Critical exponents from the Lorentzian inversion formula

Lecture at Bootstat 2021

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26 May 2021

Abstract

The Lorentzian inversion formula is a powerful tool for understanding the dynamical data of conformal field theories, specifically it can be used to extract conformal data of spinning operators from singularities of the four-point function in Lorentzian signature. In this lecture I aim to "demystify" the inversion formula by giving a concrete and explicit application of it to the Wilson–Fisher fixed-point in the ϵ expansion of ϕ^4 theory (Ising CFT). I will also discuss how it can be used to study general ϕ^p theories near their upper critical dimensions, including the non-unitary case for odd p.

1 Introduction

The Lorentzian inversion formula, introduced by Caron-Huot [1], is a dispersion relation for conformal field theories in Lorentzian kinematics. It provides a way to reconstruct the conformal data (OPE coefficients and scaling dimensions) from singularities of the Lorentzian correlator, quantified by the double-discontinuity "dDisc", in a systematic way, very schematically

$$c^2_{\phi\phi\mathcal{O}},\ \Delta_{\mathcal{O}} \overset{\text{LIF}}{\longleftarrow} \text{dDisc}[\mathcal{G}(z,\bar{z})]. \tag{1.1}$$

The discovery of the Lorentzian inversion formula played an important theoretical role in resolving the question of analyticity in spin: it shows that the conformal data for operators $\mathcal{O}_\ell$ with spin are given by a function analytic in spin. Moreover, the inversion formula can be used for practical applications, both in perturbation theory at weak or strong coupling, and non-perturbatively.

Instead of giving a broad overview, the purpose of this lecture is to provide a completely explicit example of an application of the Lorentzian inversion formula, where all computations can be worked out by pen-and-paper or short procedures in Mathematica. The lecture is mostly based on [2] and uses the conventions of [3].

We will consider a well-known and well-studied class of conformal field theories, namely $\lambda\phi^p$ *theories*, for $p \geqslant 3$. Consider the Lagrangian density of the form

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 + \lambda\phi^p + \sum_{i<p} g_i \phi^i. \tag{1.2}$$

The upper critical dimension, where ϕ^p becomes marginal, is given by

$$d_{\mathrm{c}} = \frac{2p}{p-2}. \tag{1.3}$$

For $d < d_{\mathrm{c}}$, starting from the Lagrangian density and tuning the couplings g_i, we arrive at an IR fixed-point, a CFT.[1] It is this IR CFT that we refer to by $\lambda\phi^p$ theory. The most important $\lambda\phi^p$ theories are

$p = 4$	Ising CFT	$d_{\mathrm{c}} = 4$,
$p = 3$	Lee–Yang CFT	$d_{\mathrm{c}} = 6$,
$p = 6$	tricritical Ising CFT	$d_{\mathrm{c}} = 3$.

Note that for odd p, the theories we consider are non-unitary.

In this lecture we will further set $d = d_{\mathrm{c}} - \epsilon$, i.e. to work at the Wilson–Fisher fixed-point of the respective theories. We will not make direct use of the Lagrangian description above, but instead follow a bootstrap approach and use methods inherent to conformal field theories. Thus the conformal data will constitute the fundamental observables. In particular, we will have scaling dimensions of local conformal primary operators, from which critical exponents can be computed:

$$\Delta_\phi = \frac{d-2}{2} + \gamma_\phi \qquad \longrightarrow \qquad \eta = 2\gamma_\phi, \tag{1.4}$$

$$\Delta_{\phi^2} = 2\Delta_\phi + g \qquad \longrightarrow \qquad \nu = (d - \Delta_{\phi^2})^{-1}. \tag{1.5}$$

These equations define for us the parameters γ_ϕ and g, which will depend perturbatively on ϵ and whose leading value we aim to find.

In general we will define the anomalous dimensions of composite operators with respect to Δ_ϕ rather than the value in the free theory. For instance we will consider operators of the schematic form $\phi\partial^\ell\phi$ with dimensions

$$\Delta_\ell = 2\Delta_\phi + \ell + \gamma_\ell. \tag{1.6}$$

These operators will be central to the discussion, and are called *double-twist operators.*

The plan for the lecture is the following:

1. **Preliminaries.** We will go through some preliminary definitions, familiar from the study of conformal field theories in $d > 2$ spacetime dimensions.
2. **Computation of γ_ℓ to leading order in $\lambda\phi^4$ theory.** We will then use consistency conditions to fix γ_ϕ and g, and thus the critical exponents.
3. **$\lambda\phi^p$ theory $p \neq 4$.** This section will be a bit schematic.

Note that the philosophy of the "bootstrap" using the Lorentzian inversion formula is somewhat different from the numerical bootstrap. The goal is not to construct a crossing symmetric correlator, but instead to use crossing and the inversion formula to extract information about the CFT-data, and thus the critical exponents.

[1] Of course, for p even we can impose a $\mathbb{Z}_2$ symmetry which automatically puts to zero the couplings g_i for odd i.

2 Lorentzian CFT in $d > 2$

This section reviews the necessary conventions for CFT in Lorentzian kinematics in spacetime dimensions $d > 2$.[2] Lorentzian kinematics admits an important limit, namely the *lightcone limit*, defined by $x^2 \to 0$ while keeping at least some of the components x^μ finite. In this limit, the OPE takes the form

$$\phi(x) \times \phi(0) = \sum_{\mathcal{O}} c_{\phi\phi\mathcal{O}} |x|^{-2\Delta_\phi + \tau} x^{\mu_1} \cdots x^{\mu_\ell} \mathcal{O}_{\mu_1 \cdots \mu_\ell} \tag{2.1}$$

where we have introduced the *twist* $\tau = \Delta - \ell$ of the operator $\mathcal{O}$ with dimension Δ and spin ℓ.

