Geometrical correlators in 2d CFTs: An introduction with some open problems

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Correlated percolation and phase transitions

- Consider the ferromagnetic Ising model

$$H_{\text{Ising}} = -J \sum_{\langle x,y \rangle} s(x)s(y) - H \sum_x s(x); \quad s(x) = \pm 1$$

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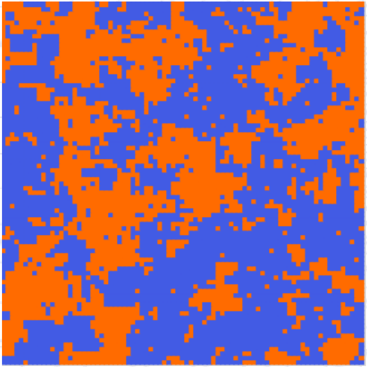

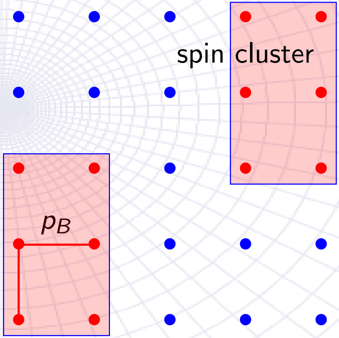

- **Fisher [1967] droplet model**: how to chose p_B to reproduce Ising critical exponents via a correlated percolation problem?

Droplets as FK clusters [Coniglio, Klein '80s]

- Consider the ferromagnetic Q-color Potts model

$$H_{\text{Potts}} = -J \sum_{\langle x,y \rangle} \delta_{s(x),s(y)}; \;\; s(x) = 1, \dots Q$$

- By rewriting the Boltzmann weight as

$$e^{-H_{\text{Potts}}} = \prod_{\langle x,y \rangle} (1 - p_B) + p_B \delta_{s(x),s(y)}; \;\; p_B = 1 - e^{-J}$$

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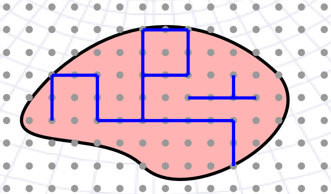
- The partition function is expanded into FK graphs [Fortuin, Kasteleyn '70s]

$$p_B^{\#\ \text{bonds}} (1 - p_B)^{\#\ \text{empty bonds}}$$

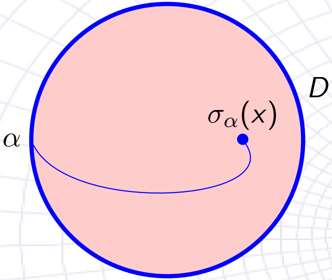
$$Z_{\text{Q}} = \sum_{\mathcal{G}}$$



$$\times\, Q^{\#}\ \text{clusters}$$

Connectivities of FK clusters

- Critical exponents of Potts are reproduced by the graph expansion



- For instance consider the $Q-1$ dim. **vector** $\sigma_\alpha(x) = \delta_{s(x),\alpha} - 1/Q$

$$\langle \sigma_\alpha(x) \rangle = \frac{Q}{Q-1} P(x \leftrightarrow \partial D)$$

- However, the percolation problem contains much **more** observables

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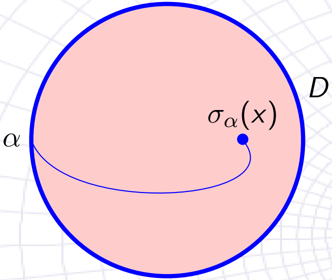

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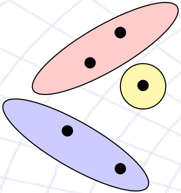
Connectivities can be defined for any real Q

$$P_{a_1 a_2 \ldots a_n}(x_1, \ldots, x_n) = \{\text{Probability that } x_i \text{ belongs to the cluster } a_i\}$$

- **Q:** How many independent connectivities are there? Can be calculated at criticality? Equivalent to solve the model for $Q \in \mathbb{R}$

Vector space of geometrical correlators [Delfino, V. '11]

- Connectivities are probability to partition a set of points into clusters



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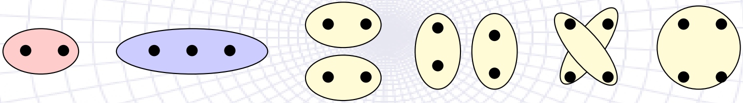
- There are **sum rules**: a basis can be formed by partitions without isolated points. For instance

