (\partial_\mu \xi_\nu = \lambda g_{\mu\nu})$$

$$SS = \int d^d x T^\mu{}_\mu$$

Generically this is not sufficient to conclude

$$T^\mu{}_\mu = 0 \quad \text{as an operator}$$

True in - Unitary, Transl + Rot invariant 2d QFT
- perturbatively in 4D

Counter examples exist

Assume $T^\mu{}_\mu = 0$

Under this assumption, action invariant under

$$SS = \frac{1}{2} \int d^d x T^{\mu\nu} (\underbrace{\partial_\mu \xi_\nu + \partial_\nu \xi_\mu}_{\text{if } = \lambda(x) g_{\mu\nu}}) = 0$$

$[g_{\mu\nu} : \text{metric}$
 $\text{rescales locally}]$

Killing equation:

$$\partial_\mu \xi_\nu + \partial_\nu \xi_\mu = \frac{2}{d} (\partial_\rho \xi^\rho) g_{\mu\nu} \quad g_{\mu\nu} = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$$

In $d=2$, passing to $\bar{z} = t + ix$, $\partial_z = \frac{\partial_t - i\partial_x}{2}$

any transformation $\bar{z} \rightarrow \bar{z} + \epsilon(z)$

with $\epsilon(z)$ holomorphic is a solution.

Among those the subgroup $SL(2, \mathbb{C}) \sim SO(3,1)$

$$z \rightarrow \frac{az+b}{cz+d} \quad a, b, c, d \in \mathbb{C}, ad - bc = 1$$

can be extended to finite transformations

(global conformal group)

$$z \rightarrow z + b. \quad \text{translation}$$

$$SL(2, \mathbb{C}): \quad z \rightarrow e^{i\theta} z \quad \text{rotation}$$

$$z \rightarrow |\lambda|^2 z \quad \text{dilatation}$$

$$z \rightarrow \frac{1}{z_0 - z} \quad \text{special conf transf}$$

In general $d > 2$:

$$\xi_\mu(x) :$$

a_μ	translations
$w_{\mu\nu} x^\nu$	($w_{\mu\nu} + w_{\nu\mu} = 0$, rotations)
λx_μ	(dilatations)
$2(b_\rho x^\rho) x^\mu - b^\mu x^\rho$	(special conf transf)

Group isomorphic to $SO(d+1, 1)$

Study the algebra:

1) Construct the generators

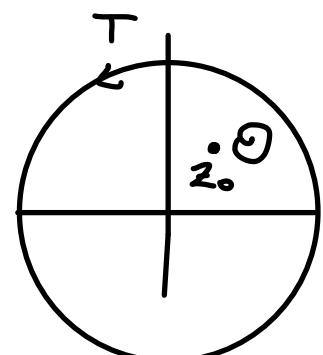
$$Q_\xi(\Sigma) = \oint_{\Sigma} T^{\mu\nu} \xi_\mu dS_\nu$$

In $d=2$ this is just:

$$Q_\xi(c) \sim \oint_c T_{zz}(z) \xi(z) dz$$

When Σ encircles a point in space this is the commutator in radial ordering

$$= -$$



Can guess the most general form of commutator

$$[Q_\xi, T_{\mu\nu}] = \alpha_1 \xi^\rho \partial_\rho T_{\mu\nu} + \alpha_2 (\partial_\rho \xi^\rho) T_{\mu\nu}$$

$$+ \alpha_3 \partial_\rho \xi_\mu T^\rho_\nu + \alpha_4 \partial_\mu \xi^\rho T_\rho_\nu$$

Integrating also $T_{\mu\nu}$ on a Σ' gives the commutation relations

2) More brute force:

Action on functions of coordinates

$$\left\{ \begin{array}{l} P_\mu = \partial_\mu \\ M_{\mu\nu} = -(x_\mu \partial_\nu - x_\nu \partial_\mu) \\ D = x^\mu \partial_\mu \\ K_\mu = 2x_\mu x^\rho \partial_\rho - x^2 \partial_\mu \end{array} \right.$$

Note: A finite conformal transformation can be written as:

$$\frac{\partial x'^\mu}{\partial x^\nu} = \Omega(x) \Lambda^\mu_\nu$$

↑ ↑
local rotation
rescaling

Conformal Algebra

$$\begin{array}{ll} P_\mu = \text{transl} & D = \text{dilat} \\ M_{\mu\nu} = \text{rotat} & K_\mu = \text{SCT} \end{array}$$

$$[D, P_\mu] = P_\mu$$

} raising / lowering operators

$$[D, K_\mu] = -K_\mu$$

$$[K_\mu, P_\nu] = 2(g_{\mu\nu}D - M_{\mu\nu}) \quad \} \text{ important for unitarity}$$

$$[M_{\mu\nu}, P_\rho] = g_{\nu\rho}P_\mu - g_{\mu\rho}P_\nu \quad \} \quad P, K \text{ } SO(d) \text{ vectors}$$

$$[M_{\mu\nu}, K_\rho] = g_{\nu\rho}K_\mu - g_{\mu\rho}K_\nu$$

$$[M_{\mu\nu}, M_{\rho\sigma}] = M_{\mu\sigma}g_{\nu\rho} - M_{\mu\rho}g_{\nu\sigma} + M_{\nu\sigma}g_{\mu\rho} - M_{\nu\rho}g_{\mu\sigma}$$

$$[M_{\mu\nu}, D] = [P_\mu, P_\nu] = [K_\mu, K_\nu] = 0$$

can be
diagonalized
simultaneously

Analog of L_0, L_1, L_{-1} and $\bar{L}_0, \bar{L}_1, \bar{L}_{-1}$ in $d=2$

$$L_0 + \bar{L}_0 \leftrightarrow D$$

$$L_{-1}, \bar{L}_{-1} \leftrightarrow P_\mu$$

$$L_0 - \bar{L}_0 \leftrightarrow M_{12}$$

$$L_1, \bar{L}_1 \leftrightarrow K_\mu$$

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m) \delta_{m+n,0}$$

Inreps of Conformal Algebras.

Since $[M_{\mu\nu}, D] = 0$ we can classify states by

- Scaling dimension Δ :

$$D |\Delta, r\rangle = \Delta |\Delta, r\rangle$$

- $SO(d)$ irrep r they belong to :

$$M_{\mu\nu} |\Delta, r\rangle = R^{(r)}(M_{\mu\nu}) |\Delta, r\rangle$$

\uparrow
representative of $M_{\mu\nu}$
in irrep r

(Analog of h, \bar{h} , (anti)holomorphic weights)



$$|\Delta+1, r'\rangle \quad \text{with } r' \in r \otimes v$$



$$|\Delta, r\rangle$$



$$|\Delta-1, r''\rangle \quad \text{with } r'' \in r \otimes v$$



Eventually Δ would become arbitrarily negative.
 Restrict to irreps of conformal algebra such that
 at some point

$$K_\mu |\Delta_{\min}, r_* \rangle = 0 \quad \text{Primary state}$$

From now on : label an irrep with quantum number $\{\Delta, r\}$ of the primary state.