For the four-point function, we shall consider the standard space-like configuration in Lorentzian CFT. We use conformal invariance to fix the position of three operators: x_1 at the origin, x_3 at a unit spacelike distance, and x_4 to infinity. Then we let x_2 move around in a diamond determined by the lightcones of x_1 and x_3. The configuration is given by figure 1. The conformal cross-ratios z and $\bar{z}$ correspond to the distances along the diagonals as indicated.

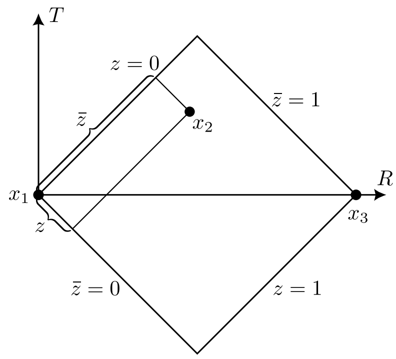


Figure 1: Kinematic setup. The point x_4 is sent to infinity along the direction R.

We will study the four-point function in the *double lightcone limit* where $x_{12}^2 \to 0$, $x_{23}^2 \to 0$, so that x_2 goes to the top of the diamond. This limit does not exist in Euclidean signature. On the level of cross-ratios, it will be necessary to break the symmetry $z \leftrightarrow 1 - \bar{z}$ slightly, and more precisely we define the double lightcone limit as

$$z \ll 1 - \bar{z} \ll 1. \tag{2.2}$$

Let us look at the conformal block decomposition in the limit $z \to 0$, keeping full $\bar{z}$ dependence. We define the correlator and its conformal block decomposition by

$$\mathcal{G}(z, \bar{z}) = (x_{12}^2 x_{34}^2)^{\Delta_\phi} \langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \rangle \tag{2.3}$$

$$= \sum_{\mathcal{O}} a_{\mathcal{O}} G_{\Delta,\ell}(z, \bar{z}), \tag{2.4}$$

[2] By limiting to $d > 2$, we will have, for unitary theories, a twist gap between the identity operator and the other operators. This will be discussed later.

where $a_{\mathcal{O}} = c^2_{\phi\phi\mathcal{O}}$ are the squared OPE coefficients, and the $G_{\Delta,\ell}(z,\bar{z})$ are the conformal blocks. In the limit $z \to 0$ the conformal blocks simplify to the form

$$G_{\Delta,\ell}(z,\bar{z}) = z^{\tau/2} k_{\frac{\Delta+\ell}{2}}(\bar{z}) + O(z^{\tau/2+1}) \tag{2.5}$$

where

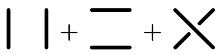
$$k_{\bar{h}}(\bar{z}) = \bar{z}^{\bar{h}} {}_2F_1(\bar{h}, \bar{h}; 2\bar{h}; \bar{z}). \tag{2.6}$$

Note that this form is independent of d; however the corrections at subleading powers in z do depend on d.

As an example, consider the conformal block decomposition for a specific theory, namely the *generalised free field* ϕ (GFF). This is the theory of a non-interacting scalar field ϕ, with arbitrary scaling dimension Δ_ϕ. It is an important theory on its own, and will also be the starting point when we consider the interacting $\lambda\phi^p$ theories.[3]

In the GFF theory, all correlators are computed by Wick contractions. For the four-point function we get

$$\mathcal{G}^{\mathrm{GFF}}(z,\bar{z}) = \tag{2.7}$$



$$= 1 + \left(\frac{z\bar{z}}{(1-z)(1-\bar{z})}\right)^{\Delta_\phi} + (z\bar{z})^{\Delta_\phi} \tag{2.8}$$

which has a conformal block decomposition in the $z \to 0$ limit as

$$\mathcal{G}^{\mathrm{GFF}}(z,\bar{z}) = 1 + \sum_{\ell=0,2,4,\ldots} a_\ell \, G_{2\Delta_\phi+\ell,\ell}(z,\bar{z}) + O(z^{\Delta_\phi+1}). \tag{2.9}$$

This can be explicitly checked order by order at small $\bar{z}$ using the conformal blocks (2.5). The squared OPE coefficients a_ℓ are known on closed form for arbitrary Δ_ϕ,[4] however we will need only the value for $\Delta_\phi = 1$, corresponding to the free scalar in $d = 4$ dimensions (the starting point of the expansion for $\lambda\phi^4$ theory). In this case

$$a_\ell = \frac{2\Gamma(\ell+1)^2}{\Gamma(2\ell+1)}, \qquad (\Delta_\phi = 1). \tag{2.10}$$

The decomposition (2.9) above tells us something about the operator content in the GFF theory. Specifically, it shows that the OPE must be of the form

$$\phi \times \phi = \mathbb{1} + \sum_{\ell=0,2,4,\ldots} \phi\partial^\ell\phi + \ldots, \tag{2.11}$$

where we have omitted operators hidden in the higher powers of z. The symbol $\phi\partial^\ell\phi$ is schematic, and by it we mean the operators

$$\phi\partial^\ell\phi \sim \phi\partial^{\{\mu_1}\cdots\partial^{\mu_\ell\}}\phi + \text{total derivatives}, \tag{2.12}$$

which are the previously mentioned *double-twist operators*. They have dimensions

$$\Delta_\ell = 2\Delta_\phi + \ell. \tag{2.13}$$

[3] More precisely, since we are working in the ϵ expansion, we will consider a value of Δ_ϕ close to the free theory, so that $\gamma_\phi \ll 1$ in equation (1.4).

[4] For generic values of Δ_ϕ one has: $a_\ell^{\mathrm{GFF}} = \frac{2(\Delta_\phi)_\ell^2}{\Gamma(\ell+1)(2\Delta_\phi+\ell-1)_\ell}$, where $(a)_k$ denotes the Pochhammer symbol. This is the $n = 0$ case of a general formula given in [4].

