Remarks on Schrödinger-invariance

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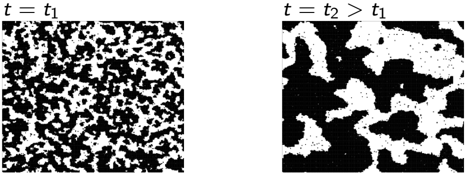
Outline

- Physical background: dynamical scaling & ageing
- Schrödinger algebra
- Two- and three-point functions and tests
- Free fields, the energy-momentum tensor and the current
- applicability to stochastic non-equilibrium field theory
- Why response functions ?

Examples are meant as illustrations, focus on dynamical symmetry concepts

Dynamical scaling out of equilibrium, after quench to $T < T_c$



Ising magnet $T < T_c$

$\longrightarrow$ ordered cluster

growth of ordered domains, of typical linear size

$$\boxed{L(t) \sim t^{1/z}}$$

dynamical exponent z: determined by equilibrium state

☞ for quenches to $T < T_c$ and without conservation laws: have $z = 2$

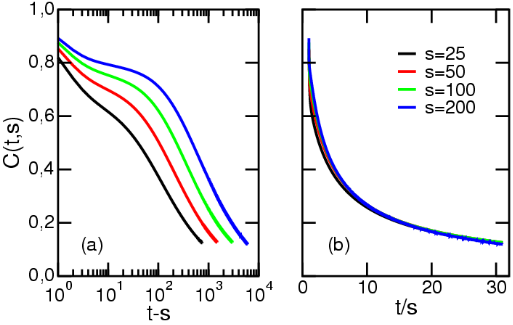
Bray, Rutenberg 1996

have dynamical scaling, although stationary states are *not* scale-invariant

Two-time observables from time-dependent order-parameter $\phi(t, \boldsymbol{r})$ show data collapse, with t: observation time, s: waiting time

two-time auto-**correlator** $$C(t,s) := \langle \phi(t,\boldsymbol{r})\phi(s,\boldsymbol{r})\rangle - \langle \phi(t,\boldsymbol{r})\rangle \langle \phi(s,\boldsymbol{r})\rangle = s^{-b} f_C\left(\frac{t}{s}\right)$$

two-time auto-**response** $$R(t,s) := \left.\frac{\delta \langle \phi(t,\boldsymbol{r})\rangle}{\delta h(s,\boldsymbol{r})}\right|_{h=0} = \left\langle \phi(t,\boldsymbol{r})\widetilde{\phi}(s,\boldsymbol{r})\right\rangle = s^{-1-a} f_R\left(\frac{t}{s}\right)$$



autocorrelator $3D$ Glauber-Ising, $T < T_c$

data collapse in scaling regime

for $t, s \gg \tau_{\rm micro}$ and $t - s \gg \tau_{\rm micro}$

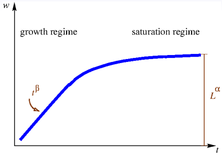
(1) *no* time-translation invariance (2) dynamical scaling (3) slow dynamics $\Rightarrow$ **ageing**

Question: derive scaling function in a model-independent way **?**

Another simple example: interface growth in EW universality class

$$\partial_t h(t,\mathbf{r}) = \nu \Delta_r h(t,\mathbf{r}) + \eta(t,\mathbf{r})$$

with η white noise, temperature T



width $w^2 = \left\langle \left(h - \overline{h}\right)^2 \right\rangle$

growth regime $w \sim t^\beta$

saturation regime $w \sim L^\alpha$

$z = \alpha/\beta$

☞ noisy diffusion ('Schrödinger') equation

linear $\Rightarrow$ exactly solvable, gives height response & correlator   long-time limit

$$R(t,s;\mathbf{r}) = r_0 (t-s)^{-d/2} \exp\left[-\frac{\mathcal{M}}{2} \frac{\mathbf{r}^2}{t-s} \right]$$

$$C(t,s;\mathbf{r}) = \frac{c_0 T}{|\mathbf{r}|^{d-2}} \left[\Gamma\left(\frac{d}{2}-1, \frac{\mathcal{M}}{2}\frac{\mathbf{r}^2}{t+s} \right) - \Gamma\left(\frac{d}{2}-1, \frac{\mathcal{M}}{2}\frac{\mathbf{r}^2}{t-s} \right) \right]$$

$$C(t;\mathbf{r}) = \frac{\bar{c}_0 T}{|\mathbf{r}|^{d}} \Gamma\left(\frac{d}{2}-1, \frac{\mathcal{M}}{4}\frac{\mathbf{r}^2}{t} \right)$$

where $\Gamma(a,x) = \int_x^\infty \mathrm{d}u\, u^{a-1} e^{-u}$ incomplete Gamma function

Röthlein, Baumann, Pleimling 06; Bustingorry, Cugliandolo, Iguain 07

☞ again data collapse, i.e. $C(t,s;\mathbf{r}) = s^{-b} F_C\left(\frac{t}{s}, \frac{\mathbf{r}^z}{(t-s)}\right)$ etc.

☞ recover the three defining properties of ageing

Question: ? can one reproduce these results from a dynamical symmetry ?

$\Longrightarrow$ interface coupled to heat bath with temperature T
$\Longrightarrow$ **difficulties with Galilei-invariance**, when $T \neq 0$

Proceed in two steps:

1. study **symmetries** of the **deterministic part**, with $T = 0$
2. use deterministic symmetries to analyse **full noisy equation**

In practice:

1. find dynamical symmetries of free diffusion equation — Lie 1881 (Jacobi 1842/43)
   $\Rightarrow$ analogies with conformal invariance — Niederer 72
2. derive **Bargman superselection** rules — Bargman 54
   $\Rightarrow$ reduction of *'noisy'* to *'deterministic'* averages

Examples of infinite-dimensional time-space transformations

group	coordinate changes		co-variance
(ortho-) conformal	$z' = f(z)$	$\bar{z}' = \bar{z}$	correlator
$(1+1)D$	$z' = z$	$\bar{z}' = \bar{f}(\bar{z})$	
Schrödinger-Virasoro	$t' = b(t)$	$\boldsymbol{r}' = \sqrt{\mathrm{d}b(t)/\mathrm{d}t}\ \boldsymbol{r}$	response
	$t' = t$	$\boldsymbol{r}' = \boldsymbol{r} + \boldsymbol{a}(t)$	
	$t' = t$	$\boldsymbol{r}' = \mathscr{R}(t)\boldsymbol{r}$	

* Schrödinger group $Sch(d)$ is maximal finite-dimensional sub-group
* dynamical symmetry of free diffusion equation or
  free Schrödinger equation under $Sch(d)$ — Jacobi 1842/43, Lie 1881

  rediscovered in physics since 1970s
* **not** the '*non-relativistic limit*' of conformal group
* time-space anisotropic dilatations $t \mapsto b^z t$, $\boldsymbol{r} \mapsto b\boldsymbol{r}$,
  with **dynamical exponent** $z = 2$
* Schrödinger-invariance predicts form of response functions (not correlators)
* applications to phase-ordering kinetics, after quench to $T < T_c$ — since 1990s

(A) Standard (projective) conformal invariance at equilibrium

CARTAN 1909

label coordinates as '*time*' t and '*space*' $\boldsymbol{r}$
in $(1+1)D$ use complex variables $w = t + \mathrm{i}r$ and $\bar{w} = t - \mathrm{i}r$

Extend global dynamical scaling to local, projective transformations

$$w \mapsto \frac{\alpha w + \beta}{\gamma w + \delta} \;\;,\;\; \bar{w} \mapsto \frac{\bar{\alpha}\bar{w} + \bar{\beta}}{\bar{\gamma}\bar{w} + \bar{\delta}} \;\;,\;\; \alpha\delta - \beta\gamma = 1 \;,\; \bar{\alpha}\bar{\delta} - \bar{\beta}\bar{\gamma} = 1$$

note: (i) translation-invariance in t, r **&** (ii) time-space rotation-invariance

Transformation of scaling operators $w \mapsto w'$ with $\dot{\beta}(w') \geq 0$ and

$$w = \beta(w') \;\;,\;\; \phi(w,\bar{w}) = \left(\frac{\mathrm{d}\beta(w')}{\mathrm{d}w'}\right)^{-\Delta} \left(\frac{\mathrm{d}\bar{\beta}(\bar{w}')}{\mathrm{d}\bar{w}'}\right)^{-\bar{\Delta}} \phi'(w',\bar{w}')$$

with $x = \Delta + \bar{\Delta}$ scaling dimension, $s = \Delta - \bar{\Delta}$ spin ('usually' $s = 0$)

at equilibrium, scalar ϕ has **a single** scaling dimension x.

infinitesimal generators $\ell_n = -w^{n+1}\partial_w - \Delta(n+1)w^n$

generators $X_n = \ell_n + \bar{\ell}_n$ and $Y_n = \ell_n - \bar{\ell}_n$ span conformal Lie algebra $\mathfrak{conf}(2)$

$$[X_n, X_m] = (n-m)X_{n+m}\ ,\ [X_n, Y_m] = (n-m)Y_{n+m}\ ,\ [Y_n, Y_m] = (n-m)X_{n+m} \qquad \text{(C)}$$

Invariant Schrödinger operator (Laplacian) $\mathcal{S} = 4\partial_w\partial_{\bar{w}}$

$$\begin{aligned} [\mathcal{S}, X_{-1}] = [\mathcal{S}, Y_{-1}] &= [\mathcal{S}, Y_0] = 0 \\ [\mathcal{S}, X_0] = -\mathcal{S}\ ,\ [\mathcal{S}, X_1] &= -2(w+\bar{w})\mathcal{S} - 8(\Delta\partial_{\bar{w}} + \bar{\Delta}\partial_w) \\ [\mathcal{S}, Y_1] &= -2(w-\bar{w})\mathcal{S} - 8(\Delta\partial_{\bar{w}} - \bar{\Delta}\partial_w) \end{aligned}$$

Lemma: If $\mathcal{S}\phi = 0$ and $\Delta = \bar{\Delta} = 0$, then $\mathcal{S}(\mathcal{X}\phi) = 0$. Cartan 1909

$\mathfrak{conf}(d)$ maps solutions of $\mathcal{S}\phi = 0$ onto solutions.

