

4-point cluster connectivities

in 2d critical Potts model

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based on [YH, Grans-Samuelsson, Jacobsen, Saleur, 2002.09071]

[YH, Jacobsen, Saleur, 2005.07258]

outline

Introduction

Potts \leftrightarrow MM on the lattice

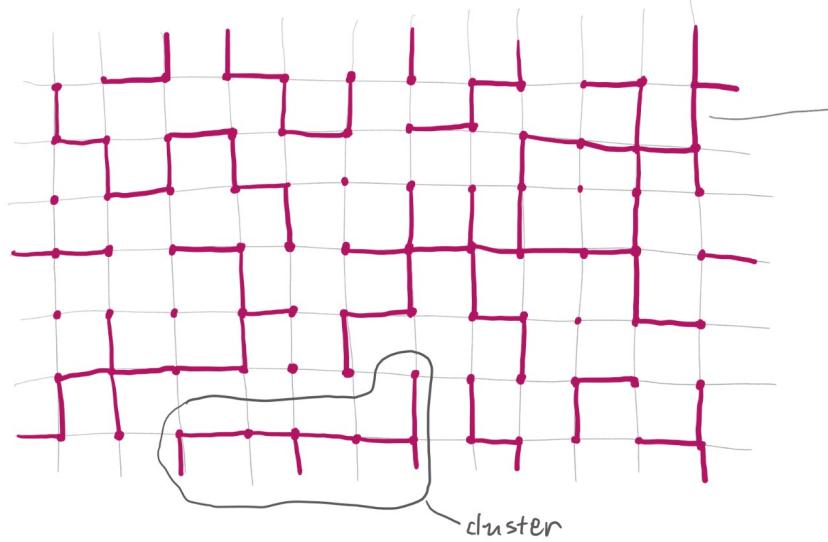
Potts spectrum

Bootstrap Potts 4-pt connectivities

interchiral conformal block

Conclusions

critical percolation



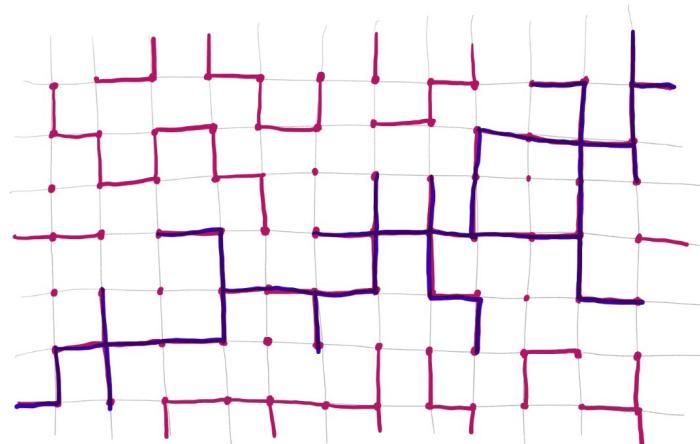
bond
close : P
open : $1-P$

} represent different
physics properties

$$\Downarrow P_c$$

geometric phase transition

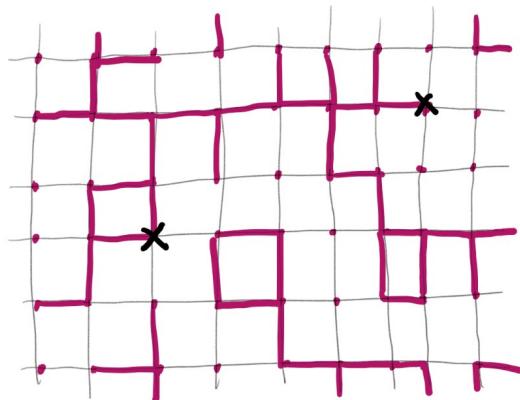
CFT description ?



mean cluster size diverges

geometric correlations

cluster connectivities



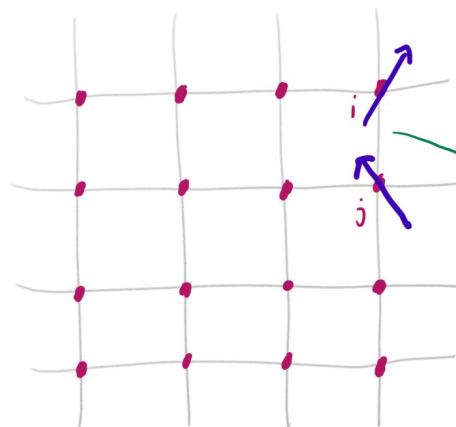
2-pt function

$$P_{aa} = \frac{\sum \text{config}}{Z}$$

how do we compute such quantity
in the field theory description ?

Potts model & FK clusters

[Fortuin, Kasteleyn, 1972]

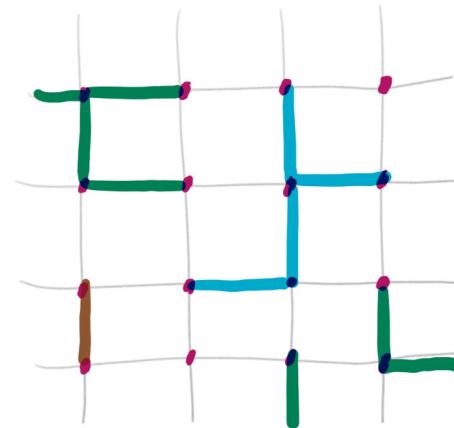
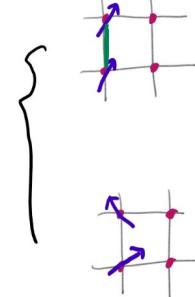


spin variable $\sigma_i = 1, \dots, Q$

$$-K \delta_{\sigma_i, \sigma_j}$$

$Q=2$: Ising

$Q=3$: 3-Potts



Q colors for clusters

$$Z_{\text{spin}} = \sum \prod_{i,j} e^{K \delta_{\sigma_i, \sigma_j}}$$

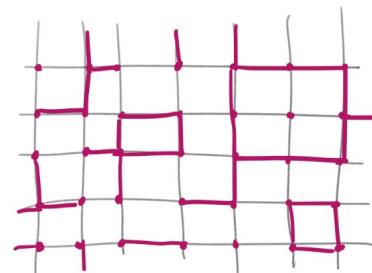
$$Z_{\text{clusters}} = \sum_{\# \text{ bonds}} v^{\# \text{ bonds}} Q^{\# \text{ clusters}}$$

$$Q \in \mathbb{Z}_{\geq 2}$$



$$Q \in \mathbb{R}$$

$$Q \rightarrow 1$$

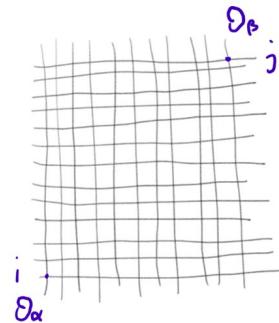


Percolation

$$\text{critical : } v = \sqrt{Q}, \quad 0 \leq Q \leq 4$$

2-point function

order parameter $\mathcal{O}_\alpha(i) = Q S_{\sigma_i, \alpha} - 1 \quad \alpha = 1, \dots, Q \quad \sum_\alpha \mathcal{O}_\alpha = 0$