In fact, the double-twist operators will also exist in the interacting theory, where they may aquire anomalous dimensions γ_ℓ. This means that in the following we will consider

$$\Delta_\ell = 2\Delta_\phi + \ell \underbrace{+\gamma_\ell}_{=0 \text{ in GFF}} . \tag{2.14}$$

The computation presented so far works equally well in Euclidean signature (indeed, the conformal block decomposition uses the limit $\bar{z} \to 0$ instead of the double lightcone limit $\bar{z} \to 1$). However, during the last 10–15 years it was realised that one can get more leverage by considering the correlator in Lorentzian kinematics [5–7]. In particular, it was observed that for large spin, a_ℓ and γ_ℓ can be determined by considering the $\bar{z} \to 1$ limit.

Moreover, the OPE coefficients a_ℓ, as well as the anomalous dimensions γ_ℓ, were in various examples of interacting theories found to be given as closed-form functions in spin ℓ, or "analytic in spin". Different methods were developed to extract these functions, however these methods were in principle only guaranteed to apply for asymptotically large spin. The question of extension to finite spin was put on a firm basis by the **Lorentzian inversion formula** published by Caron-Huot in 2017 [1].

The Lorentzian inversion formula takes the schematic form

$$C(\Delta, \ell) = \int d^2 z K_{\Delta,\ell,d}(z, \bar{z})\, \mathrm{dDisc}[\mathcal{G}(z, \bar{z})] \tag{2.15}$$

where $K_{\Delta,\ell,d}$ is a complicated kernel. The function $C(\Delta, \ell)$ knows about a_ℓ and γ_ℓ through its poles $C(\Delta, \ell) \sim \frac{a_\ell}{\Delta - \Delta_\ell}$. The inversion formula holds for spin $\ell > 0$.[5] The integrand depends on the double-discontinuity in the limit $\bar{z} \to 1$, which is defined by

$$\mathrm{dDisc}[f(\bar{z})] = f(\bar{z}) - \frac{1}{2}\left(f^{\circlearrowleft}(\bar{z}) + f^{\circlearrowright}(\bar{z})\right). \tag{2.16}$$

Here the arrows denote the analytic continuations around the branch cut which goes along the positive real $\bar{z}$ axis starting at $\bar{z} = 1$.

The Lorentzian inversion formula is very versatile, but somewhat complicated to use in the full form. It involves a two-dimensional integral and against a kernel $K_{\Delta,\ell,d}$ which takes a complicated form. Luckily, for the computation we have in mind here, there is a simplified version involving only a one-dimensional integral with a much simpler kernel. It holds under the following assumptions:

1. Perturbation theory; in our case we have $\epsilon \ll 1$.
2. Extracting data of the operators with lowest twist; for us these are the double-twist operators, with $\Delta_\ell = 2\Delta_\phi + \ell + \gamma_\ell$.

In fact these assumptions constitute an important special case that applies to various considerations, and it was denoted the **perturbative inversion formula** in [3]. It follows from the general formula under these assumptions, although we shall not give the proof here.[6]

The perturbative inversion formula is a two-step algorithm:

[5] We will only consider a correlator of identical scalar operators, which implies that spin is even. Allowing for odd spin the proper statement is that inversion formula holds for $\ell > 1$.

[6] A proof can be found in [3], section 2.5.2.

1. **Step 1.** Compute a generating function in $\ln z$ given by

$$T^{(0)}_{\bar h} + \frac{1}{2} T^{(1)}_{\bar h} \ln z + \ldots = \kappa \int_0^1 \frac{d\bar z}{\bar z^2} k_{\bar h}(\bar z) \mathrm{dDisc}[\mathcal{G}(z,\bar z)]\Big|_{z^{\Delta_\phi}}, \tag{2.17}$$

   where $\kappa = \frac{\Gamma(\bar h)^4}{\pi^2 \Gamma(2\bar h)\Gamma(2\bar h - 1)}$. This formula amounts to the following. Consider the correlator $\mathcal{G}(z,\bar z)$. Find its double-discontinuity by the definition (2.16) above. Then project onto the power z^{Δ_ϕ}, which in perturbation theory will produce powers $\ln z$, where at each order in perturbation theory only one new power of $\ln z$ will appear. Finally compute the one-dimensional integral, and store the result as prescribed by the left-hand side.

2. **Step 2.** Extract the OPE coefficients a_ℓ and anomalous dimensions γ_ℓ by the formula

$$a_\ell (\gamma_\ell)^k = T^{(k)}_{\bar h} + \frac{1}{2} \partial_{\bar h} T^{(k+1)}_{\bar h} + \ldots \Big|_{\bar h = \Delta_\phi + \ell}. \tag{2.18}$$

   For instance, if we are interested in a_ℓ, we use $k = 0$. At each order in perturbation theory, the sum in the right-hand side will truncate.

To see how this works in practice, we shall consider again the generalised free field theory (GFF), with $\Delta_\phi = 1$. We shall keep Δ_ϕ generic and use it as a regulator, taking the limit $\Delta_\phi \to 1$ at the end.