Co-variant two-point function (correlator) Polyakov 70

$$\langle \phi_1(t,r)\phi_2(0,0)\rangle = \delta_{x_1,x_2}\left(t^2+r^2\right)^{-x_1} = t^{-2x_1}f(r/t)\ ,\ f(u) \sim (1+u^2)^{-x_1} \qquad \text{(P)}$$

(B) Dynamical scaling: Schrödinger-invariance

Time-dependent behaviour characterised by **dynamical exponent** z:

$$t \mapsto t b^{-z},\ \boldsymbol{r} \mapsto \boldsymbol{r} b^{-1}$$

If $z = 2$: local scaling given by **Schrödinger group**: JACOBI 1842/43, LIE 1881

APPELL 1892, GOFF 27, KASTRUP 68, HAGEN 71, NIEDERER 72, JACKIW 72

$$t \mapsto \frac{\alpha t + \beta}{\gamma t + \delta}\ ,\ \ \boldsymbol{r} \mapsto \frac{\mathcal{D}\boldsymbol{r} + \boldsymbol{v} t + \boldsymbol{a}}{\gamma t + \delta}\ \ ;\ \ \alpha\delta - \beta\gamma = 1$$

note: (i) translation-invariance in t, r & (ii) time-space Galilei-invariance

Transformation of scaling operators $t = \beta(t')$, $\boldsymbol{r} = \boldsymbol{r}'\sqrt{\frac{\mathrm{d}\beta(t')}{\mathrm{d}t'}}$ with $\dot{\beta}(t') \geq 0$

$$\phi(t, \boldsymbol{r}) = \dot{\beta}(t')^{-x/2} \underbrace{\exp\left[-\frac{\mathcal{M}\boldsymbol{r}'^2}{4}\frac{\mathrm{d}\ln\dot{\beta}(t')}{\mathrm{d}t'}\right]}_{\text{mass term}} \phi'(t', \boldsymbol{r}')$$

Schrödinger-covariant scalar ϕ has scaling dimension x, and mass $\mathcal{M}$.

infinitesimal generators

$$X_n = -t^{n+1}\partial_t - \frac{n+1}{2} t^n r \partial_r - \frac{n(n+1)}{4} \mathcal{M} t^{n-1} r^2 - \frac{1}{2}(n+1) x t^n$$

$$Y_m = -t^{m+1/2}\partial_r - \left(m + \frac{1}{2}\right) \mathcal{M} t^{m-1/2} r$$

$$M_n = -t^n \mathcal{M}$$

also contains 'phase changes' in the wave function ! (**projective**)

non-vanishing commutators (including central extensions)

$$[X_n, X_{n'}] = (n-n')X_{n+n'} + \frac{c}{12}(n^3 - n)\delta_{n+n',0}$$

$$[X_n, Y_m] = \left(\frac{n}{2} - m\right) Y_{n+m}$$

$$[X_n, M_{n'}] = -n' M_{n+n'}$$

$$[Y_m, Y_{m'}] = (m-m')M_{m+m'}$$

with $n, n' \in \mathbb{Z}$, $m, m' \in \mathbb{Z} + \frac{1}{2}$ $\Rightarrow$ Schrödinger-Virasoro algebra $\mathfrak{sv}$

$\mathfrak{sv}$ contains 3 chiral fields, with $\dim X = 2$, $\dim Y = \frac{3}{2}$, $\dim M = 1$

$\Rightarrow$ Schrödinger algebra $\mathfrak{sch}(1) = \left\langle X_{\pm 1,0}, Y_{\pm 1/2}, M_0 \right\rangle \subset \mathfrak{sv}(1)$

Explanation of these generators:

here $d = 1$

$$
\begin{aligned}
X_{-1} &= -\partial_t && \text{time translation} \\
X_0 &= -t\partial_t - \frac{1}{2} r\partial_r && \text{dilatation} \\
X_1 &= -t^2\partial_t - tr\partial_r && \text{`special Schrödinger'} \\
Y_{-1/2} &= -\partial_r && \text{space translation} \\
Y_{1/2} &= -t\partial_r && \text{Galilei transformation}
\end{aligned}
$$

$\mathfrak{sch}(d)$ **not** semi-simple: can have **projective** representations
extra phase factors, give additional terms in the generators

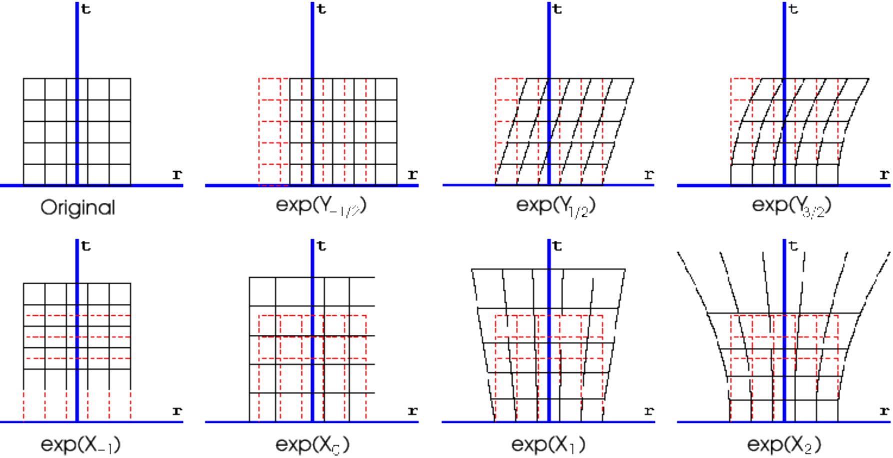
$$
\begin{aligned}
Y_{1/2} &= -t\partial_r - \mathcal{M} r \\
X_1 &= -t^2\partial_t - tr\partial_r - \frac{1}{2}\mathcal{M} r^2 \\
M_0 &= -\mathcal{M} && \text{phase shift}
\end{aligned}
$$

and also a **further generator** M_0 (central extension):

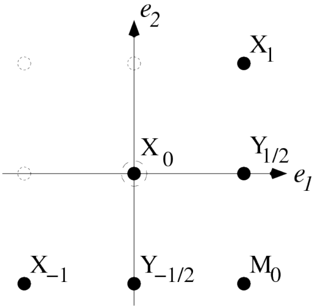
$$[Y_{1/2}, Y_{-1/2}] = M_0$$

Finally, can have a scaling dimension x: extra terms in $X_{0,1}$.

Geometric illustration of a few Schrödinger transformations:



visualisation of commutators in a root diagramme (complexified)



$\mathfrak{sch}(1) = \langle X_{\pm 1,0}, Y_{\pm 1/2}, M_0 \rangle \subset B_2$

associate root vector $\boldsymbol{x} \longleftrightarrow X$ generator

vector addition $\boldsymbol{x} + \boldsymbol{x}' \longleftrightarrow [X, X']$ commutator

if $\boldsymbol{x} + \boldsymbol{x}' \not\in \text{diagramme}$, **then** $[X, X'] = 0$
if $\boldsymbol{x} + \boldsymbol{x}' = \boldsymbol{x}'' \in \text{diagramme}$, **then** $[X, X'] \sim X''$
(modulo generators from Cartan subalgebra $\mathfrak{h}$)

subalgebras $\longleftrightarrow$ convex set under vector addition
subalgebra **isomorphisms** $\longleftrightarrow$ discrete (Weyl) symmetries of diagramme

Dynamical symmetry I: Schrödinger algebra $\mathfrak{sch}(d)$

dynamical symmetries of Langevin equation (deterministic part **!**)

Schrödinger operator in d space dimensions: $\mathcal{S} = 2\mathcal{M}\partial_t - \partial_{\boldsymbol{r}} \cdot \partial_{\boldsymbol{r}}$

(free) Schrödinger/heat equation
(noiseless) Edwards-Wilkinson equation
$\Bigg\}$: $\boxed{\mathcal{S}\phi = 0}$

$$
\begin{aligned}
[\mathcal{S}, \boldsymbol{Y}_{\pm 1/2}] &= [\mathcal{S}, M_0] = [\mathcal{S}, X_{-1}] = [\mathcal{S}, \mathcal{R}] = 0 \\
[\mathcal{S}, X_0] &= -\mathcal{S} \\
[\mathcal{S}, X_1] &= -2t\mathcal{S} + 2\mathcal{M}\left(x - \frac{d}{2}\right)
\end{aligned}
$$

infinitesimal change: $\delta\phi = \varepsilon\mathcal{X}\phi$, $\quad \mathcal{X} \in \mathfrak{sch}(d), |\varepsilon| \ll 1$

Lemma: If $\mathcal{S}\phi = 0$ and $x = x_\phi = \frac{d}{2}$, then $\mathcal{S}(\mathcal{X}\phi) = 0$. Lie 1881, Niederer '72

$\boxed{\mathfrak{sch}(d) \text{ maps solutions of } \mathcal{S}\phi = 0 \text{ onto solutions}}$.

