Operator expansions, layer susceptibility and two-point functions in BCFT

Mykola Shpot

ICMP Lviv

based on work with
Parijat Dey and Tobias Hansen

DHS, JHEP 12 (2020) 051
MS, JHEP 1 (2021) 055

Bootstat 2021

INTRODUCTION

1995 McAvity Osborn *CFTs near a boundary in general dimensions* cond-mat/9505127

g, h. To achieve this it is convenient to define a transform, $g \to \hat{g}$, by integrating G over planes parallel to the boundary[9]

[8] The two-loop results [12,19,20,21] for the surface exponents $\hat{\eta}, \hat{\eta}_n$ vanish as $N \to \infty$.
[9] This is analogous to the radon transform [22].

$$\int \mathrm{d}^{d-1}x\, G(x,x') = \frac{1}{(4yy')^{\alpha-\lambda}}\hat{g}(\rho)\,,$$
$$\rho = \frac{(y-y')^2}{4yy'}\,, \quad \lambda = \tfrac{1}{2}(d-1)\,, \tag{4.17}$$

where

$$\hat{g}(\rho) = \frac{\pi^\lambda}{\Gamma(\lambda)}\int_0^\infty \mathrm{d}u\, u^{\lambda-1} g(u+\rho)\,.$$

This transform may be inverted, $\hat{g} \to g$, by

$$g(\xi) = \frac{1}{\pi^\lambda \Gamma(-\lambda)}\int_0^\infty \mathrm{d}\rho\, \rho^{-\lambda-1}\hat{g}(\rho+\xi)$$

1995 McAvity hep-th/9507028

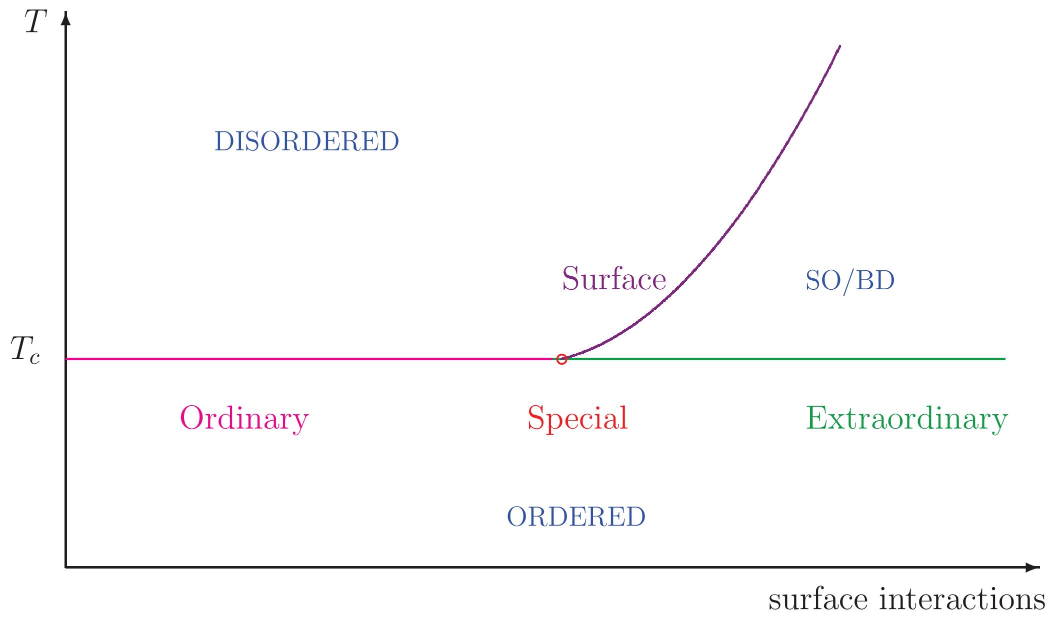
J. Phys. A: Math. Gen. 28 (1995) 6915–6930. Printed in the UK

Integral transforms for conformal field theories with a boundary

2013 Liendo Rastelli van Rees *The bootstrap program for boundary* CFT_d 1210.4258

Critical phenomena in semi-infinite systems

Phase diagram in the absence of ordering fields: $h = h_1 = 0$



EOT exists in d dimensions if $d-1$-dimensional *ST* is possible: $d \geq 3$

$h_1 \neq 0$ — Critical Adsorbtion/Normal Transition — occurs whenever a d dimensional *bulk transition* is possible: $d \geq 2$

— Metlitski 2020 $d = 3$, $n \geq 2$ — Parisen Toldin 2021 $d = 3$, $n = 3$

Extraordinary transition

- Lubensky Rubin 1975 **MF two-point function** $\rightarrow$ **DHS20** $O(\varepsilon)$

- Bray Moore 1977 **scaling;** Ohno Okabe 1984 **large** n

  $\Rightarrow$ **exact** ***parallel*** **correlation critical exponents for** L **and** T **components**

  $\left\{\eta_{\parallel} = d+2,\ \eta_{\perp} = \frac{1}{2}(d+2+\eta)\right\}^{L} \qquad \left\{\eta_{\parallel} = d,\ \eta_{\perp} = \frac{1}{2}(d+\eta)\right\}^{T}$

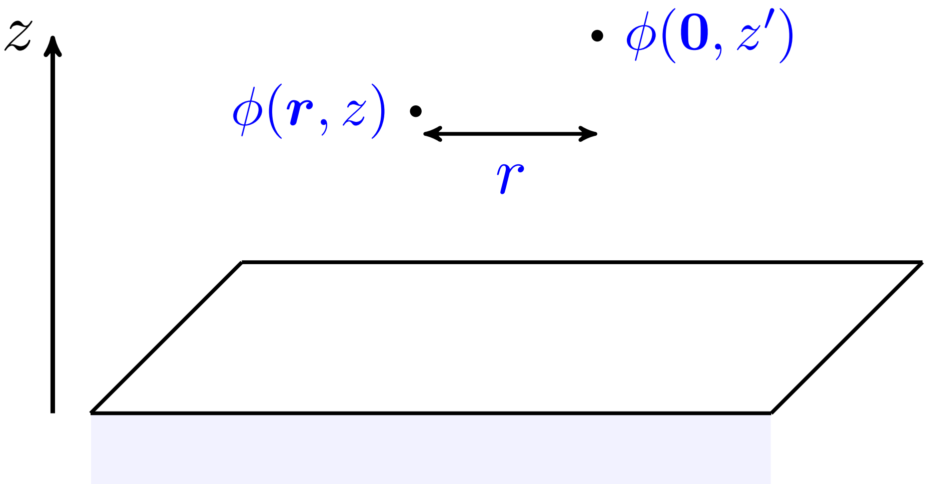
  $\Delta_{\phi} = (d-2+\eta)/2 \qquad [\,\eta/2 \equiv \gamma_{\sigma}\,] \qquad \hat{\Delta} = (d-2+\eta_{\parallel})/2$

  scaling dimension of the leading nontrivial boundary operator $\hat{O}$ $\uparrow$

- **conformal bootstrap:** Liendo Rastelli van Rees 2013, Gliozzi Liendo Meineri Rago 2015

EXP Law 2001 **Wetting, adsorption and surface critical phenomena**

Geometric setup: $\mathbb{R}^d_+ = \{\boldsymbol{x} = (\boldsymbol{r}, z) \in \mathbb{R}^d \mid \boldsymbol{r} \in \mathbb{R}^{d-1}, z \geq 0\}$



$\phi(\boldsymbol{r}, z > 0)$ — bulk fields

$\longleftarrow$ $\phi(\boldsymbol{r}, 0) \equiv \phi(r) \equiv \hat{\phi}$
boundary fields

$\uparrow$ boundary conditions $\uparrow$

bulk **OPERATORS** boundary

$O_\Delta = \phi^2,\ \phi^4,\ \phi^2\nabla^2\phi^2,\ \phi^6 \ldots$

$\hat{O}_{\hat{\Delta}} = \hat{\phi},\ \hat{\phi}^2,\ T_{zz}(r) \ldots$

LGW hamiltonian and boundary conditions

$$\mathcal{H}[\phi] = \int d^{d-1}r \int_0^\infty dz \left[\frac{1}{2} |\nabla \boldsymbol{\phi}|^2 + \frac{m_0^2}{2} |\boldsymbol{\phi}|^2 + \frac{u_0}{4!} (|\boldsymbol{\phi}|^2)^2 \right] + \int d^{d-1}r \left(\frac{c_0}{2} |\hat{\boldsymbol{\phi}}|^2 - h_1 \hat{\phi}^1 \right)$$

$$\boldsymbol{\phi} = \{\phi^i, i = 1, \underbrace{2, ..., n}_{\text{Transverse}} \}$$

