Conformal field theory of the integer quantum Hall transition: a status report

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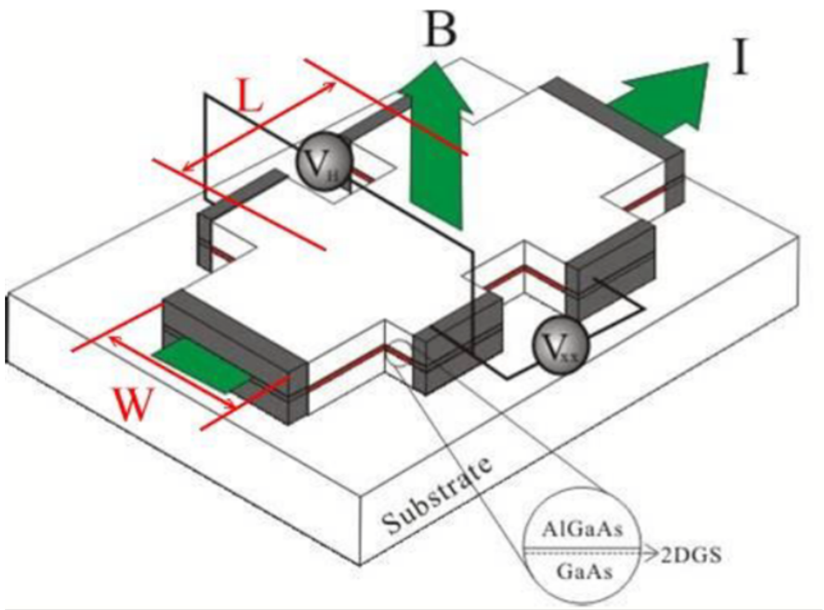
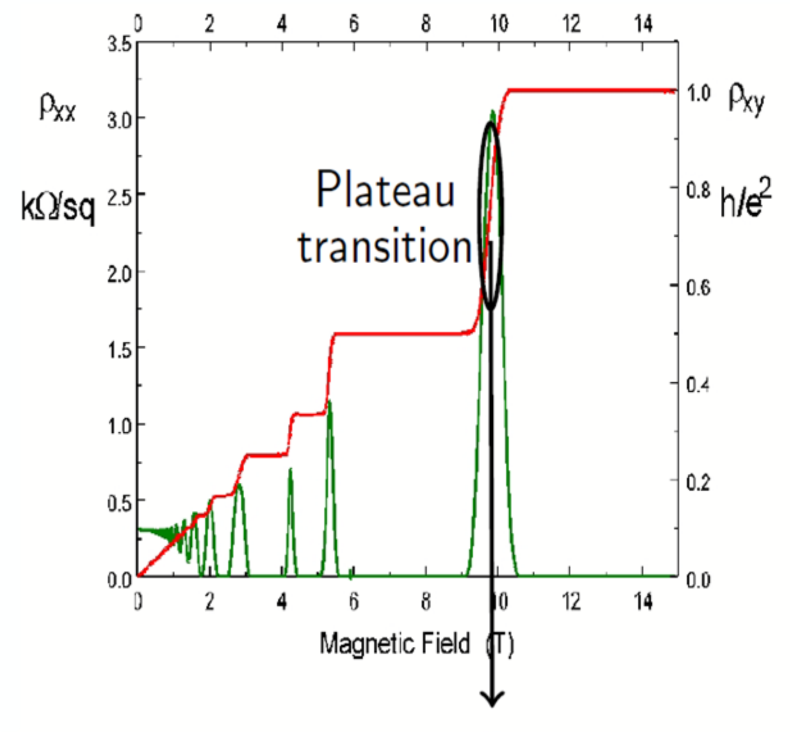
Based on: arXiv:1805.12555
arXiv:1612.04109 (with Bondesan & Wieczorek)
arXiv:1312.6172 (with B & W)

Integer Quantum Hall Effect





Two-dimensional disordered electron gas at low temperature and in a strong magnetic field

Hall resistance exhibits plateaus: $R_H = \frac{h}{ne^2}$.

Transition between plateaus is a critical phenomenon (of Anderson-localization type). Could/should be a paradigm, but so far was not understood in quantitative detail ...

