RANDOM CRITICAL POINTS
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by

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1. Introduction

The goal of these lectures is to survey various types of *disordered* or *random* critical points as potential targets for applications of the modern conformal bootstrap approach.

I say "potential" since am not sure that all random critical points are conformally-invariant, so they may not be good targets for conformal bootstrap.[(1)] But the bootstrap framework seems to be broader, as evidenced by its successful application to, say random matrices, which has nothing to do with conformal symmetry.

The survey is intended to be broad at the expense of rather shallow coverage. Thus, I will mention a lot of different topics and examples of random critical points, but will not be able to discuss most of them in detail. Examples of random models that I will review will basically fall into four large groups: 1) classical spin models, 2) quantum spin models, 3) geometric critical phenomena, and 4) disordered singe-particle fermionic problems exhibiting localization-delocalization Anderson transitions. These systems seem to be quite different and unrelated, but as I discuss physical properties of random critical points and relevant field-theoretic techniques, I hope to demonstrate that all these problems have a lot in common and are closely related. In particular, I want to explain that all these models share certain features, including multifractality of random observables. Also, I will try to exhibit mappings between different models that provide some common framework and intuition.

These notes are written from the condensed matter physics perspective, which means several things. First of all, any field theory of interest will be Euclidean, and in the conformally-invariant case the space and time will be on the same footing. Second, I will always assume that there is a microscopic lattice scale a that provides regularization of any field theory of interest in the ultraviolet (UV). Therefore, I do not worry about renormalizability or UV completeness, etc. Instead, the interest and focus is in the behavior at long distances and low energies (infrared). Finally, I will have in mind that systems of interest are finite, and their linear size L provides an IR regularization, as well as the basic scaling variable. In other words, I will consider the expectation values of local operators and their products (correlators) in a finite critical system as functions of a/L. The scaling dimension x_O of an operator O characterizes the scaling of the expectation value

$$\langle O \rangle \sim \left(\frac{a}{L}\right)^{x_O}. \tag{1.1}$$

The notation x_O (as opposed to Δ_O) is borrowed from Cardy's fantastic book [**1**], and is necessary to distinguish related but different quantities in the theory of disordered electronic systems traditionally denoted by Δ.

Whenever I use conformal field theory (CFT) results, I will stick to two dimensional CFTs ($d = 2$). I do not know much about CFTs in arbitrary d, so will have questions for the experts.

2. Disorder: general remarks

When we talk about disorder in the context of statistical mechanics or solid state physics, we mean that there are some imperfections in the lattice structure: vacancies,

[(1)] For example, there is a certain critical point describing the so-called spin quantum Hall transition in two dimensions (2D or $d = 2$) about which we know the following. Some observables (low moments of conductances) in a certain network model of this transition map exactly to 2D bond percolation problem, which is almost certainly conformally-invariant at its critical point. On the other hand, there is a strong, though indirect, numerical evidence that the spectrum of multifractal exponents for the moments of critical wave functions is inconsistent with conformal invariance. If I can explain all the terminology and issue stated in this footnote, my goal for these lecture wil have been achieved.

interstitial atoms, substitutions, dislocations, disclinations, etc. Disorder is inevitable. But is it important, interesting? Should we treat disorder as a nuisance to be suppressed or something to study and understand? This depends on the context, and in these lectures I will focus on the cases where disorder is important to the extent that it qualitatively modifies the behavior of a clean system and leads to emergence of new phases and phase transitions.

Before plunging into technical matters, let us sort out some terminology. Generally speaking, all possible imperfections are dynamical degrees of freedom: they can move, appear, and disappear. If they are treated in this way on par with other degrees of freedom (spins, electrons), we talk about *annealed* disorder. However, more often than not the time scales associated with the dynamics of imperfections are much longer than time scales that characterize other degrees of freedom. If there is such separation of time scales, we can view the imperfections as "frozen" or *quenched*. It is the quenched disorder that we will be dealing with, and I will drop the qualifier "quenched" in the following.

Disorder certainly makes life harder by breaking lattice symmetries, which makes the use of momentum space impossible in a given disordered sample. On the other hand, why should we care about the behavior of a single sample? Quoting John Herz [**2**] "it is arguable (and a well-established part of every theoretical physicist's prejudices) that unless there is some kind of universality, the problem is not very interesting anyway." This is the point of view I will adopt in these lectures. Therefore, instead of focusing on a single sample, we need to think about what happens if we repeat measurements of some quantities, such as correlation functions of local operators (we will generically call such quantities *observables*) in many nominally "identical" or "similar" samples, that is, samples that are prepared in the same way to the extent that we can control. For example, we want to make sure that all samples of a quantum Hall device come from the same large wafer, and all have the same average concentration of impurities. The actual placement of each impurity will be different in each sample, of course.

Once we follow this approach, we immediately need to think in terms of ensembles of samples, since a priori we may encounter a situation when repeated measurements of an observable in different samples produce different results. Thus, the main conceptual tool is to imagine an ensemble of physical systems described by Hamiltonians with random interactions which model important aspects of disorder, and are specified by probability distributions. Notice that while each member of an ensemble breaks lattice symmetries, the ensemble as a whole is usually chosen in a way that respects these symmetries.

Now comes the main point: in an ensemble of disordered systems, every observable should be characterized by its probability distribution, and the main goal of a theory is to compute these distributions. This is rarely possible, so more often people are already content if they can compute the average of an observable or a few of its low moments (variance, etc.). This is OK if we can argue that the distribution of a given observable is narrow. However, there are many examples of observables whose probability distributions turn out to be broad, and then we need to distinguish, for example, the behavior of the average and the typical values of an observable.

We will be interested in critical phenomena near continuous phase transitions. In this case we expect scaling behavior, one manifestation of which is the power-law dependence of correlation functions and local observables with the system size L. This is also

relevant since all numerical simulations of critical phenomena are done in finite systems, and universal quantities are extracted from finite-size scaling.

In the case of disordered systems, as a matter of principle, we need to look at the dependence of the whole probability distribution of an observable on the system size L. Then we can encounter basically two different types of behavior. On the one hand, there are extensive quantities such as the free energy, whose densities in the thermodynamic limit become deterministic. This is referred to as "self-averaging". On the other hand, there are transport quantities such as conductances and conductivities whose probability distributions may become broad in the thermodynamic limit, and then these quantities will exhibit strong sample-to-sample fluctuations.

A standard tool to study critical phenomena and scaling is the renormalization group (RG) [**1**]. In the case of critical points in clean systems the RG flows contain a finite number of scaling variables or coupling constants in a field theory. For disordered systems, if we want to address whole probability distributions of observables, we need to use some form of a *functional RG* with an infinite number of parameters [**3**]. While this is possible in some cases, I will not deal with this. Instead, I will focus on field-theoretic techniques that allow to compute only moments of observables. These include the replica and the supersymmetry (SUSY) methods that I will describe below.

Let us look at several examples of disordered systems of interest. I will use these examples to illustrate and formalize the points made above, as well as describe techniques to deal with disorder, and some known results. The examples that I chose may seem quite different and unrelated at first, but, in fact, there are connections and mappings between them.

3. Classical spin models

Classical magnets with spins s_i on sites i (a short-hand notation for the position vector $\mathbf{r}_i$) of a lattice can have disorder (randomness) in the interactions of spins J_{ij} ("bond disorder") as well as in local magnetic fields h_i ("field disorder"):

$$H = -\sum_{ij} J_{ij} s_i s_j - \sum_i h_i s_i. \tag{3.1}$$

The spins can be, for example, unit n-components vectors, and then we have a lattice version of an O(n) model. The interaction term has the global O(n) symmetry, while the magnetic fields break this symmetry. The global symmetry can also be broken by some anisotropy in the interactions between different spin components. We can also imagine "spins" that are elements of more general coset spaces G/K, where G is the global symmetry group broken down to its subgroup K by "magnetic fields" or anisotropy.

To fully specify the problem of a random magnet, we need to specify the distributions functions $P_J(\{J\})$ and $P_h(\{h\})$ for the (collections of the) random bonds and random fields. It is most natural to treat distinct J_{ij} and h_i as independent random variables. Also, to make sure that the disorder ensemble is lattice-symmetric, we require that all fields h_i are independent identically distributed (IID), and all interactions J_{ij} with the same relative distance $|i-j|$ are IID. Finally, while it is possible to consider long-range interactions and disorder, I will restrict myself to models where only spins on nearest neighbor sites interact. Pairs of nearest neighbors will be denoted as $\langle ij \rangle$. Then we

arrive at the models of the following form:

$$H = -\sum_{\langle ij \rangle} J_{ij} s_i s_j - \sum_i h_i s_i, \tag{3.2}$$

where all J_{ij} are IID random variables drawn form the distribution function $P_J(J)$, and all h_i are also IID random variables drawn from the distribution function $P_h(h)$.

When there is no disorder (the clean case), magnetic systems may exhibit symmetry-broken phases and transitions between them, happening upon changes in the temperature and other parameters. Continuous phase transitions are accompanied by universal critical phenomena. Universal properties are found in the behavior of the connected correlation functions of spins

$$G(i_1, \ldots, i_n) = \langle s_{i_1} \ldots s_{i_n} \rangle_c, \tag{3.3}$$

where the angular brackets denote the statistical average

$$\langle O \rangle \equiv \frac{1}{Z} \sum_{\{s\}} O e^{-H}, \qquad\qquad Z = \sum_{\{s\}} e^{-H}. \tag{3.4}$$

Connected correlators are obtained as derivatives of the free energy $F = -\ln Z$:

$$\langle s_{i_1} \ldots s_{i_n} \rangle_c = (-1)^n \prod_{i=1}^{n} \partial_{h_i} F \Big|_{h_i = h}. \tag{3.5}$$

The simplest of these is the order parameter, or magnetization:

$$m(h) = \langle s_i \rangle = -\frac{\partial F}{\partial h_i}\Big|_{h_i = h}. \tag{3.6}$$

The same rules apply in a disordered system except that now the free energy is a random variable that depends on a particular realization of random couplings (which I will denote simply by X): $F[X] = -\ln Z[X]$. Observables such as the correlators (3.3) are also random variables. As we discussed above, their lowest moments are the primary goal of a theory. Therefore, if I denote disorder averaging by an overline, we need to find a way to compute the quenched free energy

$$\overline{F} = -\overline{\ln Z} = -\int \mathcal{D}X\, P[X] \ln Z[X]. \tag{3.7}$$

As I mentioned above, the free energy density is self-averaging if the disorder is short-range correlated. This means that it becomes non-random in the thermodynamic limit (the volume V going to infinity):

$$f = \lim_{V \to \infty} \frac{F[X]}{V} = \overline{f}. \tag{3.8}$$

Indeed, we can break a large system into many smaller but still macroscopic subsystems, and the total free energy is the sum of the free energies of the subsystems (apart from subleading surface contributions). Each susbsystem has a different arrangement of impurities and can be viewed as a member of an ensemble of disorder realizations. Then in the thermodynamic limit the free energy density with probability one becomes equal to the average free energy density. In a finite large system with N sites the fluctuations

in the free energy density are of order $N^{-1/2}$ (the usual central limit theorem) and vanish in the thermodynamic limit. On the other hand, averages of correlators such as

$$\overline{\langle s_{i_1} \dots s_{i_n} \rangle_c} = \overline{\frac{1}{Z[X]} \sum_{\{s\}} s_{i_1} \dots s_{i_n} e^{-H[X]}} \tag{3.9}$$

are not self-averaging, since they sensitively depend on the local patterns of disorder close to the points of insertions of the spin operators.

Disconnected correlators averaged over disorder such as

$$\overline{\langle s_i \rangle \langle s_j \rangle} = \overline{\left(Z^{-1}[X] \sum_{\{s\}} s_i e^{-H[X]} \right) \left(Z^{-1}[X] \sum_{\{s\}} s_j e^{-H[X]} \right)} \tag{3.10}$$

are also important, and may have nontrivial behavior and significance. For example, spin glasses are characterized by $\overline{m} =$ and $q > 0$, where

$$q \equiv \overline{m^2} = \overline{\langle s_i \rangle^2} \tag{3.11}$$

is the so-called Edwards-Anderson order parameter.

Critical points of clean classical magnets are well described by field theories and classified into universality classes depending on the dimensionality of space d and the internal symmetry breaking pattern G/K.

In presence of disorder the interesting questions are

1. Stability of phases (Imry-Ma arguments).
2. New phases (spin glasses).
3. Stability of first order phase transitions and emergent criticality.
4. Stability of critical points (Harris criterion).
5. New types (universality classes) of critical points.

Answers to these questions depend on the details of disorder (distribution functions $P_J(J)$ and $P_h(h)$). Any attempt at answering these questions needs to address the necessity to find disorder averages such as (3.9). There are several methods that are available, and the one that is very general is the replica method.