BC: $$\left. \frac{\partial}{\partial z} \phi^1(z) \right|_{z=0} = c_0 \, \phi^1(0) - h_1$$ Longitudinal component

$$< \phi^i(z) >= 0, \qquad i = 2, ..., n$$

$$m(z) \equiv < \phi^1(z) >= \frac{\mu_0}{(2z)^{\Delta_\phi}}$$ — **OP profile:** $$\mu_0 = \begin{cases} 0, \, ORD, \; SP, \; ST \\ \neq 0, \, EOT/N \end{cases}$$

$$EOT : \begin{cases} h_1 = 0 \\ c_0 < 0 \end{cases} \qquad CA/N : \begin{cases} h_1 \neq 0 \\ \text{arbitrary } c_0 \end{cases}$$

Two-point functions in BCFT $\quad (T = T_c)$

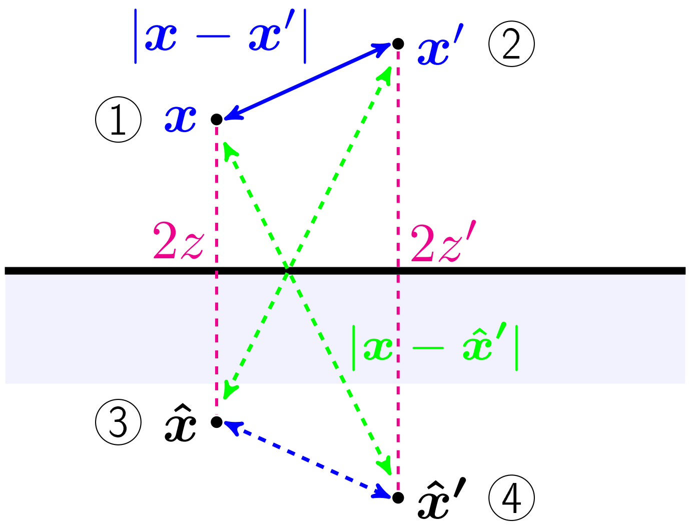
- Cardy 1984; Mc Avity Osborn 1993, 1995
- Nakayama 2013 Is boundary conformal in CFT?

$$G(\boldsymbol{x}, \boldsymbol{x}') \equiv \langle \phi(\boldsymbol{r}_1, z)\phi(\boldsymbol{r}_2, z')\rangle = \frac{g(\xi)}{(4zz')^{\Delta_\phi}} = \frac{F(v^2)}{|\boldsymbol{x} - \boldsymbol{x}'|^{2\Delta_\phi}} \qquad g(\xi) = \xi^{-\Delta_\phi} F(v^2)$$

$$\xi = \frac{r^2 + (z' - z)^2}{4zz'} = \frac{|\boldsymbol{x} - \boldsymbol{x}'|^2}{4zz'} \qquad v^2 = \frac{r^2 + (z' - z)^2}{r^2 + (z + z')^2} = \frac{|\boldsymbol{x} - \boldsymbol{x}'|^2}{|\boldsymbol{x} - \hat{\boldsymbol{x}}'|^2}$$

$$u_4 = \frac{x_{12}\, x_{34}}{x_{13}\, x_{24}} \qquad v_4 = \frac{x_{14}\, x_{23}}{x_{13}\, x_{24}}$$

$$\xi = u_4 \qquad v^2 = \frac{u_4}{v_4}$$

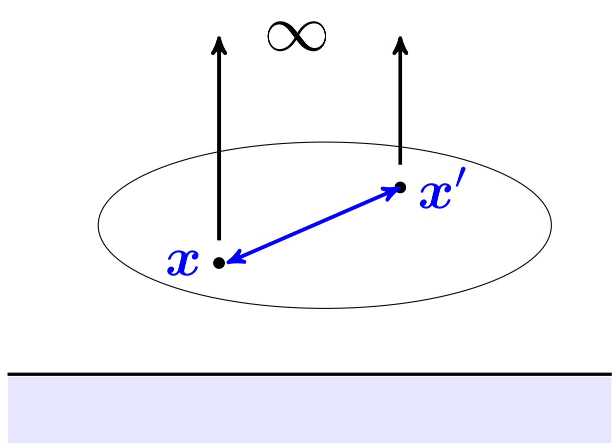


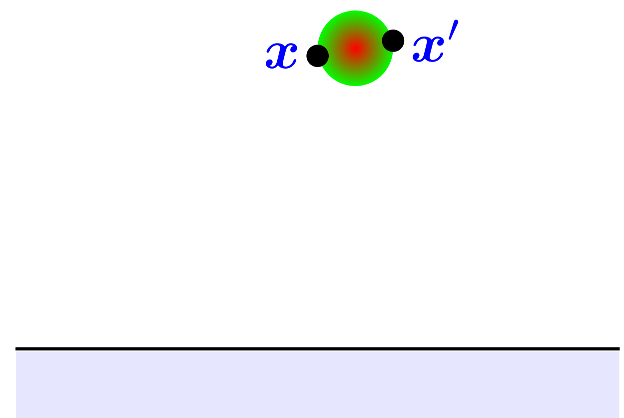
Asymptotic limits. 1. Bulk limit

$$G(\boldsymbol{x},\boldsymbol{x}') = \frac{g(\xi)}{(4zz')^{\Delta_\phi}} = \frac{F(v^2)}{|\boldsymbol{x}-\boldsymbol{x}'|^{2\Delta_\phi}} \qquad \xi = \frac{|\boldsymbol{x}-\boldsymbol{x}'|^2}{4zz'} \qquad v^2 = \frac{\xi}{\xi+1}$$

$$G(\boldsymbol{x},\boldsymbol{x}') \sim |\boldsymbol{x}-\boldsymbol{x}'|^{-2\Delta_\phi} \quad \Leftarrow \quad g(\xi) \sim \xi^{-\Delta_\phi}, \qquad \xi,\, v^2 \to 0$$

$$\xi \to 0 \quad \Leftarrow \quad \begin{cases} |\boldsymbol{x}-\boldsymbol{x}'| \ll z, z' \to \infty & \text{bulk limit} \\ |\boldsymbol{x}-\boldsymbol{x}'| \to 0 & \text{short-distance (OPE) limit} \end{cases} \qquad F(0) = C_{\phi\phi}^{\mathbb{1}}$$

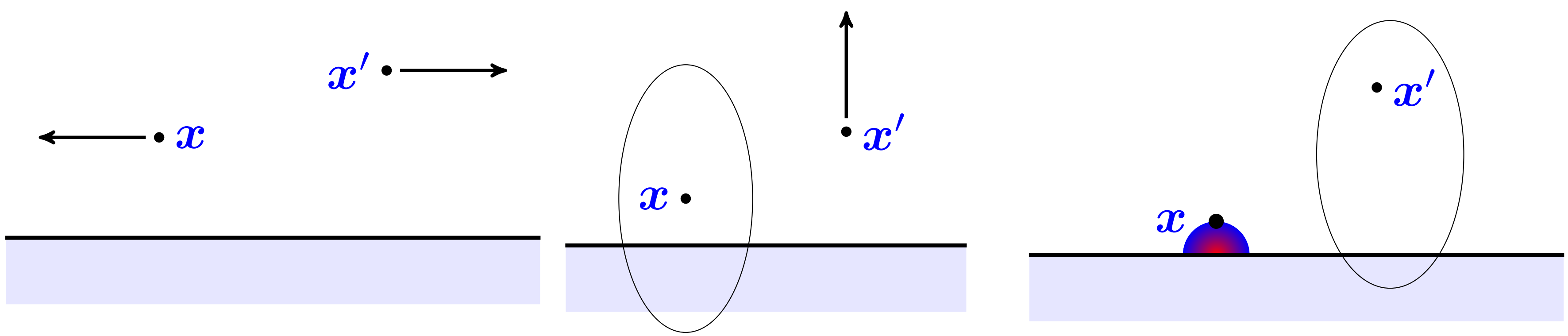




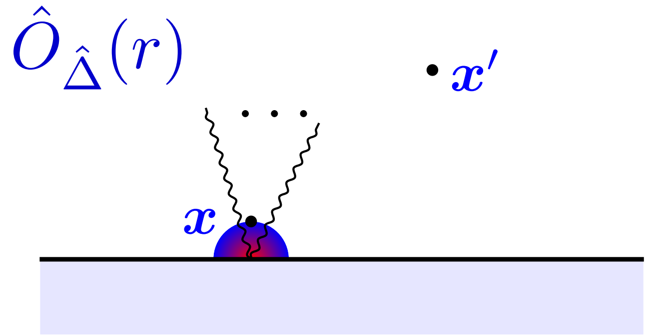
Asymptotic limits. 2. Boundary limit $\quad \xi \to \infty$