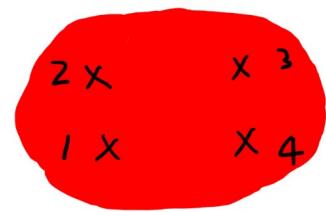
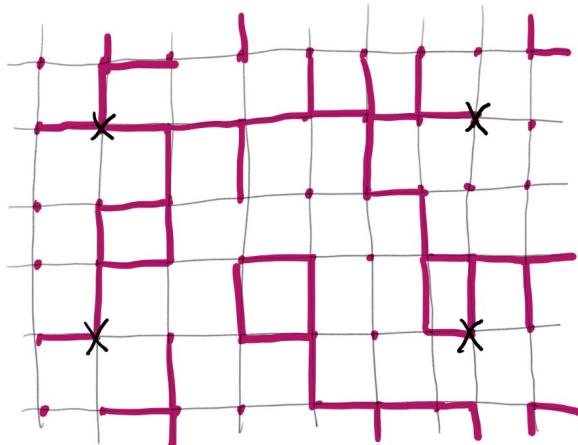
$$G_{\alpha\beta}(i, j) = \langle \mathcal{O}_\alpha(i) \mathcal{O}_\beta(j) \rangle$$

$$\frac{1}{Z} \sum_{\text{grid}} \text{weight } \mathcal{O}_\alpha(i) \mathcal{O}_\beta(j) \rightsquigarrow \frac{1}{Z} \sum_{\text{grid}} \text{weight } \mathcal{O}_\alpha(i) \mathcal{O}_\beta(j)$$

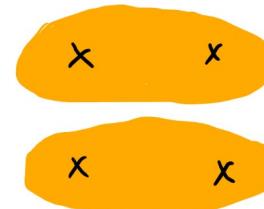
$$G_{\alpha\alpha}(i, j) = (Q-1) \quad \text{with} \quad \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \quad \text{[Delfino, Viti, 2011]}$$

Criticality $\langle \mathcal{O}_{\text{spin}}(x_1) \mathcal{O}_{\text{spin}}(x_2) \rangle_{\text{CFT}}$

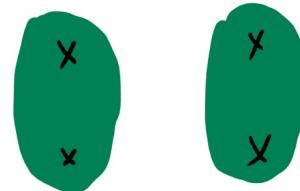
4 - point cluster connectivities



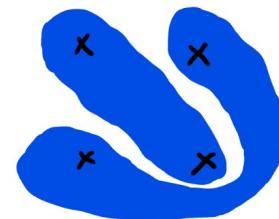
P_{aaaa}



P_{abba}



P_{aabb}



P_{abab}

$\langle \textcirclearrowleft \textcirclearrowright \textcirclearrowleft \textcirclearrowright \rangle_{\text{CFT}}$

determine these

about Potts CFT

$$0 \leq Q \leq 4$$

parametrize $\sqrt{Q} = 2 \cos \frac{\pi}{x+1} \quad x \in [1, \infty]$

central charge $c = 1 - \frac{6}{x(x+1)} \in [-2, 1]$

non-unitary
(logarithmic)

use Kac parametrization

$$h_{r,s} \leftarrow \text{can be fractions}$$

order parameter

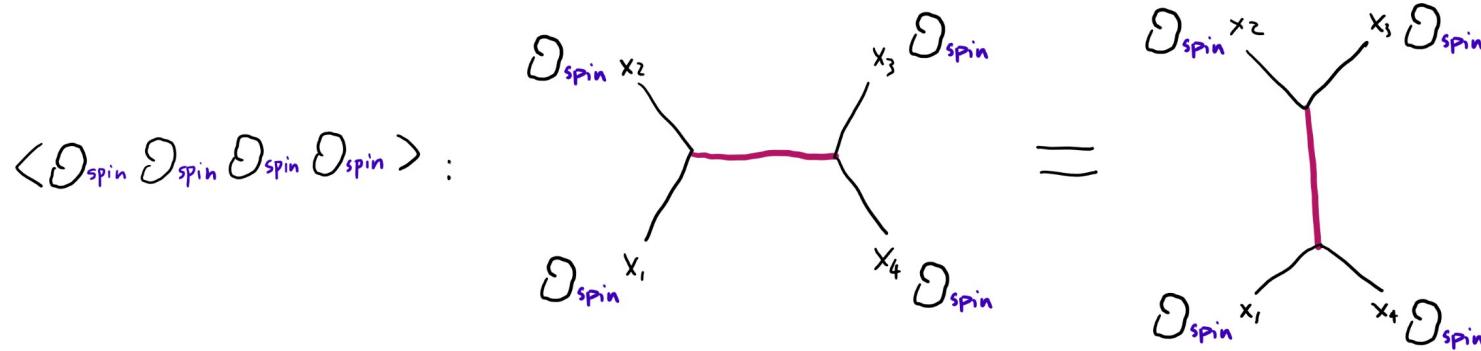
$$\mathcal{O}_{\text{spin}} : (h_{\pm,0}, h_{\pm,0})$$

non-degenerate

no decoupling null descendant

$\langle \mathcal{O}_{\text{spin}} \mathcal{O}_{\text{spin}} \mathcal{O}_{\text{spin}} \mathcal{O}_{\text{spin}} \rangle$ does not satisfy BPZ

the conformal bootstrap approach



$$\sum_{\text{spectrum}} A(x) F(x) = \sum_{\text{spectrum}} A(\tilde{x}) F(\tilde{x})$$

\uparrow

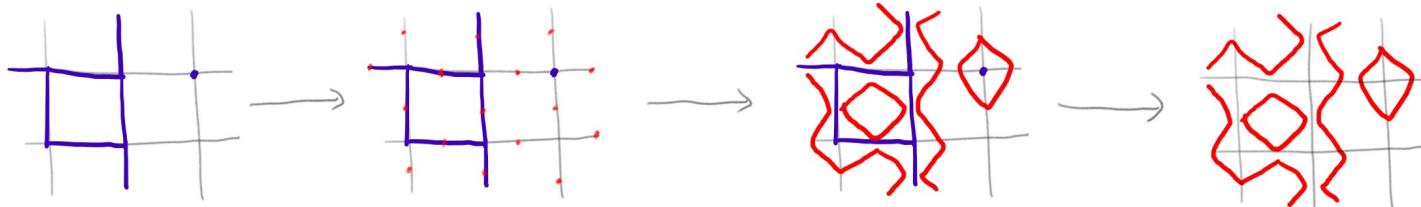
amplitude

$(C_{\text{spin spin } \phi}^{\text{geometry}})^2$

first attempt on bootstrap connectivity

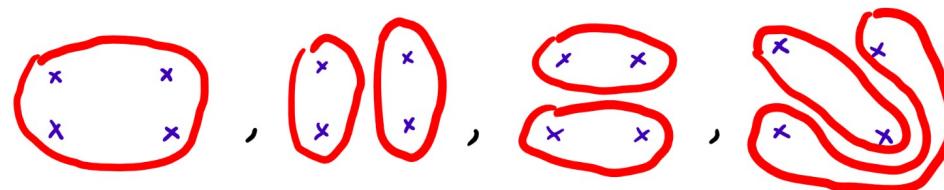
[Picco, Ribault, Santachiara , 2016]