4. Replica method and non-unitary CFTs.

To the best of my knowledge, the replica method has been invented by Sam Edwards when he was working on properties of rubbers. The first published paper where the method was used is the famous paper of Edwards and Anderson on spin glasses [**4**].

The idea of the replica method is based on the identity $Z[X] = e^{-F[X]}$. If we raise it to power n, assume $n \ll 1$ and expand, we get the limit

$$F[X] = -\ln Z[X] = \lim_{n \to 0} \frac{1 - Z^n[X]}{n} = -\frac{d}{dn} Z^n[X] \Big|_{n=0}. \tag{4.1}$$

For positive integer n the power $Z^n[X]$ can be computed by introducing n independent non-interacting identical copies, or *replicas*, of the system, all subject to the same disorder realization X. Let us labels the replicas by the index $a = 1, \dots, n$. Then the replicated Hamiltonian

$$H_n[X] = -\sum_{ij} J_{ij} \sum_{a=1}^{n} s_{i,a} s_{j,a} - \sum_i h_i \sum_{a=1}^{n} s_{i,a} \tag{4.2}$$

has the full permutation symmetry S_n among replicas. Tracing $e^{-H_n[X]}$ over all spins in all replicas gives $Z^n[X]$. This object can be averaged over disorder to give

$$Z_n = \overline{Z^n[X]}. \tag{4.3}$$

After the average over the disorder is performed, distinct replicas get coupled, and so Z_n is the partition function of a complicated interacting model.

Next we try to analytically continue the result to arbitrary (non-integer) values of n, and the take the replica limit as in Eq. (4.1):

$$\overline{F} = \lim_{n\to 0} \frac{1 - Z_n}{n}. \tag{4.4}$$

Likewise, various disorder-averaged correlation functions can be written using replicas in the following way:

$$\begin{aligned}
\overline{\langle O(r)\rangle} &= \lim_{n\to 0} \langle O_a(r)\rangle_{Z_n}, \\
\overline{\langle O(r_1)O(r_2)\rangle} &= \lim_{n\to 0} \langle O_a(r_1)O_a(r_2)\rangle_{Z_n}, \\
\overline{\langle O(r_1)\rangle\langle O(r_2)\rangle} &= \lim_{n\to 0} \langle O_{a_1}(r_1)O_{a_2}(r_2)\rangle_{Z_n}, \qquad a_1 \neq a_2,
\end{aligned} \tag{4.5}$$

where the averages on the right-hand side are over the thermal ensemble specified by the average replicated partition function Z_n, and a, a_1 and $a_2 \neq a_1$ are some specific (but arbitrary) replica indices.

The nature of the replica limit is quite subtle, since it is not obvious that quantities known for positive integer values of n admit analytical continuation. For this reason the replica method is sometimes known as the "replica trick", and people continue to debate whether it can be justified at some level of generality, and whether it can be used to compute non-perturbative results [**5**].

A famous class of models where the subtleties of the replica method are very prominent are the so-called spin glasses [**6, 7, 8, 9**], which appear when the distribution of random bonds $P_J(J)$ becomes broad in some sense, or when the geometry of the lattice makes the spin system frustrated. A mean-field theory of spin glasses can be formulated as the special infinite range model known as the Sherrington-Kirkpatric model [**10**]. This model is exactly solvable and exhibits the fascinating phenomenon of replica symmetry breaking (RBS). RBS means that the permutation symmetry S_n of the replicas is broken below some transition temperature. The RSB pattern is very nontrivial, and is described by the so-called Parisi solution [**6**]. What is remarkable is that Parisi solution has been rigorously proven to be correct by mathematicians (Talagrand [**11**] and others).

There are other cases, such as transitions on Bethe latices [**12**], where a replica solution can be compared with a SUSY solution, and the integrable theory of replicas [**13**], where the replica method is embedded into the theory of integrable models. In all cases where a comparison with an alternative and in some sense better method exists, replicas give correct non-perturbative results.

4.1. Non-unitarity: logarithmic CFTs, $c = 0$, and multifractality. — Let us now consider a random critical point described in replica method by the partition function Z_n in Eq. (4.3). For a finite value of n we can define the free energy

$$F_n = -\ln Z_n. \tag{4.6}$$

This is the free energy of the replicated theory averaged over disorder. Notice that

$$Z_0 = 1, \qquad F_0 = 0. \tag{4.7}$$

This is a very general property of theories describing random systems in the replica limit. We will see later that the same statements are valid for systems studied using SUSY formalism.

It is known at least since the late 90s [**14, 15**] that random critical points with $Z = 1$ generically exhibit logarithmic correlators described by logarithmic CFTs, see Refs. [**16, 17**] for recent accounts. The mechanism of the appearance of the logs is well understood [**18**]. In a replicated model the operator dimensions, critical amplitudes, and OPE coefficients all depend on n. When $n \to 0$, dimensions of some operators "collide" (become degenerate in the limit), and then their linear combinations form indecomposable representations of the symmetry group S_n, and lead to non-diagonalizability of the dilatation operator and the appearance *Jordan blocks*. This is very similar to the appearance of logs in the Frobenius method in the theory of ODEs when the indicial equation has roots that differ by integers. The same mechanism works in the SUSY formalism and in the interesting limits of geometric critical phenomena, see Sec. 7.

In $d = 2$ the appearance of logarithms in correlation functions is closely related to the vanishing of the central charge of the corresponding CFTs [**19, 14, 20, 21**]. Indeed, the central charge c can be extracted either from the finite-size corrections to the free energy density, or as from the expectation of the stress-energy tensor. Both these quantities identically vanish in theories where Eqs. (4.7) hold.

Note that all logarithmic and $c = 0$ CFTs are *non-unitary*, and if we hope to apply conformal bootstrap methods to them, they should be extended in this direction.

One particularly intriguing feature of random critical points is that they exhibit multifractality, which means that the moments of a random observable O have independent scaling dimensions:

$$\overline{\langle O \rangle^q} \sim L^{-x_q}. \tag{4.8}$$

In a field theory these moments are described by some local operators $\mathcal{O}_q$, and the multifractal (MF) exponents x_q are scaling dimension of these operators. However, as was noticed by Ludwig and Duplantier [**22, 23**], the MF spectra x_q are *convex* functions of the order q, and for sufficiently large $|q|$ become negative. We will see this in detail in the context of Anderson transitions later. This convexity needs to be contrasted with the concavity of scaling dimensions of composite operators (powers of a basic field) in a unitary CFT. This is another manifestation of the necessity to use non-unitary CFTs to describe random critical points.

Notice that negative dimensions of sufficiently high moments of local observables are quite physical and simply reflect that fact that these observables are non self-averaging: their distribution functions become broader and broader when the system size L increases.

5. Harris criterion and related issues

Let me illustrate the use of replicas by answering one of the questions I posed above: is a critical point in a clean system perturbatively stable upon addition of disorder? There is a simple heuristic argument due to Brooks Harris that give a criterion for stability [**24**], see [**25**] for a recent review of this criterion and related issues. Replica method allows to derive this criterion in a nice renormalization group (RG) way.

Let me consider the model (3.2) without magnetic fields and coarse-grain it. The product $s_i s_j$ of two neighboring spins becomes the local energy density operator $E(r)$ coupled to the relevant thermal scaling variable ("temperature"). In the clean case the dimension x_E of the energy operator and the RG eigenvalue y_t of the thermal variable add up to the dimension of space: $x_E + y_t = d$. By the standard RG arguments the localization length exponent ν is given by $\nu = 1/y_t$, and then we have the relation

$$x_E = d - 1/\nu. \tag{5.1}$$

The action (the Hamiltonian) for the coarse-grained model with bond disorder can be written as

$$S[J] = S^* + \int d^d r\, J(r) E(r), \tag{5.2}$$

where S^* is the RG fixed-point action of the clean system. To average of the disorder $J(r)$, we replicate the action:

$$S_n[J] = \sum_{a=1}^{n} S_a^* + \int d^d r\, J(r) \sum_{a=1}^{n} E_a(r). \tag{5.3}$$

Then we use the cumulant expansion to find

$$Z_n = \int \prod_a \mathcal{D}s_a \overline{e^{-S_n[J]}} = \int \prod_a \mathcal{D}s_a e^{-S_{\text{eff}}}, \tag{5.4}$$

where the effective action is

$$S_{\text{eff}} = \sum_a S_a^* + \kappa_1 \int d^d r \sum_a E_a(r) + \frac{1}{2} \iint d^d r d^d r' \kappa_2(r-r') \sum_{a,b} E_a(r) E_b(r') + \dots \tag{5.5}$$

$$\kappa_1 = \overline{J(r)}, \qquad \kappa_2(r-r') = \overline{J(r)J(r')} - \kappa_1^2. \tag{5.6}$$

The first cumulant κ_1 simply shifts the critical temperature, while the second cumulant is short-ranged for uncorrelated disorder, and can be replaced by $2\gamma\delta(r-r')$. Upon this replacement the operators $E_a(r)$ and $E_b(r')$ collide, and we need to look at their operator product expansion (OPE). For $a = b$ the leading operator in the OPE is just the energy itself, and the $a = b$ terms lead to further shift of the critical temperature. However, for $a \neq b$ the leading operator in the OPE is new, and since the distinct replicas are independent, we can simply write this operator as $E_a(r)E_b(r)$. Thus, the interesting perturbation in the effective action becomes

$$\delta S_{\text{eff}} = \gamma \int d^d r \sum_{a \neq b} E_a(r) E_b(r). \tag{5.7}$$

The relevance of this perturbation depends on the dimension of the perturbing operator. This ban be found using the independence of the replicas in the unperturbed theory. The two-point function of the perturbing operator at the fixed point is found by simply counting possible ways to pair replicas with the same index at the two points:

$$\Big\langle \sum_{a\neq b} E_a(r)E_b(r) \sum_{a'\neq b'} E_{a'}(r')E_{b'}(r') \Big\rangle_* = 2n(n-1)\langle E_a(r)E_a(r')\rangle_*^2 \sim \frac{2n(n-1)}{|r-r'|^{4x_E}}. \tag{5.8}$$

Form this we see that the dimension of the perturbing operator is simply $2x_E$. The corresponding RG eigenvalue is

$$y_\gamma = d - 2x_E = 2/\nu - d, \tag{5.9}$$

and we conclude that the perturbation is irrelevant ($y_\gamma < 0$) as long as

$$d\nu > 2. \tag{5.10}$$

This is the Harris criterion.

Let me make a few comments about this result and its derivation. First of all, the original derivation by Harris was much more pedestrian and compared the nominal distance to the clean transition with the RMS fluctuations of the "local" transition temperature in the correlation volume. This argument uses the central limit theorem to estimate these fluctuation, and gives the same result as replicas. We can ask what happens if we include higher cumulants of the disorder. If the distribution of the disorder is narrow in some precise sense (which is necessary for the central limit theorem to apply), the higher cumulants exist and are small. In the effective action they produce the same terms that have already been considered, or less relevant ones. However, if the disorder distribution is broad, such as the Cauchy distribution (Lorentzian), for which even the mean value does not exist, then the derivation is not justified, and we cannot say much. Another comment is that the Harris criterion can be extended and generalized in various ways: to disorder that is correlated in space with power-law second cumulant $\kappa_2(r - r') \sim |r - r'|^{-\alpha}$, to quantum critical points, and to arbitrary spatio-temporal disorder correlations, see Exercises below and [**1, 25**].

5.1. Exercises. —

1. Generalize the analysis of the relevance of bond disorder that led to the Harris criterion to show that random fields are always relevant at the clean critical point.
2. Generalize the Harris criterion to the case of correlated disorder when the second cumulant decays as
$$\kappa_2(r - r') \sim |r - r'|^{-\alpha}. \tag{5.11}$$
3. Generalize the Harris criterion to quantum systems recalling that quenched disorder does not fluctuate in time.

5.2. Relevant disorder and new random critical points. — A natural question is what happens to the clean critical point when the Harris criterion is violated and disorder is relevant? In this situation we have two relevant RG directions with positive RG eigenvalues $y_t = 1/\nu$ and $y_\gamma = 2/\nu - d$, and we expect some crossover phenomenon characterized by the crossover exponent $\phi = y_\gamma/y_t = 2 - d\nu$. But where does this crossover lead? It turns out that in many interesting cases the sharp transition survives, but is controlled by a new random fixed point. The Ising model in $d = 3$ is one such case. As we know from conformal bootstrap, the scaling dimension of the energy operator is $x_E \approx 1.412625$, and $d\nu = 3/(3 - x_E) \approx 1.8899 < 2$. Thus, the Harris criterion is violated, but only weakly. At the same time in $d = 2$ the Ising exponent $\nu = 1$ and $d\nu = 2$. In $d = 4$, the upper critical dimension for the Ising symmetry, the mean field theory is valid (apart from logarithmic corrections), and $d\nu = 4 \times 1/2 = 2$. Thus it is possible to study the new random fixed point perturbatively in $d = 2 + \epsilon$ as well as $d = 4 - \epsilon$, see [**1, 26, 27, 28**]. There is also a rigorous Chayes-Chayes-Fisher-Spencer theorem [**29**] that states that at a random critical point the correlation length exponent ν_{random} always satisfies the Harris criterion: $d\nu_{\text{random}} \geqslant 2$.