P_{aa} P_{aaa} P_{aabb} P_{abba} P_{abab} P_{aaaa}



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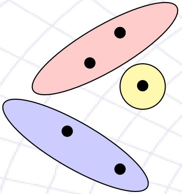

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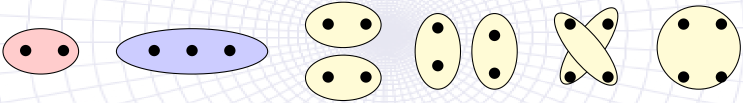

- They can be expressed **formally** in terms of S_Q inequivalent spin cfs

$$\begin{aligned}
G_{\alpha\alpha\alpha\alpha} &= Q_1(Q^2 - 3Q + 3)P_{aaaa} + Q_1^2(P_{aabb} + P_{abba} + P_{abab}), \\
G_{\alpha\alpha\beta\beta} &= (2Q - 3)P_{aaaa} + Q_1^2 P_{aabb} + P_{abba} + P_{abab} \\
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\end{aligned}$$

The puzzle of the three-point connectivity [Delfino, V. '11; Picco, Santachiara, V., Delfino '13]

- In particular, the **three point connectivity** is

$$P_{aaa}(x_1, x_2, x_3) = \frac{\langle \sigma_\alpha(x_1)\sigma_\alpha(x_2)\sigma_\alpha(x_3)\rangle}{Q(Q-1)(Q-2)}$$

- In any d, **analytic continuation** of the OPE is required at $Q = 2$. Consider $d = 2$ from now on and $T = T_c$. We postulate in the scaling limit (up to a non-universal normalization)

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$$P_{aaa}(x_1, x_2, x_3) = \langle \phi_{\Delta_\sigma}(x_1)\phi_{\Delta_\sigma}(x_2)\phi_{\Delta_\sigma}(x_3)\rangle$$

- Correct scaling of the two point function requires

$$\Delta_\sigma = 2h_{\frac{1}{2},0}, \quad h_{r,s} = \frac{(r(m+1)-sm)^2-1}{4m(m+1)}, \quad c = 1 - \frac{6}{m(m+1)}$$

- For the Q-color Potts model: $Q = 4\cos^2 \frac{\pi}{m+1}$.

The puzzle of the three-point connectivity

- From global conformal invariance, one has

$$P_{aaa}(x_1, x_2, x_3) = R(Q)\sqrt{P_{aa}(x_1, x_2)P_{aa}(x_2, x_3)P_{aa}(x_1, x_3)}$$

- By **analytically continuing** in the discrete indexes (r, s) the Dotsenko Fateev structure constant (for A-series), we argued

$$R(Q) = \sqrt{2}C^{\text{Liouville}}_{c<1}(\Delta_\sigma, \Delta_\sigma, \Delta_\sigma)$$

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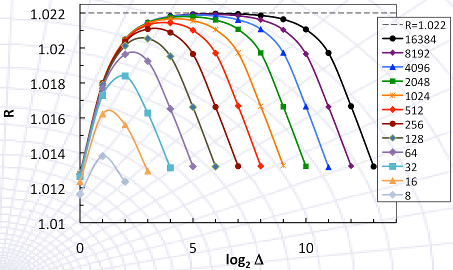
Remarks on Liouville theory with $c<1$

- $C^{\text{Liouville}}_{c<1}$ defined as **(unique)** solution of the Liouville shift equations continued to $c<1$ [Al. Zamolodchikov; Kostov, Petkova; Schomerus 06; Harlow, Maltz, Witten '11]
- Considered **'misterious'** [Al. Zamolodchikov] since $C(\Delta_1, 0, \Delta_3) \neq 0$, even if $\Delta_1 \neq \Delta_3$. In fact this implies that the theory is non-unitary [Harlow, Maltz, Witten '11] but it might exist.
- Proven to produce to **crossing symmetric** 4pt fcs. [Ribault, Santachiara '16], by taking the same spectrum as Liouville with $c>25$.