cluster \leftrightarrow loops



$$Z_{\text{cluster}} \leftrightarrow Z_{\text{loop}} = \sqrt{Q}^{\# \text{ sites}} \sum_{\substack{\text{critical} \\ \text{loop} \\ \text{diagrams}}} \sqrt{Q}^{\# \text{ loops}}$$

each loop carries weight \sqrt{Q}

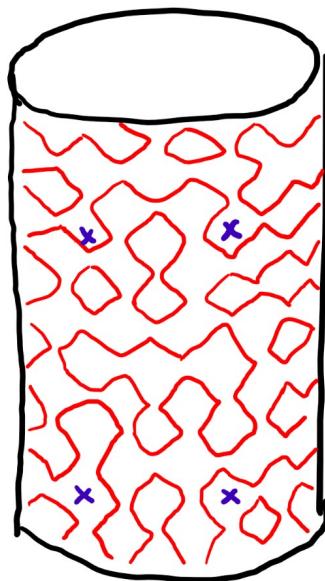


Spectrum of

[Jacobsen, Saleur, 2018]

$$I = \rangle \langle \quad e_i = \asymp \quad u = \rangle / \dots / \langle$$

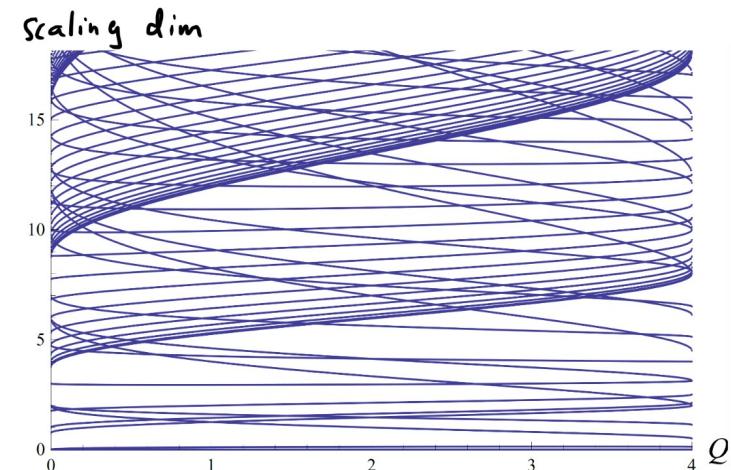
affine Temperley-Lieb algebra



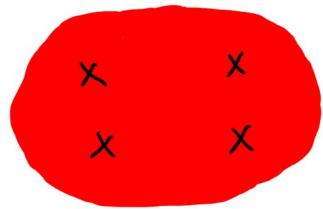
↑ transfer matrix
eigenvalues λ_i : \longrightarrow irreps \mathcal{W} \supset a tower of Virasoro primaries & descendants

combinations enter connectivity spectrum

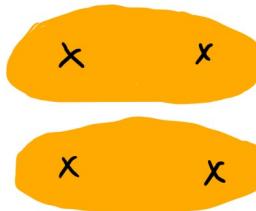
scaling dims in



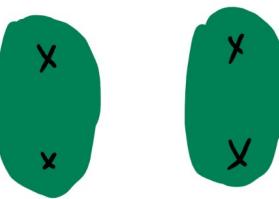
half way there



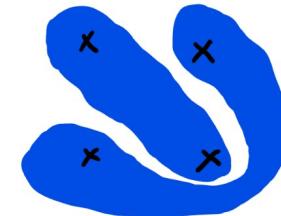
Paaaa



Pabba



Paabb



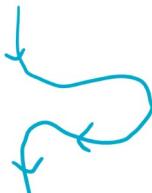
Pabab

$$\sum A F(x) = \sum A F(\bar{x})$$

spectrum



spectrum



revisit minimal models

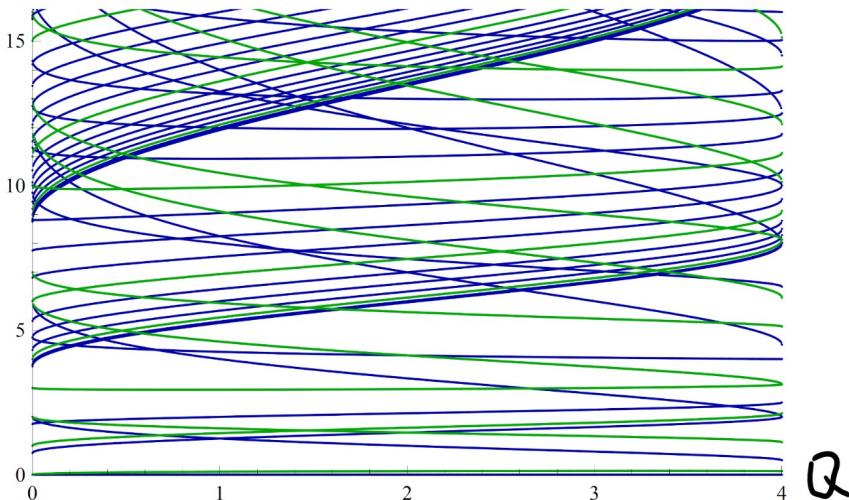
Potts vs. MM

$$\langle \hat{J} \hat{J} \hat{J} \hat{J} \rangle \underset{\substack{\Delta_J = \Delta_{\text{spin}} \\ \text{bootstrap}}} \approx \text{Monte-Carlo} + \frac{2}{Q-2}$$

[Picco, Ribault, Santachiara , 2016 & 2019]

Very simple spectrum \Rightarrow analytic continuation of type D

scaling dim



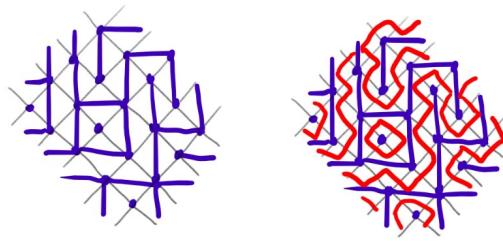
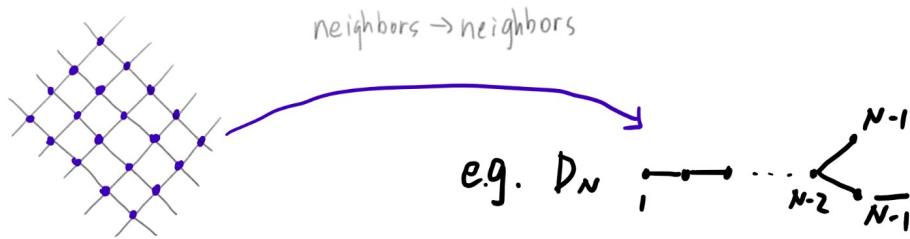
MM spectrum