It is easy to extend the replica treatment of bond disorder to the field disorder. A replica analysis of random fields shows that they are always relevant. Some clean magnetic systems can exhibit first order phase transitions that bond disorder turns

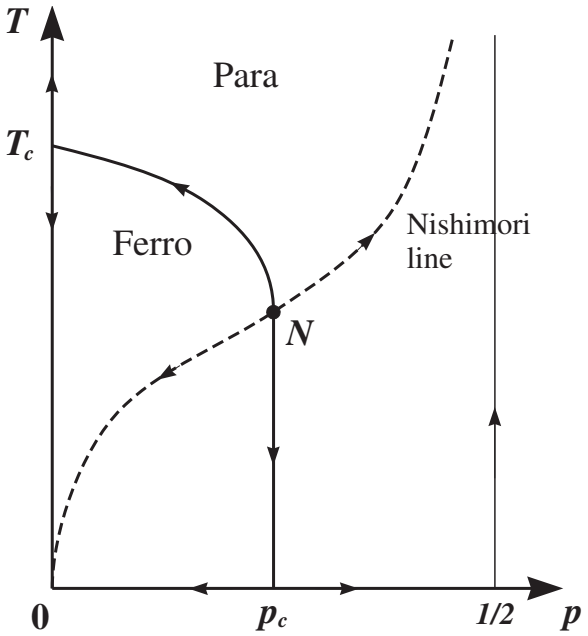

FIGURE 1. Phase diagram of the random-bond Ising model in terms of $T \propto 1/J$ and p.

into critical points. For example, for the two-dimensional Ashkin-Teller model such new critical point turns out to belong to the *clean* two-dimensional Ising universality class with logarithmic corrections [**30, 25**], see the next subsection. Moreover, random fields can even destroy ordered phases in dimensions $d \leqslant 2$. A qualitative understanding of all this is achieved by the so-called Imry-Ma argument [**31**] that in some cases can be put on a rigorous ground, both in classical and in quantum models [**32**].

5.3. Marginal case: the Ising model in $d = 2$ and disordered fermions. — What happens when $d\nu = 2$ as in the clean Ising model in $d = 2$? In this *marginal* case the Harris criterion is not sufficient to decide the fate of the clean critical point upon adding radnom bond disorder. Perturbative treatments of weak disorder predicted that it is marginally irrelevant, and causes a logarithmic flow to the clean Ising fixed point with ensuing universal logarithmic corrections to scaling, see Ref. [**33**], Chapter 10 in Ref. [**8**], and references therein. This scenario remained controversial for quite some time, but recent high-accuracy Mote-Carlo simulations [**30**] provided strong support for it.

The random-bond Ising model in $d = 2$ (2D RBIM) has a special feature that its clean critical point is described by the theory of 2D Majorana fermions (a very elegant derivation is given in Polyakov's book [**34**]). In this picture a weak bond disorder maps to the random mass of the fermions. When we replicate the theory we can combine Majorana fermions into Dirac fermions. This provides a connection to the disordered fermionic problems that I will describe in Section 8 and allows to use the SUSY method to study the 2D RBIM, see [**19**]. The SUSY method reproduces the results obtained with replicas. Jumping ahead, I can mention that the fermionic representation of the 2D RBIM belongs to class D in the Altland-Zirnbauer classification to be reviewed in Section 10.

The "fermionization" of the 2D RBIM turns out to be useful even in the case of strong disorder when the random bonds are drawn from the bimodal distribution

$$P_J(J_{ij}) = (1-p)\delta(J_{ij}-J) + p\delta(J_{ij}+J). \tag{5.12}$$

Models with such bimodal distributions exhibit spin glass phases for sufficiently large p and sufficiently low temperatures. Importantly, the bimodal distribution (5.12) belongs to a class of distributions for which the RBIM possesses a certain gauge symmetry discovered by Nishimori. This symmetry allows to obtain a lot of exact result about the model [**7**]. In particular, the gauge symmetry predicts the existence of a special *Nishimori line* in the phase diagram, that goes through a multicritical point where the three phases (ferromagnetic, paramagnetic, and spin glass) meet (in $d > 2$).

For the 2D RIBM the spin glass phase is believed to exist only at zero temperature, but the Nishimori line still intersects the ferro-para phase boundary at a multicritical Nishimori point, see the phase diagram shown in Fig. 1. The SUSY analysis of the Nishimori point and critical points in related models [**35**] suggest that they can be described by SUSY spin chains and non-linear sigma models similar to the ones appearing in the study of Anderson transitions in Section 9, but with different (enhanced) supersymmetry.

6. Quantum spin models and SYK models.

These are generalization of classical spin systems where the classical spins s_i are replaced by (non-commuting) generators $\hat{s}_i$ of a Lie algebra $\mathfrak{g} = \mathrm{Lie}\, G$ in a certain representation. In this case the Hamiltonian contains terms that do not commute, and it is interesting to look at phases of the system at zero temperature. These phases and transitions between them, called quantum phase transitions (QPTs) [**36**], are controlled by the ratios of the couplings of non-commuting terms. Generally speaking, some terms in the Hamiltonian can explicitly break the symmetry G down to a subgroup K which, in turn, can be spontaneously broken further in some ordered ground states.

In quantum systems the time t plays a role of an additional dimension. However, the scaling in time can be different from scaling in space. This difference is encoded in the so-called dynamical critical exponent z that relates the divergence of the spatial correlation length ξ close to a quantum critical point to the vanishing of the energy gap E_g in the spectrum of the system: $E_g \sim \xi^{-z}$. If $z \neq 1$, the space-time is anisotropic, which prevents the use of CFTs to describe generic QPTs. However, there are many interesting QPTs where $z = 1$. A simple example is provided by the transverse field Ising model (TFIM) in d dimensions

$$H = -Jg\sum_{\langle ij\rangle} \hat{\sigma}_i^z \hat{\sigma}_j^z - J\sum_i \hat{\sigma}_i^x, \tag{6.1}$$

where $\hat{\sigma}^\alpha$ are the three Pauli matrices representing generators of su(2) in the fundamental representation. This model is ordered for $g \gg 1$ and disordered for $g \ll 1$. In between it exhibits a QPT at a certain value g_c. This QPT has $z = 1$ and is described by the Ising universality class in $D = d + 1$ dimensions. This fact can be demonstrated as follows.

Let us consider the quantum partition function of the TFIM at a finite temperature:

$$Z_{\mathrm{qu}} = \mathrm{Tr}\, e^{-\beta H}. \tag{6.2}$$

This can be "Trotterized" (written as a discretization of a Feynman path integral in the imaginary time). The result involves the matrix elements of the short-time evolution operator $e^{-H\Delta\tau}$ in the Hilbert space $\left(\mathbb{C}^2\right)^{\otimes N}$, where N in the number of sites in the system. These matrix elements can be written in terms of Pauli matrices and turn out to be identical with the matrix elements of the transverse matrix of the classical Ising model in $D = d + 1$ dimensions. See Sachdev's book [**36**] for details.

Such classical-quantum correspondence is a central concept in the theory of QPTs and is widely applicable, even though for some quantum models, including Heisenberg models, it leads to "classical" models with Hamiltonians (actions) that contain imaginary terms coming from the Berry phases of the quantum spins.

In $2 = 1 + 1$ dimensions the classical-quantum correspondence between the 2D Ising model and $1 + 1$ TFIM can also be established by a direct fermionization of the TFIM using Jordan-Wigner transformation. The result is a 1D tight-binding free fermion Hamiltonian. The transfer matrix of the classical 2D Ising model can also be fermionized by the same Jordan-Wigner transformation [**37**], and becomes the exponential of the same 1D fermionic Hamiltonian.

Now imagine that we introduce quenched disorder in a quantum spin model. Then the corresponding classical model will also become disordered, but the disorder in the classical model is perfectly correlated along the direction that corresponds to the time in the quantum model. Such disorder is sometimes called "columnar". This columnar structure is related to the fact that the Harris criterion for quantum critical points in d dimensions is still $d\nu > 2$ (as opposed to a naively expected $(d + z)\nu > 2$).

6.1. Quantum spin chains, non-linear sigma models, and WZNW CFTs. —

6.2. Strong disorder RG and infinite randomness fixed points. —

6.3. Quantum information perspective. Entanglement and measurement-induced entanglement transitions. —

6.4. SYK models. —

7. Geometric critical phenomena.

A particular type of bond disorder in magnets leads to another very interesting subject. Let us consider

$$P_J(J_{ij}) = p\delta(J_{ij} - J) + (1 - p)\delta(J_{ij}). \tag{7.1}$$

This means that with probability p we have a bond with a non-random exchange coupling J, and with probability $1 - p$ there is no bond at all. If we now ignore the spins and focus only on the connectivity properties of clusters of bonds, we have the problem of bond percolation.

This is a very rich and well-developed subject. This simple model exhibits a geometric critical phenomenon at some special value p_c (dependent on the lattice, dimensionality, etc.), which separates the phase where all bond clusters are finite from the phase where there are infinite clusters. Many properties of clusters are described by power laws with universal critical exponents near p_c.

The bond percolation model can be obtained as a limit of a *non-random* spin model of the type (3.2) called the Q-states Potts model given by the Hamiltonian

$$H_{\text{Potts}} = -J \sum_{\langle ij \rangle} \delta_{s_i, s_j}. \tag{7.2}$$

where the spins are allowed to take Q different values, and the interaction on a bond only cares about whether the spins at the end of the bond are the same or not. It is easy to see that for $Q = 2$ the Potts model is equivalent to the Ising model. The Potts model can be reformulated in a geometric way directly in terms of clusters of bonds. The result is called the random cluster model or the Fortuin-Kastelein (FK) model. In this formulation the parameter Q can easily be continued to arbitrary complex values, and turns out that the limit $Q = 1$ gives the bond percolation. All FK models exhibit geometric critical phenomena for certain values of d and Q.

Another related family of models are the loop models obtained by a similar geometric reformulation of the *non-random* O(n) models mentioned above. In this case the geometric objects of interest are loops composed of bonds of the underling lattice, and the parameter n can be any complex number.

Notice that both classes of models — FK models and loop models — are *non-random* in the sense I considered above. Why then do I mention them? There are two reasons. First, it turns out that geometric critical phenomena are closely related to the Anderson transitions that I will describe later, and in particular, to certain network models. Second, in geometric critical phenomena one can focus on very non-trivial geometric and topological properties of extended objects, such as clusters, their boundaries and perimeters, or loops. These properties can be studied in a given realization of the underlying spin model. They are then the analogues of random observables in an ensemble of disorder realizations, and can be characterized by probability distributions. In this case the usual thermodynamic ensemble plays the role of the ensemble of disorder realizations.

An important point is that similar to the replicated theories with a non-integer number of replicas or in the replica limit, the Q-states Potts model and O(n) model are not unitary for generic values of Q and n. Moreover, even for "nice" values of the parameters when the theory is unitary, such as the Ising model at $Q = 2$, the focus on the geometric non-local observables required to extend the usual Kac table beyond its standard range appropriate for a minimal CFT. The operators $\phi_{p,q}$ that describe those non-local properties can have arbitrarily large Kac labels p and q.

7.1. Multifractal harmonic measure. — In all geometric critical phenomena we also find multifractal behavior of the harmonic measure on the boundaries of clusters or loops. Harmonic measure can be defined in any d, but for simplicity, let me restrict the discussion to $d = 2$, where it has especially nice conformal properties, and where its moments can be found from CFT.

Consider a closed simple curve C which bounds a random cluster of size L. One can imagine that the cluster is made of a conducting material and carries a total electric charge one on its boundary C. The harmonic measure of any part of C is defined as the electric charge of this part. More formally, the harmonic measure of a part of the curve C is the probability that a Brownian motion released from infinity hits the curve C in this part.