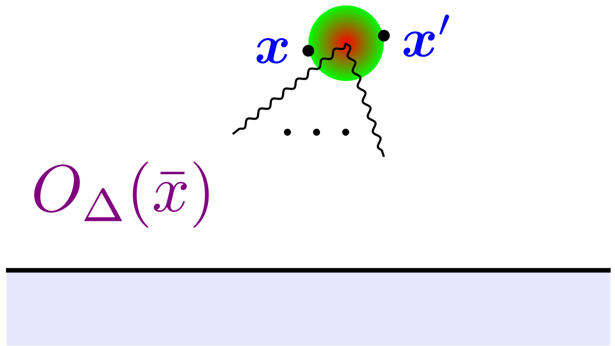
$$G(\boldsymbol{x},\boldsymbol{x}') = \frac{\boldsymbol{g(\xi)}}{(4zz')^{\Delta_\phi}} = \frac{\boldsymbol{F(v^2)}}{|\boldsymbol{x}-\boldsymbol{x}'|^{2\Delta_\phi}} \qquad g(\xi \to \infty) \sim \xi^{-\hat{\boldsymbol{\Delta}}}$$

- $r \to \infty |\ z, z'$ fixed $\quad \Rightarrow\ G(r;z,z') \sim r^{-2\hat{\boldsymbol{\Delta}}} = r^{-(d-2+\boldsymbol{\eta}_\parallel)}$
- $z' \to \infty |\ r, z$ fixed $\Rightarrow\ G(r;z,z') \sim z'^{-(\Delta_\phi+\hat{\boldsymbol{\Delta}})} = z'^{-(d-2+\boldsymbol{\eta}_\perp)} \qquad \eta_\perp = \frac{\eta_\parallel+\eta}{2}$
- $z \to 0 |\ r, z'$ fixed $\Rightarrow\ G(r;z,z') \sim z^{\hat{\boldsymbol{\Delta}}-\Delta_\phi} = z^{\boldsymbol{\eta}_\perp-\eta}$ short-distance (BOE) limit



Short-distance *operator* expansions





$\hat{b} = \hat{\Delta} + 1 - d/2$ <u>conformal blocks</u> $b = \Delta + 1 - d/2$

$$\mathcal{G}_{\mathrm{boe}}(\hat{\Delta};\xi) = \xi^{-\hat{\Delta}}\,{}_2F_1\big(\hat{\Delta},\hat{b};2\hat{b};-\xi^{-1}\big) \qquad \mathcal{G}_{\mathrm{ope}}(\Delta;\xi) = \xi^{\Delta/2}\,{}_2F_1\big(\tfrac{\Delta}{2},\tfrac{\Delta}{2};b;-\xi\big)$$

$$\boldsymbol{g(\xi)} = \sum_{\hat{\Delta}\geq 0} \mu^2_{\hat{\Delta}}\,\mathcal{G}_{\mathrm{boe}}(\hat{\Delta};\xi) \quad = \quad \xi^{-\Delta_\phi} \sum_{\Delta\geq 0} \lambda_\Delta\,\mathcal{G}_{\mathrm{ope}}(\Delta;\xi)$$

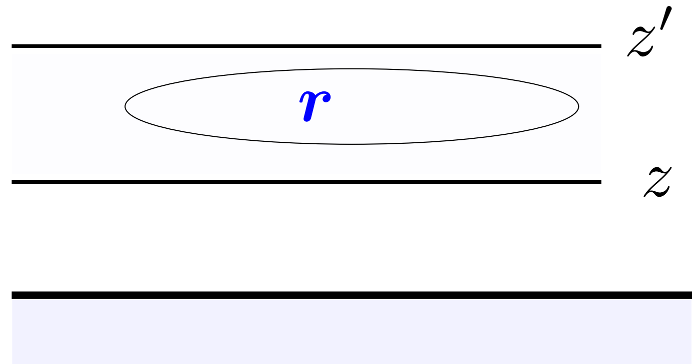
<u>bootstrap</u> $g(\xi) = \xi^{-\Delta_\phi} F(v^2)$ <u>equation</u>

Correlation function *Radon transformation** Layer suceptibility

$$G(\boldsymbol{x},\boldsymbol{x}') = \frac{\boldsymbol{g(\xi)}}{(4zz')^{\Delta_\phi}} \qquad \xi|_{r=0} = \frac{(z'-z)^2}{4zz'} \equiv \rho \qquad \chi(z,z') = \frac{\boldsymbol{R(\rho)}}{(4zz')^{\Delta_\phi-\lambda}}$$

$$\chi(z,z') = \int d^{d-1}r\, G(r;z,z')$$

$$= \tilde{G}(p=0;z,z')$$



$$\left.\begin{aligned} \boldsymbol{R(\rho)} &= \frac{\boldsymbol{\pi^{\lambda}}}{\boldsymbol{\Gamma(\lambda)}} \int_0^\infty \boldsymbol{du\, u^{-1+\lambda}\, g(u+\rho)} \\ \boldsymbol{g(\xi)} &= \frac{\boldsymbol{\pi^{-\lambda}}}{\boldsymbol{\Gamma(-\lambda)}} \int_0^\infty \boldsymbol{d\rho\, \rho^{-1-\lambda}\, R(\rho+\xi)} \end{aligned}\right\}$$

Radon transformation

$$\boldsymbol{\lambda = \frac{d-1}{2}}$$

* McAvity Osborn95 *CFTs near a boundary in general dimensions* NPB**455** 522

Simple examples. 1. Dirichlet propagator

$$G_D(r;z,z') = C_d\left(|\boldsymbol{x}-\boldsymbol{x}'|^{2-d} - |\boldsymbol{x}-\hat{\boldsymbol{x}}'|^{2-d}\right) = \frac{g(\xi)}{(4zz')^{\Delta_\phi^{(0)}}}$$

$$g(\xi) = C_d\left[\xi^{1-\frac{d}{2}} - (\xi+1)^{1-\frac{d}{2}}\right] \qquad C_d = \frac{S_d^{-1}}{d-2} \qquad \Delta_\phi^{(0)} = \frac{d-2}{2}$$

$$\tilde{G}(p;z,z') = \frac{1}{2p}\left[\mathrm{e}^{-p|z-z'|} - \mathrm{e}^{-p(z+z')}\right]$$

$$\chi(z,z') = \lim_{p\to 0}\tilde{G}(p;z,z') = \frac{1}{2}(z+z'-|z'-z|) = \mathsf{min}(z,z')$$

$$\chi(z,z') = \sqrt{4zz'}\,\frac{1}{2}\,\zeta^{\frac{1}{2}} \equiv \sqrt{4zz'}\,X(\zeta) = \sqrt{4zz'}\,R(\rho) \qquad \zeta = \frac{\mathsf{min}(z,z')}{\mathsf{max}(z,z')}$$

$$R(\rho) = X(\zeta)|_{\zeta=(\sqrt{\rho+1}-\sqrt{\rho})^2} = \left(\sqrt{\rho+1}-\sqrt{\rho}\right)/2 \qquad \rho = \frac{(z'-z)^2}{4zz'} = \xi|_{r=0}$$

$$g(\xi) = \frac{\pi^{-\lambda}}{\Gamma(-\lambda)}\int_0^\infty d\rho\,\rho^{-1-\lambda}\,R(\rho+\xi) = C_d\left[\xi^{1-\frac{d}{2}} - (\xi+1)^{1-\frac{d}{2}}\right]$$