Step 1 means computing $\mathrm{dDisc}[\mathcal{G}(z,\bar z)]$ and projecting onto z^{Δ_ϕ}. We get

$$\mathrm{dDisc}\left[1 + \left(\frac{z\bar z}{(1-z)(1-\bar z)}\right)^{\Delta_\phi} + (z\bar z)^{\Delta_\phi}\right]\Big|_{z^{\Delta_\phi}} = \mathrm{dDisc}\left[\left(\frac{\bar z}{1-\bar z}\right)^{\Delta_\phi}\right] \tag{2.19}$$

$$= 2\sin^2(\pi\Delta_\phi)\left(\frac{\bar z}{1-\bar z}\right)^{\Delta_\phi}. \tag{2.20}$$

Only the middle term survived, and we see that no powers of $\ln z$ emerge, meaning $T^{(k)}_{\bar h} = 0$ for $k \geqslant 1$. The double-discontinuity was computed by moving the $1 - \bar z$ into the exponent, and replacing $\ln(1-\bar z)$ with $\ln(1-\bar z) \pm 2\pi i$. With the double-discontinuity of the correlator at hand we proceed to compute

$$T^{(0)}_{\bar h} = \lim_{\Delta_\phi \to 1} 2\sin^2(\pi\Delta_\phi)\, \kappa \int_0^1 \frac{d\bar z}{\bar z^2} k_{\bar h}(\bar z) \left(\frac{\bar z}{1-\bar z}\right)^{\Delta_\phi} \tag{2.21}$$

$$= \frac{2\Gamma(\bar h)^2}{\Gamma(2\bar h - 1)}, \tag{2.22}$$

finishing step 1 of the algorithm. For step 2 we simply use

$$a_\ell = T^{(0)}_{\bar h}\Big|_{\bar h = \ell + 1} = \frac{2\Gamma(\ell+1)^2}{\Gamma(2\ell+1)}, \tag{2.23}$$

which, to our delight, agrees with (2.10) above. Note that this time, the result followed from only the middle term in the GFF correlator (2.8), contrary to the conformal block decomposition which needed both the final two terms.

3 Bootstrap of $\lambda\phi^4$ theory

The listener may now ask, *what makes this bootstrap?* The answer is that we need to use the crossing equation. Indeed, we shall use the crossing equation to evaluate the double-discontinuity. We get

$$\mathrm{dDisc}[\mathcal{G}(z,\bar{z})] = \mathrm{dDisc}\left[\left(\frac{z\bar{z}}{(1-z)(1-\bar{z})}\right)^{\Delta_\phi}\mathcal{G}(1-\bar{z},1-z)\right] \tag{3.1}$$

$$= \mathrm{dDisc}\left[\left(\frac{z\bar{z}}{(1-z)(1-\bar{z})}\right)^{\Delta_\phi}\sum_{\mathcal{O}'} a_{\mathcal{O}'}\underbrace{G_{\Delta',\ell'}(1-\bar{z},1-z)}_{\sim(1-\bar{z})^{\frac{\Delta'-\ell'}{2}}}\right]. \tag{3.2}$$

Note here the factor from crossing, and the $\bar{z}\to 1$ scaling of the crossed-channel block.

We may consider different choices of $\mathcal{O}'$:

- $\boldsymbol{\mathcal{O}' = \mathbb{1}}$. We get $\left(\frac{\bar{z}}{1-\bar{z}}\right)^{\Delta_\phi} \longrightarrow a_\ell^{\mathrm{GFF}}$, i.e. the identity operator in the crossed-channel gives rise to the double-twist operators.

  In fact the correlator of identical operators in any CFT contains the identity operators, so this computation show that any CFT containing a scalar operator ϕ also contains the double-twist operators $\phi\partial^\ell\phi$ with non-zero OPE coefficients.[7] In the interacting theory they do not have dimensions exactly equal to $2\Delta_\phi+\ell$, but may develop anomalous dimensions.

- $\boldsymbol{\mathcal{O}' = \phi\partial^\ell\phi}$.

  We may consider the contribution from the double-twist operators themselves. Recall that they have dimensions $\Delta' = 2\Delta_\phi + \ell' + \gamma_{\ell'}$. If we plug this definition into (3.2) and look at the singular behaviour as $\bar{z}\to 1$, we get

$$\left(\frac{1}{1-\bar{z}}\right)^{\Delta_\phi}(1-\bar{z})^{\Delta_\phi+\frac{\gamma_{\ell'}}{2}} \sim \underbrace{1+\frac{\gamma_{\ell'}}{2}\ln(1-\bar{z})}_{\mathrm{dDisc}=0}+\frac{\gamma_{\ell'}^2}{8}\underbrace{\ln^2(1-\bar{z})}_{\mathrm{dDisc}=4\pi^2}+\dots \tag{3.3}$$

  where we have indicated the double-discontinuity of each term.

Here we note a very important property of the Lorentzian inversion formula: *The contribution from $\phi\partial^\ell\phi$ is delayed.* Specifically the contribution from double-twist operators is suppressed with a factor which is proportional to their anomalous dimension squared. In perturbation theory it means that their contribution is delayed by two orders in γ_ℓ.

We can now formulate the **strategy for $\boldsymbol{\phi^4}$ theory: Invert $\boldsymbol{\mathcal{O}' = \mathbb{1}}$ and $\boldsymbol{\mathcal{O}' = \phi^2}$**. We will only work to leading non-trivial order in perturbation, where it can be shown that it is enough to consider only these two operators. We will introduce g and γ_ϕ as free parameters (recall that $\Delta_\phi = \frac{d-2}{2}+\gamma_\phi$, and $\Delta_{\phi^2} = 2\Delta_\phi + g$) and try to fix them at a later stage.

Let us execute the strategy.

First, for $\mathcal{O}'=\mathbb{1}$ we get the same as before: $\left(\frac{\bar{z}}{1-\bar{z}}\right)^{\Delta_\phi} \to T_{\bar{h}}^{(0)} \to a_\ell$. (Note that we have evaluated this to zeroth order in ϵ.)

[7] This argument relies on there being a twist gap between the identity and the first non-trivial operator, otherwise the contribution from $\mathbb{1}$ could be cancelled by other operators $\mathcal{O}'$ in (3.2).

Then for $\mathcal{O}' = \phi^2$, we use what we found in (3.3) above, with $\ell' = 0$:

$$\mathrm{dDisc}\left[\left(\frac{z\bar{z}}{1-z}\right)^{\Delta_\phi} a\frac{g^2}{8}\ln^2(1-\bar{z})\frac{\ln\bar{z}-\ln z}{\bar{z}}\right], \tag{3.4}$$

where $a = c^2_{\phi\phi\phi^2} = 2$. The last part of this expression comes from the remaining part of the conformal block which we did not include in (3.3).