Let us pick a point z_i on the curve and consider a disc of a small radius $a \ll L$ centered at z_i. It surrounds a small part of C and we define $p_i = p(z_i, a)$ to be the

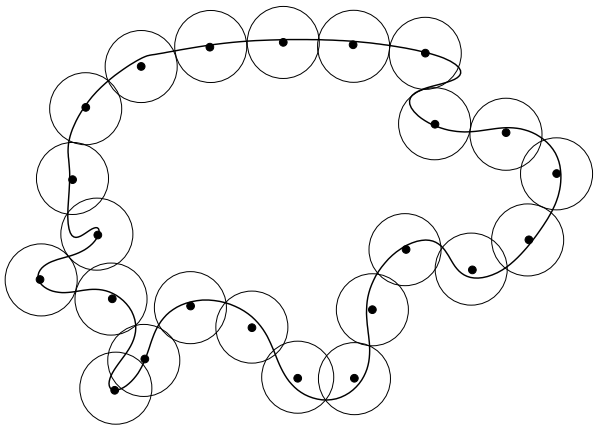

FIGURE 2. A curve covered by discs.

harmonic measure of this part that is the electric charge on it. Then we consider the moments

$$M_q = \sum_{i=1}^{N} p_i^q, \tag{7.3}$$

where q is a real power and N is the number of discs needed to cover C as in Fig. 2. For $a \ll L$ ($N \gg 1$) these moments scale as

$$M_q \sim \left(\frac{a}{L}\right)^{\tau(q)}, \quad \frac{a}{L} \to 0. \tag{7.4}$$

The function $\tau(q)$ is called the multifractal spectrum of the curve C and it encodes a lot of information about the curve C. It also has some simple properties. First of all, since all $0 < p_i \leqslant 1$, the moments M_q are well defined for any real q and the function $\tau(q)$ is non-decreasing and concave: $\tau'(q) \geqslant 0$, $\tau''(q) \leqslant 0$. Secondly, if $q = 1$ the sum in (7.3) is equal to the total charge of the cluster and therefore does not scale with a, producing the normalization condition $\tau(1) = 0$. Thirdly, if we set $q = 0$, M_0 is simply the number N of discs of radius a necessary to cover the curve C so that by definition the fractal (Hausdorff) dimension of C is $d_f = -\tau(0)$.

If the curve C were smooth we would have a simple relation $\tau(q) = q - 1$. But cluster boundaries at a critical point are fractal, and $\tau(q)$ is a non-linear function of q. In addition, the curve C is random, and so are its moments M_q. We can average them over the ensemble of curves, which is the analog of the disorder average in this case. This gives the *average* MF spectrum $\bar{\tau}(q)$. In this context the spectrum $\tau(q)$ is called the *typical* MF spectrum. These are easily related, and for a system with the central charge c can be expressed in terms of

$$\Delta(q) = \frac{\sqrt{1-c+24q} - \sqrt{1-c}}{\sqrt{25-c} - \sqrt{1-c}}, \tag{7.5}$$

which, remarkably, is the gravitationally dressed dimension q as given by the KPZ formula of 2D quantum gravity [**38**]. These results were first obtained by means of quantum gravity in [**39, 40**] and then re-derived with many generalizations in the usual CFT formalism in [**41, 42**].

7.2. 2D critical phenomena and stochastic conformal geometry: SLE, CLE, random lattices and Liouville quantum gravity. — Finally I want to mention that specifically in two dimensions the geometric critical phenomena have been very actively studied rigorously by mathematicians in the past 20-25 years. A breakthrough made by Oded Schramm in 1999 led to creation of a beautiful subject of two-dimensional probability theory and stochastic conformal geometry, which includes the famous Schramm-Loewner evolutions (SLE) and conformal loop ensembles (CLE), as well as more recent rigorous studies of Liouville quantum gravity. These very exciting developments are all very closely connected with CFT, and would deserve separate lecture courses.

8. Disordered fermionic systems

One of the most intriguing problems in modern condensed matter physics is the combined effect of correlations and disorder on electronic properties of solids. Interplay of interactions and disorder results in such diverse phenomena as metal-insulator and superconductor-insulator transitions, quantum Hall effects, mesoscopic electron transport, etc. For some of these phenomena, including the integer quantum Hall (IQH) effect, a fairly complete understanding may be achieved under the assumption that electrons do not interact.

Neglecting the interactions but retaining disorder results in problems of a single fermion in a random potential. These can be described by Hamiltonians in a continuum, such as

$$H = H_0 + U(r), \qquad H_0 = \frac{1}{2m}\Big(-i\hbar\nabla + \frac{e}{c}\mathbf{A}(r)\Big)^2, \tag{8.1}$$

where $U(r)$ is a random potential, and I included the vector potential $\mathbf{A}(r)$ to account for a possible magnetic field. Other type of models are tight-binding lattice models

$$H = \sum_{i,j}\big(t_{ij}c_i^\dagger c_j + \text{c.c.}\big) + \sum_i u_i c_i^\dagger c_i, \tag{8.2}$$

where both the hopping amplitudes t_{ij} and the on-site potentials u_i can be random. We can also generalize this by including spin, superconducting pairing correlations at a mean-field level, and more complicated band structures, including Dirac points.

There is another class of models that describe non-interacting fermions. These are the so-called network models. One of the most often used models in this class is the Chalker-Coddington (CC) network [**43**] describing the integer quantum Hall (IQH) transition. The CC network is shown in Fig. 3 and it consists of directed links and nodes arranged on a square lattice. A state of the network $|\psi\rangle$ is a vector in $\mathbb{C}^{N_L}$, where N_L is the number of links. Using the natural basis vectors $|l\rangle$ associated with each link l, we write

$$|\psi\rangle = \sum_l \psi_l |l\rangle. \tag{8.3}$$

We refer to the amplitudes ψ_l as fluxes carried by the links.

States of the network evolve in discrete time, each time step described by a unitary operator $\mathcal{U}$ acting on $\mathbb{C}^{N_L}$. The evolution operator $\mathcal{U}$ is determined by its matrix elements $\mathcal{U}_{ll'} \equiv \langle l|\mathcal{U}|l'\rangle$ which are non-zero only if l' and l are incoming and outgoing

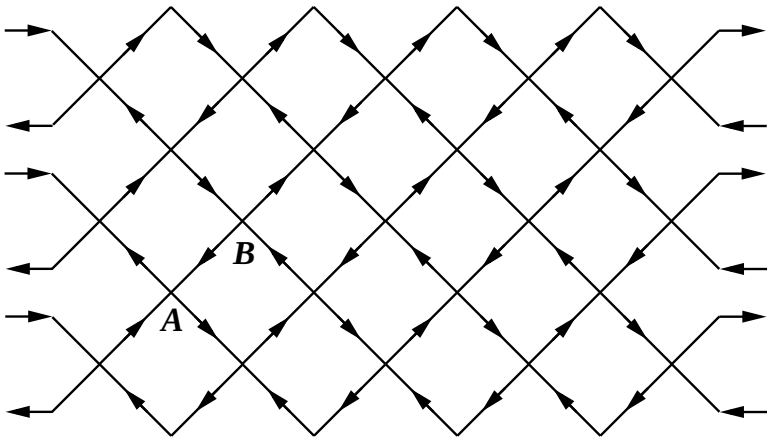

FIGURE 3. The Chalker-Coddington network model. The fluxes propagate on the links in the directions shown by the arrows. Two ideal leads are attached on the left and the right, and reflecting boundary conditions are imposed at the top and bottom.

links at the same node. In this case the matrix element is $\mathcal{U}_{ll'} = \mathcal{S}_{ll'}$, where the 2×2 (unitary) scattering matrix $\mathcal{S}$ connects incoming $(\psi_i, \psi_{\bar{\imath}})$ and outgoing $(\psi_o, \psi_{\bar{o}})$ fluxes:

$$\begin{pmatrix} \psi_o \\ \psi_{\bar{o}} \end{pmatrix} = \mathcal{S} \begin{pmatrix} \psi_i \\ \psi_{\bar{\imath}} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \psi_i \\ \psi_{\bar{\imath}} \end{pmatrix}. \tag{8.4}$$

Network models can be generalized to higher dimensions, to include spins, etc.

Single-particle disordered problems exhibit the fascinating phenomenon of Anderson localization, in which disorder leads to exponentially-localized eigenstates whose spatial extent is called the *localization length* ξ. If the states at the Fermi level of an electronic system are localized, the system is insulating. In $d \geqslant 3$ there are continua of extended states, separated (in energy) from the localized states by critical energies E_c (usually called the mobility edges). Upon variation of the Fermi energy across a critical energy E_c, a transition between metallic and insulating phases happen. In $d = 2$ and a strong magnetic field (the conditions for QH effects) extended states exist only at discrete energies separated by the cyclotron energy. Varying the Fermi energy across the energy of an extended state leads to the transition between plateaus in the IQH effect. The two insulating phases separated by an IQH transition are distinguished by the value of quantized Hall conductivity.

All such transitions, called Anderson transitions (ATs), are disorder-dominated continuous quantum phase transitions. In the vicinity of an AT the localization length ξ diverges (usually) in a power-law fashion as a function of energy:

$$\xi \sim |E - E_c|^{-\nu_\xi}. \tag{8.5}$$

The localization length exponent ν_ξ is one of the universal scaling properties whose calculation is an important goal of a theory of ATs.

Another fascinating object are the eigenstates (wave functions) directly at an AT, that is, at the critical energy E_c. They turn out to be *multufractals*. I will describe this in much more details later. For now I just want to mention that different moments of the critical wave functions exhibit different scaling behavior, and that requires us to consider a whole continuum of critical exponents called the multifractal (MF) spectrum.

The interesting questions are

1. The nature of the spectrum and correlations of eigen-energies.

2. The nature and correlations of eigenstates (wave functions).
3. Transport properties.
4. Critical properties at ATs, including MF spectra of critical wave functions.

Similar to magnetic systems, answers to these questions depend on specific detials such as the dimensionality of space d and the symmetries of the Hamiltonian. Before discussing symmetry classification of disordered systems, let me discuss an alternative to the replica method that is available in this case, the *supersymmetry* (SUSY) method.

9. Supersymmetry method and non-linear sigma models

The SUSY method is in available for systems where the basic functional integral is Gaussian. This is the case for disordered electronic systems [**44**]. Indeed, all necessary information about them is in principle contained in Green functions (GFs)

$$G^{\alpha}(E) = \frac{1}{E + i\alpha 0^{+} - H}, \tag{9.1}$$

where $\alpha = \pm 1$ for retarded (advanced) GF. In the coordinate representation

$$G^{\alpha}(r, r'; E) = \langle r | \frac{1}{E + i\alpha 0^{+} - H} | r' \rangle = \sum_{n} \frac{\psi_n(r)\psi_n^{*}(r')}{E + i\alpha 0^{+} - E_n}, \tag{9.2}$$

where ψ_n and E_n are the normalized eigenfunctions and eigenenergies of the Hamiltonian H. Various observables can be expressed in terms of GFs, including the local density of states

$$\nu(r; E) = \sum_{n} \delta(E_n - E)|\psi_n(r)|^2 = -\frac{\alpha}{\pi} \operatorname{Im} G^{\alpha}(r, r; E). \tag{9.3}$$

GFs can be written as ratios of two Gaussian functional integrals over (complex) Grassman variables χ and χ^{*}

$$G^{\alpha}(r, r'; E) = -i\alpha \frac{1}{Z} \int \mathcal{D}\chi \, \chi(r)\chi^{*}(r') \exp\Big(i\alpha \int d^{d}r \, \chi^{*}\big(E + i\alpha 0^{+} - H\big)\chi \Big), \tag{9.4}$$

where the partition function Z is

$$Z = \int \mathcal{D}\chi \, \exp\Big(i\alpha \int d^{d}r \, \chi^{*}\big(E + i\alpha 0^{+} - H\big)\chi \Big). \tag{9.5}$$

The SUSY method uses the fact that for every disorder realization $U(r)$ the partition function is a fermionic determinant, and its inverse can be written as a Gaussian integral over a complex bosonic field $\phi(r)$:

$$\frac{1}{Z} = \int \mathcal{D}\phi \, \exp\Big(i\alpha \int d^{d}r \, \phi^{*}\big(E + i\alpha 0^{+} - H\big)\phi \Big). \tag{9.6}$$

This leads to the represntation of a GF as a SUSY functional integral

$$G^{\alpha}(r, r'; E) = -i\alpha \int \mathcal{D}\phi \, \mathcal{D}\chi \, \chi(r)\chi^{*}(r') e^{-S_U} \tag{9.7}$$

with the action

$$S[U] = -i\alpha \int d^{d}r \, \Big(\phi^{*}\big(E + i\alpha 0^{+} - H\big)\phi + \chi^{*}\big(E + i\alpha 0^{+} - H\big)\chi \Big). \tag{9.8}$$

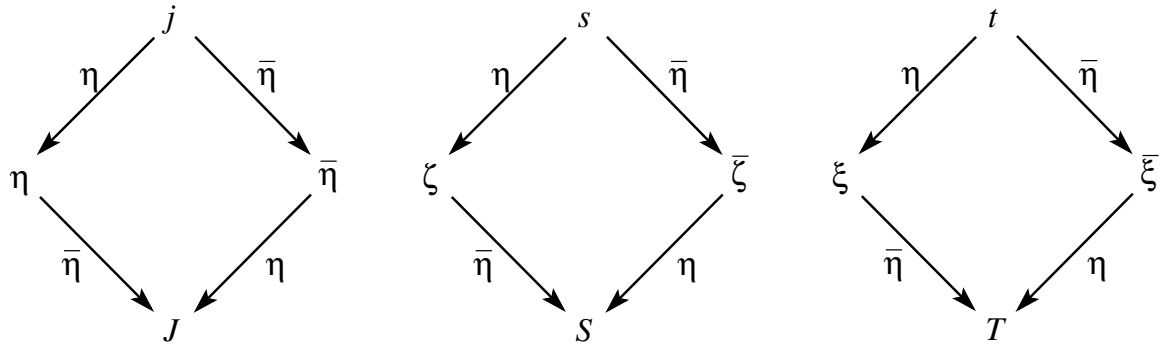

FIGURE 4. The u(1|1) supermultiplets. Left: the generators of u(1|1), middle: the action and its partners, right: the stress tensor and its partners.