[Migliaccio, Ribault, 2018]

Potts $\overset{?}{\longleftrightarrow}$ MM

cluster/loop expansion of MM

MM on the lattice : ADE RSOS lattice model [Pasquier, 1987]



clusters

loops

$$Z = \sum$$



[Kostov, 1989]

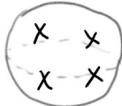
$$\langle \dots \rangle = \frac{1}{Z} \sum_{\text{clusters or loops}} \dots$$

Strategy: look at $\langle \hat{\sigma} \hat{\sigma} \hat{\sigma} \hat{\sigma} \rangle$ on the RSOS lattice

cluster interpretation

cluster expansion of MM 4-pt function

type D: $\langle \hat{\theta} \hat{\theta} \hat{\theta} \hat{\theta} \rangle \propto \sum_{aaaa} \text{ (Diagram)} + \sum_{abab} \text{ (Diagram)}$

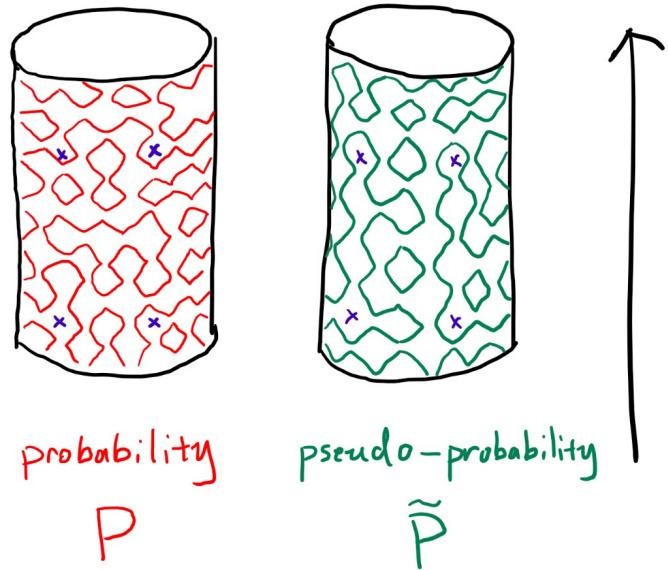


loop weights $\begin{cases} \text{contractible} \\ \text{non-contractible} \end{cases}$ \sqrt{Q} $\sum_{\substack{\text{adjacency matrix} \\ \text{eigenvalues}}} >$ Potts: all loop weight \sqrt{Q}

$$\langle \hat{\theta} \hat{\theta} \hat{\theta} \hat{\theta} \rangle \propto P_{aaaa} + \tilde{P}_{abab} \sim \text{"pseudo-probability"}$$

\tilde{P}_{abab} difference with P_{abab} involves unlikely configs

amplitude ratios on the lattice



spectrum of $\tilde{P} \sim$ spectrum of P

eigenvalue λ_i of
transfer matrix

affine Temperley Lieb
irreps ω

something amazing
about lattice amplitudes

$$\frac{A_{abab}(\lambda_i)}{A_{aaaa}(\lambda_i)}$$

depends only on Q , and $\lambda_i \in \omega$

$$\frac{\tilde{A}_{abab}(\lambda_i)}{A_{abab}(\lambda_i)}$$

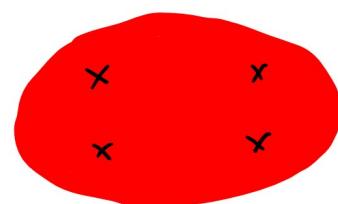
does not depend on lattice size

eg: $\frac{\tilde{A}_{abab}}{A_{abab}}(\omega_{i-1}) = \frac{2}{Q-2}$, $\frac{A_{abab}}{A_{aaaa}}(\omega_{i,1}) = \frac{2-Q}{2}$, $\frac{A_{aabb}}{A_{aaaa}}(\omega_{i,1}) = \frac{1}{1-Q}$, ...

interchiral conformal blocks

continuum limit : $\lambda_i \rightarrow$ conformal operators $(h, \bar{h}) \in$ spectrum

amplitude ratios only depend on ω → different operators in \mathcal{W}
relation is rigid



$$= \sum_{(h, \bar{h}) \in \text{spectrum}} A_{aaaa}(h, \bar{h}) F(\rangle^{h, \bar{h}} \langle)$$

P_{aaaa}

$$= \sum_{\omega} A_{aaaa}(\omega) \underbrace{\sum_{(h, \bar{h}) \in \omega} \frac{A(h, \bar{h})}{A(\omega)} F(\rangle^{h, \bar{h}} \langle)}$$

solve by bootstrap



interchiral algebra $\supset \text{Vir} \otimes \overline{\text{Vir}}$

[Gainutdinov, Read, Saleur, 2012]

interchiral conformal block

extracting Potts amplitudes from MM

$$\sum_{\omega} \left[A_{aaaa}(\omega) + \tilde{A}_{abab}(\omega) \right] \text{IF} = P_{aaaa} + \tilde{P}_{abab} \stackrel{\substack{\text{MM} \\ \text{4-pt}}}{=} \sum_{\omega} A^L(\omega) \text{IF}$$

$A_{aaaa} + \tilde{A}_{abab} = A^L$

$$= A_{aaaa} \left(1 + \frac{A_{abab}}{A_{aaaa}} \frac{\tilde{A}_{abab}}{A_{abab}} \right)$$

✓ ✓

if $A^L(\omega) \neq 0$

for $\omega \notin \text{MM spec} \quad \notin P_{aaaa} \text{ spec}$

$$A^L \rightarrow A_{aaaa}$$

$$A_{abab} \dots$$

$$A^L = 0 \cdot A_{aaaa} \neq 0$$

✓

bootstrap

$$A_{aaaa}(\omega_{0,-1}) = A^L(\omega_{0,-1}), \quad A_{abab}(\omega_{2,-1}) = \frac{Q-2}{2} A^L(\omega_{2,-1}), \quad A_{aaaa}(\omega_{4,-1}) = \frac{(Q-2)(Q^2-4Q+2)}{Q(Q-3)^2} A^L(\omega_{4,-1})$$

degeneracy → recursion

to construct \bar{F} : degenerate $E = \Phi_{2,1} : (h_{2,1}, h_{2,1})$

review: c<1 Liouville [Zamolodchikov x2, 1995]
 [Teschner, 1995] non-diagonal [Estienne, Zklef, 2015]
 generalization [Migliaccio, Ribault, 2017]

$$\Phi_{2,1} \times \Phi_{r,s} \rightarrow \Phi_{r+1,s} + \Phi_{r-1,s}$$

$$\sum_{\text{two terms}} A F^{(s)} = \begin{array}{c} \diagup \\ \diagdown \end{array} \xrightarrow{\Phi_{2,1}} = \begin{array}{c} \diagdown \\ \diagup \end{array} = \sum_{\text{two terms}} A F^{(t)}$$