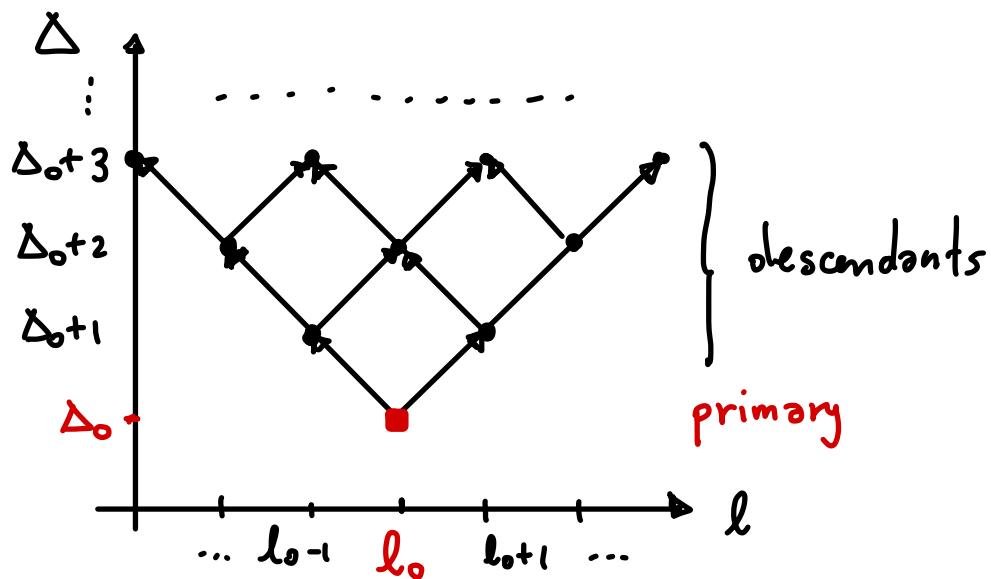
In $d=2$ the above condition defines quasi primaries

$$L_1 |h, \bar{h} \rangle = \bar{L}_1 |h, \bar{h} \rangle = 0$$

Primary states in $d=2$ satisfy a stronger condition:

$$L_n |h, \bar{h} \rangle = \bar{L}_n |h, \bar{h} \rangle = 0 \quad \forall n \geq 1$$

Irreps (ex: Spin- l tensors)



Lecture 2

Correlation functions

Main object of interest : $\langle \Theta_1(x_1) \dots \Theta_n(x_n) \rangle$
 correlation functions of local operators.

Concretely can be viewed as scaling limit of certain combinations of lattice observables.

The knowledge of all correlation functions allows us to define "operators", whose expectation value gives the correlation functions.

We will then discuss local operators as the fundamental ingredients.

- In an euclidean QFT, under rotations

Collectively
 denote
 indices
 by \mathcal{I}

$$\Theta_{\Delta, r}^{\mathcal{I}}(x) \xrightarrow{\Lambda} R^{(r)}(\Lambda)^{\mathcal{I}}{}_{\mathcal{J}} \Theta_{\Delta, r}^{\mathcal{J}}(\Lambda^{-1}x)$$

- In a scale invariant theory, the scaling dimension is a good quantum number and under rescaling (dilatations) objects transform according to their dim:

$$\Theta_{\Delta, r}^{\mathcal{I}}(x) \xrightarrow{\lambda} \lambda^{-\Delta} \Theta_{\Delta, r}^{\mathcal{I}}(\lambda^{-1}x)$$

- With a bit of work, can prove that, if

$$x \rightarrow x' = g(x) \text{ such that } \frac{\partial x'^\mu}{\partial x^\nu} = \Omega(x) \lambda^\mu_\nu$$

then local operators transform as

$$\mathcal{O}_{\Delta, r}^I(x) \longrightarrow \Omega(x)^{-\Delta} R^{(r)}(\lambda)_J^I \mathcal{O}_{\Delta, r}^J(g^{-1}(x))$$

In a CFT, correlation functions of local operators are very constrained.

Ex: opt of scalars ϕ_1, ϕ_2 with dim Δ_1, Δ_2

$$\langle \phi_1(x_1) \phi_2(x_2) \rangle = f(x_1, x_2)$$

- transl $f = f(x_{12})$ $X_{ij}^\mu = x_i^\mu - x_j^\mu$

- Rot $f = f(|x_{12}|)$

- Dilat $\lambda^{-\Delta_1-\Delta_2} f\left(\frac{|x_{12}|}{\lambda}\right) = f(|x_{12}|)$

$$\Rightarrow f = \frac{N}{|x_{12}|^{\Delta_1+\Delta_2}}$$

- SCT (or inversion) : $x^\mu \rightarrow \frac{x^\mu}{x^2}$; $\frac{\partial x'^\mu}{\partial x^\nu} = \frac{1}{x^2} \left(\delta_\nu^\mu - \frac{2x^\mu x_\nu}{x^2} \right)$

$$\phi_i(x_i) \rightarrow |x_i|^{-\Delta_i} \phi_i\left(\frac{x^\mu}{x^2}\right) \Rightarrow \Delta_1 = \Delta_2 \text{ or } N=0$$

$$\left[\text{Use } |x_1|^{-2\Delta_1} |x_2|^{-2\Delta_2} \left(\frac{1}{\frac{x_1^\mu}{|x_1|^2} - \frac{x_2^\mu}{|x_2|^2}} \right)^{\Delta_1+\Delta_2} = |x_1|^{-2\Delta_1} |x_2|^{-2\Delta_2} \cdot \frac{(|x_1| |x_2|)^{\Delta_1+\Delta_2}}{|x_{12}|^{\Delta_1+\Delta_2}} \right]$$

In conclusion:

- $\langle \phi_1(x_1) \phi_2(x_2) \rangle = \frac{\delta_{\Delta_1, \Delta_2}}{|X_{12}|^{2\Delta_1}}$ (rescaled)
 (from now assume
2pt diagonal in operator space)
- Similarly

$$\langle \phi_1(x_1) \phi_2(x_2) \phi_3(x_3) \rangle = \frac{\lambda_{\phi_1 \phi_2 \phi_3}}{|X_{12}|^{h_{123}} |X_{13}|^{h_{132}} |X_{23}|^{h_{231}}}$$

$$h_{ijk} = \Delta_i + \Delta_j - \Delta_k$$

Similar expressions in 2d

$$\langle \phi_1(z_1, \bar{z}_1) \phi_2(z_2, \bar{z}_2) \rangle = \frac{1}{(z_1 - z_2)^h (\bar{z}_1 - \bar{z}_2)^{\bar{h}}} \begin{bmatrix} \text{if} \\ h_1 = h_2 = h \\ \bar{h}_1 = \bar{h}_2 = \bar{h} \end{bmatrix}$$