The expression (3.4) will contribute to $T^{(1)}$ by

$$T^{(1)}_{\bar{h}} \sim 2 \times \text{coef. of } z^{\Delta_\phi}\ln z, \tag{3.5}$$

so we get

$$T^{(1)}_{\bar{h}} = 2\kappa\int_0^1 \frac{d\bar{z}}{\bar{z}^2}k_{\bar{h}}(\bar{z})\frac{ag^2}{8}4\pi^2(-1) \tag{3.6}$$

$$= \frac{2\Gamma(\bar{h})^2}{\Gamma(2\bar{h}-1)}\left(-\frac{ag^2}{2J^2}\right). \tag{3.7}$$

Here J^2 is the conformal spin defined by $J^2 = \frac{\Delta+\ell}{2}\frac{\Delta+\ell-2}{2}$, which the expert will recognise this as the eigenvalue of the Casimir operator on the lightcone.

The anomalous dimensions are extracted using (2.18),

$$\gamma_\ell = \frac{T^{(1)}}{T^{(0)}} = -\frac{ag^2}{2J^2}, \tag{3.8}$$

which to leading order, with $a = 2$, gives

$$\Delta_\ell = 2\Delta_\phi + \ell - \frac{g^2}{\ell(\ell+1)}. \tag{3.9}$$

This is a nice result, but this still depends on our two free parameters γ_ϕ (through Δ_ϕ) and g. However, remarkably, we can now fix them using consistency conditions.

- $\boldsymbol{\ell = 2}$. The operator at spin two is in fact the stress-tensor, which gives the constraint $\Delta_2 \overset{!}{=} d = 4-\varepsilon$. Solving for γ_ϕ we get

$$2\gamma_\phi - \frac{g^2}{6} = 0 \;\Rightarrow\; \gamma_\phi = \frac{g^2}{12}. \tag{3.10}$$

- $\boldsymbol{\ell = 0}$. There is a second constraint, less straight-forward, coming from $\ell = 0$. In the work with [2], this constraint came as a surprise to us, since the inversion formula is not guaranteed to work for $\ell = 0$. Nevertheless, we can assume that it does and thus demand that

$$\Delta_0 \overset{!}{=} \Delta_{\phi^2}. \tag{3.11}$$

  Recall that from our definitions $\Delta_{\phi^2} = 2\Delta_\phi + g = 2-\varepsilon+g$ to leading order. Comparing with (3.9) it looks like we cannot to put $\ell = 0$. However, we can resolve this problem by retrospectively upgrading the factor $\ell(\ell+1)$ to the conformal spin J^2. Inserting our definitions we get

$$2\Delta_\phi - \frac{g^2}{\frac{2-\varepsilon+g}{2}\frac{-\varepsilon+g}{2}} = 2\Delta_\phi + g + O(\varepsilon^2) \tag{3.12}$$

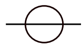
$$-g^2 = g\frac{-\varepsilon+g}{2} \tag{3.13}$$

$$g(3g-\varepsilon) = 0 \tag{3.14}$$

The final equation has two solutions: 1) free theory, and 2) $g = \epsilon/3$. The non-trivial solution implies

$$\Delta_{\phi^2} = 2\Delta_\phi + \frac{\varepsilon}{3} + \dots, \tag{3.15}$$

$$\Delta_\phi = \frac{d-2}{2} + \frac{\varepsilon^2}{108} + \dots, \tag{3.16}$$

which agree exactly the values in the Wilson–Fisher fixed-point [8]. For comparison, the leading anomalous dimension of ϕ can be computed diagrammatically by evaluating the sunset diagram: 

Let us get back to the spinning operators – why did we not need to take them as input? We have just found that the spinning operators have anomalous dimensions of order ϵ^2 by equation (3.3). This means that they will not contribute to the double-discontinuity until order ϵ^4. In fact, there are also some more exotic operators that exist in the theory at higher twist, but also these operators will not appear until order ϵ^4.

4 $\lambda\phi^p$ theory in $d_c - \epsilon$ dimensions

We will now outline the analogous computation for $\lambda\phi^p$ theories with $p \neq 4$. The story will be quite similar: we will need to consider operators appearing in the OPE

$$\phi \times \phi = \mathbb{1} + \sum_{\ell=0,2,4,\dots} \phi\partial^\ell\phi + \phi^{p-2} + \dots. \tag{4.1}$$

We can understand why the operator ϕ^{p-2} appears in the OPE by considering the diagram in figure 2.

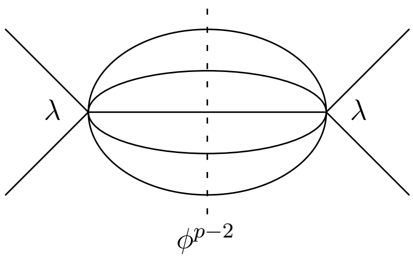


Figure 2: A diagram contributing to the ϕ four-point function in $\lambda\phi^p$ theory.

The strategy will then the following: **Invert $\mathcal{O}' = \mathbb{1}$ and $\mathcal{O}' = \phi^{p-2}$.** Again we note that the contribution from double-twist operators is delayed. The free parameters will be γ_ϕ and $c^2_{\phi\phi\phi^{p-2}} =: \lambda^2\alpha$.

Skipping the details and proceeding to the results, we get [3, section 6.3]

$$\Delta_\ell = 2\Delta_\phi + \ell - \frac{(d_c-4)^2\lambda^2\alpha}{2J^2}, \tag{4.2}$$

where, to leading order, $J^2 = \left(\frac{d-2}{2} + \ell\right)\left(\frac{d-4}{2} + \ell\right)$.