Schrödinger-covariant two-point function: derivation

two-point function $\boxed{R = R(t,s;\boldsymbol{r}_1,\boldsymbol{r}_2) := \langle \phi_1(t,\boldsymbol{r}_1)\widetilde{\phi}_2(s,\boldsymbol{r}_2)\rangle}$

physical assumption: co-variance under Schrödinger transformations (quasi-primary)
$\Rightarrow$ set of **linear** 1st-order differential eqs.: $\boxed{\mathcal{X}\mathrm{R} = 0}$; $\mathcal{X} \in \mathfrak{sch}(d)$
Each ϕ_i characterized by (i) scaling dimension x_i, (ii) mass $\mathcal{M}_i$

a) time & space translations: $R = R(\tau;\boldsymbol{r})$, $\tau = t - s$, $\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2$
b) Galilei ($1D$):

$$\begin{aligned} Y_{1/2}R &= \left[-t_1\frac{\partial}{\partial r_1} - \mathcal{M}_1 r_1 - t_2\frac{\partial}{\partial r_2} - \widetilde{\mathcal{M}}_2 r_2\right] R \\ &= \left[(-\tau\partial_r - \mathcal{M}_1 r) - r_2\left(\mathcal{M}_1 + \widetilde{\mathcal{M}}_2\right)\right] R \stackrel{!}{=} 0 \end{aligned}$$

spatial translation-invariance $\Rightarrow$ any explicit reference to r_2 must disappear !

$$\left(-\tau\partial_r - \mathcal{M}_1 r\right) R(t,\boldsymbol{r}) = 0 \qquad (1)$$

$$\left(\mathcal{M}_1 + \widetilde{\mathcal{M}}_2\right) R(t,\boldsymbol{r}) = 0 \qquad (2)$$

$$R(\tau, \boldsymbol{r}) = f(\tau) \underbrace{\exp\left[-\frac{\mathcal{M}_1}{2}\frac{\boldsymbol{r}^2}{\tau}\right]}_{\text{heat kernel}} \cdot \underbrace{\delta(\mathcal{M}_1 + \widetilde{\mathcal{M}}_2)}_{\textbf{Bargman rule}}$$

Bargman 54

N.B.: Galilei-invariance requires 'complex' fields, here the 'response field' $\widetilde{\phi}$ with $\mathcal{M}_{\widetilde{\phi}} < 0$ plays the rôle of the 'complex conjugate' of the order parameter ϕ with $\mathcal{M}_\phi > 0$

c) scaling: (use $\partial_i := \partial/\partial t_i$ and $D_i := \partial/\partial r_i$)

$$\begin{aligned} X_0 R &= \left[-t_1\partial_1 - \frac{1}{2}r_1 D_1 - t_2\partial_2 - \frac{1}{2}r_2 D_2 - \frac{1}{2}(x_1 + x_2)\right] R \\ &= \left[-\tau\partial_\tau - \frac{1}{2}r\partial_r - \frac{1}{2}(x_1 + x_2)\right] R \stackrel{!}{=} 0 \end{aligned}$$

hence $\boxed{f(\tau) = f_0 \tau^{-(x_1+x_2)/2}}$, $f_0 = \text{cste.}$

d) 'special':

$$X_1 R = \left[-t_1^2\partial_1 - t_2^2\partial_2 - t_1 r_1 D_1 - t_2 r_2 D_2 - \frac{\mathcal{M}_1}{2} r_1^2 - \frac{\widetilde{\mathcal{M}}_2}{2} r_2^2 - x_1 t_1 - x_2 t_2\right] R$$

$$= \left[\left(-\tau^2\partial_\tau - \tau r\partial_r - \frac{\mathcal{M}_1}{2} r^2 - x_1\tau\right) - \frac{1}{2} r_2^2 \underbrace{\left(\mathcal{M}_1 + \widetilde{\mathcal{M}}_2\right)}_{=0}\right.$$

$$\left. + 2t_2 \underbrace{\left(-\tau\partial_\tau - \frac{1}{2} r\partial_r - \frac{1}{2}(x_1 + x_2)\right)}_{=0} + r_2 \underbrace{\left(-\tau\partial_r - \mathcal{M}_1 r\right)}_{=0}\right] R$$

$$= \left[-\tau^2\partial_\tau - \tau r\partial_r - \frac{\mathcal{M}_1}{2} r^2 - x_1\tau\right] R(\tau, r) \stackrel{!}{=} 0$$

use the decompositions $t_1^2 - t_2^2 = (t_1 - t_2)^2 + 2t_2(t_1 - t_2)$
$t_1 r_1 - t_2 r_2 = (t_1 - t_2)(r_1 - r_2) + t_2(r_1 - r_2) + r_2(t_1 - t_2)$

combine with previous conditions: $\boxed{\tau r(x_1 - x_2) R(\tau, r) = 0}$

$\boxed{f_0 = \delta_{x_1,x_2} r_0}$, with $r_0 = \text{cste}$.

Schrödinger-covariant three-point functions

two possible forms: MH 94

$$\left\langle \phi_1(t_1,\boldsymbol{r}_1)\phi_2(t_2,\boldsymbol{r}_2)\widetilde{\phi}_3(t_3,\boldsymbol{r}_3)\right\rangle = \delta_{\mathcal{M}_1+\mathcal{M}_2+\widetilde{\mathcal{M}}_3,0}\,\exp\left[-\frac{\mathcal{M}_1}{2}\frac{\boldsymbol{r}_{13}^2}{t_{13}}-\frac{\mathcal{M}_2}{2}\frac{\boldsymbol{r}_{23}^2}{t_{23}}\right]$$
$$\times t_{13}^{-x_{13,2}/2}t_{23}^{-x_{23,1}/2}t_{12}^{-x_{12,3}/2}\,\Psi_{12,3}\left(\frac{(\boldsymbol{r}_{13}t_{23}-\boldsymbol{r}_{23}t_{13})^2}{t_{12}t_{13}t_{23}}\right)$$

$$\left\langle \phi_1(t_1,\boldsymbol{r}_1)\widetilde{\phi}_2(t_2,\boldsymbol{r}_2)\widetilde{\phi}_3(t_3,\boldsymbol{r}_3)\right\rangle = \delta_{\mathcal{M}_1+\widetilde{\mathcal{M}}_2+\widetilde{\mathcal{M}}_3,0}\,\exp\left[-\frac{\mathcal{M}_1}{2}\frac{\boldsymbol{r}_{12}^2}{t_{12}}-\frac{\mathcal{M}_1}{2}\frac{\boldsymbol{r}_{13}^2}{t_{13}}\right]$$
$$\times t_{13}^{-x_{13,2}/2}t_{23}^{-x_{23,1}/2}t_{12}^{-x_{12,3}/2}\,\Psi_{1,23}\left(\frac{(\boldsymbol{r}_{13}t_{12}-\boldsymbol{r}_{12}t_{13})^2}{t_{12}t_{13}t_{23}}\right)$$

with $t_{ab} := t_a - t_b$, $\boldsymbol{r}_{ab} := \boldsymbol{r}_a - \boldsymbol{r}_b$ and $x_{ab,c} := x_a + x_b - x_c$, $x_a = \widetilde{x}_a$

$\Psi_{12,3}$ and $\Psi_{1,23}$ are arbitrary differentiable functions

Tests of Schrödinger-covariant response

response is independent of gaussian noise

$\Rightarrow$ can use Schrödinger co-variance (deterministic !)

$$R(t,s;\boldsymbol{r}) = r_0\,\delta_{x,\widetilde{x}}\,\delta(\mathcal{M}+\widetilde{\mathcal{M}})\,(t-s)^{-x}\exp\left[-\frac{\mathcal{M}}{2}\frac{\boldsymbol{r}^2}{t-s}\right]$$

1. Edwards-Wilkinson model: has $z=2$. Exact solution has given:

$$R(t,s;\boldsymbol{r}) = r_0\,(t-s)^{-d/2}\exp\left[-\frac{\mathcal{M}}{2}\frac{\boldsymbol{r}^2}{t-s}\right]$$

$\Rightarrow$ **perfect agreement**, if one identifies $x=\widetilde{x}=d/2$.