In the SUSY form, the GFs and their products are easy to average over disorder. It is traditional to model the disorder potential $U(r)$ by a white noise with a Gaussian distribution

$$P[U(r)] \propto \exp\Big(-\frac{1}{2u} \int d^d r\, U^2(r) \Big). \tag{9.9}$$

Averaging over the disorder gives

$$\overline{\exp\Big(-i\alpha \int d^d r\, U(\phi^*\phi + \chi^*\chi) \Big)} = \exp\Big(-\frac{u}{2} \int d^d r\, (\phi^*\phi + \chi^*\chi)^2 \Big), \tag{9.10}$$

and the disorder-averaged GFs becomes

$$\overline{G^\alpha(r, r'; E)} = -i\alpha \int \mathcal{D}\phi\, \mathcal{D}\chi\, \chi(r)\chi^*(r') e^{-S_{\text{eff}}} \tag{9.11}$$

where the action is

$$S_{\text{eff}} = \int d^d r \Big(-i\alpha \big[\phi^*(E + i\alpha 0^+ - H)\phi + \chi^*(E + i\alpha 0^+ - H)\chi \big] + \frac{u}{2} (\phi^*\phi + \chi^*\chi)^2 \Big). \tag{9.12}$$

The SUSY method is easily extended to disorder averages of products of multiple GFs: for each GF we need to introduce its own "replica" of the pair (ϕ, χ). Notice, however, that no replica limit is necessary. The SUSY method can also be applied to network models.

9.1. Non-unitarity and logarithms. — Notice that in the SUSY form of a GF there is no denominator. The total partition function is unity:

$$Z_{\text{SUSY}} = 1, \tag{9.13}$$

since the fermionic and bosonic contributions to it cancel out. This is true before and after the disorder average. Thus, we see that if the SUSY method is used to study a critical point, the central charge of the resulting CFT is zero.

But there is a deeper structure in the SUSY method explained in Refs. [**14**]. The point is that both actions (9.8) and (9.12) are invariant under the action of the Lie superalgebra u(1|1), whose to bosonic generators J and j and two fermionic generators η and $\bar\eta$ form an indecomposable supermultiplet (the generator J commutes with all other generators). The members of the multiplet are connected by the adjoint action as shown in the left in Fig. 4. It turns out that any supersymmetric action S and the corresponding stress tensor T are members of similar multiplets, also shown in Fig. 4.

Symmetry Class	NLσM (n-c|c)	Compact (fermionic) space	Non-compact (bosonic) space
A (UE)	AIII|AIII	$\mathrm{U}(2N)/\mathrm{U}(N)\times\mathrm{U}(N)$	$\mathrm{U}(N,N)/\mathrm{U}(N)\times\mathrm{U}(N)$
AI (OE)	BDI|CII	$\mathrm{Sp}(4N)/\mathrm{Sp}(2N)\times\mathrm{Sp}(2N)$	$\mathrm{SO}(N,N)/\mathrm{SO}(N)\times\mathrm{SO}(N)$
AII (SE)	CII|BDI	$\mathrm{SO}(2N)/\mathrm{SO}(N)\times\mathrm{SO}(N)$	$\mathrm{Sp}(2N,2N)/\mathrm{Sp}(2N)\times\mathrm{Sp}(2N)$
AIII (chUE)	A|A	$\mathrm{U}(N)$	$\mathrm{GL}(N,\mathbb{C})/\mathrm{U}(N)$
BDI (chOE)	AI|AII	$\mathrm{U}(2N)/\mathrm{Sp}(2N)$	$\mathrm{GL}(N,\mathbb{R})/\mathrm{O}(N)$
CII (chSE)	AII|AI	$\mathrm{U}(N)/\mathrm{O}(N)$	$\mathrm{GL}(N,\mathbb{H})/\mathrm{Sp}(2N)$ $\equiv \mathrm{U}^*(2N)/\mathrm{Sp}(2N)$
C (SC)	DIII|CI	$\mathrm{Sp}(2N)/\mathrm{U}(N)$	$\mathrm{SO}^*(2N)/\mathrm{U}(N)$
CI (SC)	D|C	$\mathrm{Sp}(2N)$	$\mathrm{SO}(N,\mathbb{C})/\mathrm{SO}(N)$
BD (SC)	CI|DIII	$\mathrm{O}(2N)/\mathrm{U}(N)$	$\mathrm{Sp}(2N,\mathbb{R})/\mathrm{U}(N)$
DIII (SC)	C|D	$\mathrm{O}(N)$	$\mathrm{Sp}(2N,\mathbb{C})/\mathrm{Sp}(2N)$

TABLE 1. Sigma model spaces. Sigma model target spaces form large families of Riemannian symmetric superspaces. The last two columns list the compact and non-compact components of their 'base'.

In particular, the stess tensor T and the "top" field t form a logarithmic pair such that the dilatation operator D acts on them as a Jordan block:

$$D\begin{pmatrix} T \\ t \end{pmatrix} = \begin{pmatrix} d & 0 \\ 1 & d \end{pmatrix}\begin{pmatrix} T \\ t \end{pmatrix}. \tag{9.14}$$

Then the global conformal symmetry determines the form of the two-point functions

$$\langle T(r)T(0)\rangle = 0, \qquad \langle T(r)t(0)\rangle = \frac{b}{r^{2d}}, \qquad \langle t(r)t(0)\rangle = -\frac{2b[\ln r + O(1)]}{r^{2d}}. \tag{9.15}$$

This structure is the analog of the "collision" of operators in the replica limit. See Refs. [**20, 21, 18**] for more details.

9.2. Non-linear sigma models. —

10. Altland-Zirnbauer symmetry classification

11. Multifractality of wave functions at ATs

11.1. Multifractal spectra. — We consider exponents that characterize multifractal (MF) wave functions at Anderson localization-delocalization (LD) transitions in the bulk of a finite disordered electronic system of size L in d dimensions. To avoid the

issue of boundaries, let the system be defined on a torus. The main object of interest are the single-particle wave functions $\psi_n(r)$, normalized on the torus:

$$\int d^d r \, |\psi_n(r)|^2 = 1. \tag{11.1}$$

Our presentation here will be heuristic and will avoid all mathematical subtleties. For details and rigorous approaches see the book by Falconer [**?**]. A generic description of MF measures involves [**?**] breaking the system into little boxes B_i with linear size a, labeled by i. The number of these boxes N scales as $N = (L/a)^d$. One then calculates the probability p_i for an electron to be in the i-th box (the measure in this box) as

$$p_i = \int_{B_i} |\psi(r)|^2 d^d r, \tag{11.2}$$

and forms the so-called average generalized inverse participation ratios (IPRs)

$$\overline{P_q} = \sum_{i=1}^{N} \overline{p_i^q} = N \overline{p_i^q}. \tag{11.3}$$

(We have assumed that the system is homogeneous after disorder average.) At criticality the IPRs scale as

$$\overline{P_q} \sim \left(\frac{L}{a}\right)^{-\tau_q}, \tag{11.4}$$

where the set of exponents τ_q is usually referred to as the (average) *multifractal spectrum*.

The definition of the MF spectrum immediately implies several general properties of τ_q. First of all, $P_0 = N$ simply counts the number of boxes, while $P_1 = 1$ due to the normalization condition (11.1). Therefore,

$$\tau_0 = -d, \qquad\qquad \tau_1 = 0. \tag{11.5}$$

Next, the probabilities p_i whose moments enter the definition of $\overline{P_q}$ are bounded by $0 < p_i \leqslant 1$. Consequently, $\overline{P_q}$ must be a monotonously non-increasing function of q, since $p_i^{q_1} \geqslant p_i^{q_2}$ for $q_1 < q_2$. Then the multifractal spectrum τ_q in Eq. (11.4) must be a monotonously non-decreasing functions of q. Generally speaking, there may be a value of $q = q_t$ where τ_q has a horizontal tangent. Then it follows that $\tau_q = \tau_{q_t} = \text{const}$ for $q \geqslant q_t$. Such change in the behavior of τ_q from an increasing function to a constant is often referred to as 'freezing' or 'termination' (see Ref. [**?**] for more details). In all known cases the value q_t where such termination occurs satisfies $q_t > 0$.

It is instructive to present the two limiting cases of wave functions. First, an extended wave function's caricature is given by the uniform $|\psi|^2 = L^{-d}$. In this case it is elementary to see that $P_q = (L/a)^{-d(q-1)}$, giving the linear MF spectrum of extended states:

$$\tau_{\text{ext}} = d(q-1). \tag{11.6}$$

The other extreme case is that of a state localized within a localization volume with the linear size ξ (the localization length), which contains $N_\xi = (\xi/a)^d$ boxes. Only these boxes contribute to the IPRs, and we get $P_q = (\xi/a)^{-d(q-1)}$. There is no scaling with the system size L in this case, which is encoded in

$$\tau_{\text{loc}} = 0. \tag{11.7}$$

We also comment that numerical studies of critical wave functions are often done in lattice models, where the wave functions are defined on lattice sites i, and they are

normalized in such a way that $|\psi_i|^2$ is the probability for the electron to be on site i. In this case we can identify the lattice spacing with a, and $(L/a)^d = N$ with the total number of sites in the lattice. Then one looks at the scaling of the IPRs with the number of lattice sites:

$$\overline{P_q} = \sum_{i=1}^{N} \overline{|\psi_i|^{2q}} \sim N^{-\tau(q)/d}. \tag{11.8}$$

11.2. Multifractal analysis: geometry and probability. — MF measures can be characterized in an alternative way, focusing on their local geometry (scaling behavior) and interpreting it in terms of probability distribution of p_i or $|\psi_i|^2$. This is the point of view adopted in the so-called multifractal analysis, which we review here.

The basic idea is that even for a given sample we can bin the values of probabilities p_i and form a histogram for them. This approach makes even more sense in the case of an ensemble of samples, a natural setting for disordered systems. The histogram of p_i depends on the ratio a/L. To extract an object meaningful in the thermodynamic limit, we need to consider the local *singularity exponents* defined by $p_i = (a/L)^{\alpha_i}$. The bounds $0 < p_i \leqslant 1$ imply that the singularity exponents are bounded by $\alpha \geqslant 0$. Now, we consider the histogram of the singularity exponents and view it as the probability distribution of α.

11.3. Multifracal exponents and field theory. — Relations between general MF spectra and field theory were considered by Duplantier and Ludwig in Ref. [**?**]. For Anderson transitions such relations exist between moments of the local density of states (DOS) and the expectation values of certain operators in a sigma model, see Ref. [**?**], where it is rigorously derived using Green's function. Here we will present a heuristic argument leading to a somewhat schematic relation between wave function moments and local operators in afield theory.

Roughly speaking, identification is achieved by considering the (smeared) local density of states (DOS)

$$\nu_\eta(r, E) = \sum_n |\psi_n(r)|^2 \frac{\eta}{(E - E_n)^2 + \eta^2}, \tag{11.9}$$

where n labels the normalized energy eigenstates $\psi_n(r)$ at energy E_n. The energy levels are assumed to have a finite width η that serves as a regulator in the infinite volume limit. The global DOS $\nu_\eta(E)$ is the integral of the local DOS over the whole sample:

$$\nu_\eta(E) = \int d^d r\, \nu_\eta(r, E) = \sum_n \frac{\eta}{(E - E_n)^2 + \eta^2}. \tag{11.10}$$

In a sufficiently large system, where the mean level spacing $\delta = L^{-d}\nu_\eta(E)^{-1} \ll \eta$, many levels with close energies contribute to the local DOS. If we assume that all these wave functions with energies close to a given energy E show statistically identical scaling behavior, we can pull them out of the sum over the energies, and write schematically

$$\nu_\eta(r, E) \sim |\psi_E(r)|^2 \nu_\eta(E), \tag{11.11}$$

where $\psi_E(r)$ is a generic random wave function with energy sufficiently close to E.