Anderson transitions: strong topological insulator ⟶ band insulator

(Schnyder et al., 2008)

2d	symmetry class		3d
integer quantum Hall transition	A	AIII	Su–Schrieffer–Heeger model (particle–hole symmetry)
"Majorana" superconductor (spin–orbit scattering, T broken; Read–Green)	D	DIII	^{3}He B–phase
Broken–T BCS superconductor (spin–singlet pairing)	C	CI	BCS superconductor (T–invariant, spin–singlet)
quantum spin Hall insulator	AII	CII	Fu–Kane–Mele augmented with particle–hole symmetry
T–invariant superconductor (with spin–orbit scattering)	DIII	AII	$\mathbb{Z}_2$ strong topological insulator (Fu–Kane–Mele)

Plan of Talk

- Hyperbolic symmetry (Wegner, 1979)
- CFT of the integer quantum Hall transition
- Prospects for conformal bootstrap of Anderson transitions

What's the target space?

— Retarded & advanced Green's functions from Gaussian integration:

$$\left\langle x' \right| \frac{\pm i}{E \pm i\varepsilon - H} \left| x \right\rangle = \mathrm{Det}\big(\varepsilon \mp i(E-H)\big) \int_\phi \phi_\pm(x')\, \bar{\phi}_\pm(x)\; e^{\pm i \int \bar{\phi}_\pm (E \pm i\varepsilon - H)\phi_\pm}$$

$$\equiv G^E_\pm(x', x)$$

represent by fermion-ghost integral (Faddeev-Popov)

ϕ_+ retarded, ϕ_- advanced (r copies $\equiv$ replicas each)

— Wegner's hyperbolic symmetry ($\varepsilon \longrightarrow 0+$):

sesquilinear form $\bar{\phi}_+ \phi_+ - \phi_- \bar{\phi}_- + \bar{\psi}_+ \psi_+ + \psi_- \bar{\psi}_-$ determines the global symmetry group $\mathcal{U} \equiv U(r, r \,|\, 2r)$

(for systems in class A).

Upper critical dimension ?

Recall $\mathbb{E}\left(\frac{G_-^E(x,x) - G_+^E(x,x)}{2\pi i}\right) = \langle |\phi_-(x)|^2 + |\phi_+(x)|^2 \rangle \geq 0.$

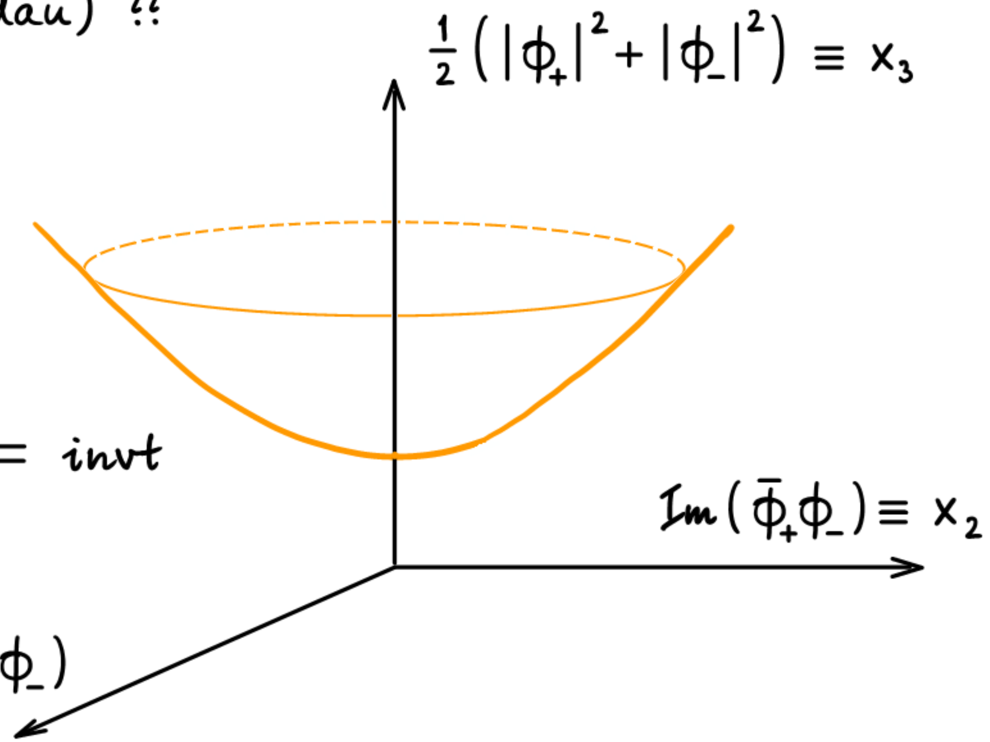
(Bare) Lagrangian ($\varepsilon \longrightarrow 0+$; fermions omitted for notational simplicity):

$$\mathcal{L} = -i\,\bar\phi_+(\nabla^2 + E)\phi_+ + i\,\bar\phi_-(\nabla^2 + E)\phi_- + \lambda^2\left(|\phi_+|^2 - |\phi_-|^2\right)^2.$$

Renormalization ? Mean-field theory (Landau) ??