2. phase-ordering kinetics, in simple magnets

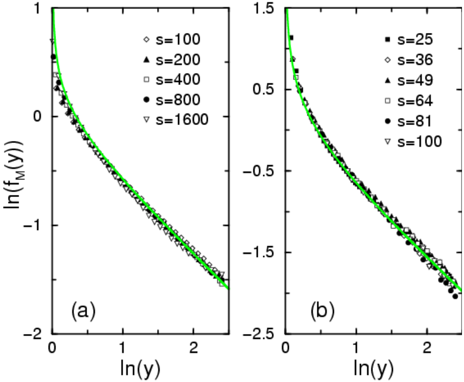
after quench to $T<T_c$ from disordered initial state

$\Rightarrow$ analysis of energy dissipation implies $z=2$ (model A) Bray, Rutenberg 94

$\Rightarrow$ can test Schrödinger-invariance in Glauber-Ising simulations, with $T<T_c$

! representations of Schrödinger algebra can also be used for **non-free fields** !

Tests of R in $2D/3D$ Glauber-Ising models



$$\chi_{\mathrm{TRM}}(t,s) = \int_0^s \mathrm{d}u\, R(t,u) = s^{-a} f_M(t/s)$$

integrated response

(thermoremanent susceptibility)

mh & Pleimling 03

$\chi_{\mathrm{TRM}}(t,s)$ for the Glauber-Ising model compared to LSI

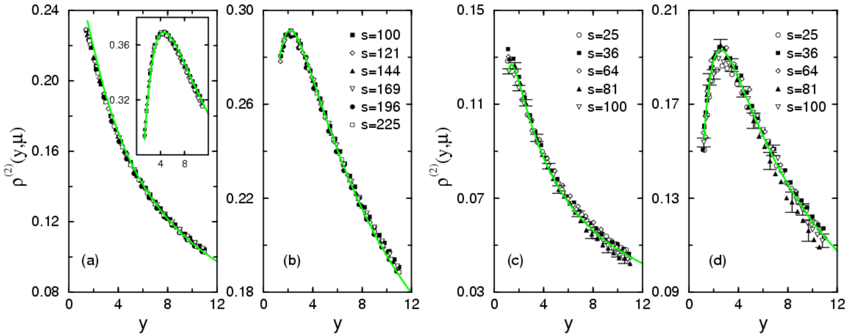
(a) $2D$, $T = 1.5$, (b) $3D$, $T = 3$ $T < T_c$, hence $z = 2$

compare data from **master equation** with local scale-symmetry

also **works** for (i) q-states $2D$ Potts model — Lorenz & Janke 07

(ii) $2D/3D$ XY model — Abriet & Karevski 04

Test time-space behaviour **(parameter-free !)**:



spatio-temporally integrated response Ising model $T < T_c$

(a,b) $2D$; $\mu = 1, 2, 4$ (c,d) $3D$; $\mu = 1, 2,$

$\int_0^s \mathrm{d}u \int_0^{\sqrt{\mu s}} \mathrm{d}r\, r^{d-1} R(t, u; \boldsymbol{r}) = s^{d/2-a} \boldsymbol{\rho}^{(2)}(t/s, \mu)$

MH & PLEIMLING, PHYS. REV. **E68**, 065101(R) (2003)

analogous results in the q-states $2D$ Potts model

LORENZ & JANKE, EUROPHYS. LETT. **77**, 10003 (2007)

(C) Schrödinger-invariant free fields

with a complex field $\phi(t,\boldsymbol{r}) \in \mathbb{C}$, have action   e.g. Janssen-de Dominicis type

$$S = \int \mathrm{d}t\,\mathrm{d}\boldsymbol{r}\,\mathscr{L} = \int \mathrm{d}t\,\mathrm{d}\boldsymbol{r}\,\left[\mathcal{M}\left(\phi^\dagger\frac{\partial\phi}{\partial t} - \phi\frac{\partial\phi^\dagger}{\partial t}\right) + \frac{\partial\phi^\dagger}{\partial\boldsymbol{r}}\cdot\frac{\partial\phi}{\partial\boldsymbol{r}}\right]$$

under transformations generated by the generators X_n, Y_m have changes

$$\delta_X S = \int \mathrm{d}t'\,\mathrm{d}\boldsymbol{r}'\,\frac{\mathcal{M}}{2} r'^2 \{\beta(t'),t'\}\phi'^\dagger\phi' \;\;,\;\; \delta_Y S = \int \mathrm{d}t'\,\mathrm{d}\boldsymbol{r}'\mathcal{M}^2(\alpha(t') - 2r')\ddot{\alpha}(t')\phi'^\dagger\phi'$$

with the Schwarzian derivative $\{\beta(t),t\} = \frac{\dddot{\beta}(t)}{\dot{\beta}(t)} - \frac{3}{2}\left(\frac{\ddot{\beta}(t)}{\dot{\beta}(t)}\right)^2$.

☞ action invariant under finite-dimensional sub-group only

Energy-momentum tensor: notation $\boldsymbol{\rho} = (t,\boldsymbol{r})$, coordinate change $\delta\boldsymbol{\rho} = \boldsymbol{\varepsilon}(\boldsymbol{\rho})$

Action transforms as

$$\delta S = \iint \mathrm{d}t\,\mathrm{d}\boldsymbol{r}\,\left(T_{\mu\nu}\partial_\mu\varepsilon_\nu + J_\mu\partial_\mu\eta\right)$$

where η is the change in the 'phase' of ϕ (to be read from the X_n, Y_m)

Implies the following consequences ⚠ schematically !

dilatation-invariance (X_0):	$2T_{00} + T_{11} + \ldots T_{dd} = 0$
Galilei-invariance ($Y_{1/2}$):	$T_{0a} + \mathcal{M} J_a = 0 \;\; ; \;\; a = 1, \ldots, d$
spatial rotation-invariance:	$T_{ab} = T_{ba} \;\; ; \;\; a, b = 1, \ldots, d$

then, under a special Schrödinger-transformation, $(1+1)D$

$$\delta_{X_1} S = \int \mathrm{d}t\, \mathrm{d}r \left[\left(\underbrace{2T_{00} + T_{11} + \ldots T_{dd}}_{=0} \right) t + \left(\underbrace{T_{0a} + \mathcal{M} J_a}_{=0} \right) r_a \right] = 0$$

invariance under { temporal & spatial translations; Galilei transformations; dilatations with $z = 2$; spatial rotations } $\Longrightarrow$ special Schrödinger-invariance

N.B.: can be extended to sub-algebras such as $\mathfrak{age}(d)$

The ‘canonical recipe’ gives the energy-momentum tensor and the current

$$T_{\mu\nu} = -\delta_{\mu\nu}\mathscr{L} + \frac{\partial \mathscr{L}}{\partial\left(\partial^{\mu}\phi\right)}\partial_{\nu}\phi + \frac{\partial \mathscr{L}}{\partial\left(\partial^{\mu}\phi^{\dagger}\right)}\partial_{\nu}\phi^{\dagger} \quad , \quad J_{\mu} = \frac{\partial \mathscr{L}}{\partial\left(\partial^{\mu}\phi\right)}\phi - \frac{\partial \mathscr{L}}{\partial\left(\partial^{\mu}\phi^{\dagger}\right)}\phi^{\dagger}$$

are conserved and obey all Ward identities with exception of trace condition. Construct improved tensor

$$\Theta_{\mu\nu} = T_{\mu\nu} + \partial^{\lambda} B_{\lambda\mu\nu}$$

which satisfies all Ward identities and is classically conserved. The current need not be improved.

N.B.: for $d = 2$ and $t \mapsto z$, $-\frac{1}{2\mathcal{M}}\Theta_{00}$ is identical to the tensor $T(z)$ of a complex fermionic free field.

(D) Stochastic field-theory out of equilibrium

theoretical approach: **Langevin equation** (model A of Hohenberg & Halperin 77)

$$2\mathcal{M}\frac{\partial\phi}{\partial t} = \Delta_{\boldsymbol{r}}\phi - \frac{\delta\mathcal{V}[\phi]}{\delta\phi} + \eta$$

order-parameter $\phi(t, \boldsymbol{r})$ non-conserved

$\mathcal{M}$: kinetic coéfficient

$\mathcal{V}$: Landau-Ginsbourg potential

η: gaussian noise, centred and with variance

$$\langle\eta(t,\boldsymbol{r})\eta(t',\boldsymbol{r}')\rangle = 2T\delta(t-t')\delta(\boldsymbol{r}-\boldsymbol{r}')$$

fully disordered initial conditions (centred gaussian noise)

Langevin equations do **not** have non-trivial dynamical symmetries !