Moments of the local DOS $\nu_\eta(r, E)$ are represented by expectation values of operators in the corresponding field theory [**?**]. We write this schematically as

$$\left[L^d \nu_\eta(r, E)\right]^q \sim \mathcal{O}_q(r). \tag{11.12}$$

This relation is understood in such a way that products of these operators can be considered, and then correlation functions of these operators in a field theory give the disorder averages of products of powers of the LDOS at different points. In view of Eq. (11.11), the same operator represents moments of the wave function $\psi_E(r)$:

$$\left[L^d \nu_\eta(E)\right]^q |\psi_E(r)|^{2q} \sim \mathcal{O}_q(r). \tag{11.13}$$

Notice that the global DOS $\nu(E)$ is self-averaging and can be replaced by its disorder average $\langle \nu(E) \rangle$.

Now we concentrate on the wave functions and the DOS *at the critical energy*, $E = E_c$, and *drop* the subscript E. The global DOS ν may vanish at criticality in the infinite system. In a finite system it has a power-law behavior

$$\nu \sim L^{-x_\nu}, \tag{11.14}$$

where the exponent x_ν vanishes in Wigner-Dyson classes but is known to be non-zero in other symmetry classes. For example, at the spin quantum Hall transition in symmetry class C [**?**, **?**, **?**], the exponent x_ν is known to be exactly $x_\nu = 1/4$ [**?**].

$\mathcal{O}_q$ is a scaling operator with the (bulk) scaling dimension x_q at the LD transition. Its one-point function (the expectation value) in a finite system of size L scales as

$$\left\langle \mathcal{O}_q(r) \right\rangle \sim L^{-x_q}. \tag{11.15}$$

Combining this with Eqs. (11.13, 11.14), we obtain the scaling of the moments of the critical wave functions:

$$\overline{|\psi(r)|^{2q}} \sim L^{-dq - x_q + q x_\nu}. \tag{11.16}$$

Comparison of Eqs. (11.16) and (11.4) implies the relations

$$\tau_q = d(q-1) + x_q - q x_\nu, \qquad\qquad x_q = \tau_q - d(q-1) + q x_\nu. \tag{11.17}$$

The values x_0 and x_1 follow from (11.5):

$$x_0 = 0, \qquad\qquad x_1 = x_\nu. \tag{11.18}$$

Often the so-called anomalous exponents Δ_q are introduces through $\tau_q = d(q-1) + \Delta_q$. These are related to x_q by

$$\Delta_q = x_q - q x_\nu, \tag{11.19}$$

and for both $q = 0$ and $q = 1$ they vanish: $\Delta_0 = \Delta_1 = 0$.

It is important for our discussion that for sufficiently large $|q|$ the dimensions x_q or Δ_q are negative (it will be demonstrated below). Thus, the one-point function (11.15) may grow with the system size. While this is a very unusual property from a (unitary) field theory point of view, it is natural when viewed from the perspective of the probability distribution of the wave function intensity.

The slope of x_q at $q = 0$ is (remember that, by definition, $\alpha_q = d\tau_q/dq$)

$$\left.\frac{dx_q}{dq}\right|_{q=0} = \alpha_0 - d + x_\nu. \tag{11.20}$$

The sign of the slope will play an important role in the following. From general properties of MF spectra (see Ref. [**?**]) it follows that $\alpha_0 > d$. Therefore, the sign of $\alpha_0 - d + x_\nu$ depends on whether $x_\nu > d - \alpha_0$ or the other way round. In the first case the slope $\alpha_0 - d + x_\nu > 0$, and it is in this case that our general arguments apply. We will assume that this is the case. For example, at the spin quantum Hall transition in class C $x_\nu = 1/4$, and (numerically) $2 - \alpha_0 \approx -0.137$.

Actually, generalizing arguments from Ref. [**?**] based on conformal invariance (in that paper we specifically focused on the case of $d = 2$) we can show that the combination in the right-hand side of the last equation is related to the typical localization length of a quasi-1D system on $S^{d-1} \times \mathbb{R}$. This shows that

$$\alpha_0 - d + x_\nu > 0. \tag{11.21}$$

Independently of the slope of x_q at $q = 0$, in the presence of termination of the τ_q spectrum at q_t, the value of x_q becomes negative for sufficiently large q. Therefore, there exists a special value of q, which we denote as q_* such that $x_{q_*} = 0$.

The operator $\mathcal{O}_{q_*}$ plays the role of the identity operator $\mathbb{1}$ in the field theory of the LD transition. We also have the Abelian OPE (which follows immediately from the definition (11.13) of the operators $\mathcal{O}_q$)

$$\mathcal{O}_{q_1}(r_1)\mathcal{O}_{q_2}(r_2) \propto r_{12}^{x_{q_1+q_2}-x_{q_1}-x_{q_2}} \mathcal{O}_{q_1+q_2}\left(\tfrac{r_1+r_2}{2}\right) + \ldots, \qquad r_{12} = |r_1 - r_2|. \tag{11.22}$$

Now choose $q_1 = q$, and $q_2 = q_* - q$. This will give

$$\mathcal{O}_q(r_1)\mathcal{O}_{2_*-q}(r_2) \sim \mathcal{O}_{q_*}\left(\tfrac{r_1+r_2}{2}\right) \sim \mathbb{1}. \tag{11.23}$$

12. Symmetries of MF spectra and generalized multifractal correlators

Now we assume the symmetry of the critical point with respect to global conformal transformations. Global conformal invariance dictates that a two-point correlator $\langle \mathcal{O}_1 \mathcal{O}_2 \rangle$ vanish unless the dimensions of the operators involved are equal. Then we see that the non-vanishing correlator $\langle \mathcal{O}_q(r_1)\mathcal{O}_{q_*-q}(r_2)\rangle$ implies that the dimensions of the operators involved are equal:

$$x_q = x_{q_*-q}. \tag{12.1}$$

Alternatively, we can say that the spectrum of dimensions x_q is symmetric across the symmetry point $q = q_*/2$. This general symmetry relation is the main result.

For the Wigner-Dyson classes $x_\nu = 0$, and the symmetry relation (12.4) leads tot he symmetry of the anomalous exponents:

$$\Delta_q = \Delta_{q_*-q}. \tag{12.2}$$

Mirlin et al. [**?**] have shown that in this case $q_* = 1$.

For Wigner-Dyson classes $\Delta_q = x_q$, and these dimensions have been shown by Mirlin et al. [**?**] to satisfy an exact symmetry relation

$$\Delta_q = \Delta_{1-q}. \tag{12.3}$$

This relation was generalized to all symmetry classes in Ref. [**?**]. There it was shown that the scaling dimensions x_q satisfy the symmetry relation

$$x_q = x_{q_*-q}. \tag{12.4}$$

Alternatively, we can say that the spectrum of dimensions x_q is symmetric across the symmetry point $q = q_*/2$. The symmetry point q_* depends on the symmetry class, as well as on the actual fixed point. The symmetry relation (12.4) implies that the operator $\mathcal{O}_{q_*}$ has vanishing dimension:

$$x_{q_*} = 0. \tag{12.5}$$

More general observables $\mathcal{O}_\lambda$ constructed of many wave functions have been considered in Ref. [**?**], where exact scaling operators have been identified. They turned out to be labeled by arbitrary Young tableaux or, equivalently, by integer partitions

$\lambda = (q_1, q_2, \ldots, q_n)$. The vector λ also has the meaning of a highest weight under the action of a Lie algebra. There is a discrete group W, called the Weyl group, acting in the weight space, and the scaling dimensions x_λ of the composite operators are invariant under the action of this group:

$$x_{w\lambda} = x_\lambda, \qquad \forall w \in W. \tag{12.6}$$

The important generators of the Weyl group are reflections in certain hyperplanes in the weight space. Their action on the weight $\lambda = (q_1, q_2, \ldots, q_n)$ in terms of its components q_j is as follows:

(i) sign inversion of $\tilde{q}_j \equiv q_j + \dfrac{c_j}{2}$ for any $j \in \{1, 2, \ldots, n\}$:

$$q_j \to -c_j - q_j, \tag{12.7}$$

(ii) permutation of $\tilde{q}_i = q_i + \dfrac{c_i}{2}$ and $\tilde{q}_j = q_j + \dfrac{c_j}{2}$ for some pair $i, j \in \{1, 2, \ldots, n\}$:

$$q_i \to q_j + \frac{c_j - c_i}{2}; \qquad\qquad q_j \to q_i + \frac{c_i - c_j}{2}. \tag{12.8}$$

The parameters c_i are the coefficients of the expansion of the bosonic part

$$\rho_b = \sum_{j=1}^{n} c_j e_j \tag{12.9}$$

of the half-sum of the positive roots ρ in a standard basis e_j. These coefficients are known for all families of symmetric superspaces. For class A they are $c_j = 1 - 2j$.

The previous case of the usual MF dimension corresponds to $\lambda = (1, 0, 0, \ldots, 0)$, and $c_1 = -q_*$.

13. Multifractal multipoint functions and RG

Equation (11.15) can be understood from an RG point of view. We need to run RG from the microscopic scale a up to the scale L, and the one-point functions at this scale becomes a number of order one. The result of the RG is the appearance of the scale factor L/a raised to the power $-x_q$.

Similar arguments can be used to determine the behavior of multi-point functions. For this we need to note that the definition (11.13) of the operators $\mathcal{O}_q$ suggests the following Abelian fusion rule:

$$\mathcal{O}_{q_1}(r_1)\mathcal{O}_{q_2}(r_2) \sim \mathcal{O}_{q_1+q_2}\big(\tfrac{r_1+r_2}{2}\big). \tag{13.1}$$

Notice that while this fusion rule seems natural, it may be incorrect, and requires an actual justification in any given model. We will assume this fusion rule from now on.

Let us consider the two-point function $\langle \mathcal{O}_{q_1}(r_1)\mathcal{O}_{q_2}(r_2)\rangle$. Now we run the RG up to the scale $r_{12} = |r_1 - r_2|$, which results in the renormalization factor $r_{12}^{-x_{q_1}-x_{q_2}}$. At this scale the two operators fuse to $\mathcal{O}_{q_1+q_2}$. Then we renormalize further up to scale L, which results in the additional factor $(L/r_{12})^{-x_{q_1+q_2}}$. Finally, at this scale the correlator is of order 1, and we get

$$\langle \mathcal{O}_{q_1}(r_1)\mathcal{O}_{q_2}(r_2)\rangle \sim r_{12}^{-x_{q_1}-x_{q_2}} \Big(\frac{L}{r_{12}}\Big)^{-x_{q_1+q_2}} = r_{12}^{x_{q_1+q_2}-x_{q_1}-x_{q_2}} L^{-x_{q_1+q_2}}. \tag{13.2}$$

The same result is obtained from the Abelian OPE (a more detailed version of the fusion rule (13.1)):

$$\mathcal{O}_{q_1}(r_1)\mathcal{O}_{q_2}(r_2) \propto r_{12}^{x_{q_1+q_2}-x_{q_1}-x_{q_2}} \mathcal{O}_{q_1+q_2}\left(\tfrac{r_1+r_2}{2}\right) + \dots \tag{13.3}$$

combined with Eq. (11.15) applied to the expectation value of $\mathcal{O}_{q_1+q_2}$.

Next consider the three-point function $\langle \mathcal{O}_{q_1}(r_1)\mathcal{O}_{q_2}(r_2)\mathcal{O}_{q_3}(r_3)\rangle$. To apply the RG argument, we need to know the hierarchy of distances between the three points. For example, let us assume that

$$r_{12} \ll r_{13} \sim r_{23} \ll L. \tag{13.4}$$

Then there will be three stages of RG and two fusions resulting in

$$\begin{aligned}
&\langle \mathcal{O}_{q_1}(r_1)\mathcal{O}_{q_2}(r_2)\mathcal{O}_{q_3}(r_3)\rangle \sim r_{12}^{-x_{q_1}-x_{q_2}-x_{q_3}} \langle \mathcal{O}_{q_1+q_2}(r_1/r_{12})\mathcal{O}_{q_3}(r_3/r_{12})\rangle \\
&\sim r_{12}^{-x_{q_1}-x_{q_2}-x_{q_3}} \left(\frac{r_{13}}{r_{12}}\right)^{-x_{q_1+q_2}-x_{q_3}} \left(\frac{L}{r_{13}}\right)^{-x_{q_1+q_2+q_3}} \\
&= r_{12}^{x_{q_1+q_2}-x_{q_1}-x_{q_2}} r_{13}^{x_{q_1+q_2+q_3}-x_{q_1+q_2}-x_{q_3}} L^{-x_{q_1+q_2+q_3}}.
\end{aligned} \tag{13.5}$$

Similar expressions are obtained in other cases when the distances between the points satisfy different inequalities.

