Extremal Flows for the Non-Positive Bootstrap (?)

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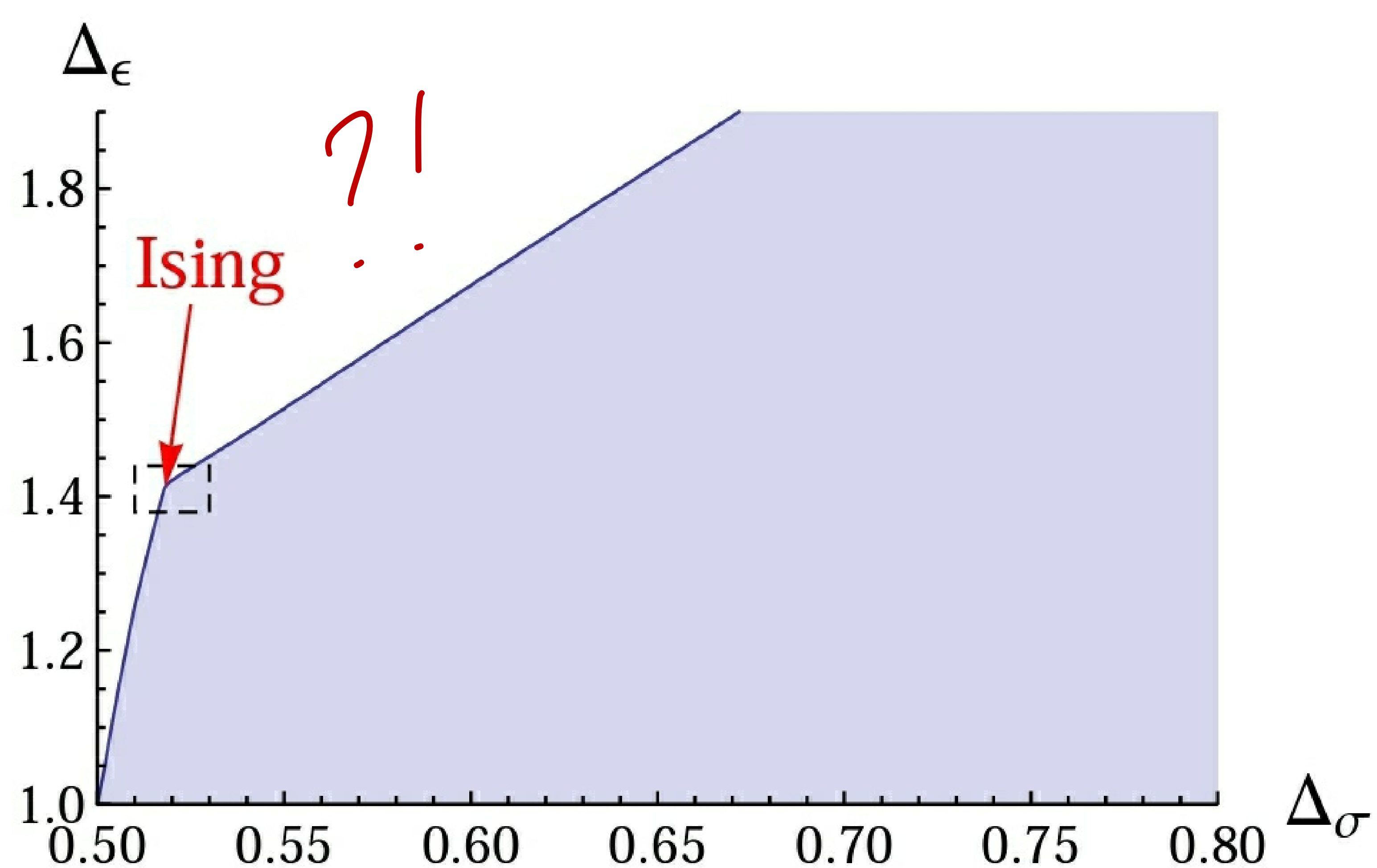
Main reference: El-Showk, MP 1605.08087 (!)

- Unitary bootstrap has been greatly successful.
- Positivity is a key ingredient.
  It implies bounds, and crucially such bounds are often saturated by interesting theories
- Why?



Positivity: $\lambda_{ij\mathcal{O}}^2 \geq 0$ $\langle\phi\phi\phi\phi\rangle = \sum$ positive

Bound saturating (extremal) CFTs will saturate as many of these inequalities as possible.

EXTREMAL CFTS ARE SPARSE*

* approximately

IDEA: Use sparseness as our starting point.

(Sparseness + positivity $\Rightarrow$ bounds)

Recap: Crossing

- Collection of correlators $G_{ijkl}(x_1, x_2, x_3, x_4)$
- Crossing equations quadratic in OPE data $\lambda_{ij\mathcal{O}}$

labels crossing eqn. $a = 1, \dots, C$

$$\sum_{\mathcal{O}} \sum_{ijkl} \lambda_{ij\mathcal{O}} \underbrace{F^{a}_{ijkl;\mathcal{O}}}_{\text{kinematic functions of cross-ratios}} \lambda_{kl\mathcal{O}} = 0$$

$\mathcal{O}$: CFT operators

$$\Leftrightarrow \quad \sum_{\mathcal{O}} \underline{\lambda}_{\mathcal{O}} \cdot \underline{\underline{F}}^{a}_{\mathcal{O}} \cdot \underline{\lambda}_{\mathcal{O}} = 0 \qquad \underline{\lambda}_{\mathcal{O}} = \begin{pmatrix} \lambda_{11\mathcal{O}} \\ \lambda_{12\mathcal{O}} \\ \vdots \end{pmatrix}$$

couplings to external states

$$\sum_{G} \underline{\lambda}_G \cdot \underline{\underline{F}}_G^a \cdot \underline{\lambda}_G = 0 \qquad \underline{\lambda}_G = \begin{pmatrix} \lambda_{11G} \\ \lambda_{12G} \\ \vdots \end{pmatrix}$$