Galilei-invariance is broken by interactions with the thermal bath

cf. dipole anisotropy of cosmic microwave background

? compare results of deterministic symmetries to stochastic models ?

take Langevin equation as classical equation of motion   JANSSEN 92, DE DOMINICIS,...

$$\langle A \rangle = \int \mathcal{D}\phi \mathcal{D}\eta \, P[\eta] \, \delta\left((2\mathcal{M}\partial_t - \Delta)\phi + \mathcal{V}'[\phi] - \eta\right) A[\phi]$$

introduce auxiliary field $\widetilde{\phi}$, integrate out **gaussian** noise η

$\Rightarrow$ arrive at **effective field-theory**, with **action** $\mathcal{J}$ and averages

$$\langle A \rangle := \int \mathcal{D}\phi \mathcal{D}\widetilde{\phi} \, A[\phi, \widetilde{\phi}] \exp(-\mathcal{J}[\phi, \widetilde{\phi}])$$

$$\mathcal{J}[\phi, \widetilde{\phi}] = \underbrace{\int \widetilde{\phi}(2\mathcal{M}\partial_t - \Delta)\phi + \widetilde{\phi}\mathcal{V}'[\phi]}_{\mathcal{J}_0[\phi,\widetilde{\phi}] \,:\, \text{deterministic}} \underbrace{- T \int \widetilde{\phi}^2 - \int \widetilde{\phi}_{t=0} C_{init} \widetilde{\phi}_{t=0}}_{+\mathcal{J}_b[\widetilde{\phi}] \,:\, \text{noise (bruit)}}$$

$\widetilde{\phi}$: response field;

$$\boxed{C(t,s) = \langle \phi(t)\phi(s) \rangle, \; R(t,s) = \langle \phi(t)\widetilde{\phi}(s) \rangle}$$

deterministic averages: $\langle A \rangle_0 := \int \mathcal{D}\phi \mathcal{D}\widetilde{\phi} \, A[\phi, \widetilde{\phi}] \exp(-\mathcal{J}_0[\phi, \widetilde{\phi}])$

masses:

$$\boxed{\mathcal{M}_\phi = -\mathcal{M}_{\widetilde{\phi}}}$$

Theorem: **IF** *$\mathcal{J}_0$ is Galilei- and spatially translation-invariant,* *then* *Bargman superselection rules hold* BARGMAN 54

$$\left\langle \phi_1 \cdots \phi_n \, \widetilde{\phi}_1 \cdots \widetilde{\phi}_m \right\rangle_0 \sim \delta_{n,m}$$

Illustration 1: computation of a response

$$\begin{aligned} R(t,s) &= \left\langle \phi(t)\widetilde{\phi}(s) \right\rangle = \left\langle \phi(t)\widetilde{\phi}(s) e^{-\mathcal{J}_b[\widetilde{\phi}]} \right\rangle_0 \\ &= \left\langle \phi(t)\widetilde{\phi}(s) \right\rangle_0 = R_0(t,s) \end{aligned}$$

Bargman rule $\Longrightarrow$ response function does **not** depend on noise !
left side: computed in stochastic models
right side: local scale-symmetry of deterministic equation

☞ **Comparison** of results of assumed deterministic $\mathfrak{age}(d)$-symmetry with explicit stochastic models/experiments **justified**.

Illustration 2:
computation of a correlator, from Bargman rule

$$C(t,s;\boldsymbol{r}) = \langle\phi(t,\boldsymbol{r})\phi(s,0)\rangle = \left\langle \phi(t,\boldsymbol{r})\phi(s,0)e^{-\mathcal{J}_b[\widetilde{\phi}]}\right\rangle_0$$

$$= \frac{\Delta_0}{2}\int_{\mathbb{R}^d}\mathrm{d}\boldsymbol{R}\,\left\langle\phi(t,\boldsymbol{r}+\boldsymbol{r}_0)\phi(s,\boldsymbol{r}_0)\widetilde{\phi}^2(0,\boldsymbol{R})\right\rangle_0 \quad \text{initial} \Rightarrow \text{phase-ordering}$$

$$+T\int_0^\infty \mathrm{d}u\int_{\mathbb{R}^d}\mathrm{d}\boldsymbol{R}\,\left\langle\phi(t,\boldsymbol{r}+\boldsymbol{r}_0)\phi(s,\boldsymbol{r}_0)\widetilde{\phi}^2(u,\boldsymbol{R})\right\rangle_0 \quad \text{thermal} \Rightarrow \text{interfaces}$$

$\mathfrak{sch}(d)$-invariance only fixes three-point function $\langle\phi\phi\widetilde{\phi}^2\rangle_0$
up to an unknown scaling function Ψ
$\Longrightarrow$ how to obtain a prediction for $f_C(y)$?

Theorem: Schrödinger-invariance $z=2 \Longrightarrow$ $\boxed{\lambda_C = \lambda_R}$

agrees with a different argument of Bray 94 in phase-ordering and with all models consider two typical cases:

1. autocorrelator $C(t,s) = C(t,s;0)$

* for **phase-ordering**, have $T=0$: $t = ys$

$$C_{\rm po}(ys,s) = \frac{\Delta_0}{2} s^{d/2-\widetilde{x}_2-x} y^{d/2-\widetilde{x}} (y-1)^{\widetilde{x}_2-x-d/2} \Psi\left(\frac{y+1}{y-1}\right)$$

* for **interfaces**, have $\Delta_0 = 0$:

$$C_{\rm int}(ys,s) = T s^{d/2+1-x-\widetilde{x}_2} y^{\widetilde{x}_2-x-d/2} \int_0^1 {\rm d}\theta\ \left[(y-\theta)(1-\theta)\right]^{d/2-\widetilde{x}_2} \Psi\left(\frac{y+1-2\theta}{y-1}\right)$$

where $\Psi(w) = \int_{\mathbb{R}^d} {\rm d}\boldsymbol{R}\ \exp\left[-\frac{\mathcal{M}w}{2}\boldsymbol{R}^2\right] \psi(\boldsymbol{R}^2)$

! treat $\widetilde{\phi}^2$ as **composite scaling operator**, with scaling dimension $2\widetilde{x}_2$ **!**

for free fields: $\widetilde{x}_2 = \widetilde{x}$ and $\Psi(w) = \Psi_0\, w^{\omega}$ $\Rightarrow$ scaling fixes $\omega = d/2 - \lambda_C$

agrees with EW model, if $x = \widetilde{x} = d/2$ RÖTHLEIN *et al.* 06

2. equal-time correlator $C(t,\boldsymbol{r}) = C(t,t;\boldsymbol{r})$

three-point function has **singularity** when $t-s\to 0$

treat by ansatz $\Psi_{12,3}(A)=\Psi_0 A^{\omega}$ and fix ω to have regular limit $t-s=\varepsilon\to 0$

rederive Ward identities for 3-point function $\langle\phi(t,\boldsymbol{r})\phi(t,0)\widetilde{\phi}^2(u,\boldsymbol{R})\rangle_0 \Rightarrow$ same result

$$\begin{aligned}
C(t,\boldsymbol{r}) &= \frac{T\Psi_0}{(|\boldsymbol{r}|^2)^{x-\widetilde{x}}}\int_0^t \mathrm{d}u\, u^{-2\widetilde{x}}\int_{\mathbb{R}^d}\mathrm{d}\boldsymbol{R}\,\exp\left(-\frac{\mathcal{M}}{2u}\left[(\boldsymbol{r}-\boldsymbol{R})^2+\boldsymbol{R}^2\right]\right)\\
&= \frac{T\Psi_0}{(|\boldsymbol{r}|)^{2(x-\widetilde{x})}}\int_0^t \mathrm{d}u\, u^{-2\widetilde{x}}\int_{\mathbb{R}^d}\mathrm{d}\boldsymbol{R}\,\exp\left[-\frac{\mathcal{M}}{2u}\left[\left(\frac{\boldsymbol{r}}{2}-\boldsymbol{R}\right)^2+\left(\frac{\boldsymbol{r}}{2}+\boldsymbol{R}\right)^2\right]\right]\\
&= \frac{T\Psi_0}{(|\boldsymbol{r}|)^{2(x-\widetilde{x})}}\int_0^t \mathrm{d}u\, u^{-2\widetilde{x}}\int_{\mathbb{R}^d}\mathrm{d}\boldsymbol{R}\,\exp\left[-\frac{\mathcal{M}}{4u}\boldsymbol{r}^2\right]\exp\left[-\frac{\mathcal{M}}{u}\boldsymbol{R}^2\right]\\
&= \frac{T\Psi_0}{(|\boldsymbol{r}|)^{2(x-\widetilde{x})}}\left(\frac{\pi}{\mathcal{M}}\right)^{d/2}\int_0^t \mathrm{d}u\, u^{d/2-2\widetilde{x}}\exp\left[-\frac{\mathcal{M}}{4}\frac{\boldsymbol{r}^2}{u}\right]\\
&= T\bar{c}_0\,|\boldsymbol{r}|^{d-2x-2\widetilde{x}}\,\Gamma\left(2\widetilde{x}-\frac{d}{2}-1,\frac{\mathcal{M}}{4}\frac{\boldsymbol{r}^2}{t}\right)
\end{aligned}$$

for simplicity, we used $\widetilde{x}_2=\widetilde{x}$ $\Gamma(a,x)$: incomplete Gamma function

agrees with EW model, if one identifies $x=\widetilde{x}=d/2$.

also agrees with numerical simulations in 'Family model' of interfaces RÖTHLEIN *et al.* 06

(E) Non-equilibrium dynamical scaling: Ageing-invariance

Time-dependent scaling with **dynamical exponent** z: $t \mapsto t b^{-z}$, $\boldsymbol{r} \mapsto \boldsymbol{r} b^{-1}$

! No time-translation-invariance **out of equilibrium !**

For $z = 2$: local scaling given by **Ageing group**:

$$t \mapsto \frac{\alpha t}{\gamma t + \delta} \;\;,\;\; \boldsymbol{r} \mapsto \frac{\mathcal{D}\boldsymbol{r} + \boldsymbol{v} t + \boldsymbol{a}}{\gamma t + \delta} \;\;;\;\; \alpha\delta = 1$$

Transformation of scaling operators $t = \beta(t')$, $\boldsymbol{r} = \boldsymbol{r}' \sqrt{\frac{\mathrm{d}\beta(t')}{\mathrm{d}t'}}$ with $\beta(0) = 0$ and $\dot{\beta}(t') \geq 0$

$$\phi(t, \boldsymbol{r}) = \left(\frac{\mathrm{d}\beta(t')}{\mathrm{d}t'} \right)^{-x/2} \left(\frac{\mathrm{d}\ln \beta(t')}{\mathrm{d}t'} \right)^{-\xi} \exp\left[-\frac{\mathcal{M}\boldsymbol{r}'^2}{4} \frac{\mathrm{d}\ln \dot{\beta}(t')}{\mathrm{d}t'} \right] \phi'(t', \boldsymbol{r}')$$

out of equilibrium, have **2 distinct** scaling dimensions, x and ξ.