Is also easy to generalize these expressions to higher multipoint functions. It should be clear from the discussion above that a generic n-point MF function $\langle \prod_{i=1}^n \mathcal{O}_{q_i}(r_i)\rangle$, where the pairwise distances r_{ij} are all much smaller than the system size L will scale with the system size as

$$\langle \mathcal{O}_{q_1}(r_1)\dots\mathcal{O}_{q_n}(r_n)\rangle \sim L^{-x_{q_1+\dots+q_n}}. \tag{13.6}$$

14. A summary of CFT results

Here I use the relevant information from Section 4.3.1 in [**?**].

A basic notion in conformal field theory (CFT) is that of a quasiprimary operator $\phi(r)$. Such operators transform in a very specific way under global conformal transformations, and the results summarized in this section are derived assuming the quasiprimary nature of all operators involved.

In a CFT the expectation value of any nontrivial quasiprimary field in the infinite plane is zero:

$$\langle \phi(r)\rangle_{\text{CFT}} = 0. \tag{14.1}$$

Only the identity operator has the obvious property

$$\langle \mathbb{1}\rangle_{\text{CFT}} = 1. \tag{14.2}$$

If we assume translation, rotation, and scale invariance, then the two-point function of fields with dimensions x_1 and x_2 is (see Eq. (4.51) in [**?**])

$$\langle \phi_1(r_1)\phi_2(r_2)\rangle_{\text{CFT}} = C_{12}\, r_{12}^{-x_1-x_2}, \qquad r_{12} = |r_1 - r_2|. \tag{14.3}$$

If one additionally assumes invariance under special conformal transformations, the two-point function is restricted further to be (see Eq. (4.55) in [**?**])

$$\langle \phi_1(r_1)\phi_2(r_2)\rangle_{\text{CFT}} = \begin{cases} C_{12}\, r_{12}^{-2x_1}, & \text{if } x_1 = x_2, \\ 0, & \text{if } x_1 \neq x_2. \end{cases} \tag{14.4}$$

Similar arguments based on the (finite) conformal symmetry in any dimensionality lead to the following expressions for three- and four-point functions:

$$\langle\phi_1(r_1)\phi_2(r_2)\phi_3(r_3)\rangle_{\rm CFT} = C_{123}\, r_{12}^{x_3-x_1-x_2} r_{13}^{x_2-x_1-x_3} r_{23}^{x_1-x_2-x_3}, \tag{14.5}$$

$$\langle\phi_1(r_1)\dots\phi_4(r_4)\rangle_{\rm CFT} = \prod_{i<j}^{4} r_{ij}^{x/3-x_i-x_j} F(X_1,X_2), \qquad x=\sum_{i=1}^{4} x_i. \tag{14.6}$$

In the last formula for the four-point function the undetermined function F depends on the two cross-ratios

$$X_1 = \frac{r_{12}r_{34}}{r_{13}r_{24}}, \qquad\qquad X_2 = \frac{r_{12}r_{34}}{r_{14}r_{23}}. \tag{14.7}$$

In two dimensions (2D), where the points are specified by complex coordinates z, correlation the two- and three-point functions factorize into holomorphc and antiholomorphic factors (with possibly different holomorphic and antiholomorphic dimensions h and $\bar h$):

$$\langle\phi_1(z_1,\bar z_1)\phi_2(z_2,\bar z_2)\rangle_{\rm CFT} = C_{12}\, z_{12}^{-2h}\bar z_{12}^{-2\bar h}, \qquad \text{if } h_1=h_2=h \text{ and } \bar h_1=\bar h_2=\bar h, \tag{14.8}$$

$$\begin{aligned}\langle\phi_1(z_1,\bar z_1)\phi_2(z_2,\bar z_2)\phi_3(z_3,\bar z_3)\rangle_{\rm CFT} &= C_{123} z_{12}^{h_3-h_1-h_2} z_{13}^{h_2-h_1-h_3} z_{23}^{h_1-h_2-h_3} \\ &\times \bar z_{12}^{\bar h_3-\bar h_1-\bar h_2} \bar z_{13}^{\bar h_2-\bar h_1-\bar h_3} \bar z_{23}^{\bar h_1-\bar h_2-\bar h_3}.\end{aligned} \tag{14.9}$$

For four points in two dimensions the two cross-ratios are not independent, since the four points involved are in the same plane. Denoting $z_{ij}=z_i-z_j$, we have

$$\eta = \frac{z_{12}z_{34}}{z_{13}z_{24}}, \qquad 1-\eta = \frac{z_{14}z_{23}}{z_{13}z_{24}}, \qquad \frac{\eta}{1-\eta} = \frac{z_{12}z_{34}}{z_{14}z_{23}}. \tag{14.10}$$

Then the four-point function can be written as

$$\langle\phi_1(z_1,\bar z_1)\dots\phi_4(z_4,\bar z_4)\rangle_{\rm CFT} = \prod_{i<j}^{4} z_{ij}^{h/3-h_i-h_j} \bar z_{ij}^{\bar h/3-\bar h_i-\bar h_j} F(\eta,\bar\eta), \quad h=\sum_{i=1}^{4} h_i, \quad \bar h=\sum_{i=1}^{4}\bar h_i. \tag{14.11}$$

15. Relation of MF multipoint functions and CFT correlators

The previous discussion immediately implies that not all MF multipoint functions have representation as CFT correlators. For example, even the one-point function (11.15) is consistent with Eq. (14.1) only if $x_q > 0$. But as we have mentioned already, for a whole infinite range of q the dimensions $x_q < 0$. In addition, as we have seen, there is a non-trivial operator $\mathcal{O}_{q*}$, distinct from the identity operator, whose dimension is zero, and whose expectation value in the infinite system is one:

$$\langle\mathcal{O}_{q_*}\rangle = 1. \tag{15.1}$$

This indicates that we cannot expect all aspects of MF multipoint functions to be describable by a CFT.

Let us look at the two-point function (13.2). We notice the following: we can choose $q_2=-q_1$, and then the dimension $x_{q_1+q_2}=x_0=0$. This implies that with this choice the two-point function stops depending on the system size:

$$\langle\mathcal{O}_{q_1}(r_1)\mathcal{O}_{-q_1}(r_2)\rangle \sim r_{12}^{-x_{q_1}-x_{-q_1}}\left(\frac{L}{r_{12}}\right)^{-x_0} = r_{12}^{-x_{q_1}-x_{-q_1}}. \tag{15.2}$$

Then the system size can be taken to infinity, and the two-point function can be compared with a two-point function in a critical field theory. We see that the above form is consistent with requirements of rotational, translational and scale invariance, Eq. (14.3), but not with conformal invariance, Eq. (14.4), since the dimensions $x_{q_1} \neq x_{-q_1}$.

However, the presence of the special operator $\mathcal{O}_{q_*}$ allows us to make another choice. Indeed, if we choose the special value $q_2 = q_* - q_1$, the dimension $x_{q_1+q_2} = x_{q_*} = 0$. In this case we have

$$\langle \mathcal{O}_{q_1}(r_1)\mathcal{O}_{q_*-q_1}(r_2)\rangle \sim r_{12}^{-x_{q_1}-x_{q_*-q_1}}\left(\frac{L}{r_{12}}\right)^{-x_{q_*}} = r_{12}^{-2x_{q_1}}. \tag{15.3}$$

This two-point is independent of the system size and is also consistent with the CFT form (14.4), since, due to the symmetry relation (12.4) the two operators involved have the same dimensions.

This discussion and the scaling (13.6) suggest that CFT may apply only to such multipoint MF functions $\langle \mathcal{O}_{q_1} \dots \mathcal{O}_{q_n}\rangle$ where the sum of the indices q_i is equal to q_* (a sort of "neutrality" condition):

$$\sum_{i=1}^{n} q_i = q_*. \tag{15.4}$$

An alternative choice $\sum_i q_i = 0$ leads to MF multipoint functions that are well defined in the infinite system, but are inconsistent with CFT.

For example, if we make the choice (15.4) in the three-point function (13.5), we get

$$\langle \mathcal{O}_{q_1}(r_1)\mathcal{O}_{q_2}(r_2)\mathcal{O}_{q_*-q_1-q_2}(r_3)\rangle \sim r_{12}^{x_{q_1+q_2}-x_{q_1}-x_{q_2}} r_{13}^{-2x_{q_1+q_2}}. \tag{15.5}$$

If we specify the general CFT expression (14.5) to our case and use the symmetry relation (12.4) we get

$$\langle \mathcal{O}_{q_1}(r_1)\mathcal{O}_{q_2}(r_2)\mathcal{O}_{q_*-q_1-q_2}(r_3)\rangle_{\rm CFT} \propto r_{12}^{x_{q_1+q_2}-x_{q_1}-x_{q_2}} r_{13}^{x_{q_2}-x_{q_1}-x_{q_1+q_2}} r_{23}^{x_{q_1}-x_{q_2}-x_{q_1+q_2}}. \tag{15.6}$$

If we now assume that the arrangement of points satisfies the first inequality in (13.4), we can replace $r_{23} \approx r_{13}$, and the three-point function simplifies to

$$\langle \mathcal{O}_{q_1}(r_1)\mathcal{O}_{q_2}(r_2)\mathcal{O}_{q_*-q_1-q_2}(r_3)\rangle_{\rm CFT} \sim r_{12}^{x_{q_1+q_2}-x_{q_1}-x_{q_2}} r_{13}^{-2x_{q_1+q_2}}, \tag{15.7}$$

which is the same as Eq. (15.5).

On the other hand, the choice $\sum_i q_i = 0$ leads to inconsistent expressions. The MF three-point function (13.5) in this case becomes

$$\langle \mathcal{O}_{q_1}(r_1)\mathcal{O}_{q_2}(r_2)\mathcal{O}_{-q_1-q_2}(r_3)\rangle \sim r_{12}^{x_{q_1+q_2}-x_{q_1}-x_{q_2}} r_{13}^{-x_{q_1+q_2}-x_{-q_1-q_2}}, \tag{15.8}$$

while the CFT expression (14.5) reduces to a different form

$$\langle \mathcal{O}_{q_1}(r_1)\mathcal{O}_{q_2}(r_2)\mathcal{O}_{-q_1-q_2}(r_3)\rangle_{\rm CFT} \sim r_{12}^{x_{-q_1-q_2}-x_{q_1}-x_{q_2}} r_{13}^{-2x_{-q_1-q_2}}. \tag{15.9}$$

In the above discussion we used an implicit assumption that the operators $\mathcal{O}_q$ are quasiprimary.

Before proceeding, let me point out another type of relations between CFT correlators and MF multipoint functions. Let us consider the two-point function (15.3) again. Let us now place the operator $\mathcal{O}_{q_*-q_1}$ at a large distance L (which is not the system size) from the point $r_1 = 0$. Then I can rewrite the two-point function as

$$L^{x_{q_1}}\langle \mathcal{O}_{q_1}(0)\mathcal{O}_{q_*-q_1}(L)\rangle_{\rm CFT} \sim L^{-x_{q_1}}. \tag{15.10}$$

This should be compared with Eq. (11.15). Similarly, if we place the operator $\mathcal{O}_{q_*-q_1-q_2}$ in the three-point function (15.6) at a distance $L \gg r_{12}$ from the points r_1 and r_2, we can replace $r_{13} \approx r_{23} \approx L$, and the we get

$$\langle \mathcal{O}_{q_1}(r_1)\mathcal{O}_{q_2}(r_2)\mathcal{O}_{q_*-q_1-q_2}(L)\rangle_{\rm CFT} \sim r_{12}^{x_{q_1+q_2}-x_{q_1}-x_{q_2}} L^{-2x_{q_1+q_2}}. \tag{15.11}$$

This can be rewritten also as

$$L^{x_{q_1+q_2}}\langle \mathcal{O}_{q_1}(r_1)\mathcal{O}_{q_2}(r_2)\mathcal{O}_{q_*-q_1-q_2}(L)\rangle_{\rm CFT} \sim r_{12}^{x_{q_1+q_2}-x_{q_1}-x_{q_2}} L^{-x_{q_1+q_2}}, \tag{15.12}$$

which is exactly the same as Eq. (13.2). Thus, a three-point CFT function can be related both to a three-point MF function in an infinite system and a two-point MF function in a finite system.

Such relations continue for higher correlators. In particular, we can consider the four-point function $\langle \mathcal{O}_{q_1}(r_1)\mathcal{O}_{q_2}(r_2)\mathcal{O}_{q_3}(r_3)\mathcal{O}_{q_*-q_1-q_2-q_3}(L)\rangle_{\rm CFT}$, which in a certain limit can be related to the three-point function (13.5).