Picture an orbit of the symmetry group $SU(1,1)$:

$$x_3^2 - x_1^2 - x_2^2 = \frac{1}{4}\left(|\phi_+|^2 - |\phi_-|^2\right)^2 = \text{invt}$$



[Compare with the mean-field scenario for a compact symmetry group.]

Systems with a large local density of states $\mathrm{Im}\, \langle x | \frac{1/\pi}{E - i\varepsilon - H} | x \rangle$.

— Can derive nonlinear σ model (Wegner, Hikami, Efetov, ...)

with target space $\mathcal{U}/K$ where $\mathcal{U} \equiv U(r, r | r + r)$

$$\underbrace{U(r|r)_+}_{\text{}} \times U(r|r)_- \equiv K \subset \mathcal{U}.$$

Parametrize field as $Q = u \, \Sigma_3 \, u^{-1}$, $\quad \Sigma_3 = \sigma_3 \otimes 1_{r|r}$, $\quad u \in \mathcal{U}$.

$(Q^2 = 1)$ $\qquad$ ret/adv $\otimes$ SUSY

— Conventional scenario of Anderson transition:

metal (extended states) $\curvearrowright$ symmetry $\mathcal{U}$ broken spontaneously,

insulator (localized states) $\curvearrowright$ $\mathcal{U}$ unbroken (symmetry restoration).

— Expansion in $\epsilon = d - 2$ (Hikami, Wegner), but no upper critical dimension.

Systems with a small density of states.

Conventional nonlinear σ model <u>cannot</u> be derived.

?

[➡ Integer quantum Hall transition (MZ, arXiv: 1805.12555)]

Strong disorder $\rightsquigarrow$ $\bar{\phi}_+ \phi_+ - \phi_- \bar{\phi}_- + \bar{\psi}_+ \psi_+ + \psi_- \bar{\psi}_- = 0$! ("light cone")

Change basis $\Sigma_3 = \sigma_3 \otimes 1_{r|r} \longrightarrow \Sigma_2 = \sigma_2 \otimes 1_{r|r}$ ("light cone coordinates").

Target space (after bosonization) is <u>nilpotent</u> $\mathcal{U}$-orbit $\mathcal{U}/P$

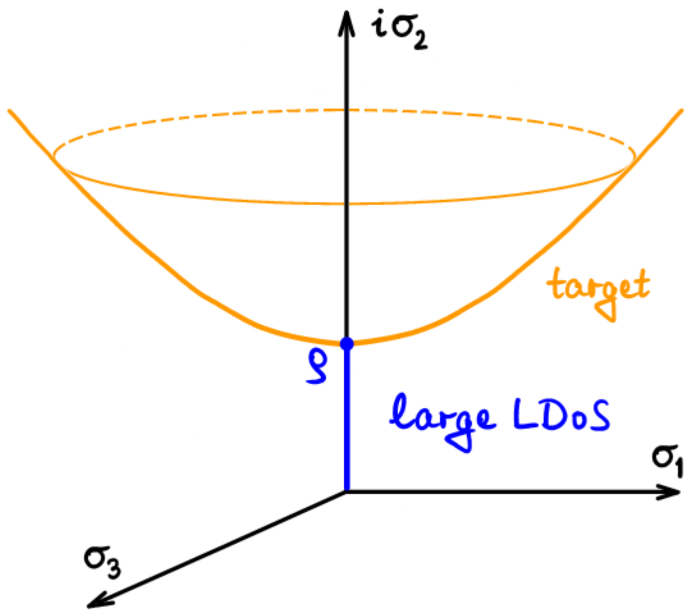
where $P = \left\{ \begin{pmatrix} A & B \\ 0 & A \end{pmatrix} \right\}$.