NB: if TTI (equilibrium criticality), then $\xi = 0$.

Dynamical symmetry II: ageing algebra $\mathfrak{age}(d)$

$1D$ Schrödinger operator: $\mathcal{S} = 2\mathcal{M}\partial_t - \partial_r^2 + 2\mathcal{M}\left(x + \xi - \frac{1}{2}\right) t^{-1}$

☞ generalised 'Schrödinger equation': $\boxed{\mathcal{S}\phi = 0}$

extra potential term arises in several models, **without** time-translations

(e.g. $1D$ Glauber-Ising, spherical & Arcetri models)

Lemma: If $\mathcal{S}\phi = 0$, then $\mathcal{S}(\mathcal{X}\phi) = 0$. Niederer 74

$\boxed{\mathfrak{age}(d) \text{ maps solutions of } \mathcal{S}\phi = 0 \text{ onto solutions}}$.

As before: $\mathfrak{age}(d)$-covariant two-point function

$$\boxed{R(t,s;\mathbf{r}) = r_0 s^{-1-a}\left(\frac{t}{s}\right)^{1+a'-\lambda_R/z}\left(\frac{t}{s}-1\right)^{-1-a'}\exp\left(-\frac{\mathcal{M}_1}{2}\frac{\mathbf{r}^2}{t-s}\right)}$$

with $1+a = \frac{x_1+x_2}{2}$, $a' - a = \xi_1 + \xi_2$, $\lambda_R = 2(x_1+\xi_1)$, $\mathcal{M}_1 + \mathcal{M}_2 = 0$

N.B.: for auto-response (i.e. $\mathbf{r} = \mathbf{0}$) also valid for $z \neq 2$; simply replace $\frac{\lambda_R}{2} \mapsto \frac{\lambda_R}{z}$

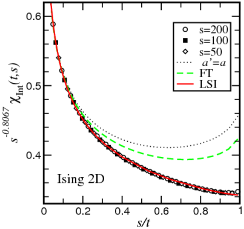
☞ also obtain prediction for autoresponse $R(t,s;\mathbf{0})$ *at* criticality $T = T_c$

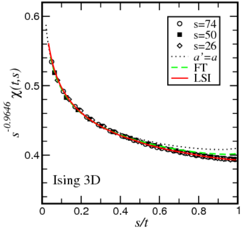
Examples of ageing-covariant two-point functions

(a) $1D$ **Glauber-Ising model**, $T=0$, ϕ: **magnetisation** reproduces the $\mathfrak{age}(1)$-covariant autoresponse with $a=0,\ a'=-\frac{1}{2},\ \lambda_R=1,\ z=2$

$\Rightarrow$ independent scaling dimensions: $\boxed{x=\frac{1}{2},\ \widetilde{x}=\frac{3}{2},\ \xi=0,\ \widetilde{\xi}=-\frac{1}{2}}$.

(b) $2D/3D$ kinetic Glauber-Ising model, at $T=T_c>0$





LSI with $a\neq a'$:
Ising data (momentum space !) at $T=T_c$
two-loop ε-expansion (FT)
$\rightarrow$ resummation needed ?

Have $a'-a=-1/2$ in $1D$ (exact); $a'-a=-0.187(20)$ in $2D$; $a'-a=-0.022(5)$ in $3D$

Pleimling & Gambassi, Phys. Rev. **B71**, 180401 ('05); mh, Enss, Pleimling, J. Phys. **A39**, L589 ('06)

(c) **kinetic spherical model** equation, at $T\leq T_c$ Godrèche & Luck '00

$$\partial_t\phi(t,\boldsymbol{r})=\Delta_{\boldsymbol{r}}\phi(t,\boldsymbol{r})-\mathfrak{z}(t)\phi(t,\boldsymbol{r})+\text{noise}\ ,\quad \mathfrak{z}(t)\sim t^{-1}$$

Observation: the **hidden assumption** $a=a'$ often **invalid** out of equilibrium.
Observables **cannot** always be identified with scaling operators.

• Why responses ? Dualised Schrödinger algebra $\mathfrak{sch}(d)$:

idée: treat the mass $\mathcal{M}$ as a variable, define 'dual' coordinate ζ GIULINI 96

$$\phi(t,\boldsymbol{r}) = \phi_{\mathcal{M}}(t,\boldsymbol{r}) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathrm{d}\zeta\, e^{-\mathrm{i}\mathcal{M}\zeta}\, \widehat{\phi}(\zeta,t,\boldsymbol{r})$$

trade projective representation for 'true' representation in dual space

$$\begin{aligned} X_n &= \mathrm{i}\frac{n(n+1)}{4}t^{n-1}\boldsymbol{r}^2\partial_\zeta - t^{n+1}\partial_t - \frac{n+1}{2}\, t^n\boldsymbol{r}\cdot\partial_{\boldsymbol{r}} - (n+1)\frac{x}{2}t^n \\ \boldsymbol{Y}_m &= \mathrm{i}\left(m+\frac{1}{2}\right)t^{m-1/2}\boldsymbol{r}\partial_\zeta - t^{m+1/2}\partial_{\boldsymbol{r}} \\ M_n &= \mathrm{i}t^n\partial_\zeta \end{aligned}$$

MH & UNTERBERGER 03

Generators live at the **boundary** of $(d+3)$-dim. Lorentzian space

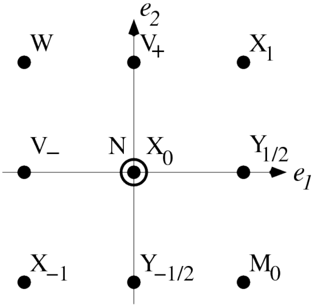
e.g. MINIC & PLEIMLING 08, FUERTES & MOROZ 09, LEIGH & HOANG 09,. . .

The Schrödinger/heat equation becomes $\mathcal{S}\widehat{\phi}=0$, explicitly

$$\mathcal{S}\widehat{\phi} = 2\mathrm{i}\frac{\partial^2\widehat{\phi}}{\partial\zeta\partial t} + \frac{\partial^2\widehat{\phi}}{\partial\boldsymbol{r}^2} = \left(2M_0X_{-1} + Y_{-1/2}^2\right)\widehat{\phi} = 0$$

visualisation of extension of $\mathfrak{sch}(1)$ from a root diagramme

$$\mathfrak{sch}(1) = \langle X_{\pm1,0}, Y_{\pm1/2}, M_0 \rangle \subset B_2 \cong \mathfrak{conf}(3)$$



new coordinates $\boldsymbol{\xi} = (\xi_{-1}, \xi_0, \xi_1)$

$$\zeta = \frac{1}{2}(\xi_0 + \mathrm{i}\xi_{-1}),\ t = \frac{1}{2}(-\xi_0 + \mathrm{i}\xi_{-1}),\ r = \sqrt{\frac{\mathrm{i}}{2}}\,\xi_1$$

Schrödinger/heat equation

$\partial_\mu \partial^\mu \Psi(\boldsymbol{\xi}) = 0$ with $\psi(\zeta, t, r) = \Psi(\boldsymbol{\xi})$

has conformal dynamical symmetry

$\Rightarrow$ include new generators $V_\pm, W, N$, extend $\mathfrak{sch}(d) \subset \mathfrak{conf}(d+2)_{\mathbb{C}}$

Burdet, Perrin, Sorba '73

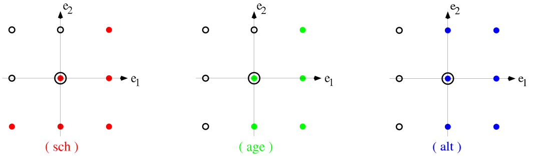
Lemma: If $\mathcal{S}\psi = 0$ and $x = x_\psi = \frac{1}{2}$, then $\mathcal{S}(\mathcal{X}\psi) = 0$.