Tests and applications of Liouville 3pt function

- The conjecture has been checked by MC. For instance at $Q = 1$, [Ziff '11], for three points on an equilateral triangle embedded into a torus.



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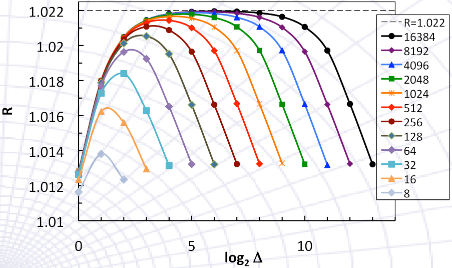
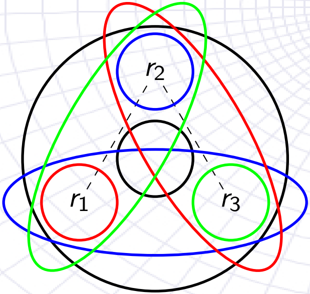




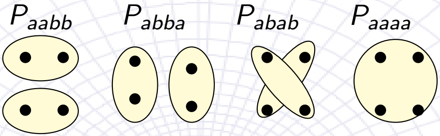
- A full interpretation was given in terms of partition functions of critical loop ensembles [Ikhlef, Jacobsen, Saleur '16]

$$C_{c<1}^{\text{Liouville}}(\Delta_1, \Delta_2, \Delta_3) \propto Z_{n_1,n_2,n_3}$$

- The loop model interpretation solved the 'mistery': $C(\Delta_1, 0, \Delta_3) \neq 0$.

The four-point FK connectivities: this afternoon

- We have seen that there are four lin. independent **4-pt connectivities**



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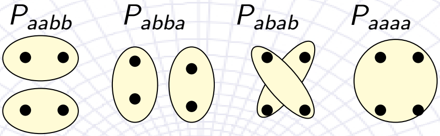


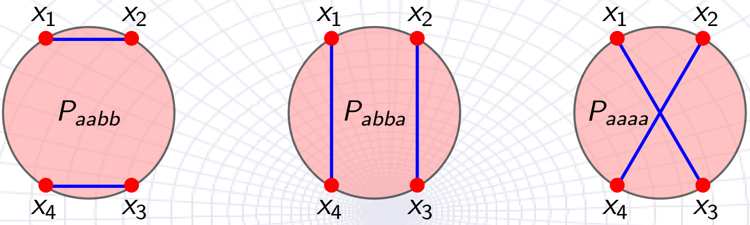
- They have been recently calculated by **conformal bootstrap** [Yifei talk]. In particular from the bootstrap solution and geometry

$$\lim_{x_1 \to x_2} P_{aaaa}(x_1, x_2, x_3, x_4) \simeq \frac{(\sqrt{2} C^{\text{Liouville}}_{c<1})^2}{|x_{12}|^{2\Delta_\sigma} |x_{34}|^{2\Delta_\sigma}} |\eta|^{2\Delta_\sigma} + \dots$$

- Thus allowing also to check the Liouville 3pt function with > 20 **digits** of accuracy [Nivesvivat, Ribault '20]
- In the following, I'll sketch a simpler analytic solution valid for points on the **boundary** [Gori, V. '18]

The boundary case: Ising FK connectivities

- To simplify even futher, consider the **Ising case**. By **planarity**, when the points are on the boundary there are 3 independent 4pt functions



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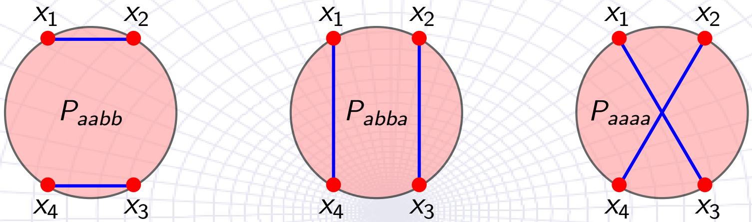

- **Boundary conditions** are free. By scaling arguments and the FK mapping, they should be extracted from

$$\langle \phi_{\Delta_\sigma}(x_1)\phi_{\Delta_\sigma}(x_2)\phi_{\Delta_\sigma}(x_3)\phi_{\Delta_\sigma}(x_4)\rangle|_{\mathcal{S}}; \quad \Delta_\sigma = h_{1,3} = \tfrac{1}{2}, \quad \mathrm{spin}(\phi_{\Delta_\sigma}) = \tfrac{1}{2}$$