Again we get a consistency condition at spin two:

- $\boldsymbol{\ell = 2}$: Assuming the stress-tensor, we put $\Delta_2 \overset{!}{=} d = d_\mathrm{c} - \epsilon$ to get

$$\lambda^2 \alpha = \frac{d_\mathrm{c}(d_\mathrm{c}+2)}{(d_\mathrm{c}-4)^2} \gamma_\phi \tag{4.3}$$

  With $d_\mathrm{c} = 6$ this reduces to $\lambda^2 a = 12\gamma_\phi$.

In general, unfortunately, there will still be one undetermined free parameter, γ_ϕ, after using this equation. However, we can evaluate the resulting function at spin zero to get

$$\Delta_{\phi^2} = 2\Delta_\phi + \frac{2d_\mathrm{c}(d_\mathrm{c}+2)}{(d_\mathrm{c}-2)(4-d_\mathrm{c})} \gamma_\phi, \tag{4.4}$$

which does agree with the a general result for multicritical theories [9]. We have then parametrised the anomalous dimensions of all bilinear operators in terms of one parameter, γ_ϕ.

In the most interesting case of $\lambda\phi^3$ theory, with $d_\mathrm{c} = 6$, we get

$$\Delta_\phi = 2 - \frac{\varepsilon}{2} + \gamma_\phi, \tag{4.5}$$

$$\Delta_0 = 4 - \varepsilon + 2\gamma_\phi - 12\gamma_\phi. \tag{4.6}$$

In this case ($p = 3$) it was observed by Gonçalves [10] that there is an additional relation, a shadow relation

$$\Delta_\phi + \Delta_0 \overset{!}{=} d \tag{4.7}$$

$$6 - \frac{3\varepsilon}{2} - 9\gamma_\phi = 6 - \varepsilon \tag{4.8}$$

$$\gamma_\phi = -\frac{\varepsilon}{18}. \tag{4.9}$$

This agrees precisely with the known anomalous dimension in the Lee–Yang CFT [11].

5 Summary and outlook

The general philosophy of our application of the Lorentzian inversion formula can be summarised as

$$a_\ell, \gamma_\ell \longleftarrow \mathrm{dDisc}\left[\left(\frac{z\bar{z}}{(1-z)(1-\bar{z})} \right)^{\Delta_\phi} \sum_{\mathcal{O}'} a_{\mathcal{O}'} G_{\Delta_{\mathcal{O}'}, \ell_{\mathcal{O}'}}(1-\bar{z}, 1-z) \right] \tag{5.1}$$

$$\sim \sum_{\mathcal{O}'} a_{\mathcal{O}'} \sin^2 \frac{\pi}{2}(\tau' - 2\Delta_\phi). \tag{5.2}$$

We will now comment on various further aspects.

5.1 Perturbative inversion formula

- The computation for $\lambda\phi^4$ theory can be extended to order ϵ^4, see [2], and to theories with global symmetry [12].

- We may also use the Lorentzian inversion formula in the $1/N$ expansion [13], see for instance the new $1/m$ expansion for theories with $MN = \mathrm{O}(m)^n \rtimes \mathrm{S}_n$ symmetry [14], mentioned in the talk by A. Stergiou.
- Note that the inversion formula can also be used to compute corrections to the OPE coefficients, and from them the corrections to the central charges. See exercise 4 below.

5.2 Numerical application

There is another approach that was mentioned in the lecture by D. Poland and the discussion session by S. Rychkov, namely the numerical application of the inversion formula. This can be done at finite ϵ, and has been evaluated for 3d Ising [15] and O(2) [16] CFTs.[8] One needs to consider the inversion of two types of contributions.

- **Individual operators.** $\mathcal{O}' = \mathbb{1}, \phi^2, t_{ij}, T^{\mu\nu}, J^\mu$, etc. Operators at low spin can be inverted one-by-one, and it is possible to compute the result of numerical inversion integral in a complicated but precise form.[9] The problem of considering the inversion of individual operators is referred to in the literature as crossing kernels, see e.g. [19, 20].
- **$\mathcal{O}' = \phi\partial^\ell\phi$ and other twist families**. Families of operators with spin require more work. To deal with them one first has to compute the sum over crossed-channel operators, before extracting the double-discontinuity and computing the inversion.

In general, the numerical application of the inversion integral requires as input quite detailed knowledge of the spectrum. It would be interesting to see if this can be performed for the Lee–Yang CFT.

5.3 Theoretical considerations

There are some theoretical issues with the computations outlined in this lecture.

- Why was it possible we extend to $\ell = 0$ when the inversion formula is only guaranteed to work for $\ell > 0$? This is not completely understood. Perhaps we can attempt to answer the question by considering what the alternative would look like? Solutions to crossing truncated in spin were found in the important paper [21], and generalised to arbitrary d in [22]. Using the result from [22] we note that if the external operator is close to the unitarity bound, a solution with finite support in spin is only allowed for $d = 4$. See also the discussion in [23].
- The extension to non-unitary theories. The original derivation of the Lorentzian inversion formula in [1] is limited to unitary conformal field theories. However, the important properties required are not unitarity *per se*, but properties that

[8] See also [17] for a similar computation for the Ising CFT predating the inversion formula, and [18] for a comparison and discussion.