16. CFT and exact parabolicity of MF spectra

In a recent work [**?**] Bondesan and Zirnbauer considered scaling operators $\mathcal{O}_q(z,\bar z)$ in the context of the integer quantum Hall transition in 2D. This critical point belongs to class A, so $q_* = 1$ and $x_q = \Delta_q$. In this case the operators $\mathcal{O}_q$ are not expected to have conformal spin, so $h_q = \bar h_q = \Delta_q/2$.

The main results of Bondesan and Zirnbauer is the exact parabolic form of the dimensions Δ_q:

$$\Delta_q = bq(1-q), \tag{16.1}$$

with constant b left undetermined. They have also found an explicit form of a certain four-point function that is, essentially, the correlator

$$\langle \mathcal{O}_{q_1}(z_1,\bar z_1)\dots\mathcal{O}_{1-q_1-q_2-q_3}(z_4,\bar z_4)\rangle_{\rm CFT} = \prod_{i<j}^{4} |z_{ij}|^{\Delta/3-\Delta_{q_i}-\Delta_{q_j}} F(\eta,\bar\eta), \tag{16.2}$$

$$\Delta = \Delta_{q_1} + \Delta_{q_2} + \Delta_{q_3} + \Delta_{q_1+q_2+q_3}. \tag{16.3}$$

More precisely, in Ref. [**?**] the "neutrality" condition (15.4) is achieved by considering $n-1$ insertions of the operators $\mathcal{O}_q$ plus one operator π_c describing a point contact. This operator, having the character of an integral over the continuum of q values, always allows for a fusion channel with the correct total "charge" $q_* = 1$.

Let me briefly summarize the argument of Ref. [**?**]. The authors make two essential assumptions: 1) the operators $\mathcal{O}_q$ are not only quasiprimary, but actually primary, having simple transformation law for any conformal transformation in 2D; 2) the operators $\mathcal{O}_q$ satisfy (the 2D variant of) the Abelian OPE (13.3):

$$\mathcal{O}_{q_1}(z_1,\bar z_1)\mathcal{O}_{q_2}(z_2,\bar z_2) \propto |z_{12}|^{\Delta_{q_1+q_2}-\Delta_{q_1}-\Delta_{q_2}} \mathcal{O}_{q_1+q_2}(z_2,\bar z_2) + \dots \tag{16.4}$$

The Abelian fusion immediately implies that there is only one conformal block in the correlator (16.2), that is $F(\eta,\bar\eta) = |f(\eta)|^2$. The holomorphic function $f(\eta)$ is present in the holomorphic factor of the correlator (16.2):

$$G(z_1,\dots,z_4) = \langle \mathcal{O}_{q_1}(z_1)\dots\mathcal{O}_{1-q_1-q_2-q_3}(z_4)\rangle_{\rm CFT} = \prod_{i<j}^{4} z_{ij}^{h/3-h_{q_i}-h_{q_j}} f(\eta), \tag{16.5}$$

$$h = h_{q_1} + h_{q_2} + h_{q_3} + h_{q_1+q_2+q_3}. \tag{16.6}$$

There are three possible fusion channels in this correlator, where each of the charges q_1, q_2, and q_3 fuses with the neutralizing charge $1-q_1-q_2-q_3$. These three channels lead to the simple power-law singularities of the functions $f(\eta)$ at the points $\eta = 1, 0$ and ∞. For example, fusing q_1 with q_2, and q_3 with $1-q_1-q_2-q_3$ (so that $|\eta| \ll 1$) gives

$$G(z_1,\ldots,z_4) \sim z_{12}^{h_{q_1+q_2}-h_{q_1}-h_{q_2}} z_{34}^{h_{q_1+q_2}-h_{q_3}-h_{q_1+q_2+q_3}} z_{13}^{-2h_{q_1+q_2}}. \tag{16.7}$$

On the other hand, the right-hand side of Eq. (16.5) where the distances between points are chosen appropriately ($|z_{12}|, |z_{34}| \ll |z_{13}| \approx |z_{14}| \approx |z_{23}| \approx |z_{24}|$) becomes

$$G(z_1,\ldots,z_4) \sim f(\eta)\, z_{12}^{h/3-h_{q_1}-h_{q_2}} z_{34}^{h/3-h_{q_3}-h_{q_1+q_2+q_3}} z_{13}^{-2h/3}. \tag{16.8}$$

Comparison of the two expressions gives that near $\eta = 0$

$$f(\eta) = \eta^{h_{q_1+q_2}-h/3}[a_0 + O(\eta)]. \tag{16.9}$$

The other two fusion channels give the forms of the singularities near the other two singular points:

$$f(\eta) = (1-\eta)^{h_{q_2+q_3}-h/3}[b_0 + O(1-\eta)], \tag{16.10}$$

$$f(\eta) = (1/\eta)^{h_{q_1+q_3}-h/3}[c_0 + O(1/\eta)]. \tag{16.11}$$

Explain the branch points, branch cuts, and monodromy.

The three found singularities of a single holomorphic function are only consistent if

$$f(\eta) = \eta^{h_{q_1+q_2}-h/3}(1-\eta)^{h_{q_2+q_3}-h/3}[a_0 + a_1\eta + \cdots + a_M\eta^M], \tag{16.12}$$

where M must be a non-negative integer which is actually fixed by the singularity at $\eta = \infty$:

$$M = h - h_{q_1+q_2} - h_{q_1+q_3} - h_{q_2+q_3} \geqslant 0. \tag{16.13}$$

In fact, choosing $q_1 = q_2 = q_3 = 0$ we see that $M = 0$. This fixes the conformal block

$$f(\eta) = a_0\eta^{h_{q_1+q_2}-h/3}(1-\eta)^{h_{q_2+q_3}-h/3}. \tag{16.14}$$

Moreover, the condition $M = 0$ gives the following functional equation for the dimensions h_q:

$$h_{q_1+q_2+q_3} - h_{q_1+q_2} - h_{q_1+q_3} - h_{q_2+q_3} + h_{q_1} + h_{q_2} + h_{q_3} = 0, \tag{16.15}$$

or, for the full dimensions Δ_q,

$$\Delta_{q_1+q_2+q_3} - \Delta_{q_1+q_2} - \Delta_{q_1+q_3} - \Delta_{q_2+q_3} + \Delta_{q_1} + \Delta_{q_2} + \Delta_{q_3} = 0. \tag{16.16}$$

This equation immediately implies $\Delta_0 = 0$, as it should be. Setting $q_1 = q$, $q_2 = q_3 = \epsilon$ gives

$$\Delta_{q+2\epsilon} - 2\Delta_{q+\epsilon} - \Delta_{2\epsilon} + \Delta_q + 2\Delta_\epsilon = 0. \tag{16.17}$$

Now we expand up to second order in ϵ:

$$\begin{aligned}&\Delta_q + 2\Delta_q'\epsilon + 2\Delta_q''\epsilon^2 - 2\Delta_q - 2\Delta_q'\epsilon - \Delta_q''\epsilon^2 - 2\Delta_0'\epsilon - 2\Delta_0''\epsilon^2 + \Delta_q + 2\Delta_0'\epsilon + \Delta_0''\epsilon^2 \\ &\quad = (\Delta_q'' - \Delta_0'')\epsilon^2 = 0.\end{aligned} \tag{16.18}$$

Thus $\Delta_q'' = \Delta_0'' = \text{const}$, which implies that the function Δ_q is a quadratic polynomial. A general quadratic polynomial that vanishes at $q = 0$ and satisfies the symmetry property (12.4) is exactly of the form given in Eq. (16.1).

16.1. Generalizations. — Now we can generalize the arguments by Bondesan and Zirnbauer in two directions.

First, the arguments seem not to be specific to the IQH critical point. Indeed, the Abelian fusion is obvious from the definition of the scaling operators $\mathcal{O}_q$ (especially when viewed as "plane waves" on the target space of a sigma model). Conformal invariance and the primary nature of the operators must be assumed in any case, but this seems to be equally natural for the IQH transition as well as for other 2D critical points of disordered electrons. Then we conclude that whenever we have the symmetry relation (12.4), that is, in the five classes identified in Refs. [**?**, **?**], we also should have the exact relation

$$x_q = bq(q_* - q) \tag{16.19}$$

for the simple MF spectra.

Secondly, we can obtain a generalization of this parabolic form for all general scaling operators $\mathcal{O}_\lambda$ that we have considered in [**?**]. Indeed, there is still a general operator that is analogous to $\mathcal{O}_{q_*}$ with vanishing dimension. In this case this operator is $\mathcal{O}_{-\rho_b} = \mathcal{O}_{(-c_1,-c_2,\dots)}$. Presumably, a point contact can play the role of this operator in the setting of a network model.

The operators $\mathcal{O}_\lambda$ still satisfy the Abelian fusion:

$$\mathcal{O}_\lambda(r_1)\mathcal{O}_{\lambda'}(r_2) \propto r_{12}^{x_{\lambda+\lambda'}-x_\lambda-x_{\lambda'}} \mathcal{O}_{\lambda+\lambda'}\left(\tfrac{r_1+r_2}{2}\right) + \dots \tag{16.20}$$

Again, this is obvious from the form of these operators as exponential functions on the target space of a sigma model.

We also assume conformal invariance. Then the arguments of Bondesan and Zirnbauer can immediately be generalized to give the following equation for the dimensions:

$$x_{\lambda+\lambda'+\lambda''} - x_{\lambda+\lambda'} - x_{\lambda+\lambda''} - x_{\lambda'+\lambda''} + x_\lambda + x_{\lambda'} + x_{\lambda''} = 0. \tag{16.21}$$

Let me denote by $e_i = (0,\dots,1,\dots,0)$ (unit in the i-th place) the standard basis in the weight space, and choose $\lambda' = \epsilon' e_i$, $\lambda'' = \epsilon'' e_j$:

$$x_{\lambda+\epsilon' e_i+\epsilon'' e_j} - x_{\lambda+\epsilon' e_i} - x_{\lambda+\epsilon'' e_j} - x_{\epsilon' e_i+\epsilon'' e_j} + x_\lambda + x_{\epsilon' e_i} + x_{\epsilon'' e_j} = 0. \tag{16.22}$$

Then I expand this to second order in ϵ' and ϵ'', denoting $\partial_i = \partial/\partial q_i$:

$$x_{\lambda+\epsilon' e_i} = x_\lambda + \partial_i x_\lambda \epsilon' + \frac{1}{2}\partial_i^2 x_\lambda \epsilon'^2, \qquad x_{\lambda+\epsilon'' e_j} = x_\lambda + \partial_j x_\lambda \epsilon'' + \frac{1}{2}\partial_j^2 x_\lambda \epsilon''^2,$$

$$x_{\lambda+\epsilon' e_i+\epsilon'' e_j} = x_\lambda + \partial_i x_\lambda \epsilon' + \partial_j x_\lambda \epsilon'' + \frac{1}{2}\partial_i^2 x_\lambda \epsilon'^2 + \frac{1}{2}\partial_j^2 x_\lambda \epsilon''^2 + \partial_i\partial_j x_\lambda \epsilon'\epsilon''. \tag{16.23}$$

The zeroth and the first orders are satisfied identically. In the second order we have

$$\begin{aligned} &\frac{1}{2}\partial_i^2 x_\lambda \epsilon'^2 + \frac{1}{2}\partial_j^2 x_\lambda \epsilon''^2 + \partial_i\partial_j x_\lambda \epsilon'\epsilon'' - \frac{1}{2}\partial_i^2 x_\lambda \epsilon'^2 - \frac{1}{2}\partial_j^2 x_\lambda \epsilon''^2 - (\lambda \leftrightarrow 0) \\ &= (\partial_i\partial_j x_\lambda - \partial_i\partial_j x_0)\epsilon'\epsilon'' = 0. \end{aligned} \tag{16.24}$$

This gives

$$\partial_i\partial_j x_\lambda = \partial_i\partial_j x_0 = \text{const}, \tag{16.25}$$

which implies that x_λ is a quadratic function of q_i. Thus, it is also a quadratic function of $\tilde{q}_i$. But in these variable it must be even due to the first type of the Weyl reflections (sign inversions of $\tilde{q}_i$), which restricts it to

$$x_\lambda = \sum_i b_i \tilde{q}_i^2 + B. \tag{16.26}$$

The other type of Weyl reflections (permutations of different $\tilde{q}_i$'s) forces all coefficients b_i to be equal: $b_i = -b$. Finally, the constant B is found from the requirement $x_0 = 0$, which gives $B = b\sum_i c_i^2/4$, and

$$x_\lambda = -b\sum_i q_i(q_i + c_i) = -b(\lambda, \lambda + \rho_b). \tag{16.27}$$

Thus, the generalized dimensions are proportional to the value of the quadratic Casimir operator in the representation labeled by the highest weight λ. Now choosing $\lambda = qe_1$, we see that the constant b is the same as in the simple MF spectrum (16.19).

17. Network models

17.1. SUSY method for networks. —

17.2. Mappings to classical stat mech and geometric randomness problems. —

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