In $d > 2$, things get complicated due to index structures when considering non-scalar objects

For instance:

central charge in $d > 2$

$$\langle T^{\mu\nu}(x_1) T^{\rho\sigma}(x_2) \rangle = \frac{C_T}{S_d |X_{12}|^{2d}} \left(I^{\mu\rho} I^{\nu\sigma} + I^{\mu\sigma} I^{\nu\rho} - \frac{2}{d} g^{\mu\nu} g^{\rho\sigma} \right)$$

$$I^{\mu\nu}(x_{12}) = g^{\mu\nu} - 2 \frac{x_{12}^\mu x_{12}^\nu}{|X_{12}|^2}$$

$$S_d = \text{volume sphere} = \frac{\pi^{d/2}}{\Gamma(\frac{d}{2} + 1)}$$

Similarly, all two point functions are fixed.

Same for 3pt:

$$\langle \phi_1(x_1) \phi_2(x_2) T^{\mu\nu}(x_3) \rangle = \frac{\lambda_{\phi_1 \phi_2 T} (\bar{Z}_{123}^\mu \bar{Z}_{123}^\nu - g^{\mu\nu} \bar{Z}_{123}^2)}{|x_{12}|^{h_{123}} |x_{13}|^{h_{132}} |x_{23}|^{h_{231}}}$$

$$\bar{Z}_{ijk}^\mu = \frac{x_{ik}^\mu}{x_{ik}^2} - \frac{x_{jk}^\mu}{|x_{jk}|^2}$$

$$\text{Ward identity: } \lambda_{\phi_i \phi_j T} = - \frac{d \Delta_i}{(d-1) S_d} \delta_{\Delta_i \Delta_j}$$

Similarly, all 3pt functions are fixed by a finite number of 3pt-function coefficients

$$\langle O_1^I(x_1) O_2^J(x_2) O_3^K(x_3) \rangle = \frac{\sum_{a=1}^{n_3} \lambda_{a0_10_20_3}^{(a)} T_{(a)}^{IJK}}{|x_{12}|^{h_{123}} |x_{13}|^{h_{132}} |x_{23}|^{h_{231}}}$$

n_3 is fixed by $SO(d)$ representation theory.

Higher point correlation functions are not fixed:

Ex $\langle \phi(x_1) \dots \phi(x_4) \rangle = \frac{g(\mu, \nu)}{|x_{12}|^{2\Delta_\phi} |x_{34}|^{2\Delta_\phi}}$

$$\mu = \frac{|x_{12}|^2 |x_{34}|^2}{|x_{13}|^2 |x_{24}|^2} \quad \nu = \frac{|x_{14}|^2 |x_{23}|^2}{|x_{13}|^2 |x_{24}|^2}$$

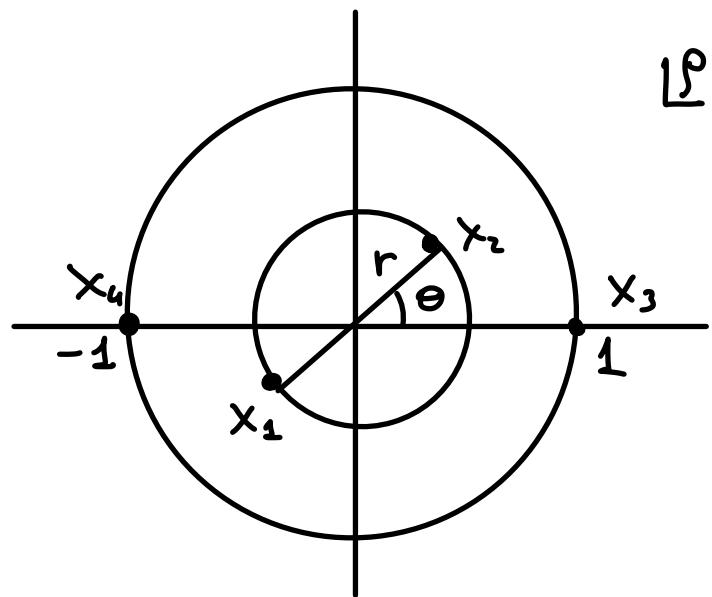
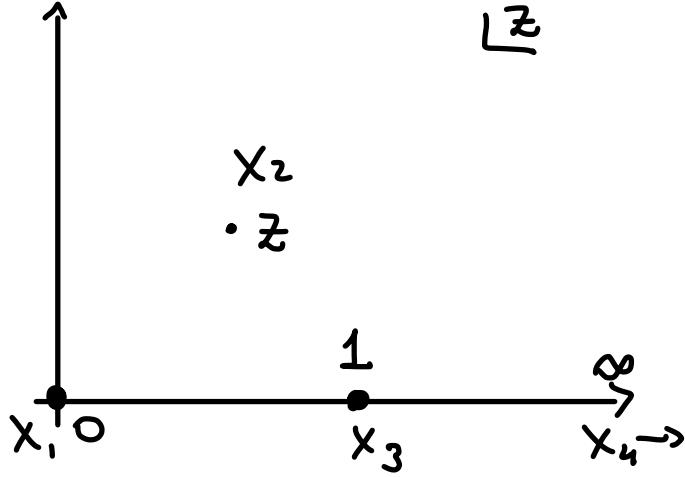
Sometimes we will use alternative variables

$$\mu = z \bar{z} \quad \nu = (1-z)(1-\bar{z})$$

(in euclidean $\bar{z} = z^*$, but we will often consider them as complex and independent)

Alternatively

$$z = \frac{4f}{(1+f)^2} \quad f = r e^{i\theta}$$



Note : in any dimension there are exactly 2 conformal invariants made with q points.

In conclusion , the covariance properties of the correlation function alone do not allow to fix higher point correlation functions.