$\mathfrak{conf}(d+2)_{\mathbb{C}}$ maps solutions of $\mathcal{S}\psi = 0$ onto solutions

Parabolic subalgebras of B_2

Parabolic subalgebra: Cartan subalgebra $\mathfrak{h} \oplus \{\text{positive roots}\}$.
positive roots: all roots to the right of a straight line through $\mathfrak{h}$

Classification of parabolic subalgebras of $B_2 \cong \mathfrak{conf}(3)_{\mathbb{C}}$:



extended Schrödinger
$\widetilde{\mathfrak{sch}}(1) := \mathfrak{sch}(1) + \mathbb{C}N$

extended ageing
$\widetilde{\mathfrak{age}}(1) := \mathfrak{age}(1) + \mathbb{C}N$
$=$ **minimal** standard parabolic subalgebra

extended conformal Galilean
$\widetilde{\mathrm{CGA}}(1) := \mathrm{CGA}(1) + \mathbb{C}N$

○ Find $\mathfrak{sch}(1)$-covariant dual two-point function $\widehat{F} = \langle \widehat{\phi}_1 \widehat{\phi}_2 \rangle$, $\quad x_1 = x_2$

$\zeta_- = \frac{1}{2}(\zeta_1 - \zeta_2)$, $t = t_1 - t_2$, $r = r_1 - r_2$

$$\widehat{F}(\zeta_-, t, r) = |t|^{-x_1} \widehat{f}\left(\frac{2\zeta_- t + \mathrm{i} r^2}{|t|} \right) \quad \overset{N}{\Longrightarrow} \quad \widehat{f}(u) = \widehat{f}_0 u^{-x_1 - \xi_1 - \xi_2}$$

Causality for $\widetilde{\mathfrak{sch}}(1)$: use $\zeta = \zeta_1 - \zeta_2$, invert dualisation

$$
\begin{aligned}
F &= \frac{1}{2\pi}\int_{\mathbb{R}^2} \mathrm{d}\zeta_1 \mathrm{d}\zeta_2 \, e^{-\mathrm{i}\mathcal{M}_1\zeta_1 - \mathrm{i}\mathcal{M}_2\zeta_2} \, |t|^{-x} \, \widehat{f}\left(\frac{2(\zeta_1-\zeta_2)t + \mathrm{i}r^2}{|t|} \right) \\
&= \frac{|t|^{-x}}{4\pi} \underbrace{\int_{\mathbb{R}} \mathrm{d}\eta \, e^{-\mathrm{i}(\mathcal{M}_1+\mathcal{M}_2)\eta/2}}_{4\pi\delta(\mathcal{M}_1+\mathcal{M}_2)} \int_{\mathbb{R}} \mathrm{d}\zeta \, e^{-\mathrm{i}(\mathcal{M}_1-\mathcal{M}_2)\zeta/2} \, \widehat{f}\left(2\mathrm{sign}\,(t) \left(\zeta + \frac{\mathrm{i}}{2} \frac{r^2}{\mathrm{sign}\,(t)\,|t|} \right) \right) \\
&= \delta(\mathcal{M}_1+\mathcal{M}_2)|t|^{-x}\widehat{f}_0 \int_{\mathbb{R}} \mathrm{d}\zeta \, e^{-\mathrm{i}\mathcal{M}_1\zeta} \left(2\mathrm{sign}\,(t)\right)^{-x-\xi} \left(\zeta + \frac{\mathrm{i}r^2}{2\mathrm{sign}\,(t)|t|} \right)^{-x-\xi} \\
&= \delta(\mathcal{M}_1+\mathcal{M}_2) \left(2\mathrm{sign}\,(t)\right)^{-x-\xi} \mathcal{M}_1^{x+\xi-1} |t|^{-x} \widehat{f}_0 \underbrace{\int_{\mathbb{R}+\frac{\mathrm{i}\mathcal{M}_1}{2}\frac{r^2}{t}} \mathrm{d}\zeta \, e^{-\mathrm{i}\zeta} \zeta^{-x-\xi}}_{I_{\pm}^{(0)}(x+\xi)} \; e^{-\frac{\mathcal{M}_1}{2}\frac{r^2}{t}} \\
&= \delta(\mathcal{M}_1+\mathcal{M}_2)|t|^{-x} \underbrace{2^{-x-\xi}\mathcal{M}_1^{x+\xi-1}\widehat{f}_0 I_{+}^{(0)}(x+\xi)}_{=:\,F_0} \, e^{-\frac{\mathcal{M}_1}{2}\frac{r^2}{t}} \Theta(t) \qquad \text{if } x+\xi>0
\end{aligned}
$$

physical convention $\mathcal{M}_1 > 0 \Rightarrow$ causality condition $t = t_1 - t_2 > 0$

☞ co-variant F should be interpreted as (causal) **reponse function !**

N.B.: recall that a response $F = F(t_1, t_2) = \left. \frac{\delta\langle\phi(t_1)\rangle}{\delta h(t_2)} \right|_{h=0}$ vanishes for $t_1 < t_2$

$\Rightarrow$ Physical consequence: causality as required for responses $\widetilde{\mathfrak{sch}}(d)$

in dual space, use conformal invariance $\langle \Psi_1(\boldsymbol{\xi}_1)\Psi_2(\boldsymbol{\xi}_2)\rangle = \Psi_0 \delta_{x_1,x_2} |\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2|^{-2x_1}$

$$\langle \psi_1(\zeta_1, t_1, \boldsymbol{r}_1)\psi_2(\zeta_2, t_2, \boldsymbol{r}_2)\rangle = \langle \Psi_1(\boldsymbol{\xi}_1)\Psi_2(\boldsymbol{\xi}_2)\rangle = \psi_0 \delta_{x_1,x_2} (t_1 - t_2)^{-x_1} \left(\zeta_1 - \zeta_2 + \frac{\mathrm{i}}{2}\frac{(\boldsymbol{r}_1 - \boldsymbol{r}_2)^2}{t_1 - t_2}\right)^{-x_1}$$

Physical convention: positive mass $\mathcal{M} > 0$ of field ϕ

If scaling dimension $x_1 > 0$, **then** derive causal form (2P):

$$\begin{aligned}\left\langle \phi_1(t_1, \boldsymbol{r}_1)\phi_2^*(t_2, \boldsymbol{r}_2)\right\rangle &= \int_{\mathbb{R}^2} \mathrm{d}\zeta_1 \mathrm{d}\zeta_2\, e^{-\mathrm{i}\mathcal{M}_1\zeta_1 + \mathrm{i}\mathcal{M}_2\zeta_2} \left\langle \psi_1(\zeta_1, t_1, \boldsymbol{r}_1)\psi_2(\zeta_2, t_2, \boldsymbol{r}_2)\right\rangle \\ &= \phi_0\, \delta_{x_1,x_2}\, \delta_{\mathcal{M}_1,\mathcal{M}_2}\, \mathcal{M}_1^{1-x_1}\, \Theta(t_1 - t_2)\, (t_1 - t_2)^{-x_1} \exp\left(-\frac{\mathcal{M}_1}{2}\frac{(\boldsymbol{r}_1 - \boldsymbol{r}_2)^2}{t_1 - t_2}\right)\end{aligned}$$

If scaling dimensions $x_1 > 0$, and $x_2 > 0$, **then** derive causal form (3P):

$$\begin{aligned}\left\langle \phi_1(t_1, \boldsymbol{r}_1)\phi_2(t_2, \boldsymbol{r}_2)\phi_3^*(t_3, \boldsymbol{r}_3)\right\rangle &= \mathcal{C}_{12,3}\, \delta(\mathcal{M}_1 + \mathcal{M}_2 - \mathcal{M}_3) \\ \times &\quad \Theta(t_1 - t_3)\,\Theta(t_2 - t_3)\, (t_1 - t_2)^{-x_{12,3}/2} (t_1 - t_3)^{-x_{13,2}/2} (t_2 - t_3)^{-x_{23,1}/2} \\ \times &\quad \exp\left[-\frac{\mathcal{M}_1}{2}\frac{(\boldsymbol{r}_1 - \boldsymbol{r}_3)^2}{t_1 - t_3} - \frac{\mathcal{M}_2}{2}\frac{(\boldsymbol{r}_2 - \boldsymbol{r}_3)^2}{t_2 - t_3}\right] \\ \times &\quad \Psi_{12,3}\left(\frac{1}{2}\frac{[(\boldsymbol{r}_1 - \boldsymbol{r}_3)(t_2 - t_3) - (\boldsymbol{r}_2 - \boldsymbol{r}_3)(t_1 - t_3)]^2}{(t_1 - t_2)(t_2 - t_3)(t_1 - t_3)}\right)\end{aligned}$$

Causality requires at least the parabolic sub-algebras of $\mathfrak{conf}(d+2)_{\mathbb{C}}$

An infinite-dimensional extension of $\widetilde{\mathfrak{sch}}(1)$

extended Schrödinger-Virasoro algebra

$$\boxed{\widetilde{\mathfrak{sv}}(1) := \langle X_n, Y_m, M_n, N_n \rangle_{n\in\mathbb{Z}, m\in\mathbb{Z}+\frac{1}{2}} \supset \mathfrak{sv}(1)}$$

additional non-vanishing commutators, beyond those of $\mathfrak{sv}(1)$:

$$[X_n, N_{n'}] = -n' N_{n+n'} \ , \ \ [Y_m, N_n] = -Y_{m+n'} \ , \ \ [M_n, N_{n'}] = -2N_{n+n'}$$

admissible central extensions: $n, n' \in \mathbb{Z}$

$$\begin{aligned} [X_n, X_{n'}] &= (n-n')X_{n+n'} + \frac{c}{12}(n^3 - n)\delta_{n+n',0} \\ [N_n, N_{n'}] &= \kappa n \delta_{n+n',0} \\ [X_n, N_{n'}] &= -n' N_{n+n'} + \alpha n^2 \delta_{n+n',0} \end{aligned}$$

maximal finite-dimensional sub-algebra: $\widetilde{\mathfrak{sch}}(1) = \mathfrak{sch}(1) + \mathbb{C}N_0$