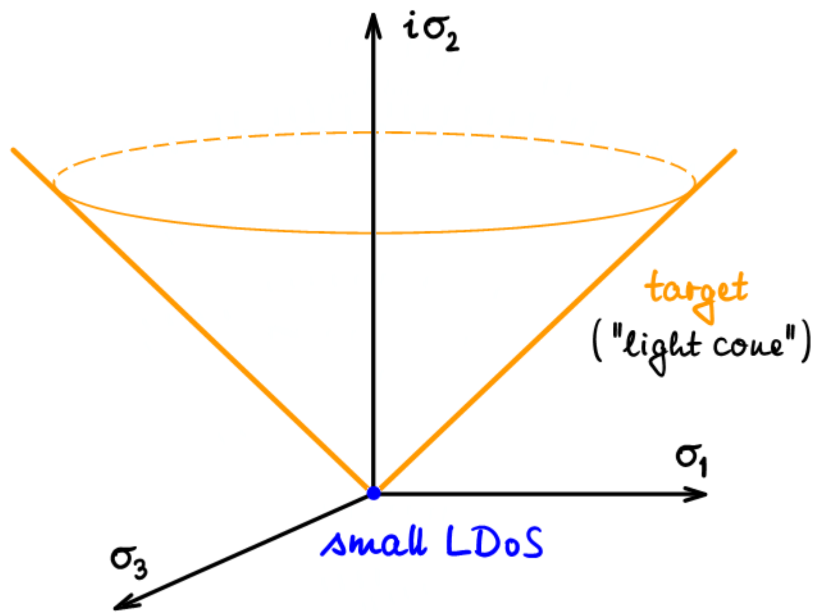
Field $Q = u \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} u^{-1}$ satisfies $Q^2 = 0$.

Parametrization: $u = nh$, $n = \begin{pmatrix} 1 & 0 \\ C & 1 \end{pmatrix}$, $h = \begin{pmatrix} g_L & 0 \\ 0 & g_R \end{pmatrix}$.

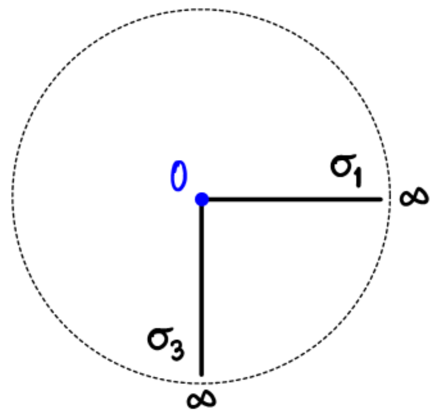
$$Q = n \begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix} n^{-1}, \quad M = g_L g_R^{-1}.$$

Geometric picture (cartoon for Lie $SU(1,1) \cong sl_2(\mathbb{R})$)





Top view (Poincaré disk)



$$\rho^2 \left(1 \cdot d\varphi^2 + \sinh^2\varphi \, d\alpha^2\right)$$

$x_1 = \rho \sinh\varphi \cos\alpha$ etc.

$$0 \cdot d\varphi^2 + e^{2\varphi} d\alpha^2$$

$x_1 = e^{\varphi} \cos\alpha$ etc.

$$\text{metric} = dx_3^2 + dx_1^2 - dx_2^2$$

Conformal field theory
of the integer quantum Hall transition

— Scenario for the IQH transition.

Spontaneous symmetry breaking occurs (the field gets stuck on a "light ray").

CFT of RG-fixed point is $GL(r|r)_{n,\gamma}$ Wess–Zumino–Witten model with level $n = 4$ and marginal deformation $\gamma = 1$:

$$S^{\mathrm{WZW}}_{n,\gamma}[M] = \frac{in}{4\pi}\int_\Sigma \Big(\mathrm{STr}\, M^{-1}\partial M \wedge M^{-1}\bar\partial M + \tfrac{1}{3} d^{-1}\, \mathrm{STr}\, (M^{-1} dM)^{\wedge 3} \Big) - \frac{i\gamma}{4\pi}\int_\Sigma \mathrm{STr}\,(M^{-1}\partial M) \wedge \mathrm{STr}\,(M^{-1}\bar\partial M).$$

Target = Riemannian symmetric superspace of type $A|A$;

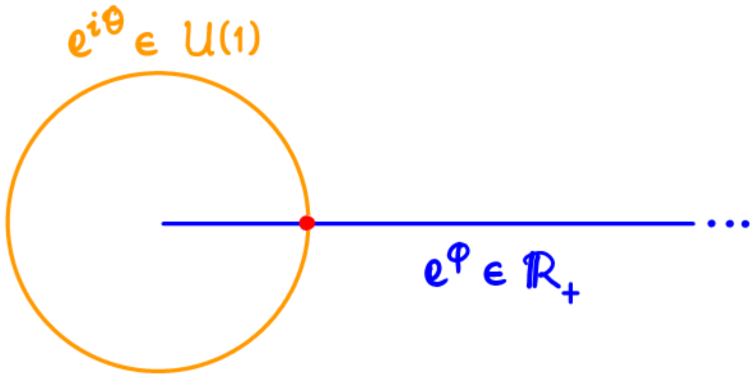
base manifold = $\mathrm{Herm}^+(r) \times U(r)$.