Radial Quantization

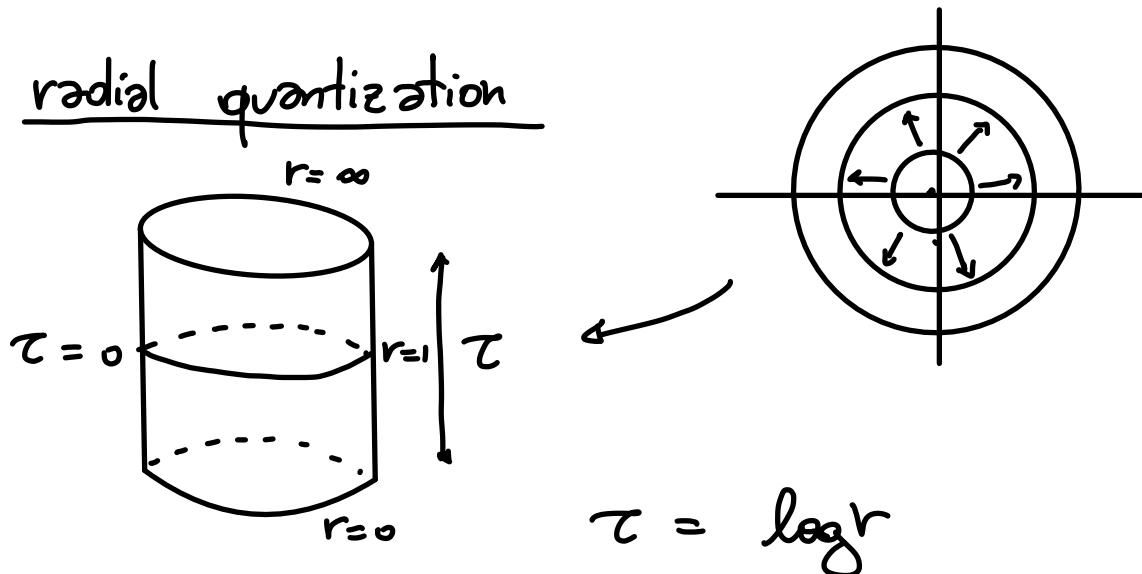
In order to make progresses we need to start using properties of the QFT. In particular we need to use the existence of a space of states and introduce a notion of "time direction", inner product , and conjugation.

We will recall the $d=2$ story and proceed by analogy .

i) First we observe that it is convenient to foliate the space in surfaces invariant under rotations

Also, the evolution operator from one slice to the next, better not change the property of being a primary.

Hence: radial quantization



A radially quantized theory behaves as an ordinary euclidean QFT on a cylinder $\mathbb{R} \times S_{d-1}$

\uparrow time \uparrow compact space.

We can immediately import familiar concepts:

- | | |
|--|---|
| <ul style="list-style-type: none"> • Time ordered $\langle \dots \rangle$ • At each slice τ we have a space of states:
$\phi_0\rangle \longleftrightarrow$ boundary condition
at $\tau = -\infty$ of P.I. $\int \frac{(\phi)}{16\pi}$ | <ul style="list-style-type: none"> • Radially ordered $\langle \dots \rangle$ • On each sphere we have a state:
$0\rangle \sim$ operator at origin |
|--|---|

$$\Gamma \text{Logic: } \int dy \int \mathcal{D}[x] e^{-S/\hbar}$$

$x(t_i) = x_i$
 $x(t_f) = y$

$$\int dy |y\rangle \langle y|_{x_f} = \underbrace{|x_i, t_f\rangle}_{\text{evolved}}$$

Hence for each state there is a boundary condition on the operator configuration space (i.e. an operator)

states \rightarrow operators

Given an operator, we can insert it in a P.I. to create an object living on the sphere with right quantum numbers

operators \rightarrow states.



- Conjugated states:

$$(|\phi\rangle)^* = \langle \phi|$$

state defined on a slice by operators inserted at later times

- Reflected under inversion

$$(|\phi(x)\rangle)^+ = \langle \phi(x)|$$

$$= \langle 0| \underbrace{I \cdot \phi(x) \cdot I}_{1 \times 1^{\text{D}}}$$

$$\phi\left(\frac{x^n}{x^2}\right)$$

Lecture 3

Operator Product Expansion (OPE)

First consequence of having a space of states : consider the state produced by the insertion of multiple operators :

$O_1(x) O_2(0) |0\rangle$ = state $\in \mathcal{H}$, not in an irrep
 \Rightarrow can be expanded in a complete basis

$$= \sum_{\Delta, r, \alpha} \lambda_{\Delta, r, \alpha} \underbrace{| \Delta, r, \alpha \rangle}_{\text{for each state there is an operator}} = \sum_{\Delta, r, \alpha} \lambda_{\Delta r \alpha}^{(x)} O_{\Delta, r, \alpha}(0) |0\rangle$$

$$= \sum_{\substack{\text{O}_k \\ \text{primary}}} C_{O_1 O_2 O_k}^I(x) \left(O_k^I(0) + \underset{P_\mu; \dots P_{\mu_n}}{\underset{\uparrow}{\text{descendants}}} \right) |0\rangle$$

$$= \sum_{\substack{\text{O}_k \\ \text{primary}}} C_{O_1 O_2 O_k}^I(x, \alpha) O_k^I(0) |0\rangle$$

Converges because we expanded the state in a basis of eigenvectors of H_{vir}, D .

In $d=2$ this form of the OPE is rarely used.
 Most of the times we just quote universal behaviors:

$$T(z) \times \phi(\omega) \sim \frac{h_\phi}{(z-\omega)^2} \phi(\omega) + \frac{\partial \phi}{(z-\omega)} + \text{reg}$$

$$T(z) \times T(\omega) \sim \frac{c/2}{(z-\omega)^4} + \frac{2T(\omega)}{(z-\omega)^2} + \frac{\partial T(\omega)}{(z-\omega)} + \text{reg}$$

or Fusion rules of minimal models

$$\phi_{(r,s)} \times \phi_{(m,n)} = \sum_{\substack{k=1+r-m \\ k+r+m=1 \bmod 2}}^{k_{\max}} \sum_{\substack{l=1+s-n \\ l+s+n=1 \bmod 2}}^{l_{\max}} \phi_{(k,l)}$$

$$k_{\max} = \min(r+m-1, 2p' - 1 - r - m)$$

$$l_{\max} = \min(s+n-1, 2p - 1 - s - n)$$

Important difference :

- The second case is not really an OPE: is a group
- They statement just stating the decomposition of maps

Same as angular momentum composition

$$L + S = \sum_{J=\min(0, |L-S|)}^{L+S} J$$

Not true as an operator relation

- First one is similar but neglects all intensity info contained in regular terms.

Also, we deal with quasi primary that generically have powers $\frac{1}{(z-w)^{2h}}$

Consistency with conformal invariance partially fix the function $C_{O_1 O_2 O_K}^I(x, \bar{x})$:

$$O_1(x) \times O_2(0) \sim \sum_{O_K} \frac{\lambda_{O_1 O_2 O_K}}{|x|^{\Delta_1 + \Delta_2 - \Delta_K}} \left(O_K(0) + \dots \right)$$

3pt function coeff (OPE coeff)

fix by conf. symm.