known solution
from BPZ

$$\Rightarrow \frac{A_{r+1,s}}{A_{r-1,s}} \quad \text{recursion}$$

c<1 Liouville : $\Phi_{1,2} : (h_{1,2}, h_{1,2})$ also degenerate

$$\Rightarrow \frac{A_{r,s+1}}{A_{r,s-1}} \Rightarrow \text{analytic solution not in Potts}$$

constructing $F(\succ^w)$

relation of fields in the same \mathcal{W} = $\frac{A_{r+1,s}}{A_{r,s}}$

can be obtained from
two other $\langle \mathcal{E} \cdots \rangle$

$$\frac{C_{(r+1,s)(\frac{1}{2},0)(\frac{1}{2},0)}^2}{C_{(r,s)(\frac{1}{2},0)(\frac{1}{2},0)}^2} :$$

$$\frac{C_{(r+1,s)(\frac{1}{2},0)(\frac{1}{2},0)}}{C_{(r,s)(\frac{1}{2},0)(\frac{1}{2},0)}} \times \text{other stuff} = \text{known stuff}$$

$$\begin{array}{c} (\frac{1}{2},0) \quad (\frac{1}{2},0) \\ \diagdown \qquad \diagup \\ \mathcal{E} \qquad (-\frac{1}{2},0) \end{array} = \begin{array}{c} (\frac{1}{2},0) \quad (\frac{1}{2},0) \\ \diagdown \qquad \diagup \\ (r+1,s) \qquad C_{(\frac{1}{2},0)(\frac{1}{2},0)(r+1,s)} \\ | \\ \mathcal{E} \qquad (r,s) \end{array}$$

$$F(\succ^w) = \sum_{(h,\bar{h}) \in \mathcal{W}} \text{recursion} \times \text{Virasoro block}$$

interchiral conformal bootstrap

$$A_{aaaa}(\omega) \quad \text{Diagram: Two red ovals, each containing a black horizontal bar with four external legs.} = \quad A_{aaaa}(\omega)$$

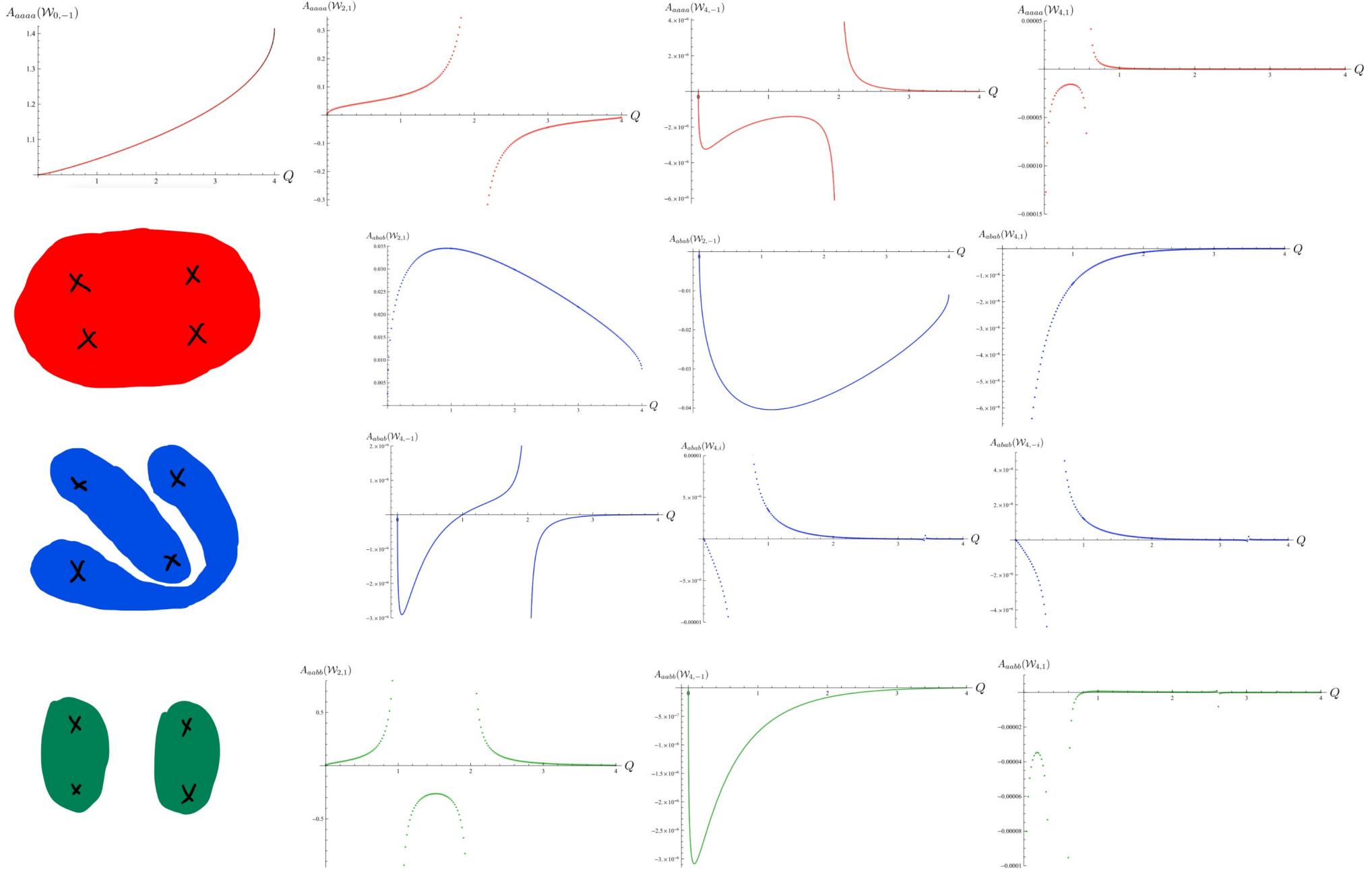
$$A_{aab b}(\omega) \quad \text{Diagram: Two green ovals, each containing a black horizontal bar with four external legs.} = \quad A_{abba}(\omega)$$

$$A_{abab}(\omega) \quad \text{Diagram: Two blue ovals, each containing a black horizontal bar with four external legs.} = \quad A_{abab}(\omega)$$