Introduce polar coordinates:

$$\sum_{G} \underset{\substack{\cap \\ \mathbb{R}}}{\lambda_G^2} \underbrace{\left(\underline{n}_G \cdot \underline{\underline{F}}_G^a \cdot \underline{n}_G \right)}_{\text{kinematics}} = 0$$

orientation in $\phi_i \phi_j$ space

$$\Leftrightarrow \sum_{\underbrace{q}} \lambda_q^2 \, F_q^a(z,\bar{z}) = 0$$

Quantum numbers : $\begin{cases} \Delta, \underline{n} & : \text{ continuous} \\ \ell, Q, \ldots & : \text{ discrete} \end{cases}$

· To make progress we must (usually) consider finite set of crossing constraints

$$\mathcal{F}^a_q(z,\bar{z}) \longrightarrow \vec{\mathcal{F}}_q = \Big\{ \underbrace{\omega_n}_{\text{basis functionals}} [\mathcal{F}^a_q] \quad n=1,\dots,N \Big\}$$

e.g. $\omega_n = \partial_z^{2n+1} \big|_{z=1/2}$

$$\sum_q \lambda_q^2 \underbrace{\vec{\mathcal{F}}_q}_{N \text{ dimensional vector}} = 0$$

Linear equations!

- ## Simplest bootstrap:

$$\sum_\Delta \lambda_\Delta^2 F_\Delta(z) = 0$$

$$\rightarrow \sum_\Delta \lambda_\Delta^2 \vec{F}_\Delta = -\vec{F}_0 - (\dots) \equiv \vec{T}$$

- Linear system of N equations. A generic set of N $\vec{F}_{\Delta_i}$ will provide a solution (albeit for $\lambda^2_{\Delta_i} \in \mathbb{R}$)

- Sparseness: (Gliozzi)

Can relax somewhat, sometimes

$$\sum_{i=1}^{N/2} \lambda_i^2 \vec{F}_{\Delta_i} = \vec{T}$$

$\left\{\begin{array}{l} N \text{ equations for} \\ N/2 \;\; \lambda_i^2 \\ + \\ N/2 \;\; \Delta_i \end{array}\right.$

- Sparseness: $$\sum_{i=1}^{N/2} \lambda_i^2 \, \bar{F}^0_{\Delta_i} = \bar{T}^0$$
- Solution w/ positive λ^2_i, Δ_i (or even real) not guaranteed! Bug / feature
- How to solve?

1) "Method of determinants"

$$\begin{array}{cc} \overset{N/2+1}{\longleftrightarrow} & \\ \begin{pmatrix} (F_{\Delta_1})_1 & \cdots & (F_{\Delta_{N/2}})_1 & T_1 \\ (F_{\Delta_1})_2 & \cdots & (F_{\Delta_{N/2}})_2 & T_2 \\ \vdots & \cdots & \vdots & \vdots \\ (F_{\Delta_1})_N & \cdots & (F_{\Delta_{N/2}})_N & T_N \end{pmatrix} & \updownarrow N \end{array}$$

All $(N/2+1)\times(N/2+1)$ minors must vanish

(M. Meineri's Talk)

2) $$\sum_{i=1}^{N/2} \lambda_i^2 \, \vec{F}_{\Delta_i} + \sum_{j=1}^{N/2} \mu_i \, \vec{W}_i = \vec{T}$$

$\hookrightarrow$ auxiliary vectors

Convenient choice:

$$\vec{W}_i = \partial_\Delta \vec{F}_{\Delta_i}$$

$$M \equiv \left(F_{\Delta_1} \middle| \cdots \middle| F_{\Delta_{N/2}} \middle| \partial_\Delta F_{\Delta_1} \middle| \cdots \middle| \partial_\Delta F_{\Delta_{N/2}} \right)$$

$$M^{-1} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_{N/2} \\ \beta_1 \\ \vdots \\ \beta_{N/2} \end{pmatrix}$$

Basis of dual space: linear functionals

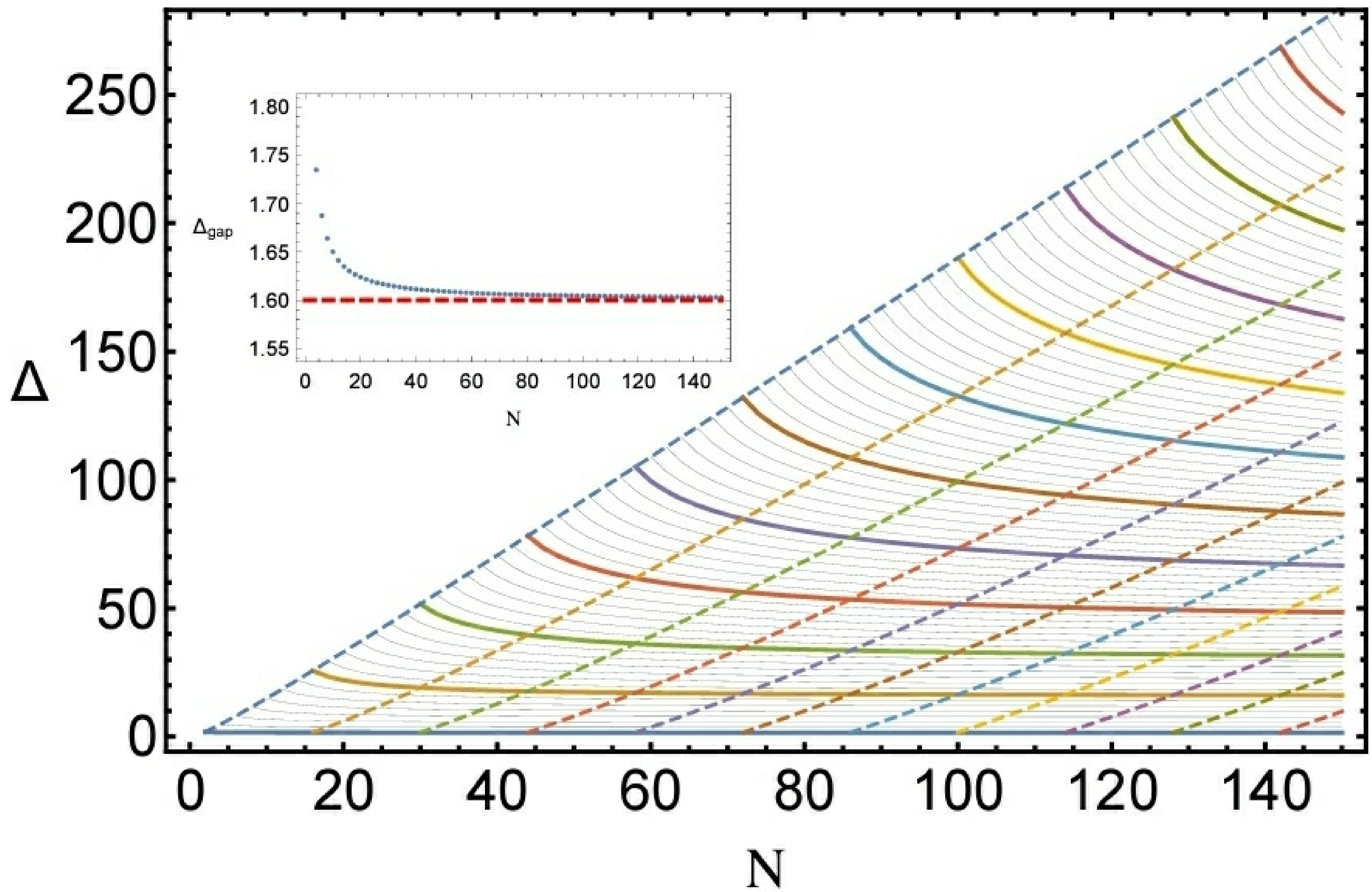
$$\mu_i = \underbrace{\beta_i[T]}_{\text{(ratio of determinants)}} \stackrel{!}{=} 0 \qquad \lambda_i^2 = \alpha_i[T]$$