2. EOT, *Landau* approximation [Eisenriegler1984 J Chem Phys **81** 4666]

$$G_E(p; z < z') = \frac{1}{2p}\Big[W(-pz) - W(pz)\Big]\,W(pz') \qquad W(x) = \mathrm{e}^{-x}\Big(1+\frac{3}{x}+\frac{3}{x^2}\Big)$$

$$\chi_E(z,z') = \lim_{p\to 0} G_0(p; z, z') = \frac{1}{5}\frac{[\min(z,z')]^3}{[\max(z,z')]^2} = \sqrt{4zz'}\,\frac{1}{10}\,\zeta^{\frac{5}{2}}$$

$$\mathsf{R} \quad \Rightarrow \quad G_E(r; z, z') = (4zz')^{-\Delta_\phi}\, g_E(\xi)$$

$$g_E(\xi) = \underbrace{\xi^{1-\frac{d}{2}} - (\xi+1)^{1-\frac{d}{2}}}_{g_D(\xi)} + \frac{12}{4-d}\left[\xi^{2-\frac{d}{2}} + (\xi+1)^{2-\frac{d}{2}} + \frac{4}{6-d}\Big(\xi^{3-\frac{d}{2}} - (\xi+1)^{3-\frac{d}{2}}\Big)\right]$$

[Lubensky Rubin 1975]

$$g_E^{(d=4)}(\xi) = \frac{1}{\xi} - \frac{1}{\xi+1} + 12 + 6(1+2\xi)\ln\frac{\xi}{\xi+1}$$

[LR75, ..., Liendo Rastelli van Rees 2013]

Radon transformation and BOE blocks [DHS20]

$$\xi = \frac{r^2 + (z' - z)^2}{4zz'} \qquad\qquad \zeta = \frac{\min(z, z')}{\max(z, z')}$$

$$G_{\mathbf{con}}(r; z, z') = (4zz')^{-\Delta_\phi}\, g_{\mathbf{con}}(\xi) \quad \longleftrightarrow \quad \chi(z, z') = (4zz')^{\lambda - \Delta_\phi}\, X(\zeta)$$

one-to-one correspondence between BOE blocks and ζ powers

$$\boldsymbol{g}_{\mathbf{con}}(\boldsymbol{\xi}) = \textstyle\sum_{\hat{\Delta}>0} \boldsymbol{\mu}^2_{\hat{\Delta}}\, \boldsymbol{\mathcal{G}}_{\mathbf{boe}}(\hat{\boldsymbol{\Delta}}; \boldsymbol{\xi}) \quad \overset{\boldsymbol{R}}{\longleftrightarrow} \quad \boldsymbol{X}(\boldsymbol{\zeta}) = \textstyle\sum_{\hat{\Delta}>0} \boldsymbol{c}_{\hat{\Delta}}\, \boldsymbol{\zeta}^{\hat{\boldsymbol{\Delta}} - \boldsymbol{\lambda}}$$

in scaling functions of correlator $g_{\mathbf{con}}(\xi)$ and layer susceptibility $X(\zeta)$

⋆ Enhanced scaling form: $\chi(z, z') = (4zz')^{\lambda - \Delta_\phi}\, \zeta^{\hat{\Delta}^{(l.n.)} - \lambda}\, Y(\zeta)$ [MS20]

$Y(\zeta \to 0)$ — regular

$\to$ Guess: $G(r; z, z') = \dfrac{(1 - v^2)^{\hat{\Delta} - \Delta_\phi}}{|\boldsymbol{x} - \boldsymbol{x}'|^{2\Delta_\phi}}\, \tilde{F}(v^2)$ $\qquad \tilde{F}(v^2 \to 1)$ — regular

3. BOE decompositions via *Radon transformation*

$$g(\xi) = (\xi + 1)^{-a} \qquad \xi = \frac{r^2}{4zz'} + \rho \qquad \rho = \frac{1}{4}(\zeta + \zeta^{-1} - 2)$$

$$R(\rho) = \frac{\pi^\lambda}{\Gamma(\lambda)} \int_0^\infty du\, u^{-1+\lambda} g(u+\rho), \;\; \lambda > 0 \quad \Rightarrow \quad R(\rho) = \pi^\lambda (a)_{-\lambda} (\rho + 1)^{\lambda - a}$$

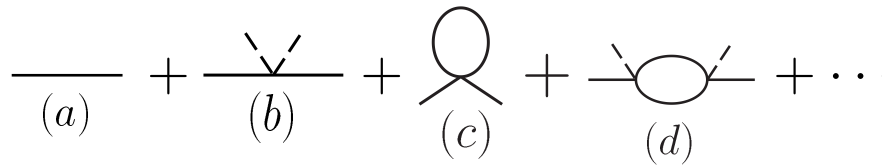
$$X(\zeta) \sim \zeta^{a-\lambda}(1+\zeta)^{-2(a-\lambda)} = \sum_{n \geq 0} (-1)^n \frac{(2a-2\lambda)_n}{n!} \zeta^{a-\lambda+n} \qquad [\,DHS\,20\,]$$

$$(\xi+1)^{-a} = \sum_{n\geq 0} \frac{(a)_n (2a-2\lambda)_n}{(-4)^n (a-\lambda)_n n!} \xi^{-a-n} {}_2F_1\Big(a+n, a-\lambda+\frac{1}{2}+n; 2a-2\lambda+1+2n; -\xi^{-1}\Big)$$

$\lambda = \frac{d-1}{2}$ $\rightarrow$ Herzog Huang JHEP2017 189 deduced "with a little bit of guess work"

EXTRAORDINARY TRANSITION **1Loop:** $O(\varepsilon = 4 - d)$

$$\chi(z, z') =$$



$$\Rightarrow$$

$$\left.\begin{aligned} \chi_L(z, z') &= \sqrt{4zz'}\; \zeta^{\frac{5-\varepsilon}{2}}\, C_d \big[1 + \varepsilon\, h(\zeta)\big] \\ \chi_T(z, z') &= \sqrt{4zz'}\; \zeta^{\frac{3-\varepsilon}{2}}\, \tilde{C}_{d-1} \big[1 + \varepsilon\, j(\zeta)\big] \end{aligned}\right\} \boldsymbol{\chi(z, z') = (4zz')^{\lambda - \Delta_\phi}\, \zeta^{\hat{\Delta} - \lambda}\, Y(\zeta)}$$

$$h(\zeta) = h_0(\zeta) + h_1(\zeta) + h_1(-\zeta)$$

$$h_0(\zeta) = \frac{1}{140(n+8)} \left[203n + 3140 - 10(7n + 96)\zeta^{-2} + 20(7n + 128)\zeta^{-4}\right]$$

$$h_1(\zeta) = \frac{(21n + 204)\,\zeta(1 + \zeta^2) + 4(7n + 74)\,\zeta^2 - 72(1 + \zeta^4)}{42(n+8)\,\zeta^6}\,(1 - \zeta)^3\, \ln(1 - \zeta)$$

$\underline{\zeta \to 0}$ — **BOE limit** (exponentiated with $\varepsilon \to 0$) $\underline{\zeta \to 1}\,, -1$ — **OPE limit**

Power expansions of $\chi_{L,T}(z,z')$

$$\chi_L(z,z') = (4zz')^{\lambda-\Delta_\phi}\,\zeta^{-\lambda}\sum_{\hat\Delta=d,\,k} c_{\hat\Delta}\zeta^{\hat\Delta} + O(\varepsilon^2) \qquad k = 6, 8, 10, \ldots$$

$$c_d = \frac{1}{10}\Big[1+\frac{76-n}{60\,(n+8)}\,\varepsilon\Big] \qquad c_k = 2\,\frac{k\,(k-3)\,(n+8)-2\,(5n+76)}{(k-5)_3(k)_3\,(n+8)}\,\varepsilon$$

$$\chi_T(z,z') = (4zz')^{\lambda-\Delta_\phi}\,\zeta^{-\lambda}\sum_{\hat\Delta=d-1,\,k} \tilde c_{\hat\Delta}\zeta^{\hat\Delta} + O(\varepsilon^2) \qquad k = 7, 9, 11, \ldots$$

$$\tilde c_{d-1} = \frac{1}{6}\Big[1+\frac{2n+15}{6\,(n+8)}\,\varepsilon\Big] \qquad c_k = 4\,\frac{(k+2)(k-5)}{(k-4)_6\,(n+8)}\,\varepsilon$$