- The choice of the intermediate channels $\mathcal{S}$ is fixed by **simmetries** and **degeneracy** of the Virasoro algebra representation $h_{1,3} = 1/2$ at $c = 1/2$

The boundary case: Ising FK connectivities

- Indeed, the OPE is fixed by the condition of level three null vector decoupling

$$\phi_{1,3} \cdot \phi_{1,3} = 1 + \phi_{1,3} + \phi_{1,5}$$

- In the minimal BCFT, the spin at the boundary is by duality a **free Majorana fermion** $\Psi = \phi_{2,1}$

$$\Psi \cdot \Psi = 1.$$

$Q = 2$ $\mathbb{Z}_2$ Ising

$s \backslash r$	1	2	3	4	5
5	$\frac{5}{2}$	1	$\frac{1}{6}$	0	$\frac{1}{2}$
4	$\frac{21}{16}$	$\frac{5}{16}$	$\frac{1}{6}$	$-\frac{1}{48}$	$\frac{5}{16}$
3	$\frac{1}{2}$	0	$\frac{1}{6}$	1	$\frac{5}{2}$
2	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{35}{48}$	$\frac{33}{16}$	$\frac{65}{116}$
1	0	$\frac{1}{2}$	$\frac{5}{3}$	$\frac{7}{2}$	6

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- **Algebraically**, the two Virasoro reps. are distinguished by the decoupling of the **level two null vector** of $\phi_{1,3}$ at $c = 1/2$

$$(L_{-1}^2 - \tfrac{3}{4} L_{-2})|\phi_{1,3}\rangle = \begin{cases} 0 \Rightarrow \text{Single channel OPE} \\ |\chi\rangle \Rightarrow \text{three channels OPE} \end{cases}$$

The boundary case: Ising FK connectivities

- The additional OPE channel $\phi_{1,3} \cdot \phi_{1,3} \to \phi_{1,3}$ is **singular**.
- Its regularization produces, as well known, **logarithmic singularities** in the corresponding conformal blocks; see also [Santachiara, V. '14; He, Jacobsen, Saleur '20; Nivesvivat, Ribault 20]
- For the boundary case, the regularization prescription is selected by the **logarithmic solution** of a third order ODE.

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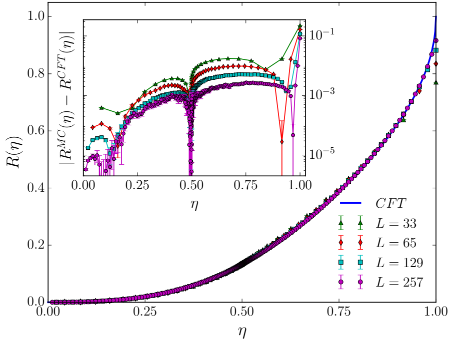


- Analytic determination of all three linearly ind. connectivities

$$R(\eta) = \frac{P_{aabb}}{P_{aabb} + P_{abba} + P_{aaaa}}$$

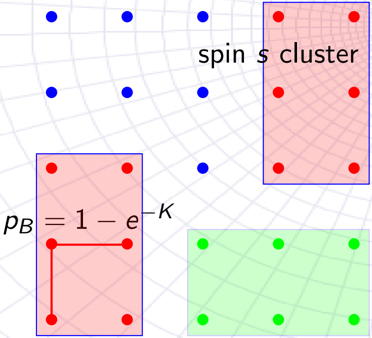
- $\eta = \frac{x_{12}x_{34}}{x_{13}x_{24}}$.