- In this simple example, can always find solution for all N. Possible to bootstrap non-unitary solution.
- Non-linear equations can be solved by making good enough guess + Newton's method.

(Better functional bases, easier to guess)

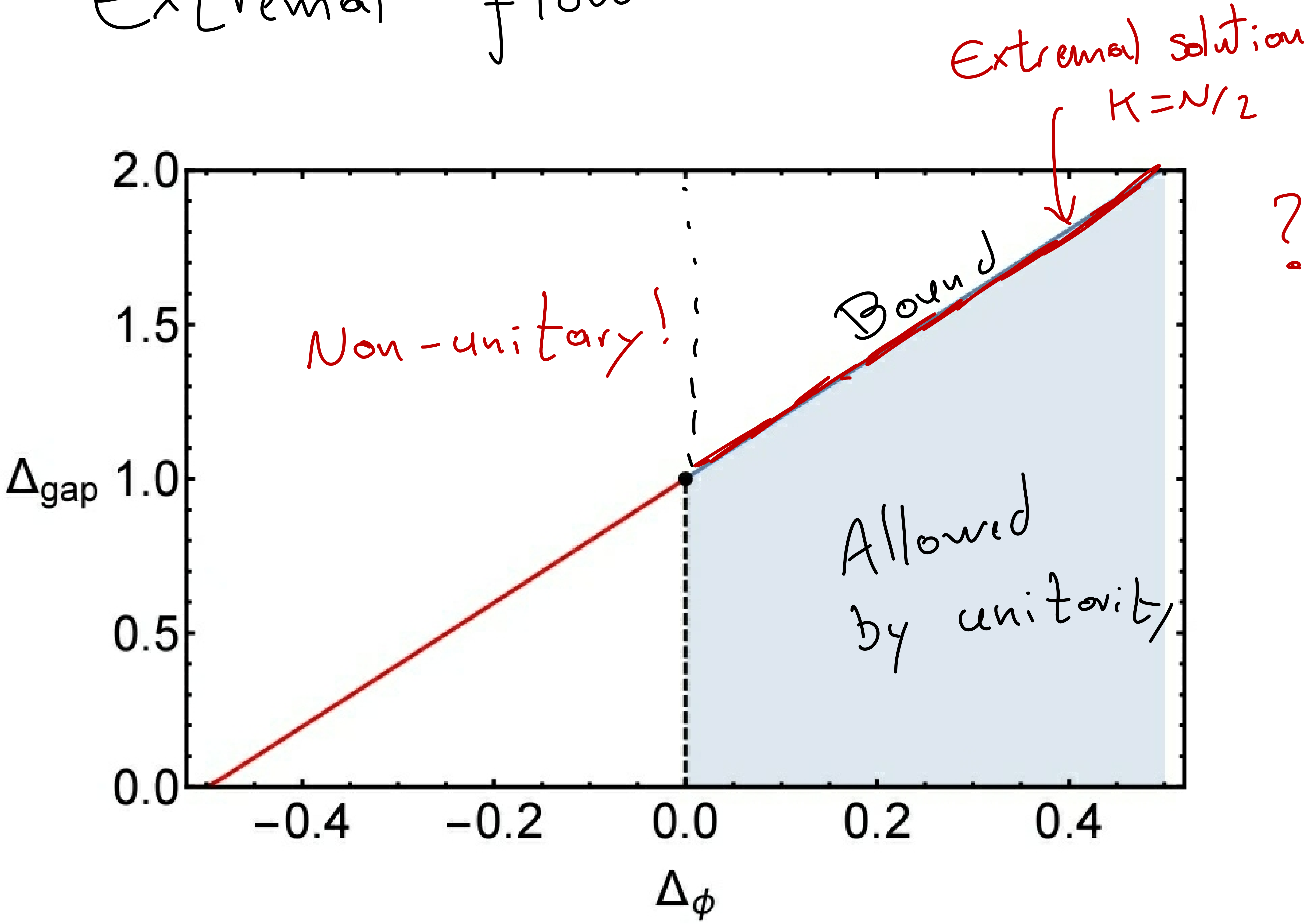
- For fixed N, can continuously deform solutions by varying some continuous variable $\longrightarrow$ "EXTREMAL FLOWS"

• "Upgrading": Constructing extremal solution w/ Newton's method + guess



→ Same method used by Hartman et al for modular b.s.