$\Big\{\hat\Delta; c_{\hat\Delta}\,, \tilde c_{\hat\Delta}\Big\}$ — BCFT data for the layer susceptibility

BOE decompositions for correlation functions

$$\zeta^{-\lambda+\hat{\Delta}} \quad \overset{\boldsymbol{R}}{\longleftrightarrow} \quad \sigma_{\hat{\Delta}}\, \mathcal{G}_{\text{boe}}(\hat{\Delta};\xi)$$

$$\mathcal{G}_{\text{boe}}(\hat{\Delta};\xi) = \xi^{-\hat{\Delta}}\, {}_2F_1\big(\hat{\Delta}, \hat{b}; 2\hat{b}; -\xi^{-1}\big)$$

$$\hat{\Delta} = d \downarrow \boldsymbol{T_{zz}} \qquad \hat{b} = \hat{\Delta} + 1 - \frac{d}{2}$$

$$g_L{}^{\text{con}}(\xi) = \sigma_d c_d\, \mathcal{G}_{\text{boe}}(4-\varepsilon;\xi) + \underbrace{\sum_{k=6,\text{even}}^{\infty} \sigma_k\, c_k\, \mathcal{G}_{\text{boe}}(k;\xi)}_{O(\varepsilon)} + O(\varepsilon^2)$$

$$g_T{}^{\text{con}}(\xi) = \sigma_{d-1}\tilde{c}_{d-1}\, \mathcal{G}_{\text{boe}}(3-\varepsilon;\xi) + \overbrace{\sum_{k=7,\text{odd}}^{\infty} \sigma_k\, \tilde{c}_k\, \mathcal{G}_{\text{boe}}(k;\xi) + O(\varepsilon^2)}$$

$\hat{\Delta} = d-1 \uparrow$ the analogue of the displacement operator

for the broken rotation current $J_\mu^{[1i]}$ [Marco Meineri]

$\left\{\hat{\Delta}; \sigma_{\hat{\Delta}} c_{\hat{\Delta}}, \sigma_{\hat{\Delta}} \tilde{c}_{\hat{\Delta}}\right\}$ — **BCFT data for the two-point function**

Explicit results

$$g^{\mathbf{con}}_{L,T}(\xi) = g^{(0)}_{L,T}(\xi) + \varepsilon\, g^{(1)}_{L,T}(\xi) + O(\varepsilon^2)$$

$$g^{(0)}_L(\xi) = \frac{1}{2\xi} - \frac{1}{2(\xi+1)} + 6 + 3(2\xi+1)\ln\frac{\xi}{\xi+1} \qquad g^{(0)}_T(\xi) = \frac{1}{2\xi} + \frac{1}{2(\xi+1)} + \ln\frac{\xi}{\xi+1}$$

$$g_L{}^{(1)}(\xi) = \frac{1+\ln\left[\xi(\xi+1)\right]}{4\xi(\xi+1)} + 6\ln\xi + 3(2\xi+1)\ln\xi\cdot\ln\frac{\xi}{\xi+1}$$
$$+\left[\frac{72\xi(2\xi+3)+n+80}{4(n+8)}\ln\frac{\xi}{\xi+1} - 2\,\frac{(5n+52)\xi+2(2n+19)}{n+8}\right]\ln\frac{\xi}{\xi+1}$$
$$+\,2\,\frac{n+14}{n+8}\left[1+3(2\xi+1)\mathrm{Li}_2\left(-\frac{1}{\xi}\right)\right]$$

(i) OPE limit: $\xi\to 0:\quad g^{\mathrm{con}}_{L,T}(\xi) \sim \xi^{-1+\frac{\varepsilon}{2}} + O(\varepsilon^2) = \xi^{-\Delta_\phi} + O(\varepsilon^2)$

(ii) BOE: $\xi\to\infty:\quad g^{\mathrm{con}}_L(\xi) \sim \xi^{-4+\varepsilon} = \xi^{-d} \qquad g^{\mathrm{con}}_T(\xi) \sim \xi^{-3+\varepsilon} = \xi^{-(d-1)}$

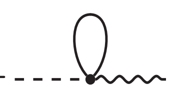
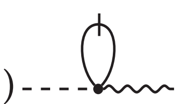
Bootstrap equation

$$g(\xi) = \sum_{\hat{\Delta}\geq 0} \mu^2_{\hat{\Delta}}\, \mathcal{G}_{\rm boe}(\hat{\Delta};\xi) := \xi^{-\Delta_\phi} \sum_{\Delta\geq 0} \lambda_\Delta\, \mathcal{G}_{\rm ope}(\Delta;\xi)$$

$$G(x,x') = <\phi(x)\phi(x')> = \frac{g(\xi)}{(4zz')^{\Delta_\phi}}$$ — <u>full</u> two-point function

$$g(\xi) = \mu_0^2 + g_L{}^{\rm con}(\xi) + (n-1)\, g_T{}^{\rm con}(\xi)$$

One-point function

$\langle \phi_L \rangle =$  $+$  $+ (n-1)$  $+ \cdots = \dfrac{\mu_0}{(2z)^{\Delta_\phi}}$

$$\Rightarrow \qquad \mu_0^2 = 2\,\frac{n+8}{\varepsilon} - \frac{n^2+46n+244}{n+8} + O(\varepsilon)$$

Bulk-channel expansion

$b = \Delta + 1 - \frac{d}{2}$

$$F_{\rm ope}(\xi) \equiv \sum_{\Delta \geq 0} \lambda_\Delta \, \mathcal{G}_{\rm ope}(\Delta;\xi) = \sum_{\Delta \geq 0} a_\Delta \, \xi^{\frac{\Delta}{2}} \, g_{\rm ope}(\Delta;\xi) := \xi^{\Delta_\phi} g(\xi)$$

$g_{\rm ope}(\Delta;\xi) = \beta(\Delta;\varepsilon)\,{}_2F_1\left(\frac{\Delta}{2},\frac{\Delta}{2};b;-\xi\right)$ $\quad a_\Delta = \frac{\lambda_\Delta}{\beta(\Delta;\varepsilon)}$ $\quad \beta(\Delta;\varepsilon) = \frac{\Gamma(\frac{\Delta}{2})\Gamma(\frac{\Delta}{2}+2-\frac{d}{2})}{\Gamma(b)}$

ε expansion of $F_{\rm ope}(\xi)$

$$\boldsymbol{F_{\rm ope}(\xi) = \varepsilon^{-1} F^{(-1)}_{\rm ope}(\xi) + F^{(0)}_{\rm ope}(\xi) + \varepsilon\, F^{(1)}_{\rm ope}(\xi) + O(\varepsilon^2)}$$

$$\Delta := \Delta_k = 2 + 2k + \gamma_k^{(1)}\varepsilon + \gamma_k^{(2)}\varepsilon^2 + O(\varepsilon^3) \qquad \longleftrightarrow \qquad |\phi|^{2(1+k)}$$

$$a_\Delta := a_{\Delta_k} = a_k^{(-1)}\varepsilon^{-1} + a_k^{(0)} + a_k^{(1)}\varepsilon + O(\varepsilon^2) \qquad k = -1, 0, 1, 2...$$