(if O_K has indexes, must be contracted with x^μ)

OPE subject to selection rules :

- only certain $SO(d)$ irrep can appear : ex $\phi_i \times \phi_j \sim$ spin-l (integer)
- if global symmetries g , only irreps $r' \in r_1 \otimes r_2$ can appear.

Valid inside correlation functions away from other || insertions.

Using the OPE any correlation function can be reduced to a sum (eventually infinite) of smaller correlation functions.

Ingredients :

- Quantum number of operators
- OPE coefficients

$\left. \begin{matrix} \text{CFT} \\ \text{data} \end{matrix} \right\}$

Constraints on CFT - data

As mention in intro, we will restrict to theories that satisfy the property of reflection positivity

History of this condition:

- Osterwalder - Schrader listed it as an axiom to prove that the analytic continuation of euclidean QFT is an unitary QFT with Minkowsky signature.
- For CFT this condition can be relaxed to positivity of Lpt in reflection symmetric configurations.

$$\frac{\Theta \cdot x_N}{\Theta^+ \cdot x_s} \text{ plane}$$

existence of Θ is
an assumption

image
under reflection

- This condition translates in the request that a state has positive norm

$$\langle \psi | \psi \rangle \geq 0$$

Notice that by states we mean both primaries and descendants.

- Consistency with conformal symmetry gives non trivial constraints.

In $d=2$ one has similar story: for instance

$$\langle h | L_n L_{-n} | h \rangle = \| L_{-n} | h \rangle \|^2 \geq 0$$

$$\left(2hn + \frac{c}{12} n(n^2 - 1) \right) \langle h | h \rangle \Rightarrow c > 0 \\ (n=1) \Rightarrow 2h \geq 0$$

More in general: Kac determinant

In $d \geq 2$ one has similar consequences:

$$\Delta \geq \Delta_{\min}(r, d)$$

$$\lambda_{0,0,0_3} \text{ real} \Rightarrow \lambda_{0,0,0_3}^2 \geq 0$$

EXAMPLES:

$\mathcal{O}_{\mu_1 \dots \mu_d} \equiv$ traceless symm tensors of spin- l of $SO(d)$

$$\Delta = l = 0 \quad (\text{identity})$$

$$\Delta \geq \frac{d-2}{2} \quad (l=0)$$

$$\Delta \geq l+d-2 \quad (l>0)$$

More Unitarity bounds:

Ex: $d=3$ (irreps parametrized by J , $2J \in \mathbb{N}$)

$$\begin{cases} \Delta \geq \frac{1}{2} & (J=0) \\ \Delta \geq 1 & (J=\frac{1}{2}) \\ \Delta \geq J+1 & (J>\frac{1}{2}) \end{cases}$$

Ex: $d=4$ (irreps parametrized by $\underset{J}{SO(4)} \sim \underset{\bar{J}}{SU(2)} \times \underset{\bar{J}}{SU(2)}$
 $(2J, 2\bar{J} \in \mathbb{N})$

$$\begin{cases} \Delta \geq 1 & (J=\bar{J}=0) \\ \Delta \geq \frac{J}{2} + 1 & (J>0, \bar{J}=0) \\ \Delta \geq \frac{J+\bar{J}}{2} + 2 & (J \neq 0, \bar{J} \neq 0) \end{cases}$$

As far as OPE coefficients:

$$\begin{array}{ccc} O_1 \cdot x_i^r & & x_2 \cdot O_2 \\ & \vdots & \\ & (O_{\Delta, \ell})^r \cdot x_i^r & \vdots O_{\Delta, \ell} \\ & \vdots & \vdots \\ O_2 \cdot x_j^r & \xrightarrow[\text{lim } \rightarrow \infty]{} & x_3 \cdot O_2 \end{array}$$

$$\langle O_1 O_2 O_{\Delta, \ell}^r O_{\Delta, \ell} O_2 O_1 \rangle \geq 0 \quad \text{reflection positivity}$$

$\brace{ \text{cluster decoupling}}$

$$\langle O_1 O_2 O_{\Delta, \ell}^r \rangle \langle O_{\Delta, \ell} O_2 O_1 \rangle \Rightarrow \lambda_{O_1 O_2 O_{\Delta, \ell}}^2 \stackrel{\#}{=} \text{positive number known}$$

Conformal blocks

Using the OPE we can reduce any correlation function to sums of lower- n correlation functions:

Ex: 4pt of identical scalars:

$$\langle \phi(x_1) \phi(x_2) \phi(x_3) \phi(x_4) \rangle =$$

$$= \sum_{O_K \in \phi \times \phi} \frac{\lambda_{\phi \phi O_K}^2}{x_{12}^{2\Delta_\phi - \Delta_K} x_{34}^{2\Delta_\phi - \Delta_K}} C_{\phi \phi O_K}^I(x_{12}, \partial_2) C_{\phi \phi O_K}^T(x_{34}, \partial_4) \langle O_K^I(x_2) O_K^T(x_4) \rangle$$

Comparing with the general expectation: $\frac{g(\mu, \nu)}{x_{12}^{2\Delta_\phi} x_{34}^{2\Delta_\phi}}$
 we obtain the decomposition:

$$\underbrace{g(\mu, \nu)}_{\text{invariant}} = \sum_{O_K \in \phi \times \phi} \lambda_{\phi \phi O_K}^2 g_{\Delta_K, l_K}(\mu, \nu)$$

\uparrow \uparrow
 different irreps \Rightarrow must be invariant
 don't mix as well

The function $g_{\Delta_K, l_K}(\mu, \nu)$ are called conformal blocks and encode the contributions of an irrep to a four point function.

Computing the exact form of the conformal blocks was a long standing problem.

In '84 Zamolodchikov found a recursion relation to compute the Virasoro conformal block

(Used the analytic structure of Virasoro block as function of the complex variable $c = \text{central charge}$)

'01 - '04 Dolan & Osborn solved a differential equation satisfied by conformal block exactly, showing that in even d they have a simple form in z, \bar{z} coordinates

$$\text{Ex: } d=2 \quad g_{\Delta, \epsilon}(z, \bar{z}) = K_{\Delta+\epsilon}(z) K_{\Delta-\epsilon}(\bar{z}) + (z \leftrightarrow \bar{z})$$

$$K_\beta(x) = x^{\beta/2} {}_2F_1(\beta/2, \beta/2, \beta; x)$$

$$d=4 \quad g_{\Delta, \epsilon}(z, \bar{z}) = \frac{1}{(-2)^\epsilon} \frac{z \bar{z}}{z - \bar{z}} \left(K_{\Delta+\epsilon}(z) K_{\Delta-\epsilon}(\bar{z}) - (z \leftrightarrow \bar{z}) \right)$$

:

$$d=6$$

:

Odd dimensions so far have eluded a simple solution like the one above (nice problem).