- Extremal flow



(unitarity $\Rightarrow \Delta_\phi \geq 0$)

- Unfortunately, difficulties arise in the presence of several channels in crossing eqn.

e.g. $$\sum_{\ell} \sum_{\Delta} \lambda^2_{\Delta,\ell} \vec{F}_{\Delta,\ell} = \vec{T}^{\,0}$$

or, in BCFT, bulk + bdy channels.

- Solutions w/ N/2 operators will usually not exist (w/ real Δ). Problem can be alleviated by having # > N/2 w/ extra constraints (e.g. fixed vectors for which we know Δ). But this rapidly becomes insufficient ... (Meineri's talk)

Example - $N=4$

$$\lambda_1 \vec{F}_{\Delta_1} + \lambda_2 \vec{F}_{\Delta_2} + \mu_1 \partial_\Delta \vec{F}_{\Delta_1} + \mu_2 \partial_\Delta \vec{F}_{\Delta_2} = -\vec{F}_0$$

$$(\beta_1)_a \propto \varepsilon_{a b_1 b_2 b_3} (F_{\Delta_1})^{b_1} (F_{\Delta_2})^{b_2} (\partial_\Delta F_{\Delta_2})^{b_3}$$

- $\vec{\beta}_i \cdot \vec{F}_0 = 0 \qquad i=1,2 \qquad (\Longleftrightarrow \mu_i = 0)$

Recall $\vec{F}_\Delta = \begin{pmatrix} \omega_1[F_\Delta] \\ \vdots \\ \omega_N[F_\Delta] \end{pmatrix}$ $\left(\omega_i \overset{e.g.}{=} \partial_z^{2i+1} F_\Delta(z)\big|_{1/2}\right)$

and hence $\vec{\alpha}_i$ or $\vec{\beta}_i$ are simply linear combinations of ω_i

$$\beta_i = \beta_{i1}\,\omega_1 + \beta_{i2}\,\omega_2 + \dots \qquad \vec{\beta}_i \equiv (\beta_{i1}, \dots, \beta_{iN})$$

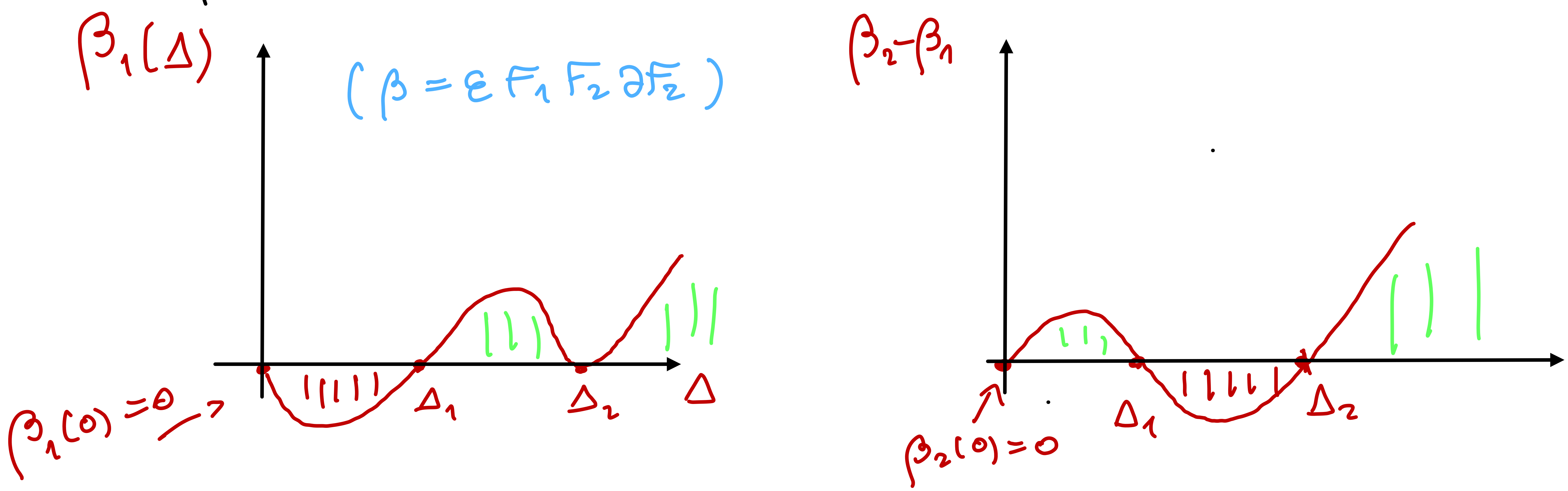
$$\vec{\beta}_i \cdot \vec{F}_\Delta = \sum_{i=1}^{N} \beta_{i1}\, \omega_i(\Delta)$$

- Crossing constraints are then:

$$\sum_\Delta \lambda^2_\Delta \, \omega_i(\Delta) = 0 \iff \begin{cases} \sum_\Delta \lambda^2_\Delta \, \beta_i(\Delta) = 0 \\ \sum_\Delta \lambda^2_\Delta \, \alpha_i(\Delta) = 0 \end{cases} \quad i = 1 \ldots N/2$$

$i = 1 \ldots N$

Suppose a solution exists w/ real Δ_1, Δ_2:



$\lambda^2_\Delta \geq 0 \implies$ Bounds!

(assuming positivity of $\beta_i(\Delta)$)

(green contributions must cancel red contributions)

- There is therefore a close link between extremal (sparse) solutions, and bootstrap bounds.

- We will use this link as guidance for "correcting" Gliozzi's "method". This modified method can in principle work for all N

For unitary ~~non-unitary~~ ...? ← (non-trivial!)