ε expansion: $F_{\rm ope}(\xi) = \varepsilon^{-1} F^{(-1)}_{\rm ope}(\xi) + F^{(0)}_{\rm ope}(\xi) + \varepsilon\, F^{(1)}_{\rm ope}(\xi) + O(\varepsilon^2)$

$$F^{(-1)}_{\rm ope}(\xi) = \sum_{k=0}^{\infty} \xi^{k+1} < a_k^{(-1)} > g_{\rm ope}(2k+2;\xi)\big|_{\varepsilon=0} \qquad\qquad < x_k > = \sum_j x_{k,j}$$

$$F^{(0)}_{\rm ope}(\xi) = \sum_{k=-1}^{\infty} \xi^{k+1} \left(\tfrac{1}{2} < a_k^{(-1)} \gamma_k^{(1)} > \ln \xi + < a_k^{(-1)} > \partial_\varepsilon + < a_k^{(0)} > \right) g_{\rm ope}(2k+2;\xi)$$

$$F^{(1)}_{\rm ope}(\xi) = \sum_{k=-1}^{\infty} \xi^{k+1} \Big[\tfrac{1}{8} < a_k^{(-1)} (\gamma_k^{(1)})^2 > \ln^2 \xi + < a_k^{(1)} > + < a_k^{(0)} > \partial_\varepsilon + \tfrac{1}{2} < a_k^{(-1)} > \partial_\varepsilon^2$$

$$+ \tfrac{1}{2} (< a_k^{(-1)} \gamma_k^{(2)} > + < a_k^{(0)} \gamma_k^{(1)} > + < a_k^{(-1)} \gamma_k^{(1)} > \partial_\varepsilon) \ln \xi \Big] g_{\rm ope}(2k+2;\xi)$$

$$\boxed{\varepsilon^{-1} F^{(-1)}_{\rm ope}(\xi) + F^{(0)}_{\rm ope}(\xi) + \varepsilon\, F^{(1)}_{\rm ope}(\xi) = \xi^{\Delta_\phi} \left[\mu_0^2 + g_L{}^{\rm con}(\xi) + (n-1)\, g_T{}^{\rm con}(\xi) \right]}$$

$$\longrightarrow \qquad \underset{-1<\xi<0}{\rm Disc} \ln \xi = 2\pi i \qquad\qquad \underset{-1<\xi<0}{\rm Disc} \ln^2 \xi = 4\pi i \ln(-\xi)$$

Bulk CFT data 1. OPE coefficients

$$a_k = a_k^{(-1)}\varepsilon^{-1} + a_k^{(0)} + a_k^{(1)}\varepsilon + O(\varepsilon^2) \qquad k = -1, 0, 1, 2...$$

$$< a_0^{(-1)} >= 2(n+8) \qquad < a_{k\geq 1}^{(-1)} >= 4(n+8)$$

$$< a_{-1}^{(0)} >= \frac{n}{2} \qquad < a_0^{(0)} >= -\frac{n^2+74n+408}{2(n+8)}$$

$$< a_{k\geq 1}^{(0)} >= 24k - 2\,\frac{n^2+46n+244}{n+8} - (n+8)\left(\frac{3}{k} + 4H_{k-1}\right)$$

$$H_k = \sum_{l=1}^{k} \frac{1}{l} \qquad - \qquad \text{harmonic numbers}$$

$a_k^{(1)}$ remains undetermined

Bulk CFT data 2. Scaling dimensions

$$\Delta_k = 2 + 2k + \gamma_k^{(1)}\varepsilon + \gamma_k^{(2)}\varepsilon^2 + O(\varepsilon^3) \qquad k = 0, 1, 2...$$

$$\gamma_k^{(1)} = 6\,\frac{k^2-1}{n+8} \qquad \gamma_0^{(2)} = \frac{(n+2)(13n+44)}{2(n+8)^3}$$

$$\gamma_{k\geq 1}^{(2)} = -2\,\frac{6k^2(n+20)+13n+50}{(n+8)^2}\,H_{k-1} + \frac{36\,k^4 - 3\,k^2(n+44) - 13n - 50}{k\,(n+8)^2} + \frac{k^2\big(n(11n+314)+1628\big) - 2\,\big(n(2n+77)+398\big)}{(n+8)^3}$$

$$\Delta_0 = \Delta_{\phi^2} = 2 - \frac{6\varepsilon}{n+8} + \frac{(n+2)(13n+44)}{2(n+8)^3}\,\varepsilon^2 + O(\varepsilon^3) \qquad \left[\Delta_{\phi^2} = d - y_t = d - \frac{1}{\nu}\right]$$

$$\Delta_1 = \Delta_{\phi^4} = 4 - 3\,\frac{3n+14}{(n+8)^2}\,\varepsilon^2 + O(\varepsilon^3) \qquad \left[\Delta_{\phi^4} = d - y_{u_0} = d + \omega\right]$$

$$\Delta_\phi = 1 - \frac{\varepsilon}{2} + \frac{n+2}{4\,(n+8)^2}\,\varepsilon^2 + O(\varepsilon^3)$$

Checks: Δ_k with $k=0,1,2$ agree with

- $$\Delta_{\phi^{2(k+1)}} = 2k+2+\frac{6(k^2-1)}{n+8}\varepsilon - \frac{k+1}{(n+8)^3}\Big[-\frac{1}{2}(13n+44)(n+2) + k\big(34(k-1)(n+8)+11n^2+92n+212\big)\Big]\varepsilon^2 + O(\varepsilon^3)$$

Derkachov Manashov *On the stability problem in the $O(n)$ nonlinear sigma model* Phys Rev Lett 1997 **79** 1423

$$\Delta_{\psi^{k+1}} = 2k+2-\frac{2^d\Gamma(\frac{d+1}{2})\sin(\pi\frac{d}{2})}{\pi^{\frac{3}{2}}\Gamma(\frac{d}{2}+1)}(k+1)\Big[(k-1)(d-2)+\frac{k}{2}(d-4)(d-1)\Big]\frac{1}{n}+O\Big(\frac{1}{n^2}\Big)$$

$$\underset{\varepsilon\to 0}{=} 2k+2+6(k^2-1)\frac{\varepsilon}{n}-\frac{1}{2}\left(22k^2+9k-13\right)\frac{\varepsilon^2}{n}+O(\varepsilon^3)+O\Big(\frac{1}{n^2}\Big)$$

Lang Rühl *The critical $O(n)$ σ-model at dimension $2<d<4$: Fusion coefficients and anomalous dimensions* Nucl Phys B 1993 **400** 597

Outlook 1. Bootstrap equation for the layer susceptibility

$$\frac{g(\xi)}{(4zz')^{\Delta_\phi}} = G(\boldsymbol{x},\boldsymbol{x}') = \frac{F(v^2)}{|\boldsymbol{x}-\boldsymbol{x}'|^{2\Delta_\phi}}$$

$$\boldsymbol{R} \Updownarrow \qquad\qquad \Updownarrow \textbf{?}$$

$$(4zz')^{\lambda-\Delta_\phi}\, X(\zeta) = \chi(z,z') = \left(\frac{z+z'}{2}\right)^{2(\lambda-\Delta_\phi)} Z(y)$$

$$\zeta \to 0 \downarrow \qquad\qquad \underline{\zeta \to 1} \downarrow y \to 0$$

$$\textbf{!}\ (4zz')^{\lambda-\Delta_\phi} \sum_{\hat{\Delta}>0} c_{\hat{\Delta}}\, \zeta^{\hat{\Delta}-\lambda} = \chi(z,z') = \left(\frac{z+z'}{2}\right)^{2(\lambda-\Delta_\phi)} \sum_{\Delta\geq 0} b_\Delta\, y^\Delta\ \textbf{!}$$

$$DHS \Updownarrow \qquad\qquad \Updownarrow \textbf{?}$$

$$\sum_{\hat{\Delta}\geq 0} \mu^2_{\hat{\Delta}}\, \mathcal{G}_{\rm boe}(\hat{\Delta};\xi) = g(\xi) = \xi^{-\Delta_\phi} \sum_{\Delta\geq 0} \lambda_\Delta\, \mathcal{G}_{\rm ope}(\Delta;\xi)$$

<u>Suggestion:</u> $\zeta = \dfrac{\mathrm{min}(z,z')}{\mathrm{max}(z,z')} = \dfrac{z+z'-|z-z'|}{z+z'+|z-z'|} \equiv \dfrac{1-y}{1+y} \qquad y = \dfrac{|z-z'|}{z+z'}$