Still a puzzle: Potts spin clusters

- Consider the coupled Potts models (Potts dilute Potts model)

$$H_{PP} = -J\sum_{\langle x,y\rangle}\delta_{s(x),s(y)} - K\sum_{\langle x,y\rangle}\delta_{\tau(x)\tau(y)}\delta_{s(x),s(y)},\ \ \tau(x) = 1,\ldots,P.$$

- Its FK graph expansion, leads to a **correlated** percolation problem in the $P \to 1$ limit

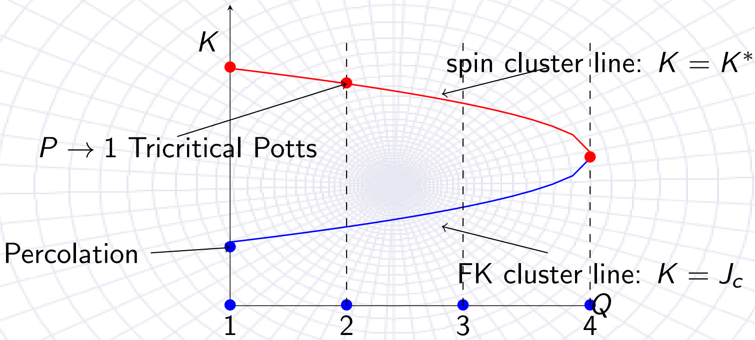


$$\begin{cases} K \to \infty : \text{spin clusters} \\ K \to J : \text{FK clusters} \end{cases}$$

- It turns out that in $d = 2$, they both percolate at $J = J_c$; not true in $d = 3$.

Still a puzzle: Potts spin clusters

- By fixing $J = J_c$, we can perform an **RG analysis** in the coupling K [Coniglio, Peruggi 80's] and take the $P \to 1$ limit



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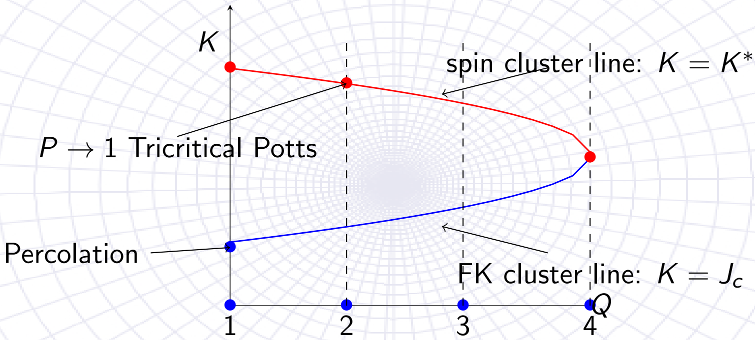
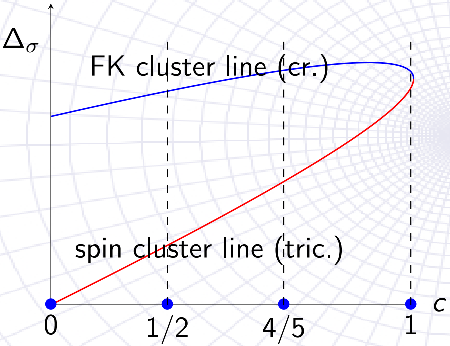


- The **RG picture** suggests that spin cluster are obtained by analytic continuation of FK clusters along the **tricritical branch** [Nienhuis et al., 79] of the Potts model.

Still a puzzle: Potts spin clusters

- This guess produces the correct **fractal dimension** $2-\Delta_\sigma$ of the clusters [by MC]. Δ_σ=dim. spin operator along the critical/tricritical branch.



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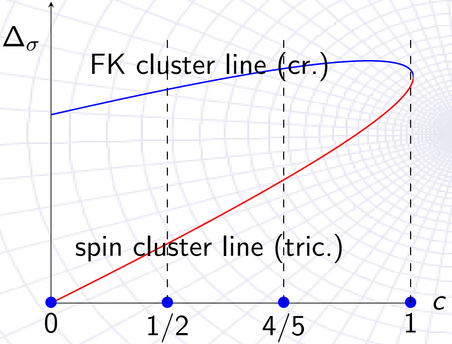

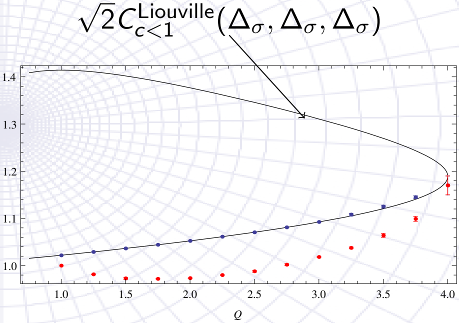

- Does the same analytic continuation hold for the **three-point connectivity** of Potts spin clusters? No.
- **Q:** What is then the CFT of critical 2d Potts spin clusters?