On the other hand alternative methods have been developed, most notably a recursion relation à la Zamolodchikov

'14 Ks Poland, Simmons - Duffin

:

$$g_{\Delta, \epsilon}(r, \eta) = (4r)^\Delta \sum_{n=0}^{\infty} \sum_{J=J_{\min}}^{l+n} C_J \underbrace{(\cos \theta)}_{\gamma} \underbrace{\omega(\Delta, n, J)}_{\text{rational functions of } \Delta}$$

In concrete applications one truncates the series to a certain order N . Then the conformal block and its derivatives can be approx with arbitrary accuracy

$$\partial_r^n \partial_\eta^m g_{\Delta, e}(r_*, \eta_*) = (4r_*)^\Delta \left(\frac{P_N^{mn}(\Delta)}{Q_N(\Delta)} + O(r_*^{N-n}) \right)$$

One important property is that $Q_N(\Delta) \geq 0$ for Δ 's obeying the unitarity bounds.

— δ —

Back to 4pt Functions:

$$\langle \underbrace{\phi(x_1) \phi(x_2)}_{\text{exchanged } x_2 \longleftrightarrow x_4} \phi(x_3) \phi(x_4) \rangle = \langle \phi(x_1) \phi(x_2) \underbrace{\phi(x_3) \phi(x_4)}_{\text{exchanged } x_2 \longleftrightarrow x_4} \rangle$$

$$\frac{1}{x_{12}^{2\Delta\phi} x_{34}^{2\Delta\phi}} \sum_{O_K \in \phi \times \phi} \lambda_{\phi\phi O_K}^2 g_{\Delta_{\phi\phi}, O_K}(\mu, \nu) = \frac{1}{x_{14}^{2\Delta\phi} x_{23}^{2\Delta\phi}} \sum_{O_K} \lambda_{\phi\phi O_K}^2 g_{\Delta_K, O_K}(\nu, \mu)$$

Relabeling and reshuffling:

$$\boxed{\sum_{\Delta, e} \lambda_{\Delta, e}^2 (\mu^{-\Delta\phi} g_{\Delta, e}(\mu, \nu) - \nu^{-\Delta\phi} g_{\Delta, e}(\nu, \mu)) = 0}$$

Crossing equation.

Similar logic for correlation functions of any n operators

Main difference:

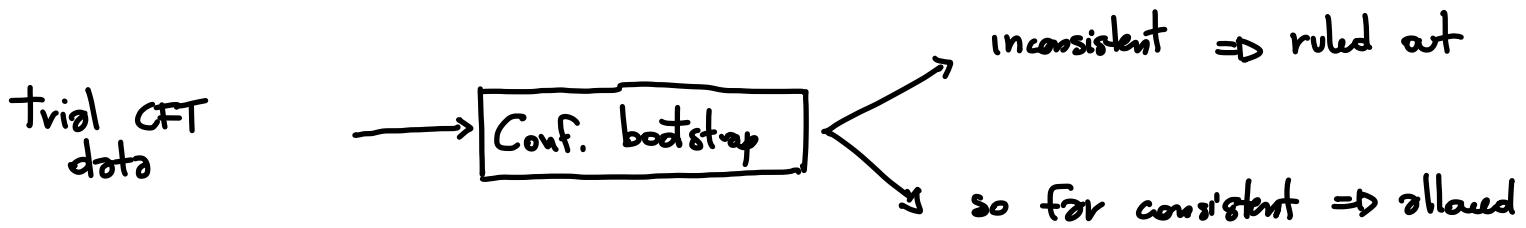
- generically additional crossing eqns. (and different OPEs involved)
- Conformal blocks get increasingly complicated as external operators are non scalars (and depend generically on Δ_2, Δ_{3n})
- If a configuration is not reflection positive no OPE squared b/w $\lambda_1, \lambda_2 \geq 0$ (Ex: $\langle \underline{\phi}_1 \underline{\phi}_1 \underline{\phi}_2 \underline{\phi}_2 \rangle \sim \sum \lambda_{11} \lambda_{22} \dots$)

In conclusion, for us a CFT (euclidean) will be a collection of local operators parametrized by a collection of quantum numbers $\{\Delta_i, r_i\}$ (Spectrum) together with the 3pt functions (OPE coeff.) of any triplets λ_{ijk} , subject to unitarity constraints, such that any 4pt is crossing symmetric.

Lecture 4

The modern conformal bootstrap approach arises from a simple observation: we do not need to solve exactly the bootstrap equation to extract informations.

We can start asking a more humble question. Given a trial spectrum, (and eventually OPE coeff), can we efficiently check if it is consistent with a finite number of crossing eqns?



A practical algorithm: (simplest one)

- 1) Assume there exists a scalar ϕ with dimension Δ_ϕ
- 2) Choose a trial spectrum S (and eventually OPE coeff) such that

$$\phi \times \phi \sim \sum_{\Delta, l \in S} \frac{\lambda_{\Delta, l}}{x_{12}^*} (\mathcal{O}_{\Delta, l} + \dots)$$

- 3) Consider the crossing eq arising from $\langle \phi \phi \phi \phi \rangle$

$$\sum_{\Delta, l \in S} \lambda_{\Delta, l}^2 F_{\Delta, l}^{\Delta \phi}(z, \bar{z}) = 0$$

- 4) Check consistency of S with above crossing eq. How?

Look for a linear functional α such that:

- $\alpha \left[\sum_{\Delta, l} \lambda_{\Delta, l}^2 F_{\Delta, l}^{\Delta \phi} \right] = \sum_{\Delta, l} \lambda_{\Delta, l}^2 F_{\Delta, l}^{\Delta \phi}$ (swappability)
- $\alpha \left[F_{\Delta, l}^{\Delta \phi}(z, \bar{z}) \right] \geq 0$
- $\alpha \left[F_{0, 0}^{\Delta \phi}(z, \bar{z}) \right] = 1 \quad \text{recall } \lambda_{0, 0}^2 = 1 \text{ by const}$

If found, then

$$\underbrace{\sum_{\Delta, l} \lambda_{\Delta, l}^2 \alpha \left[F_{\Delta, l}^{\Delta \phi} \right]}_{\geq 0} + 1 = 0$$

inconsistency
(no choice of $\lambda_{\Delta, l}$)
real would make it possible

The question at this point is to find an efficient way to test existence of such functionals α .