[9] The result is of the form $\sum_{p=0}^{\infty}\sum_{q=-p}^{p} c(z)\mathcal{A}_{p,q}\Omega$, where all three factors depend on the involved conformal data. The $\mathcal{A}_{p,q}$ describe the coefficients of the subcollinear conformal blocks, which are not known on closed form but can be computed systematically. Ω is given as a sum of two terms, which each is a ratio of Gamma functions times a hypergeometric ${}_4F_3$ at unit argument. See e.g. [16] for details.

follow from unitarity, specifically 1) boundedness in the Regge limit, 2) twist gap. The twist gap is not necessary for all applications of the inversion formula, but necessary to prove the existence of double-twist operators. The first condition may hold in a wider range of theories, which may include some non-unitary theories. Indeed, at least for small ϵ the the $\lambda\phi^p$ theories considered here do seem to satisfy these conditions also for odd p. In addition, it has been noted that all $\lambda\phi^p$ theories, including the Ising CFT on fact are non-unitarity away from integer dimension [24], however this does not affect any of our conclusions.

- For studying the non-unitary theories, it would be interesting if one can use a combination of Gliozzi's method [25] and the Lorentzian inversion formula to track the $\lambda\phi^p$ theories from their ε expansion into the finite ε region.

5.4 Some general remarks on $\lambda\phi^p$ theories

- The $\lambda\phi^p$ theories are quite well-studied for even $p = 2\theta$, which is the unitary case. Some early references are [26, 27], and some results for the O(N) symmetric case may be found in [28]. One observable that is known for general $p = 2\theta$ is the leading anomalou dimension of ϕ:

$$\gamma_\phi = \frac{2(\theta-1)^2\Gamma(\theta+1)^6}{\Gamma(2\theta+1)^3}\epsilon^2 = \frac{\epsilon^2}{108}, \frac{\epsilon^2}{1000}, \frac{9\epsilon^2}{171500}, \ldots, \quad \theta = 2, 3, 4, \ldots. \tag{5.3}$$

  The anomalous dimensions (4.2) of the spinning operators in the multicritical theories, which we now have rederived, were computed in [29] using multiplet recombination methods.

- The theories for even $p = 2\theta$ connect in $d = 2$ to the unitary minimal models $\mathcal{M}_{\theta+2,\theta+1}$, with central charges $c = 1 - \frac{6}{(\theta+1)(\theta+2)}$, see for instance [30], section 7.4.7.

- The case $p = 3$, the Lee–Yang CFT, is well-studied [11, 31]. Some numerical results in intermediate dimensions $2 < d < 6$ were computed in [32]. In $d = 2$ the Lee–Yang CFT is the minimal model $\mathcal{M}_{2,5}$ with central charge $c = -\frac{22}{5}$.[10]

- The theories for higher odd $p = 2t + 1$ are less studied. The case $p = 5$ is interesting, see [9]. Moreover, see [35] for a ϕ^5 theory with global symmetry that may be unitary, denoted `Pentagon`. The anomalous dimension γ_ϕ for odd $p = 2t+1$ is not known on an explicit closed form, but can be computed case by case [9]:

$$\gamma_\phi = -\frac{\epsilon}{18}, -\frac{\epsilon}{1530}, \quad t = 1, 2. \tag{5.4}$$

[10] An important observable in this model is the edge exponent σ, defined by $\sigma = \frac{\Delta_\phi}{d-\Delta_\phi}$ and known to order ϵ^5 [33]. Another observable is $a = C^2_{\phi\phi\phi}$. The exact values for these observables in the 2d minimal model $\mathcal{M}_{2,5}$ are $\Delta_\phi = -\frac{2}{5}$, $\sigma = -\frac{1}{6}$ and $a = -\frac{\Gamma(\frac{6}{5})^2\Gamma(\frac{1}{5})\Gamma(\frac{2}{5})}{\Gamma(\frac{4}{5})^3\Gamma(\frac{3}{5})} = -3.65312\ldots$ [31, 34].

6 Exercises

Exercise 1. Confirm the double-discontinuities used in the lecture,

a) $\mathrm{dDisc}\left[\ln^2(1-\bar{z})\right] = 4\pi^2$,

b) $\mathrm{dDisc}\left[\left(\frac{\bar{z}}{1-\bar{z}}\right)^\alpha\right] = 2\sin^2(\pi\alpha)\left(\frac{\bar{z}}{1-\bar{z}}\right)^\alpha$.

Hint: Use that the circles ↺, ↻ amount to replacing $\ln(1-\bar{z})$ by $\ln(1-\bar{z}) \pm 2\pi i$.

Exercise 2. Use Mathematica to confirm the integral

$$\int_0^1 \frac{d\bar{z}}{\bar{z}^2} k_{\bar{h}}(\bar{z}) \left(\frac{\bar{z}}{1-\bar{z}}\right)^\alpha = \frac{\Gamma(2\bar{h})\Gamma(1-\alpha)^2\Gamma(\bar{h}-1+\alpha)}{\Gamma(\bar{h})^2\Gamma(\bar{h}+1-\alpha)}, \quad \alpha < 0, \bar{h} > 1-\alpha. \tag{6.1}$$

Then take $\alpha = \Delta_\phi$ and perform the limit $\Delta_\phi \to 1$ to confirm equation (2.21) used in the main text. Note that some analytic continuations are needed. *Hint:* you may need the identity ${}_2F_1(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-b-a)}{\Gamma(c-b)\Gamma(c-a)}$ valid when $a+b<c$.

Exercise 3. Assume that $g = O(\epsilon)$ and work to order ϵ^2. Then show the form (3.4), i.e. that of the double-discontinuity from the ϕ^2 operator takes the form

$$\left(\frac{z\bar{z}}{(1-z)(1-\bar{z})}\right)^{\Delta_\phi} G_{2\Delta_\phi+g,0}(1-\bar{z},1-z)\Big|_{z^{\Delta_\phi}}$$
$$= \bar{z}^{\Delta_\phi}\frac{g^2}{8}\ln^2(1-\bar{z})\frac{\ln\bar{z}-\ln z}{\bar{z}} + O(\epsilon^3) + \text{terms with dDisc=0}. \tag{6.2}$$

Hint: At order ϵ^2, all ϵ dependence must be contained in g^2, so you may use the $4d$ conformal block $G_{\Delta,\ell}(z,\bar{z}) = \frac{z\bar{z}}{z-\bar{z}}\left(k_{\frac{\Delta+\ell}{2}}(z)k_{\frac{\Delta-\ell-2}{2}}(\bar{z}) - k_{\frac{\Delta+\ell}{2}}(\bar{z})k_{\frac{\Delta-\ell-2}{2}}(z)\right)$.