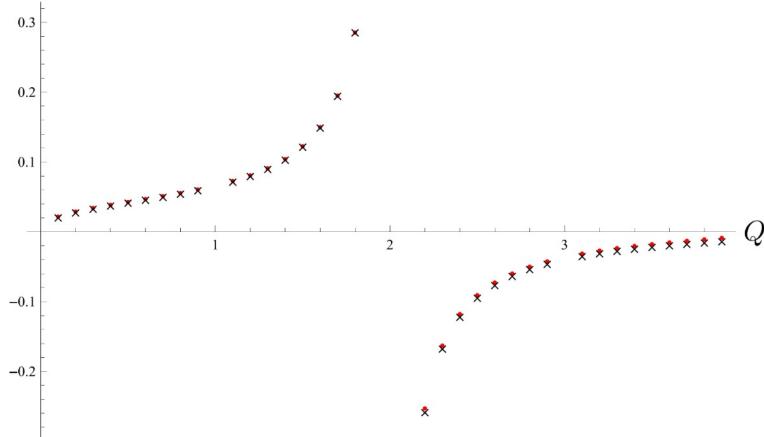
$$A_{abba}(\omega) \quad \text{Diagram: Two orange ovals, each containing a black horizontal bar with four external legs.} = \quad A_{aabb}(\omega)$$

solve for $A(\omega)$ numerically

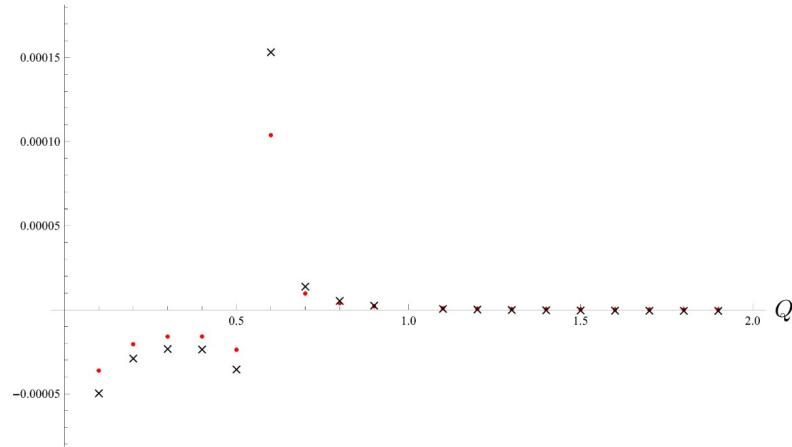
results



$$\frac{A_{aaaa}(\mathcal{W}_{2,1})}{A_{aaaa}(\mathcal{W}_{0,-1})}$$



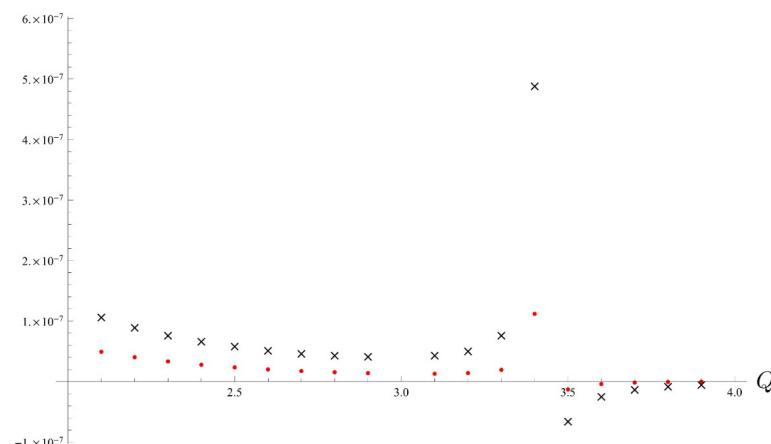
$$\frac{A_{aaaa}(\mathcal{W}_{4,1})}{A_{aaaa}(\mathcal{W}_{0,-1})}$$



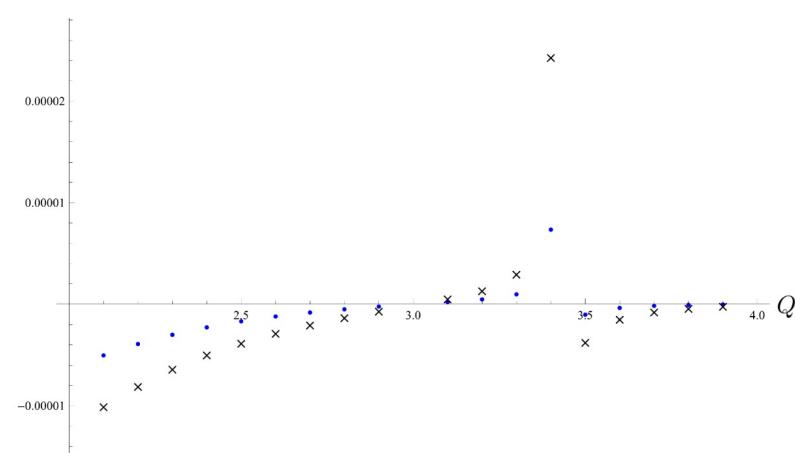
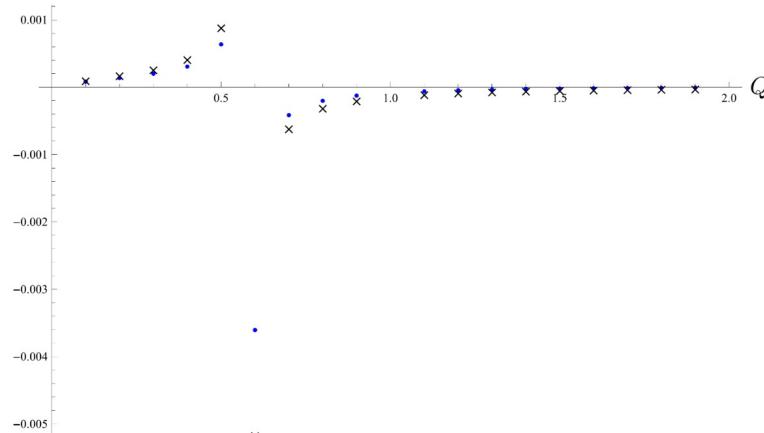
bootstrap vs. lattice