CFT is non-unitary ($c = 0$); WZW-field M has scaling dimension zero.

Operator product expansions of $T(z)$ with $J(z)$ with $M(z,\bar z)$ are known.

[No first-principles derivation published yet; but predictions match observations.]

— Explicit expressions for $r = 1$.



Gauss decomposition:

$$M = \begin{pmatrix} 1 & 0 \\ \eta & 1 \end{pmatrix} \begin{pmatrix} e^{\varphi} & 0 \\ 0 & e^{i\theta} \end{pmatrix} \begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix}$$

Invariant metric $\mathrm{STr}\,(M^{-1}dM)^2 = d\varphi^2 + d\theta^2 + 2\, e^{\varphi - i\theta}\, d\xi\, d\eta$ ↝

Berezin integration measure $DM = \frac{1}{2\pi}\, e^{-\varphi + i\theta}\, d\varphi\, d\theta \otimes \frac{\partial^2}{\partial\xi\,\partial\eta}$.

<u>Note</u>: $\int DM\; e^{-t\,\mathrm{STr}\,(M + M^{-1})} = 1$ (volume $= 1$).

[Compare with Lie supergroup targets]

$$S^{\mathrm{WZW}}_{n,\gamma} = \frac{in}{4\pi} \int \left(\partial\varphi \wedge \bar\partial\varphi + \partial\theta \wedge \bar\partial\theta + 2\, e^{\varphi - i\theta}\, \partial\xi \wedge \bar\partial\eta \right)$$
$$- \frac{i\gamma}{4\pi} \int \left(\partial\varphi - i\partial\theta\right) \wedge \left(\bar\partial\varphi - i\bar\partial\theta\right).$$

Integrate out fermions ↝ $S_{\mathrm{eff}} = \frac{in}{4\pi} \int \left(\partial\varphi \wedge \bar\partial\varphi + \partial\theta \wedge \bar\partial\theta \right) - \frac{Q}{8\pi} \int (\varphi - i\theta)\, \mathcal{R}$

"background charge" $Q = -1$.

Prospects for conformal bootstrap of Anderson transitions

Prospects ?

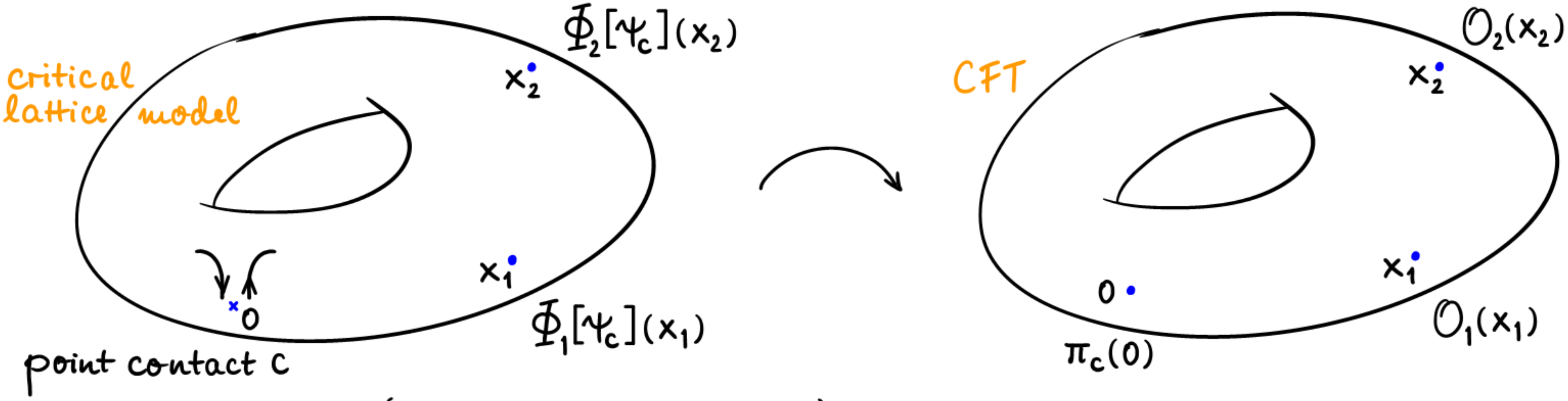
Challenge: $\langle 0 | \mathcal{O}_1(x_1)\, \mathcal{O}_2(x_2) \cdots \mathcal{O}_r(x_r) | 0 \rangle$ makes no sense here!