- Recall general crossing equs:

$$\sum_q \lambda_q^2 F_q^a(z,\bar z) = T^a(z,\bar z)$$

- Bounds: (example)

$$\boxed{\min_{\Lambda \in S} \Lambda \cdot T \quad \text{s.t.} \quad \begin{array}{l} \Lambda \cdot F_q \geqslant 0 \quad \forall q \in \mathcal{L} \\ \Lambda \cdot F_{q_0} = 1 \end{array}}$$

$$\left(\lambda_{q_0}^2 = \Lambda \cdot T - \text{pos} \leq \Lambda \cdot T \right)$$

Semi-inf linear program

$$\Updownarrow$$

$$\max_{\lambda_q^2} \lambda_{q_0}^2 \quad \text{s.t.} \quad \lambda_{q_0}^2 \vec{F}_{q_0} + \sum_{q \in \mathcal{L}} \lambda_q^2 \vec{F}_q = \vec{T}$$

Duality:

$$\boxed{\min_{\Lambda \in S} \Lambda \cdot T = \max_{\lambda_q^2} \lambda_{q_0}^2}$$

- Optimality Conditions (Karuhn - Kush - Tucker)

- $\exists\, q_i\,,\ \lambda_i^2 \geq 0 \qquad i = 1, \dots, k \qquad k \in N-1$

$$1.\quad \sum_{i=1}^{K} \lambda_i^2\, \overline{F}_{q_i}^{\,0} + \lambda_{q_0}^2\, \overline{F}_{q_0}^{\,0} = \overline{T}^{\,0}$$

$$2.\quad \Lambda \cdot \overline{F}_{q_i} = 0\,, \qquad \Lambda \cdot \overline{F}_{q_0} = 1$$

$$* \; 3.\quad \Lambda \cdot \nabla_q \overline{F}_{q_i} \begin{cases} = 0 & \text{if } q_i \in \operatorname{int} \mathcal{L} \\ = \text{inward Pointing vector} & \text{if } q_i \in \partial \mathcal{L} \end{cases}$$

EXTREMAL FUNCTIONALS $\Leftrightarrow$ EXTREMAL SOLUTION

Understanding KKT:

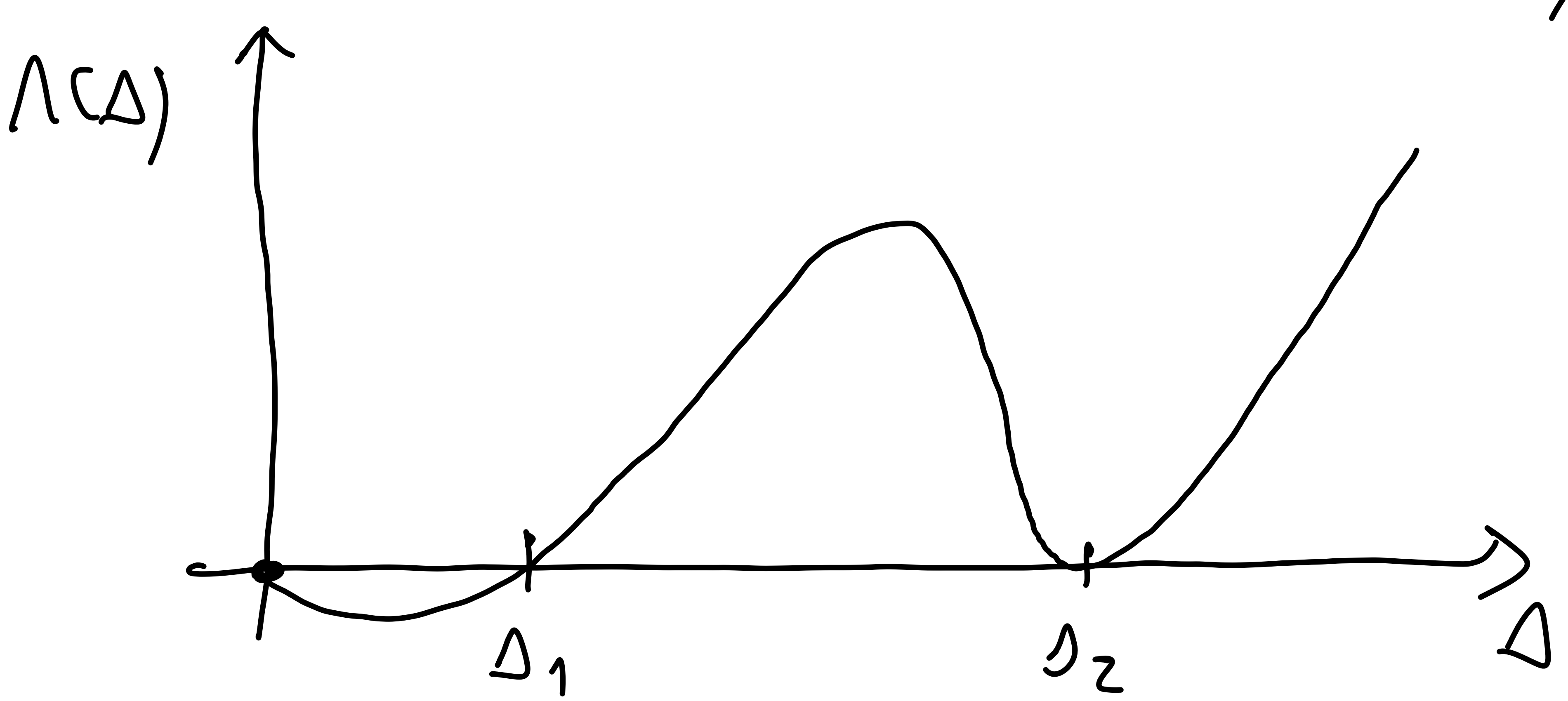
1. $$\sum_i^k \lambda_i F_i = T$$

Solution presents obstruction to further optimisation

2. $$\Lambda \cdot F_i = 0$$

Functional must be compatible w/ solution above.

3. $$\Lambda \cdot \partial_\Delta F_i = 0$$ → necessary for positivity



$$\mathcal{S} = \{\Delta : \Delta \geq \Delta_g = \Delta_1\}$$

• Solving KKT:

- Suppose $K = N/2$. Then conditions 2,3 can be solved trivially, functionals come for free. $\left(\beta \sim \varepsilon F_1 F_2 \partial F_2 F_3 \partial F_3 \dots \right)$

- If $K > N/2$ we get new constraints.

\# of constraints = \# d.o.f.