\times — lattice

\bullet } bootstrap

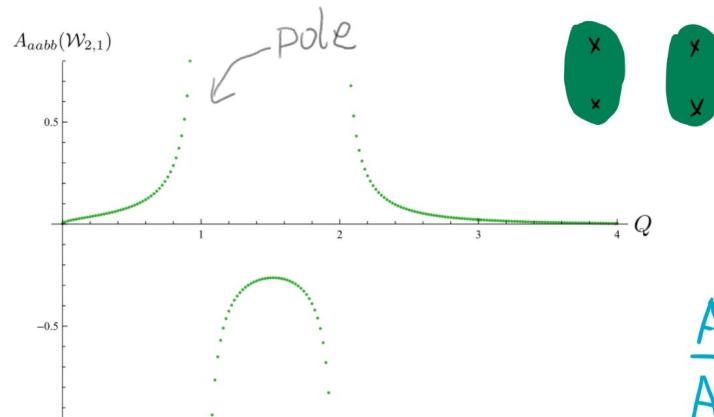


$$\frac{A_{abab}(\mathcal{W}_{4,i})}{A_{abab}(\mathcal{W}_{2,-1})}$$

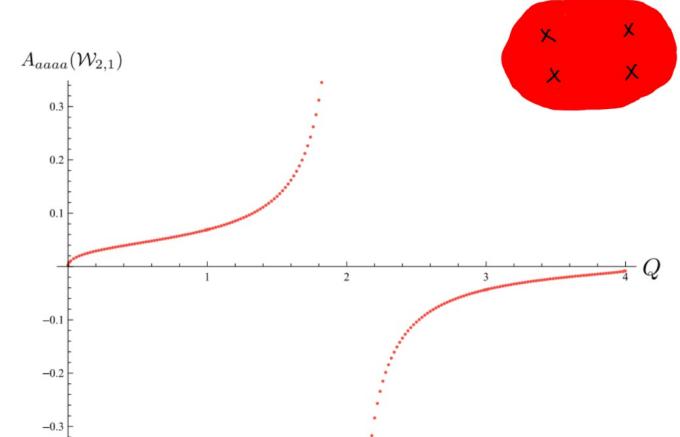


Singularities & exact amplitudes

$\mathcal{W}_{2,1} : (r,s) = (0,2), (1,2), \dots$



$$\frac{A_{aabb}}{A_{aaaa}}(\mathcal{W}_{2,1}) = \frac{1}{1-Q}$$



P_{aabb} spec: $\mathcal{W}_{2,1}, \dots, \overline{\mathcal{W}_{0,q^2}}$



$(r,s) = (1,1), (2,1), (3,1), \dots$

P_{aaaa} spec: $\mathcal{W}_{2,1}, \dots$

analytic structure in $Q \leftarrow$ spectrum

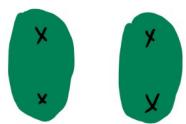
Singularities & exact amplitudes

at $Q=1$: $h_{1,1} = \bar{h}_{1,1} = h_{1,-2}$ degenerate conformal dim

$$F_{h_{1,1}}(z) = \dots + \underbrace{\frac{R_{11}}{h_{1,1} - h_{1,-2}}}_{\sim F_{h_{1,-2}}^{\text{reg}}(z)} F_{h_{1,-2}}(z) \quad \text{similarly for } \bar{F}_{h_{1,1}}(\bar{z})$$

$$\frac{1}{\#_{Q-1}}$$

combining the left & right : $F_{h_{1,1}}(z) \bar{F}_{h_{1,1}}(\bar{z}) = \dots + \frac{\#}{Q-1} \left(\underbrace{F_{h_{1,-2}}^{\text{reg}}(z) F_{h_{1,-2}}(\bar{z})}_{(h_{1,-2}, h_{1,-2}) \in \mathcal{W}_{2,1}} + \text{c.c.} \right)$

 — smooth function in Q

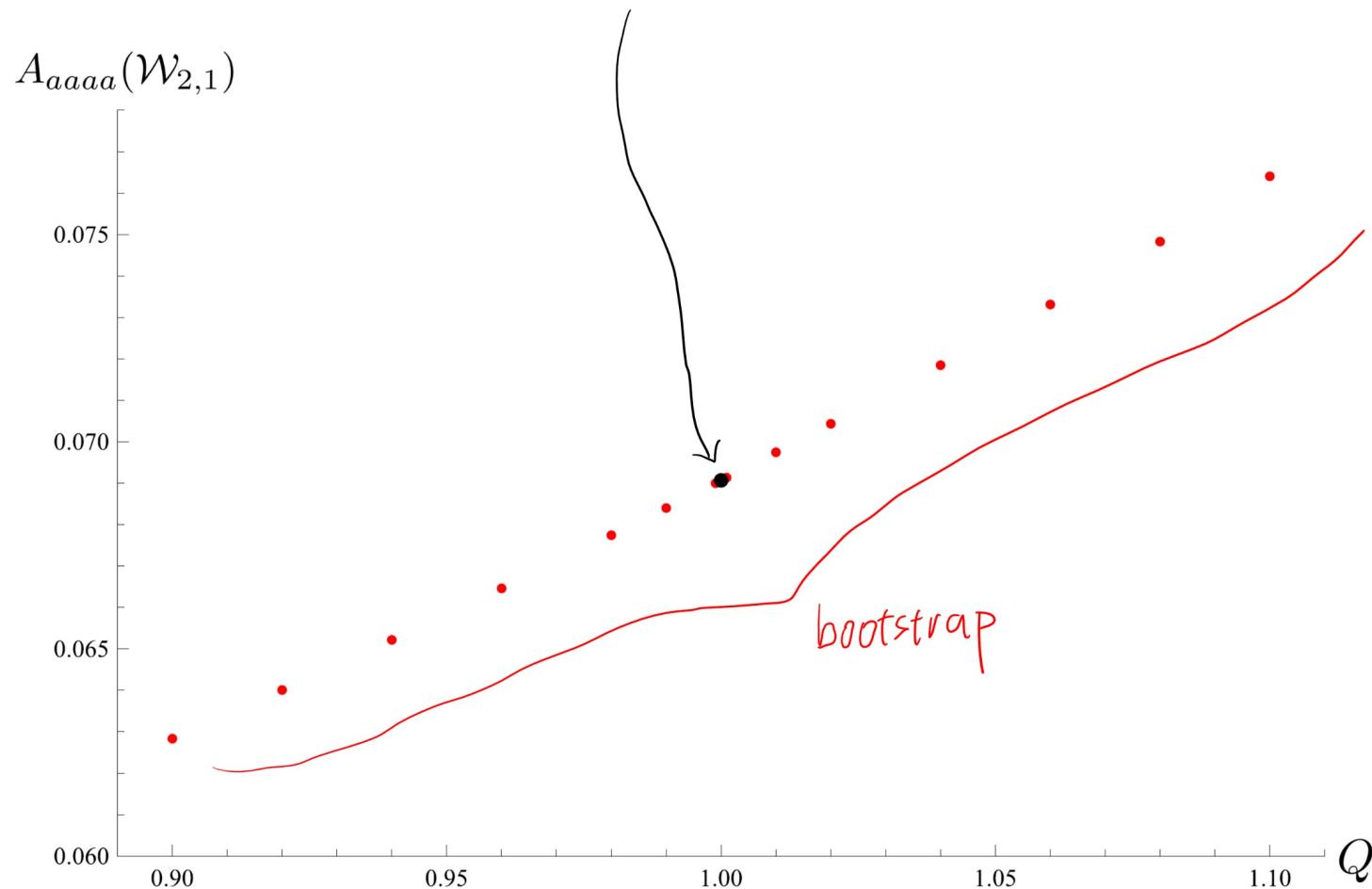
$$\Rightarrow A_{aabb}(\mathcal{W}_{2,1}) = \frac{\#}{Q-1} \quad \text{in contrast } A_{aaaa}(\mathcal{W}_{2,1}) \text{ has no pole at } Q=1$$

$$\Rightarrow \frac{A_{aabb}}{A_{aaaa}}(\mathcal{W}_{2,1}) = \frac{1}{1-Q} \quad \rightarrow \text{exact amplitude } A_{aaaa} \text{ at } Q=1$$