One set of functionals that has proven very useful:

$$\alpha_\lambda = \sum_{\substack{m+n \leq \lambda \\ m, n \\ m \leq n}} a_{mn} \partial_z^m \partial_{\bar{z}}^n \Big|_{z=\bar{z}=1/2}$$

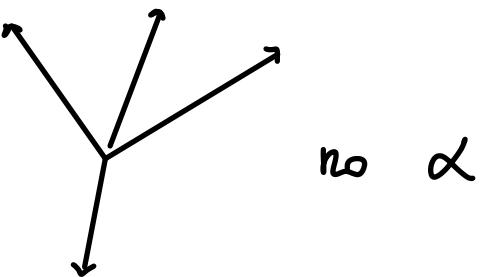
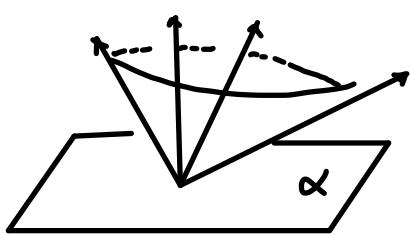
As $\lambda \rightarrow \infty$ the basis is complete. Alternatively people have considered integrating against some Kernel. Pros & cons.

Pictorially one can visualize the first λ derivatives of $F_{D,e}^{\Delta t}$ as a vector

$$\bar{F}_{D,e}^{\Delta t} = \begin{pmatrix} F_{D,e}^{\Delta t}(1/2, 1/2) \\ \vdots \\ \partial_z^n \partial_{\bar{z}}^m F_{D,e}^{\Delta t}(1/2, 1/2) \end{pmatrix}$$

Similarly by taking the first λ derivative of crossing, eqn:

$$\sum \lambda_{D,e}^2 \bar{F}_{D,e}^{\Delta t} = 0$$



Existence of α is a Convex optimization problem
Solvable with standard algorithms (semi-definite programming)

Recall $\partial_r^n \partial_\varepsilon^m g_{\text{de}}(r_*, \eta_*) = (4r_*)^\Delta \frac{P_N^{nm}(\Delta)}{Q_N(\Delta)}$

Hence $\alpha [F_{\text{de}}] \geq 0 \iff$ Linear combination of derivatives of g_{de}
 Each derivative is a rational function
 with positive prefactor and positive $Q_N(\Delta)$
 $\Rightarrow \underbrace{\sum a_{nm} P^{nm}(\Delta) \geq 0}_{\text{exactly what SDPB does.}} \forall \Delta \geq \Delta_*$

5) If spectrum S is consistent, choose $S' \subset S$ and repeat;
 otherwise choose $S'' \supset S$ and repeat.

6) Stop when satisfied \therefore 5') add more crossing eqns
 — & — 5'') increase the search
 space by taking
 larger λ

Comments on results

- Excluded spectra by a given α_λ remain excluded by using more general functionals $\alpha_{\lambda' > \lambda}$
- Considering more correlation functions can only exclude more
- No statistical errorbar, only excluded (rigorously) regions.

Application : Ising model in $d=2$.

1) Remember the first lecture we said the ISING fixed point has the following features

- \mathbb{Z}_2 symmetry
- even scalar deformation ε : $\{\Delta_\varepsilon, l=0\}$
- odd scalar deformation σ : $\{\Delta_\sigma, l=0\}$

2) Choose a trial spectrum: (OPEs)

$$S_+ \quad \sigma \times \sigma \sim \mathbb{1} + \cancel{\sigma} + \varepsilon + (\text{scalars with } \Delta > \Delta_\varepsilon) \quad \begin{matrix} \text{(write it as a} \\ \text{fusion rule, i.e.} \\ \text{no OPE coeff and} \\ \text{x-dependence.)} \end{matrix}$$

$$+ (\text{non-scalars})$$

$$S_+ \quad \varepsilon \times \varepsilon \sim \mathbb{1} + \varepsilon + (\text{scalars with } \Delta > \Delta_\varepsilon)$$

$$+ (\text{non-scalars})$$

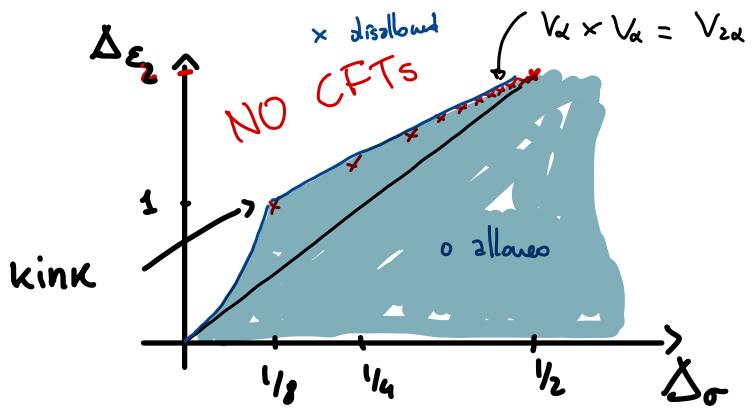
$$S_- \quad \sigma \times \varepsilon \sim \cancel{\mathbb{1}} + \sigma + \cancel{\varepsilon} + (\text{scalars with } \Delta > \Delta_\sigma)$$

$$(\text{non-scalars})$$

3) Choose a qpt : $\langle \sigma(x_1) \dots \sigma(x_n) \rangle$

$$\rightarrow \sum_{\Delta, \ell \in S_+} \lambda_{\Delta, \ell}^2 F_{\Delta, \ell}^{\Delta \sigma}(u, v) = 0$$

4) For given $\Delta_\sigma, \Delta_\varepsilon$ we can check the consistency of S_+ with crossing.



Will find an allowed region
and hopefully a disallowed one.

Allowed: we know CFT existing
 \Rightarrow the are values allowed
 (since we are considering a sub
 problem we expect the region
 to be extended)

Disallowed: eventually by making S too small, the positive properties of qpt will make it impossible to satisfy crossing.

$$\phi_{(1,2)} \times \phi_{(1,2)} \sim \phi_{(1,1)} + \phi_{(1,3)}$$

$$h_{rs} = \frac{[(m+1)r - ms]^2}{4m(m+1)}$$

$h_{1,1} = 0$
 $h_{1,2} = \frac{m-2}{4(m+1)}$ ← (x in figure)
 $h_{1,3} = \frac{m-1}{m+1}$

$$C = 1 - \frac{6}{m(m+1)}$$

Moreover, there is a Vertex operator algebra obtained by
a scalar field:

$$\langle \varphi(x)\varphi(y) \rangle = \log |x-y|^2 \quad \text{not } \propto \text{primal}$$

$$\langle i\partial\varphi(z)i\partial\varphi(w) \rangle = \frac{1}{(z-w)^2} \quad \text{primary}$$

$$V_\alpha(z, \bar{z}) = e^{i\alpha \varphi(z, \bar{z})} \quad \text{primary of } \Delta_\alpha = \frac{\alpha^2}{2}$$