For a workable system of (CFT) correlation functions, need adapted framework (replacing the L^2-inner product of unitary / reflection-positive theories).

$\longrightarrow$ Draw inspiration from analogy with

- L. Schwartz: distributions $\otimes$ test functions $\longrightarrow \mathbb{R}$
- general setting of StatMech: states pair with observables

Proposal (Bondesan, Wieczorek & Z; PRL 2014):



$$\mathbb{E}\left(\Phi_1[\psi_c](x_1)\, \Phi_2[\psi_c](x_2) \cdots \right) = \langle \pi_c(0)\, \mathcal{O}_1(x_1)\, \mathcal{O}_2(x_2) \cdots \rangle$$

Example: model of IQHT $\longrightarrow$ $GL(r|r)_{n=4,\,\gamma=1}$ WZW model

Let $r=1$:

Φ	$\mathcal{O}$
$\lvert\psi(x)^2\rvert^q$	$e^{q\varphi}(x)$
$\lvert\mathrm{Det}_{p\times p}(\psi_k(x_\ell))\rvert^2$	$e^{-ip\theta}(x)$

Spectrum of primary fields: $\mathcal{O}_{p,q} = e^{q\varphi - ip\theta}$; $q \in \frac{1}{2} + i\mathbb{R}$, $p = 0, 1, 2, \ldots, n$

with scaling dimensions $\Delta_{p,q} = \frac{1}{n}\Big(p(p-1) - q(q-1)\Big)$ > 0 $(\gamma = 1)$

$+ \frac{1-\gamma}{n^2}(p^2 + q^2)$ $(\gamma \neq 1)$

Completeness: point contact operator $\pi_c =$ "$\delta_{Id}(M)$" has a Fourier expansion in terms of the fields $\mathcal{O}_{p,q}$.

Remark. Compare with 3-point fctn of quantum Liouville theory:

$$\langle 0 | V_{\lambda_1} V_{\lambda_2} V_{\lambda_3} | 0 \rangle \propto \underbrace{\int^{\infty} d\varphi \; e^{+\varphi + \sum_k (-\frac{1}{2} + i\lambda_k)\varphi}}_{\text{zero-mode integral}} \quad \text{exists,}$$

but in our case $\int^{\infty} d\varphi \; e^{-\varphi + \sum_k (+\frac{1}{2} + i\lambda_k)\varphi}$ does <u>not</u> exist!

Examples (continued; from BWZ, PRL 2014)

— 2-point function: $\mathbb{E}\left(|\psi_c(x)^2|^q \right) = \left\langle \pi_c(0) \; e^{q\varphi}(x) \right\rangle$

$$\sim |x|^{-(\Delta_q + \Delta_{q^*-q})} = |x|^{-2\Delta_q}$$

$$q^* = -Q = +1$$

— 3-point function: $\mathbb{E}\left(|\psi_c(x_1)^2|^{q_1} |\psi_c(x_2)^2|^{q_2} \right) = \left\langle \pi_c(0) \; e^{q_1\varphi}(x_1) \; e^{q_2\varphi}(x_2) \right\rangle$

$$\sim |x_1|^{-\Delta_{q_1} - \Delta_{q^*-q_1-q_2} + \Delta_{q_2}} \; |x_2|^{-\Delta_{q_2} - \Delta_{q^*-q_1-q_2} + \Delta_{q_1}}$$

$$|x_1 - x_2|^{-\Delta_{q_1} - \Delta_{q_2} + \Delta_{q^*-q_1-q_2}}$$

Summary / Outlook

- Expect different field theories for small vs large LDoS
  (note: Ioffe-Regel criterion)

- Upper critical dimension?
  (note: Vollhardt-Wölfle self-consistent theory of Anderson localization)

- Need adapted framework!
  (For system-size independent correlation functions,
  replace eps-regularization by point contacts.)

Thank you!