Outlook 2. Correlation functions to $O(\varepsilon^2)$

Diehl Dietrich 1981

To determine $G^R(\rho; z, z', u^*, 0)\ldots$ one still has to do a Fourier transform in one parallel and two perpendicular momenta... of

$$G^{(2)R^*}_{\mathrm{loc}}(\mathbf{q}_1) = \mu^{-2}(|\mathbf{q}_1|/\mu)^{-2+\eta}\left[1+\eta\left(\frac{C_E}{2}-\frac{13}{8}\right)\right]+O(\varepsilon^3),$$

$$G^{(2)R^*}_{\mathrm{nloc}}(\mathbf{p}_1, k_1, k_2) = \mu^{-3}\left[\left(\frac{\mathbf{p}_1^2+k_1^2}{\mu^2}\right)\left(\frac{\mathbf{p}_1^2+k_2^2}{\mu^2}\right)\right]^{-1}\cdot\frac{\pi}{4}\cdot\frac{n+2}{n+8}\cdot\varepsilon$$
$$\cdot\left\{\left[\left|\frac{k_1+k_2}{\mu}\right|^{1-\frac{6\varepsilon}{n+8}}-\left|\frac{k_1-k_2}{\mu}\right|^{1-\frac{6\varepsilon}{n+8}}\right]\right.$$
$$\cdot\left[1+\frac{-(n^2/2)+4n+34}{(n+8)^2}\,\varepsilon-\frac{6\ln 2}{n+8}\,\varepsilon\right]$$
$$+\frac{6\varepsilon}{n+8}\left[\frac{n+2}{12}\left(\left|\frac{k_1+k_2}{\mu}\right|-\left|\frac{k_1-k_2}{\mu}\right|\right)\right.$$
$$+\frac{n+2}{12}\left(4\,\frac{k_1k_2}{2|\mathbf{p}_1|\mu}\operatorname{arccot}\frac{|k_1+k_2|+|k_1-k_2|}{2|\mathbf{p}_1|}\right.$$
$$\left.-\left|\frac{2p_1}{\mu}\right|\arctan\frac{|k_1+k_2|-|k_1-k_2|}{|2\mathbf{p}_1|}\right)$$
$$+\frac{k_1k_2}{\mu\cdot|\mathbf{p}_1|}\left(\pi-2\arctan\frac{|k_1-k_2|+|k_1+k_2|}{2|\mathbf{p}_1|}\right)$$
$$+\left(1-\frac{n+2}{12}\right)\cdot\frac{1}{2}\left(\left|\frac{k_1+k_2}{\mu}\right|-\left|\frac{k_1-k_2}{\mu}\right|\right)$$
$$\cdot\ln\frac{(2\mathbf{p}_1)^2+(|k_1+k_2|-|k_1-k_2|)^2}{\mu^2}$$
$$+\frac{n+2}{12}\cdot\frac{1}{2}\left(\left|\frac{k_1+k_2}{\mu}\right|-\left|\frac{k_1-k_2}{\mu}\right|\right)$$
$$\left.\left.\cdot\ln\frac{(2\mathbf{p}_1)^2+(|k_1+k_2|+|k_1-k_2|)^2}{\mu^2}\right]\right\}+O(\varepsilon^3).$$

Ohno Okabe 1985

$$G_\rho(z_1, z_2) = (z_1z_2)^{(2-d-\eta)/2}f(v)$$

non-dimensional argument

$$v = \frac{z_1^2+z_2^2+\rho^2}{2z_1z_2} = \frac{\bar{r}^2+r^2}{\bar{r}^2-r^2}.$$

$$f(v) = \frac{1}{(2\pi)^{d/2}}\left[\exp[\tfrac{1}{2}(2-d)\pi\mathrm{i}]\,\frac{Q^{(d-2)/2}_{\mu-1/2}(v)}{(v^2-1)^{(d-2)/4}}\right.$$
$$\left.+\eta\,\frac{y}{v^2-1}\int^v\frac{\mathrm{d}v}{y}\int_\infty^v\mathrm{d}v\,\frac{y}{v^2-1}\int_\infty^v\frac{\mathrm{d}v}{y}\int_\infty^v\mathrm{d}v\left(\frac{y}{v^2-1}\right)^3+\mathrm{O}(\varepsilon^3)\right]$$

where Q^σ_ν is the associated Legendre function of second kind and y is t

$$y = \begin{cases} 1 & \text{for the ordinary transition} \\ v & \text{for the special transition.} \end{cases}$$

This is the first presentation of the correlation function in real space.

Outlook 3. Correlation functions and critical exponents: $O(\varepsilon^2)$

$\rightarrow$ 2019 Bissi Hansen Söderberg Analytic bootstrap for boundary CFT

Procházka Söderberg

$$G(x,x') = \frac{(\xi+1)^{\gamma-\hat{\gamma}}}{|\boldsymbol{x}-\boldsymbol{x}'|^{2\Delta_\phi}} \mp \frac{\xi^{\gamma-\hat{\gamma}}}{|\hat{\boldsymbol{x}}-\boldsymbol{x}'|^{2\Delta_\phi}}$$

— the scaling form of

Lubensky Rubin 1975:

$$\Gamma^*(\vec{\mathrm{x}},\vec{\mathrm{x}}') = \frac{1}{|\vec{\mathrm{x}}-\vec{\mathrm{x}}'|^{d-2+\eta}} \left(\frac{|\vec{\mathrm{x}}-\nu\vec{\mathrm{x}}'+2\lambda\hat{e}_\perp|^2}{4(z+\lambda)(z'+\lambda)} \right)^{\tilde{\eta}} - \frac{1}{|\vec{\mathrm{x}}-\nu\vec{\mathrm{x}}'+2\lambda\hat{e}_\perp|^{d-2+\eta}} \left(\frac{|\vec{\mathrm{x}}-\vec{\mathrm{x}}'|^2}{4(z+\lambda)(z'+\lambda)} \right)^{\tilde{\eta}}$$

$\Rightarrow$ Use $\chi(z,z')$ to compute correlation functions to $O(\varepsilon^2)$?

$\Rightarrow$ Use $\chi(z,z')$ to compute critical exponents to $O(\varepsilon^3)$?

Outlook 4. Other functions and models, Feynman integrals

2020 Herzog, Kobayashi The $O(N)$ model with ϕ^6 potential in $\mathbb{R}^2 \times \mathbb{R}^+$

2021 Gimenez-Grau, Liendo, van Vliet Superconformal boundaries in $4-\varepsilon$ dimensions

2020 Loebbert, Miczajka, Müller, Münkler Yangian bootstrap for massive Feynman integrals arXiv:2010.08552

$d = D+1$

$$\int_{\boldsymbol{p}'}^{(D)} \frac{1}{(p'^2+m_1^2)^\alpha \left[(\boldsymbol{p}'+\boldsymbol{p})^2+m_2^2\right]^\beta}$$

The combination of the above inversion with D-dimensional translations yields the special conformal transformations in $D+1$ dimensions,

$$x^{\hat{\mu}} \mapsto \frac{x^{\hat{\mu}} + c^{\hat{\mu}} x_{\hat{\nu}} x^{\hat{\nu}}}{1 + 2c_{\hat{\nu}} x^{\hat{\nu}} + c_{\hat{\rho}} c^{\hat{\rho}} x_{\hat{\nu}} x^{\hat{\nu}}}, \tag{2.10}$$

albeit with the extra-dimensional component c^{D+1} set to zero, since I_n is not invariant under the respective translation. These transformations are generated by the conformal generator

$$I_{11} \sim p^{-\varepsilon} \left(\frac{p^2}{p^2+(m_1+m_2)^2} \right)^{\frac{\varepsilon}{2}} F_1 \left(\ldots; \frac{p^2}{p^2+(m_1+m_2)^2}, \frac{p^2+(m_1-m_2)^2}{p^2+(m_1+m_2)^2} \right)$$

$$\boldsymbol{p} \to \boldsymbol{r},\ m_1 \to z,\ m_2 \to z' \quad \to \quad u^2 = \frac{r^2}{r^2+(z+z')^2}, \quad v^2 = \frac{r^2+(z-z')^2}{r^2+(z+z')^2}$$

$$I_{11}^{(D=2)} \sim \frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|^2}\, v \ln \frac{1+v}{1-v}$$

[MS'07JMP] A massive Feynman integral...