Exercise 4 (**Hard**)**.** In this exercise you will compute the order $\epsilon^2 \sim g^2$ correction α_ℓ to the OPE coefficients,

$$c^2_{\phi,\phi,\phi\partial^\ell\phi} = a_\ell^{\mathrm{GFF}} + \alpha_\ell + O(\epsilon^3), \tag{6.3}$$

where a_ℓ^{GFF} is given in footnote 4 above with $\Delta_\phi = 1 - \frac{\epsilon}{2} + \gamma_\phi$. The result will then be used to find the leading correction to the central charge C_T.

To find the OPE coefficients, compute first both $T^{(0)}_{\bar{h}}$ and $T^{(1)}_{\bar{h}}$ to order g^2. Then use (2.18) to find the OPE coefficients:

$$c^2_{\phi,\phi,\phi\partial^\ell\phi} = T^{(0)}_{\bar{h}} + \frac{1}{2}\partial_{\bar{h}}T^{(1)}_{\bar{h}}\Big|_{\bar{h}=\Delta_\phi+\ell}. \tag{6.4}$$

Hint: For the order g^2 correction to $T^{(0)}_{\bar{h}}$ you need the integral (see exercise 5)

$$\int_0^1 \frac{d\bar{z}}{\bar{z}^2} k_{\bar{h}}(\bar{z}) \ln\bar{z} = -\frac{\Gamma(2\bar{h})}{\Gamma(\bar{h})^2}\frac{1}{\bar{h}^2(\bar{h}-1)^2}. \tag{6.5}$$

Answer: $\alpha_\ell = \frac{\Gamma(\ell+1)^2}{\Gamma(2\ell+1)}\frac{ag^2}{\ell(\ell+1)}\left(\frac{1}{\ell+1} + S_1(2\ell) - S_1(\ell)\right)$, where S_1 denotes the harmonic numbers.[11]

[11] This result only holds working at order ϵ^2.

Next, use the conformal Ward identity $c^2_{\phi\phi T^{\mu\nu}} = \frac{d^2\Delta_\phi^2}{4(d-1)^2 C_T}$, and the values $a = 2$, $g = \frac{\epsilon}{3}$ and $\gamma_\phi = \frac{\epsilon^2}{108}$, to find the correction to the central charge. *Answer:* $\frac{C_T}{C_{T,\text{free}}} = 1 - \frac{5}{324}\epsilon^2 + O(\epsilon^3)$.[12]

Exercise 5. Compute the integrals (6.1) and (6.5) in Mathematica using the integral representation $k_{\bar h}(\bar z) = \frac{\Gamma(2\bar h)}{\Gamma(\bar h)^2}\bar z^{\bar h}\int_0^1 \frac{\mathrm{d}t}{t(1-t)}\left(\frac{t(1-t)}{1-t\bar z}\right)^{\bar h}$.

Exercise 6. This exercise will show the following important result from the papers [6,7]:

Proposition. In any unitary CFT in $d > 2$ dimensions, the anomalous dimensions γ_ℓ of the double-twist operators $\phi\partial^\ell\phi$ have a leading scaling for large ℓ that takes the form

$$\gamma_\ell \sim -\frac{a_{\min} C_0}{J^{\tau_{\min}}} \tag{6.6}$$

for some positive constant C_0, where $a_{\min} = c^2_{\phi\phi\mathcal{O}_{\min}}$ and $\tau_{\min}$ are the (squared) OPE coefficient and twist of the operator $\mathcal{O}_{\min} \neq \mathbb{1}$ with smallest twist in the $\phi \times \phi$ OPE.[13]

a) Deduce first the scaling $J^{-\tau_{\min}}$. *Hint:* The scaling at large $J^2 = \bar h(\bar h - 1)$ is determined by the negative exponent α of powers $(1-\bar z)^{-\alpha}$, or equivalently (why?) by the negative exponent α of powers $(\frac{1-\bar z}{\bar z})^{-\alpha}$. You can therefore use (6.1) to invert such a power.

b) Argue why it is necessary to have twist gap between $\mathbb{1}$ and $\mathcal{O}_{\min}$ for the proposition to be satisfied.

c) Determine the constant C_0. *Answer:* $C_0 = \frac{2\Gamma(\Delta_\phi)^2\Gamma(\tau_{\min}+2\ell_{\min})}{\Gamma(\Delta_\phi - \frac{\tau_{\min}}{2})^2\Gamma(\frac{\tau_{\min}}{2}+\ell_{\min})^2}$. *Hint:* For the purpose of this exercise, it is enough to consider the form (2.5) (with $z \leftrightarrow 1-\bar z$) for the conformal block of $\mathcal{O}_{\min}$.[14]

Acknowledgements

I thank the organisers of "Bootstat 2021" for giving me the opportunity to speak at this great workshop. I also thank Omar Zanusso for useful discussions regarding multicritical theories.

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[12] For the theory of a free scalar field in d dimensions, $C_{T,\text{free}} = \frac{d}{d-1}$. You may check that this value follows from $a = 2$ and $g = \gamma_\phi = 0$.

[13] We assume that there is a unique operator with minimal twist. By the unitarity bounds, is either a scalar operator, or the stress-tensor.

[14] Note that in the limit $z \to 0$, $k_{\bar h'}(1-z) = -\frac{\Gamma(2\bar h')}{\Gamma(\bar h')^2}\ln z + \text{regular}$.

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