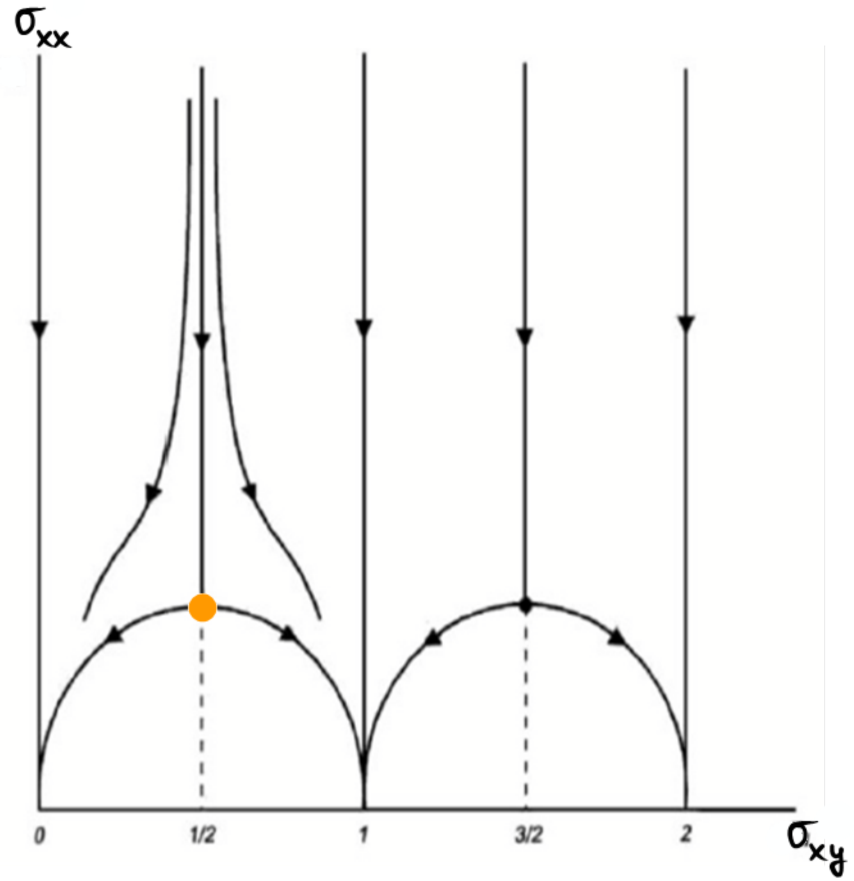
Nonlinear sigma model (Pruisken et al., 1983)

$$\mathcal{L} = \underbrace{\frac{\sigma_{xx}}{8} \mathrm{STr}\, \partial_\mu Q\, \partial_\mu Q}_{\text{weak localization}} + \underbrace{\frac{\sigma_{xy}}{8} \epsilon_{\mu\nu} \mathrm{STr}\, Q\, \partial_\mu Q\, \partial_\nu Q}_{\theta\text{-term}}$$

Wegner–Efetov SUSY method ⇝ target space is complex Grassmann manifold $\mathcal{U}/K$ with global symmetry group $\mathcal{U} = \mathrm{U}(r, r|2r)$:

$$Q = u \Sigma_3 u^{-1}, \qquad \Sigma_3 = \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Pruisken–Khmelnitskii RG flow diagram (conjectured, 1983)



Affleck's argument (PRL 1985).

Current one-form $= J_\mu * dx^\mu$.

Current-current correlation function in the plane:

$$\langle J_\mu(x) J_\nu(x') \rangle = C_{\mu\nu}(x-x').$$

At the critical point, expect rotational invariance:

$$C_{\mu\nu}(x) = x_\mu x_\nu f(|x|^2) + \delta_{\mu\nu} |x|^2 g(|x|^2).$$

Scale invariance $\curvearrowright$ $C_{\mu\nu}(x) = \dfrac{x_\mu x_\nu f_0 + \delta_{\mu\nu}|x|^2 g_0}{|x|^4}$.

Current conservation $(\partial^\mu C_{\mu\nu} = 0)$ $\curvearrowright$ $C_{\mu\nu}(x) = \dfrac{x_\mu x_\nu - \delta_{\mu\nu}|x|^2/2}{|x|^4} f_0$.

Make decomposition $J_\mu * dx^\mu = J + \bar{J} = j^{10} * dz + j^{01} * d\bar{z}$.

It follows that $\langle j^{10}(z,\bar{z})\, j^{10}(w,\bar{w}) \rangle = \dfrac{n}{(z-w)^2}$,

$$\langle j^{01}(z,\bar{z})\, j^{01}(w,\bar{w}) \rangle = \frac{n}{(\bar{z}-\bar{w})^2},$$

and $\langle j^{10}(z,\bar{z})\, j^{01}(w,\bar{w}) \rangle = 0$.

Fixed point: Wess-Zumino-Witten model

WZW field $M : \Sigma \to X$. Action functional:

$$S_n^{WZW}[M] = \frac{in}{4\pi} \int_\Sigma \left(\mathrm{STr}\, M^{-1}\partial M \wedge M^{-1}\bar\partial M + \tfrac{1}{3} d^{-1} \mathrm{STr}\, (M^{-1} dM)^{\wedge 3} \right).$$

Target = Riemannian symmetric superspace X of type $A|A$,

base manifold $\mathrm{Herm}^+(r) \times U(r)$.