Ex: $N = 3$, $K = 2$ $\Rightarrow$ 4 d.o.f.

$$\lambda_1 \vec{F}_{\Delta_1} + \lambda_2 \vec{F}_{\Delta_2} = \vec{T} \Rightarrow \text{3 equs}$$

$$\vec{\beta} \propto \vec{F}_{\Delta_1} \times \vec{F}_{\Delta_2} \Rightarrow \vec{\beta} \cdot \vec{F}_{\Delta_1} = \vec{\beta} \cdot \vec{F}_{\Delta_2} = 0 \checkmark$$

$$\boxed{\vec{\beta} \cdot \partial_\Delta \vec{F}_{\Delta_2} = 0} \Rightarrow \text{1 equ.}$$

- A typical bootstrap solution will have $K > N/2$. Tangency conditions crucial to determine extremal flow uniquely.
- The number $K - N/2$ is a measure of how sparse the solution is.
- Singularities (e.g. "kinks") correspond e.g. to jumps in $K - N/2$.

PROPOSAL: Impose tangency conditions even in absence of unitarity or positivity

1. $\sum_{i=1}^{K} \lambda_i^2 \vec{F}_{q_i} + \lambda_{q_0}^2 \vec{F}_{q_0} = \vec{T}$

2. $\Lambda \cdot F_{q_i} = 0 \quad , \quad \Lambda \cdot F_{q_0} = 1$

* 3. $\Lambda \cdot \nabla_q F_{q_i} \begin{cases} = 0 & \text{if } q_i \in \operatorname{int} \mathcal{L} \\ = \text{inward pointing vector} & \text{if } q_i \in \partial \mathcal{L} \end{cases}$

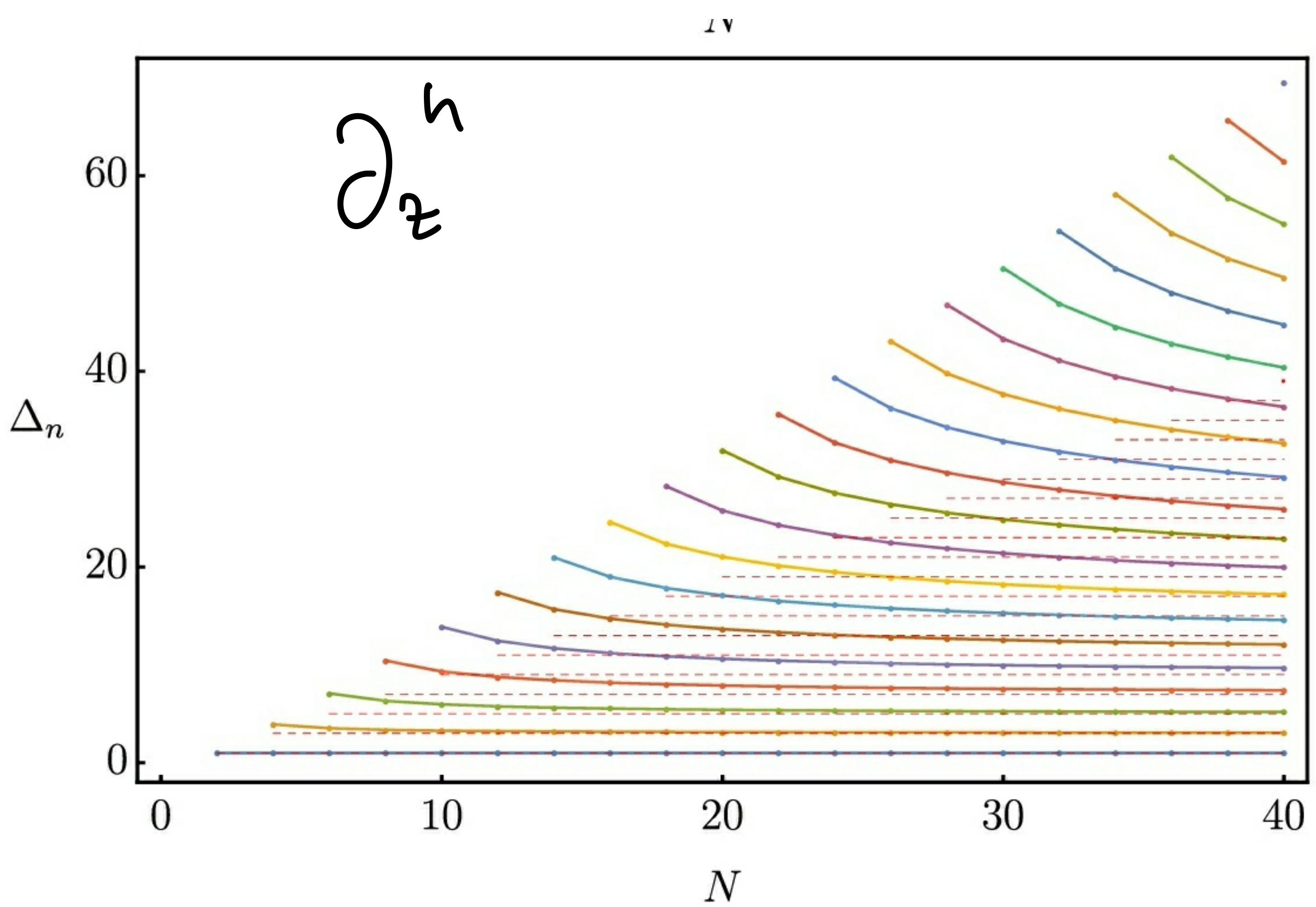
These are complicated non-linear equations in $q_i = (\Delta_i, \ell_i, \dots)$ and OPE coeffs.