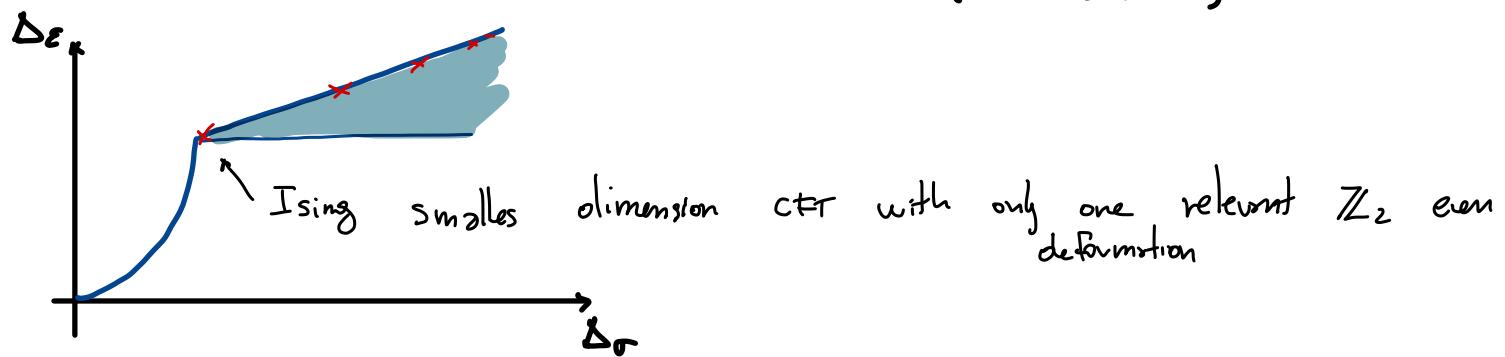
$$\langle V_{\alpha_1}(z_1, \bar{z}_1) V_{\alpha_2}(z_2, \bar{z}_2) V_{\alpha_3}(z_3, \bar{z}_3) \rangle = \prod_{i < j} |z_i - z_j|^{4\alpha_i \alpha_j}$$

$$\text{if } \alpha_1 + \alpha_2 + \alpha_3 = 0$$

So if $\alpha = \alpha_1 = \alpha_2 \Rightarrow \alpha_3 = -2\alpha \Rightarrow V_{\alpha_3}$ has $\dim \frac{4\alpha^2}{2}$
 (line / in figure)

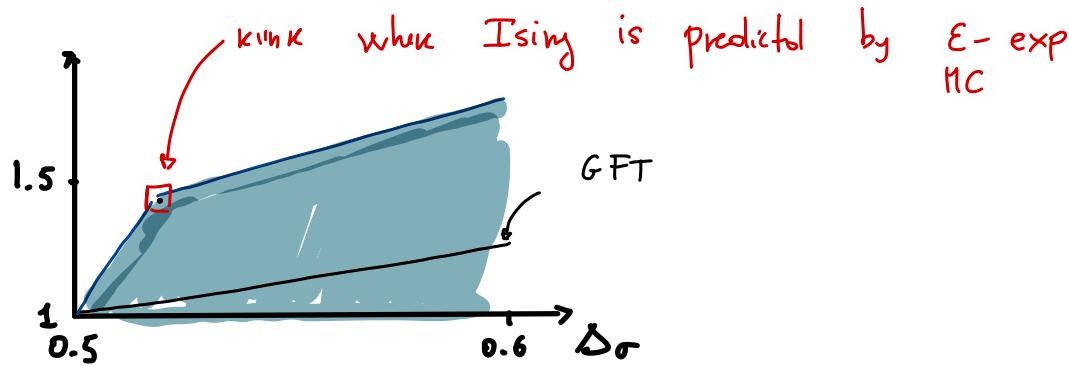
5) Repeat with more stringent spectra.

$$\text{Ex: } \sigma \times \sigma \sim 11 + \Delta_\varepsilon + (\text{other scalars irrelevant}) \\ \Delta \geq 2 \\ + (\text{non-scalars})$$



Question: minimal set of assumptions to restrict the space of consistent values to a close, isolated region?

Repeat in 3d : exact same story:



In this case additional assumptions do help

- $\sigma \times \sigma \sim \mathbb{1} + \varepsilon + (\text{irrelevant scalars}) + (\text{non-scalars})$
- $\varepsilon \times \varepsilon \sim \mathbb{1} + \varepsilon + \quad \parallel \quad + \quad \parallel$
- $\sigma \times \varepsilon \sim \sigma \quad + \quad \parallel \quad + \quad \parallel$

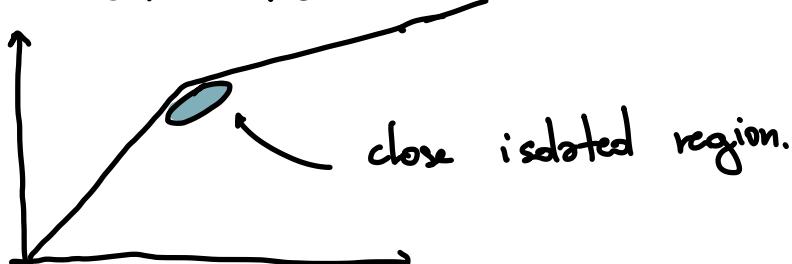
Consider $\langle \sigma \sigma \sigma \sigma \rangle$, $\langle \varepsilon \varepsilon \varepsilon \varepsilon \rangle$, $\langle \sigma \varepsilon \sigma \varepsilon \rangle$ together.

- Eventually can also include in trial spectrum info about OPE coeff, since they appear directly in crossing

$$\sum \lambda_{\sigma,\varepsilon}^2 F_{\sigma,\varepsilon} + \underbrace{\left(F_{\mathbb{1}} + \lambda_{\sigma \sigma \varepsilon}^2 F_{\sigma \varepsilon, \sigma} \right)}_{\text{known}} = 0$$

- the larger the search space, the stronger the condition (but hard to "efficiently navigate")

With all this



$$\Delta_\sigma, \Delta_\varepsilon \in \boxed{\quad}_{\min}^{\max}$$

At this point one can check if using more sophisticated functionals (ex α_λ , λ large) can do better.

The more we increase λ the more the allowed region shrinks.

Most precise determination of critical exponents:

$$\Delta_\sigma = \frac{1 + \gamma}{2} = 0.5181489(10)$$

$$\Delta_\varepsilon = 3 - \frac{1}{\nu} = 1.412625(10)$$