Deformation:

$$S_{n,\gamma}^{WZW}[M] = S_n^{WZW}[M] - \frac{i\gamma}{2\pi} \int_\Sigma \mathrm{STr}(M^{-1}\partial M) \wedge \mathrm{STr}(M^{-1}\bar\partial M).$$

Left and right translations $M \mapsto u_L M u_R^{-1}$ generate current algebra $\widehat{gl}(r|r)_{n,\gamma}$ with holomorphic conserved current:

$$J^Y = n\, \mathrm{STr}(Y \partial M \cdot M^{-1}) - \gamma\, \mathrm{STr}(Y)\, \mathrm{STr}(\partial M \cdot M^{-1}), \quad \bar\partial J^Y = 0,$$

and anti-holomorphic conserved current $\bar J^Y$, $\partial \bar J^Y = 0$.

<u>Remark</u>: WZW field is dual to NLσM field.

Operator product expansions

OPE of current with fundamental field :

$$J^{\alpha}{}_{\beta}(z)\, M^{\gamma}{}_{\delta}(w,\bar{w}) = (-1)^{|\beta|+1}\frac{\delta^{\gamma}{}_{\beta}}{z-w}\, M^{\alpha}{}_{\delta}(w,\bar{w}) + \ldots \;\curvearrowright\; \text{current-OPE:}$$

$$J^{A}(z)\, J^{B}(w) = -n\,\frac{\mathrm{STr}(AB)}{(z-w)^2} + \gamma\,\frac{\mathrm{STr}A\cdot\mathrm{STr}B}{(z-w)^2} + \frac{J^{[A,B]}(w)}{z-w} + \ldots$$

The (Sugawara) energy-momentum tensor determined by

$$T(z)\, J(w) = \frac{J(w)}{(z-w)^2} + \frac{\partial J(w)}{z-w} + \ldots \quad \text{is Virasoro with } c=0:$$

$$T_{\widehat{gl}(r|r)_{n,\gamma}} = -\frac{(-1)^{|\alpha|}}{2n}\left(J^{\alpha}{}_{\beta}\, J^{\beta}{}_{\alpha}\right) + \frac{1-\gamma}{2n^2}\left(J^{\alpha}{}_{\alpha}\, J^{\beta}{}_{\beta}\right)(-1)^{|\alpha|+|\beta|}.$$

OPE of energy-momentum tensor with fundamental field :

$$T(z)\, M(w,\bar{w}) = h\,\frac{M(w,\bar{w})}{(z-w)^2} + \frac{\partial M(w,\bar{w})}{z-w} + \ldots, \quad h = \frac{1-\gamma}{2n^2}.$$

The observed phenomenology is obtained for $n=4$ and $\gamma=1$.

Derivation of CFT (novel method – sketch)

– Second quantization $\begin{cases} \mathrm{Det}^{-1}(1-\mathcal{U}) = \mathrm{Tr}\,\rho_B(\mathcal{U}) & \text{bosons,} \\ \mathrm{Det}(1-\mathcal{U}) = \mathrm{STr}\,\rho_F(\mathcal{U}) & \text{fermions.} \end{cases}$

Factorization: $\rho(\mathcal{U}_r \mathcal{U}_s) = \rho(\mathcal{U}_r)\,\rho(\mathcal{U}_s)$.

– Howe's representation by twisted convolution of Gauss kernels:

Cauchy transform $\frac{1+\mathcal{U}}{1-\mathcal{U}} \equiv a_\mathcal{U} \curvearrowright \rho(\mathcal{U}) = \int_\psi \exp\left(-\tfrac{1}{2}\tilde{\psi}\, a_\mathcal{U}\, \psi\right) T_\psi$.

– Generating function after disorder averaging:

$$Z = \int_\psi e^{-\frac{1}{2}\sum_l |\tilde{\psi}(l)\psi(l)|}\, e^{-\frac{1}{2}\tilde{\psi}\frac{1+\mathcal{U}_s}{1-\mathcal{U}_s}\psi}$$

$$\left(\tilde{\psi}\psi = |z_+|^2 - |z_-|^2 + \bar{\zeta}_+\zeta_+ - \bar{\zeta}_-\zeta_-\right).$$

– Nonabelian bosonization $\psi\tilde{\psi} \sim \begin{pmatrix} J & M \\ M^{-1} & \bar{J} \end{pmatrix}$

$(n = 4 \curvearrowright$ four Dirac nodes in $\mathcal{U}_s)$.