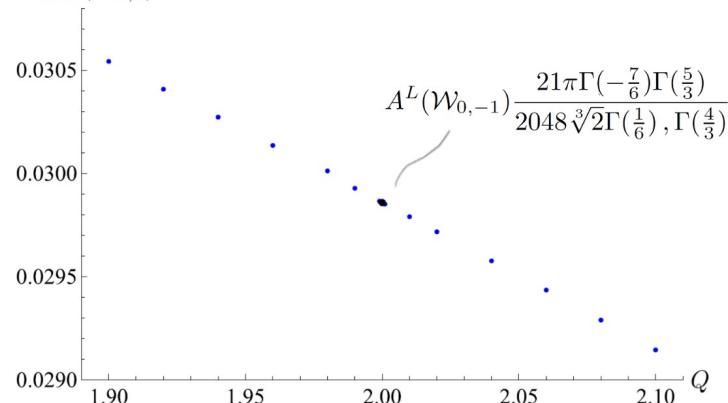
take residue

Singularities & exact amplitudes

$$A_{aaaa}(\mathcal{W}_{2,1})|_{Q=1} = \frac{5\pi\Gamma(-\frac{5}{4})\Gamma(\frac{7}{4})}{144\sqrt{3}\Gamma(\frac{1}{4})\Gamma(\frac{5}{4})}$$

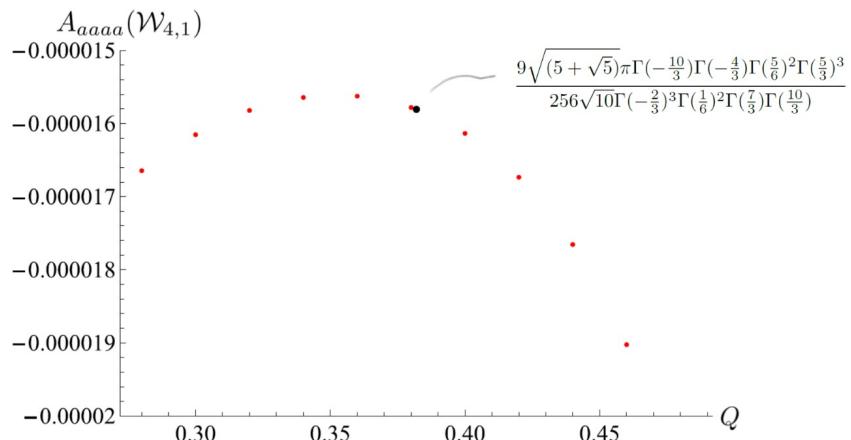


$A_{abab}(\mathcal{W}_{2,1})$



Spectrum — amplitude ratio

analytic structure in \mathcal{Q}



$A_{aaaa}(\mathcal{W}_{4,1})$

$$\frac{\sqrt{(5-\sqrt{5})}\pi\Gamma(-\frac{11}{4})\Gamma(-\frac{7}{4})\Gamma(\frac{5}{8})^2\Gamma(\frac{5}{4})^3\Gamma(\frac{15}{8})^4}{10\sqrt{10}\Gamma(-\frac{7}{8})^4\Gamma(-\frac{1}{4})^3\Gamma(\frac{3}{8})^2\Gamma(\frac{11}{4})\Gamma(\frac{15}{4})}$$

2.6×10^{-8}

2.5×10^{-8}

2.4×10^{-8}

2.3×10^{-8}

2.2×10^{-8}

2.1×10^{-8}

2.0×10^{-8}

1.9×10^{-8}

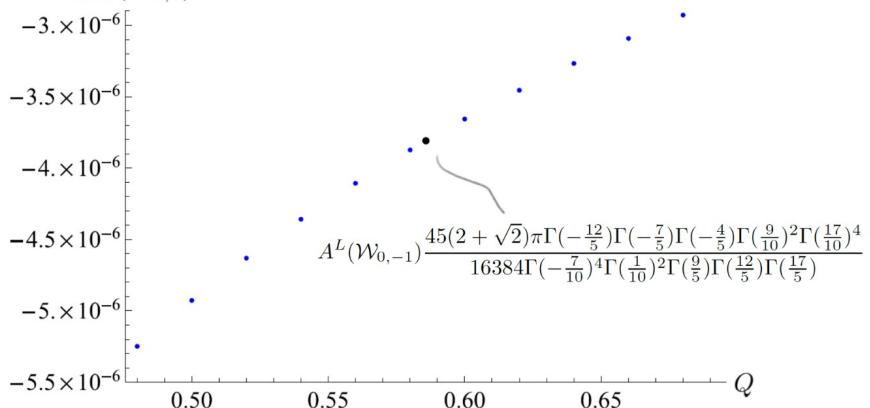
2.55

2.60

2.65

2.70

$A_{abab}(\mathcal{W}_{4,1})$



$A_{abab}(\mathcal{W}_{4,1})$

-2×10^{-10}

-4×10^{-10}

-6×10^{-10}

-8×10^{-10}

-1×10^{-9}

-1.2×10^{-9}

-1.4×10^{-9}

-1.6×10^{-9}

3.35

3.40

3.45

3.50

$$A^L(\mathcal{W}_{0,-1}) \frac{823543(\sqrt{2}-2)\Gamma(-\frac{6}{7})\Gamma(-\frac{3}{7})\Gamma(\frac{1}{7})\Gamma(\frac{11}{14})^2\Gamma(\frac{19}{14})^3\Gamma(\frac{27}{14})^2}{6871947673600\sqrt[3]{2}\Gamma(-\frac{13}{14})^2\Gamma(-\frac{5}{14})^2\Gamma(\frac{3}{14})\Gamma(\frac{6}{7})\Gamma(\frac{10}{7})\Gamma(\frac{17}{7})}$$

"renormalized" Liouville recursion

non-diagonal
 $C < 1$ Liouville : degenerate $(h_{1,2}, h_{1,2}) \Rightarrow \frac{A^L(\omega_{j+1,-})}{A^L(\omega_{j-1,-})}$

not in Potts!

} \Rightarrow analytic solution

+

degenerate $(h_{2,1}, h_{2,1})$

in Potts : a "renormalized" version of Liouville recursion

$$\frac{A_{aaaa}(\omega_{4,-1})}{A_{aaaa}(\omega_{0,-1})} = \frac{(Q-2)(Q^2-4Q+2)}{Q(Q-3)^2} \frac{A^L(\omega_{4,-1})}{A^L(\omega_{0,-1})}$$

$$\frac{A_{abab}(\omega_{4,-1})}{A_{abab}(\omega_{2,-1})} = \frac{(Q-1)(Q-4)(Q^2-4Q+2)}{2Q(Q-3)^2} \frac{A^L(\omega_{4,-1})}{A^L(\omega_{2,-1})}$$

$$\frac{A_{aaaa}(\omega_{4,1})}{A_{aaaa}(\omega_{2,1})} = \frac{(Q-2)^2}{(Q-1)^2(Q^2-4Q+2)} \frac{A^L(\omega_{4,1})}{A^L(\omega_{2,1})}$$

\Rightarrow analytically solvable ?

rational functions in Q

Conclusions

- analytic understanding may not be far symmetry . "renormalized" Liouville recursion
- log CFT of 2d loop models

$O(n)$ [Gorbenko, Zan, 2020]

Potts [Nivesvivat, Ribault, 2020]

[Grans-Samuelsson, Liu, YH, Jacobsen, Saleur, 2020]

- $C=0$ LCFT work in progress