C. Roger & J. Unterberger, Ann. H. Poincaré **7**, 1477 (2006) and Springer Lecture Notes (2011)

Some further reading:

1. MH & M. Pleimling, *Non-equilibrium phase transitions, vol. 2: Ageing and dynamical scaling . . .*, Springer (Heidelberg 2010)

$2^{\rm nd}$ ed. in preparation

2. J. Unterberger, C. Roger, *The Schrödinger-Virasoro algebra: Mathematical structure and dynamical Schrödinger symmetries*, Springer (Heidelberg 2012)

in-depth analysis of many mathematical aspects

* MH, *Dynamical symmetries and causality in non-equilibrium phase transitions*, Symmetry **7**, 2108 (2015) [arXiv:1509.03669]
* MH, *From dynamical scaling to local scale-invariance: a tutorial*, Eur. Phys. J. Spec. Topic **226**, 605 (2017) [arxiv:1610.06122]

Appendix

Example for the t^{-1}-term in Langevin eq.: **Arcetri model**

continuous slopes $u_{\mathbf{i}} \in \mathbb{R}^d$, replace RSOS condition by 'spherical' constraint for $d > 0$ phase transition $T_c(d) > 0$, exponents not mean-field if $d < 2$

spherical constraint: $\left\langle \sum_{\mathbf{i}\in\Lambda} u_{\mathbf{i}}^2 \right\rangle = d\mathcal{N}$ MH & DURANG 15, MH 15

Langevin equation, with Lagrange multiplier $\mathfrak{z}(t)$ & centered gaussian noise $\eta_{\mathbf{i}}(t)$

$$\frac{\partial u_a(t,\mathbf{r})}{\partial t} = \nu\Delta u_a(t,\mathbf{r}) + \mathfrak{z}(t)u_a(t,\mathbf{r}) + \partial_a\eta(t,\mathbf{r}) \;\;,\;\; \langle \eta(t,\mathbf{r})\eta(s,\mathbf{r}')\rangle = 2\nu T\delta(t-s)\delta(\mathbf{r}-\mathbf{r}')$$

set $g(t) := \exp\left(2\int_0^t \mathrm{d}t'\, \mathfrak{z}(t')\right)$, spherical constraint gives Volterra eq.

$$g(t) = f(t) + 2T\int_0^t \mathrm{d}\tau\, f(t-\tau)g(\tau) \;\;,\;\; f(t) = \frac{de^{-4t}I_1(4t)}{4t}\left(e^{-4t}I_0(4t)\right)^{d-1}$$

find for $T \leq T_c$: $g(t) \overset{t\to\infty}{\sim} t^{-F} \Leftrightarrow \boxed{\mathfrak{z}(t) \sim \frac{F}{2}t^{-1}}$

quite analogous to spherical model of a ferromagnet GODRÈCHE & LUCK 00, PICONE & MH 04

Examples of infinite-dimensional time-space transformations (bis)

group	coordinate changes		co-variance
ortho-conformal	$z' = f(z)$	$\bar{z}' = \bar{z}$	correlator
$(1+1)D$	$z' = z$	$\bar{z}' = \bar{f}(\bar{z})$	
Schrödinger-Virasoro	$t' = b(t)$	$\boldsymbol{r}' = (\mathrm{d}b(t)/\mathrm{d}t)^{1/2}\,\boldsymbol{r}$	response
	$t' = t$	$\boldsymbol{r}' = \boldsymbol{r} + \boldsymbol{a}(t)$	
	$t' = t$	$\boldsymbol{r}' = \mathscr{R}(t)\boldsymbol{r}$	
conformal galilean	$t' = b(t)$	$\boldsymbol{r}' = (\mathrm{d}b(t)/\mathrm{d}t)\,\boldsymbol{r}$	correlator
	$t' = t$	$\boldsymbol{r}' = \boldsymbol{r} + \boldsymbol{a}(t)$	
	$t' = t$	$\boldsymbol{r}' = \mathscr{R}(t)\boldsymbol{r}$	

* arises from quantum gravity

Bondi, Metzner, Sachs 1965
Havas, Plebanski 1978

* **is** the non-relativistic limit of conformal group
* has dynamical exponent $z = 1$
  i.e. applications in hydrodynamics, ...

$\mathscr{R}(t) \in SO(d)$

* conformal galilean invariance predicts form of correlators

On Galilei transformations

in Schrödinger algebra $\mathfrak{sch}(d)$: $\quad Y_{\frac{1}{2}} = -t\partial_r - \mathcal{M}r$

in conformal galilean algebra $\mathrm{CGA}(d)$: $\quad Y_0 = -t\partial_r - \gamma$

$\Rightarrow$ imply different transformations of scaling operators

$$\begin{cases} \mathfrak{sch}(d): & \varphi'(t,\mathbf{r}) = \exp\left(-\mathcal{M}\mathbf{v}\cdot\mathbf{r} + \frac{\mathcal{M}}{2}\mathbf{v}^2 t^2\right)\varphi(t,\mathbf{r}-\mathbf{v}t) \\ \mathrm{CGA}(d): & \varphi'(t,\mathbf{r}) = \exp\left(-\mathbf{v}\cdot\gamma\right) \qquad \varphi(t,\mathbf{r}-\mathbf{v}t) \end{cases}$$

* Schrödinger algebra is **not** semi-simple
* $Y_{\frac{1}{2}}$ with spatial translations $Y_{-\frac{1}{2}} = -\partial_r$ $\Rightarrow$ Bargman super-selection rules
  and classical central extension, since $[Y^j_{\frac{1}{2}}, Y^{j'}_{-\frac{1}{2}}] = -\mathcal{M}\delta^{j,j'} \neq 0$
* Y_0 commutes with spatial translations $Y_{-1} = -\partial_r$ $\qquad [Y_0, Y_{-1}] = 0$

$\Rightarrow$ physical applications depend on the choice of representation

Scaling relation $\lambda_C = d - z\Theta$ with slip exponent Θ

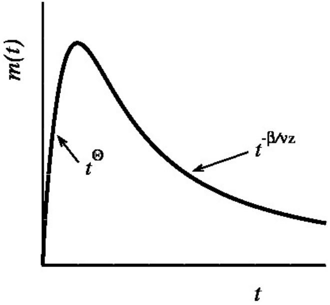
critical system **at** $T = T_c$, with an initial magnetisation $m_0 > 0$

find two **distinct** scaling regimes:

$$m(t) \sim \begin{cases} t^{\Theta} & ; \text{ if } t \ll t_m \\ t^{-\beta/(\nu z)} & ; \text{ if } t \gg t_m \end{cases}$$

with $t_m \sim m_0^{-1/(\Theta+\beta/\nu z)}$

$\Theta :=$ slip exponent



Theorem: (Janssen, Schaub, Schmittmann 89) *Scaling relation with critical autocorrelation exponent* $\lambda_C = \lambda_R$*:*

$$\lambda_C = \lambda_C(T_c) = d - z\Theta$$

λ_C and Θ are **independent** of equilibrium critical exponents re-derive this scaling relation from local scale-invariance:

take initial magnetisation m_0 into accout, hence effective action

$$\mathcal{J}[\phi,\widetilde{\phi}] = \mathcal{J}_0[\phi,\widetilde{\phi}] + \mathcal{J}_b[\widetilde{\phi}] + \mathcal{J}_{\rm ini}[\widetilde{\phi}], \quad \boxed{\mathcal{J}_{\rm ini}[\widetilde{\phi}] = -\int_{\mathbb{R}^d} \mathrm{d}\boldsymbol{r}\; m_0 \widetilde{\phi}(0,\boldsymbol{r})}$$

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Use Bargman's superselection rules

$$\begin{aligned} \langle m(t)\rangle &= \langle \phi(t,0)\rangle = \left\langle \phi(t,0) e^{-\mathcal{J}_b[\widetilde{\phi}]-\mathcal{J}_{\rm ini}[\widetilde{\phi}]}\right\rangle_0 \\ &= m_0 \int_{\mathbb{R}^d} \mathrm{d}\boldsymbol{r}\; \underbrace{\left\langle \phi(t,0)\widetilde{\phi}(0,\boldsymbol{r})\right\rangle_0}_{R(t,0;\boldsymbol{r})} \end{aligned}$$

response function $R(t,0;\boldsymbol{r}) = t^{-\lambda_R/z}\mathcal{F}(\boldsymbol{r}t^{-1/z})$, for $t \ll t_m$. Hence

$$m(t) = t^{(d-\lambda_R)/z} m_0 \int_{\mathbb{R}^d} \mathrm{d}\boldsymbol{u}\, \mathcal{F}(\boldsymbol{u}) \stackrel{!}{\sim} t^{\Theta}$$

only term linear in m_0 survives for $t \ll t_m$ $\Rightarrow$ $\Theta = (d-\lambda_R)/z$.

Reproduces JSS-relation, since $\lambda_C = \lambda_R$. QED