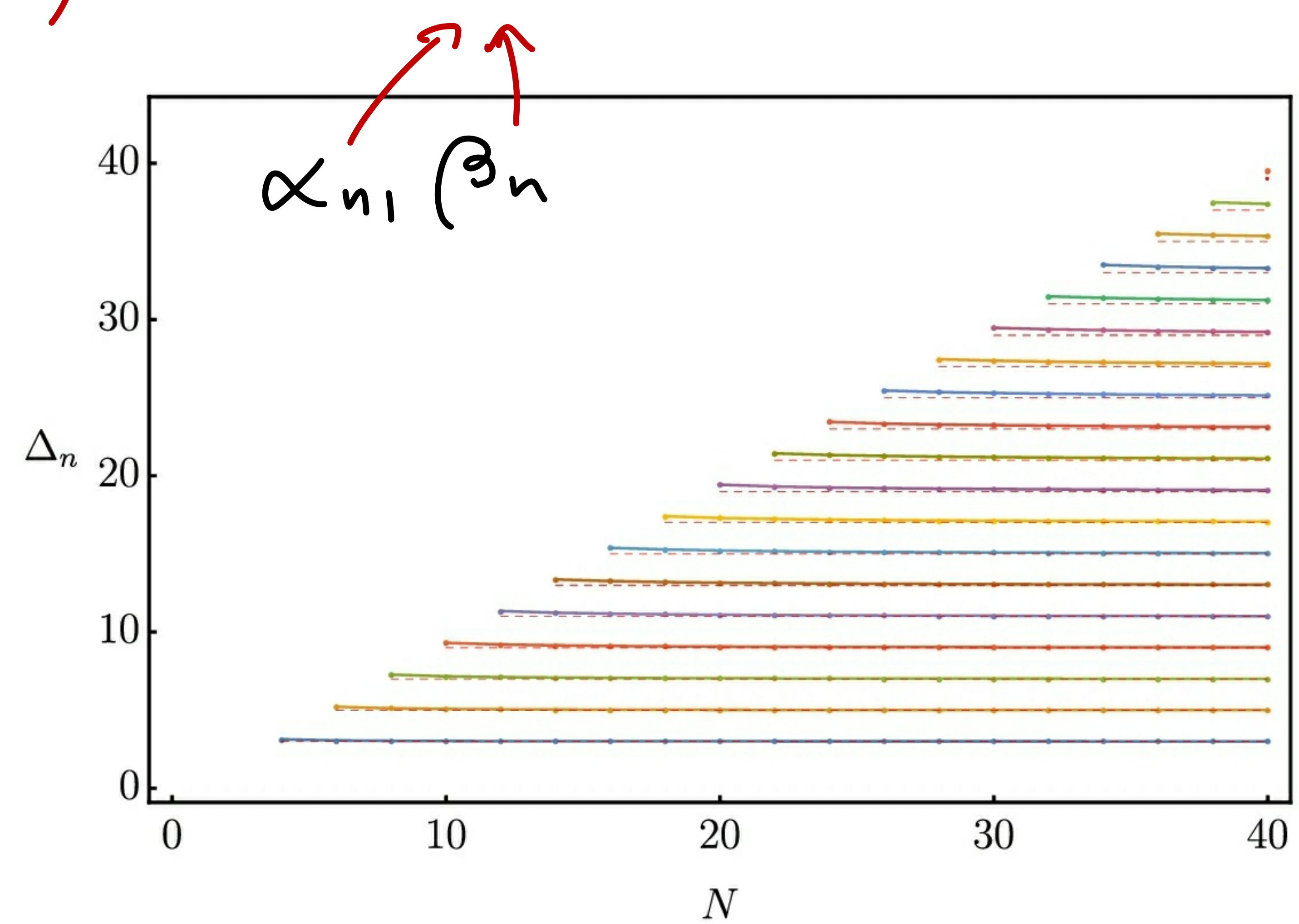
- Hard to solve in general: need to specify many discrete choices in q_i, K, and then perform multidim. search in continuous variables Δ_i.

• Using appropriate functional bases likely to be crucial.

(K = N/2 case)

MP, Mazac





• Bases capture correct asymptotics. (MP, Zan)

$\Rightarrow$ Efficient decoupling of UV/IR (in Δ)

• Higher-dim bases proposed
- MP
- Mazac, Rastelli, Zhou
- Gopakumar, Sinha, Zahed