SUPPLEMENTARY STAFF

$m(z)$

Flow equation: $$\ell \frac{d}{d\ell} \bar{u}(\ell) = \beta_u[\bar{u}(\ell)]$$

Solutions: $$\bar{u}(\ell \to 0) \quad \approx \quad \begin{cases} u^* & \text{for } d < 4 \\ \frac{1}{\beta_2 |\ln \ell|} + O\left(\frac{\ln|\ln \ell|}{|\ln \ell|^2}\right) & \text{for } d = 4 \\ u \ell^{d-4} & \text{for } d > 4 \end{cases}$$

$$d = 4: \qquad m(z) \sim \frac{z^{-1}}{\sqrt{u(\ell)}}, \qquad \ell = (\mu z)^{-1}, \qquad m(z) \sim z^{-1} \sqrt{|\ln \mu z|}$$

BOE asymptotics: $\xi \to \infty$

$$g_L^{\rm con}(\xi) \sim \xi^{-\hat{\Delta}} \qquad \hat{\Delta} = d:$$

$\hat{O} = T_{zz}$ — energy-momentum tensor

$$g_T^{\rm con}(\xi) \sim \xi^{-\hat{\Delta}} \qquad \hat{\Delta} = d - 1:$$

$\hat{O}$ — the analogue of the displacement operator for the broken rotation current $J_\mu^{[1i]}$

The conservation equation for this current is broken by a delta function on the boundary which is multiplied by a scalar boundary operator that is a vector of the preserved $O(N-1)$ subgroup. Similar to the displacement operator, this operator should obey a Ward identity that relates its coupling μ_{d-1} to the bulk field ϕ^1 with the one-point function coefficient μ_0 of this field. It would be interesting to derive this Ward identity to confirm the nature of this operator. Similar protected defect operators appeared for instance in the context of a BPS defect which breaks part of the R-symmetry in a supersymmetric theory [Marco Meineri]

Simple examples

[Herzog Huang JHEP 2017 189]

$$\frac{\xi^{-a}}{2}\left[1+\left(\frac{\xi}{\xi+1}\right)^{a}\right]=\sum_{\substack{n\geq 0\\ \text{even}}}\mu_n^2\,\mathcal{G}_{\mathbf{boe}}(\underbrace{a+n}_{\hat{\Delta}_n};\xi) \quad (1)$$

$$\frac{\xi^{-a}}{2}\left[1-\left(\frac{\xi}{\xi+1}\right)^{a}\right]=\sum_{\substack{n\geq 1\\ \text{odd}}}\mu_n^2\,\mathcal{G}_{\mathbf{boe}}(\overbrace{a+n}^{\hat{\Delta}_n};\xi) \quad (2)$$

$$\mathbf{(1)+(2):}\quad \xi^{-a}=\sum_{n\geq 0}\mu_n^2\,\mathcal{G}_{\mathbf{boe}}(\hat{\Delta}_n;\xi) \qquad \mathcal{G}_{\mathrm{boe}}(\hat{\Delta};\xi)=\xi^{-\hat{\Delta}}{}_2F_1\left(\hat{\Delta},\hat{\beta};2\hat{\beta};-\xi^{-1}\right)$$

$$\mathbf{(1)-(2):}\quad (\xi+1)^{-a}=\sum_{n\geq 0}(-1)^n\,\mu_n^2\,\mathcal{G}_{\mathbf{boe}}(\hat{\Delta}_n;\xi) \qquad \hat{\beta}=a+1-\frac{d}{2}$$

$$\boldsymbol{\mu_n^2=\frac{(a)_n(\hat{\beta})_n}{(2\hat{\beta}-1+n)_n n!}}$$

Generalizations

[Fields Wimp'60 *Math Comp* 15 390]

$$\frac{1}{2}\left[1+\left(\frac{\xi}{\xi+1}\right)^a\right] = \sum_{\substack{n\geq 0\\ \text{even}}} p_n\, \xi^{-n}\, {}_2F_1\left(a+n, b+n; c+2n; -\xi^{-1}\right) \qquad (3)$$

$$\frac{1}{2}\left[1-\left(\frac{\xi}{\xi+1}\right)^a\right] = \sum_{\substack{n\geq 1\\ \text{odd}}} p_n\, \xi^{-n}\, {}_2F_1\left(a+n, b+n; c+2n; -\xi^{-1}\right) \qquad (4)$$

(3) + (4) : $$1 = \sum_{n\geq 0} p_n\, \xi^{-n}\, {}_2F_1\left(a+n, b+n; c+2n; -\xi^{-1}\right)$$

(3) − (4) : $$(\xi+1)^{-a} = \sum_{n\geq 0} (-1)^n\, p_n\, \xi^{-a-n}\, {}_2F_1\left(a+n, b+n; c+2n; -\xi^{-1}\right)$$

$$\boldsymbol{p_n = \frac{(a)_n (b)_n}{(c-1+n)_n n!}}$$

(a), b, c − f r e e

BOE and OPE decompositions of unity

(i) BOE: $c = 2b \longrightarrow 1 = \sum_{n\geq 0} p_n|_{c=2b}\, \xi^{-n}\, {}_2F_1\left(a+n, b+n; 2b+2n; -\xi^{-1}\right)$

(ii) OPE: $b{=}a,\ \xi^{-1}{\to}\xi \longrightarrow 1 = \sum_{n\geq 0} p_n|_{b=a}\, \xi^{n}\, {}_2F_1\left(a+n, a+n; c+2n; -\xi\right)$

$$\sum_{n\geq 0} p_n|_{c=2b}\, \xi^{-n}{}_2F_1(\underbrace{a{+}n}_{\hat{\boldsymbol{\Delta}}_{\boldsymbol{n}}}, b{+}n; 2b{+}2n; -\xi^{-1}) = \sum_{n\geq 0} p_n|_{b=a}\, \xi^{n}{}_2F_1(\underbrace{a{+}n}_{\boldsymbol{\Delta_n/2}}, a{+}n; c{+}2n; -\xi)$$

$$\boldsymbol{\mathcal{G}_{\mathrm{boe}}(\hat{\Delta};\xi) = \xi^{-\hat{\Delta}}{}_2F_1\left(\hat{\Delta}, \hat{\beta}; 2\hat{\beta}; -\xi^{-1}\right)} \qquad \boldsymbol{\mathcal{G}_{\mathrm{ope}}(\Delta;\xi) = \xi^{\Delta/2}{}_2F_1\left(\tfrac{\Delta}{2}, \tfrac{\Delta}{2}; \beta; -\xi\right)}$$

$$\sum_{\hat{\Delta}_n} \frac{(a)_n (b)_n}{(2b-1+n)_n n!}\, \mathcal{G}_{\mathbf{boe}}(\hat{\Delta}_n;\xi) = \xi^{-2a} \sum_{\Delta_n} \frac{(a)_n^2}{(c-1+n)_n n!}\, \mathcal{G}_{\mathbf{ope}}(\Delta_n;\xi)$$

$$\boldsymbol{\hat{\Delta}_n = \{a+n \,|\, n \in \mathbb{N}_0\}} \qquad \boldsymbol{\Delta_n = \{2a+2n \,|\, n \in \mathbb{N}_0\}}$$

BOE and OPE decompositions of $(\boldsymbol{\xi}+\mathbf{1})^{-\boldsymbol{a}}$

BOE: $c{=}2b \to (\xi{+}1)^{-a}{=}\sum_{n\geq 0}(-1)^n\, p_n|_{c=2b}\,\xi^{-a-n}{}_2F_1(a{+}n, b{+}n; 2b{+}2n; -\xi^{-1})$

OPE: $b{=}a,\ \xi^{-1}{\to}\xi \to (\xi{+}1)^{-a} = \sum_{n\geq 0}(-1)^n q_n|_{b=a}\,\xi^n{}_2F_1\,(a{+}n, a{+}n; c{+}2n; -\xi)$

$$q_n = \frac{(a)_n(c-b)_n}{(c-1+n)_n n!}$$

$$\sum_{\hat{\Delta}_n}\mu^2_{\hat{\Delta}_n}\,\mathcal{G}_{\mathbf{boe}}(\hat{\Delta}_n;\xi) = \xi^{-a}\sum_{\Delta_n}\lambda_n\,\mathcal{G}_{\mathbf{ope}}(\Delta_n;\xi)$$

$$\mu^2_{\hat{\Delta}_n} = \frac{(-1)^n(a)_n(b)_n}{(2b-1+n)_n n!} \qquad \lambda_n = \frac{(-1)^n(a)_n(c-a)_n}{(c-1+n)_n n!}$$

$$\hat{\boldsymbol{\Delta}}_{\boldsymbol{n}} = \{\boldsymbol{a}+\boldsymbol{n}|\,\boldsymbol{n}\in\mathbb{N}_{\mathbf{0}}\} \qquad \boldsymbol{\Delta}_{\boldsymbol{n}} = \{\mathbf{2}\boldsymbol{a}+\mathbf{2}\boldsymbol{n}|\,\boldsymbol{n}\in\mathbb{N}_{\mathbf{0}}\}$$