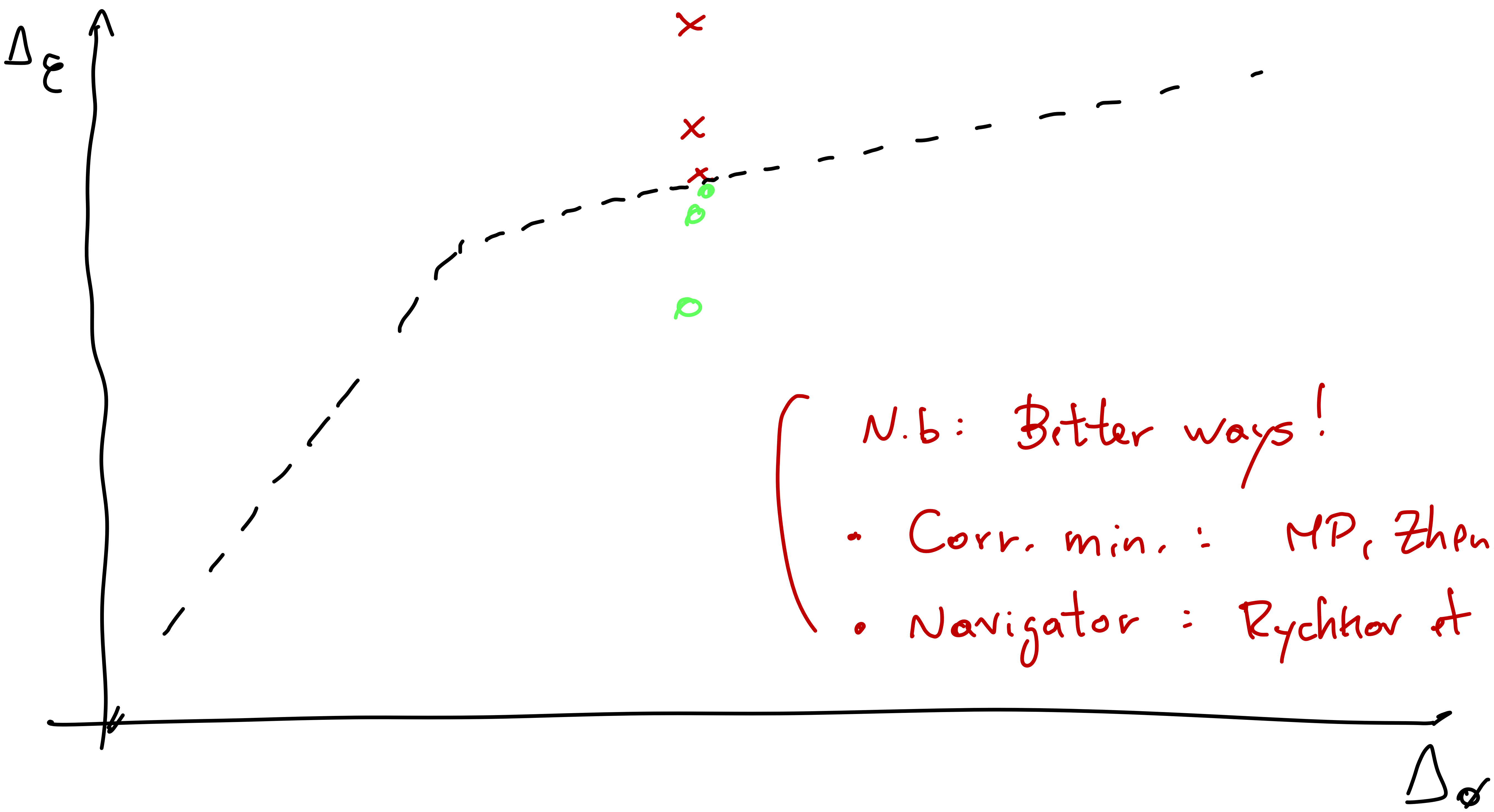
• Non-Positive Extremal Flows

• Different approach: use the fact that numerical bootstrap constructs (unitary) solutions to these equations to get initial point. Then, deform continuously (i.e. use Newton's method) to obtain other solutions until a non-unitary region is reached.

(Note: such flows are of course also useful for usual unitary bootstrap, since they avoid repeated optimisation runs)

- Typical workflow:

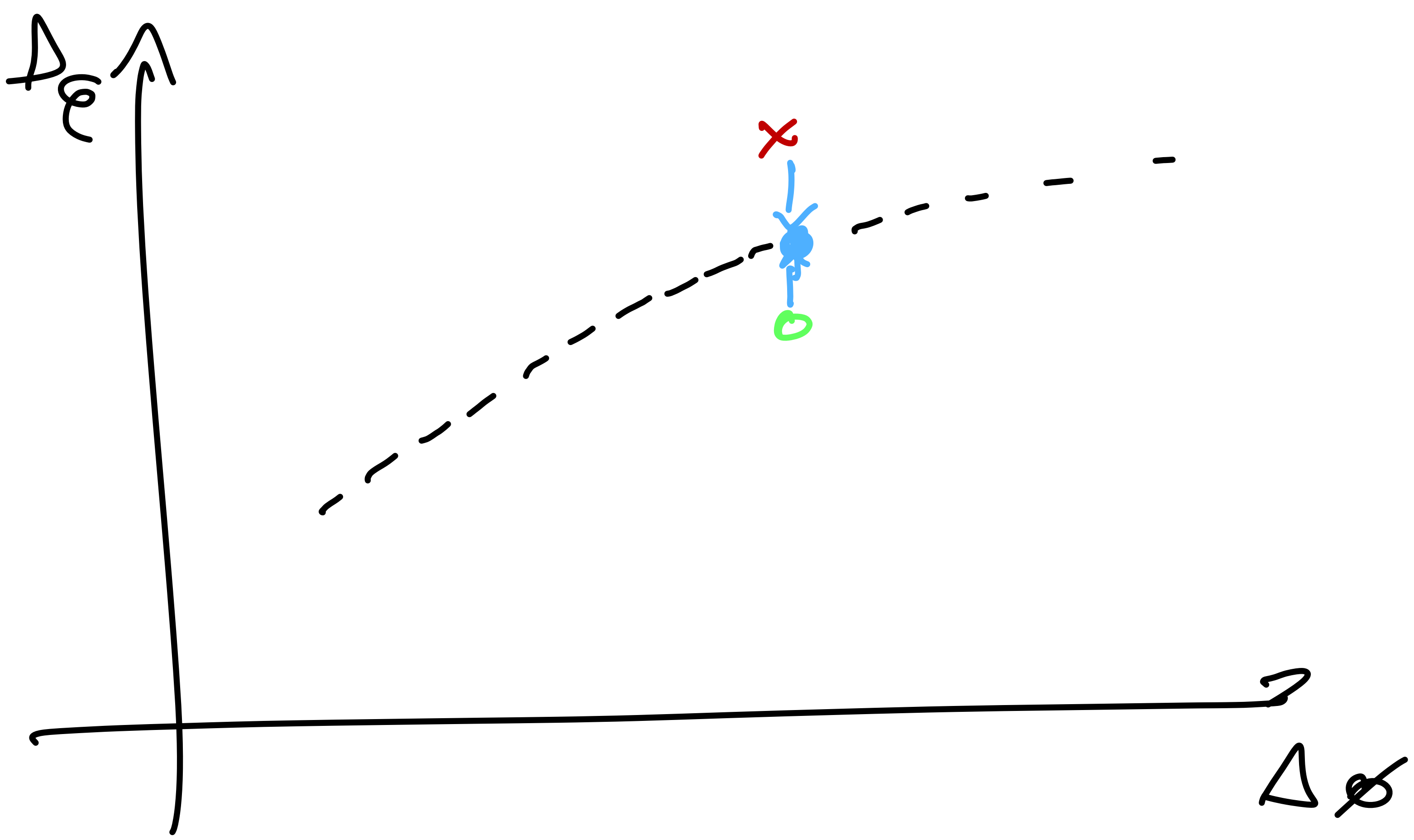
1. Construct an approximate functional/spectrum pair using SDPB or other



(N.b: Better ways!

- Corr. min.: MP, Zheng
- Navigator: Rychkov et al)

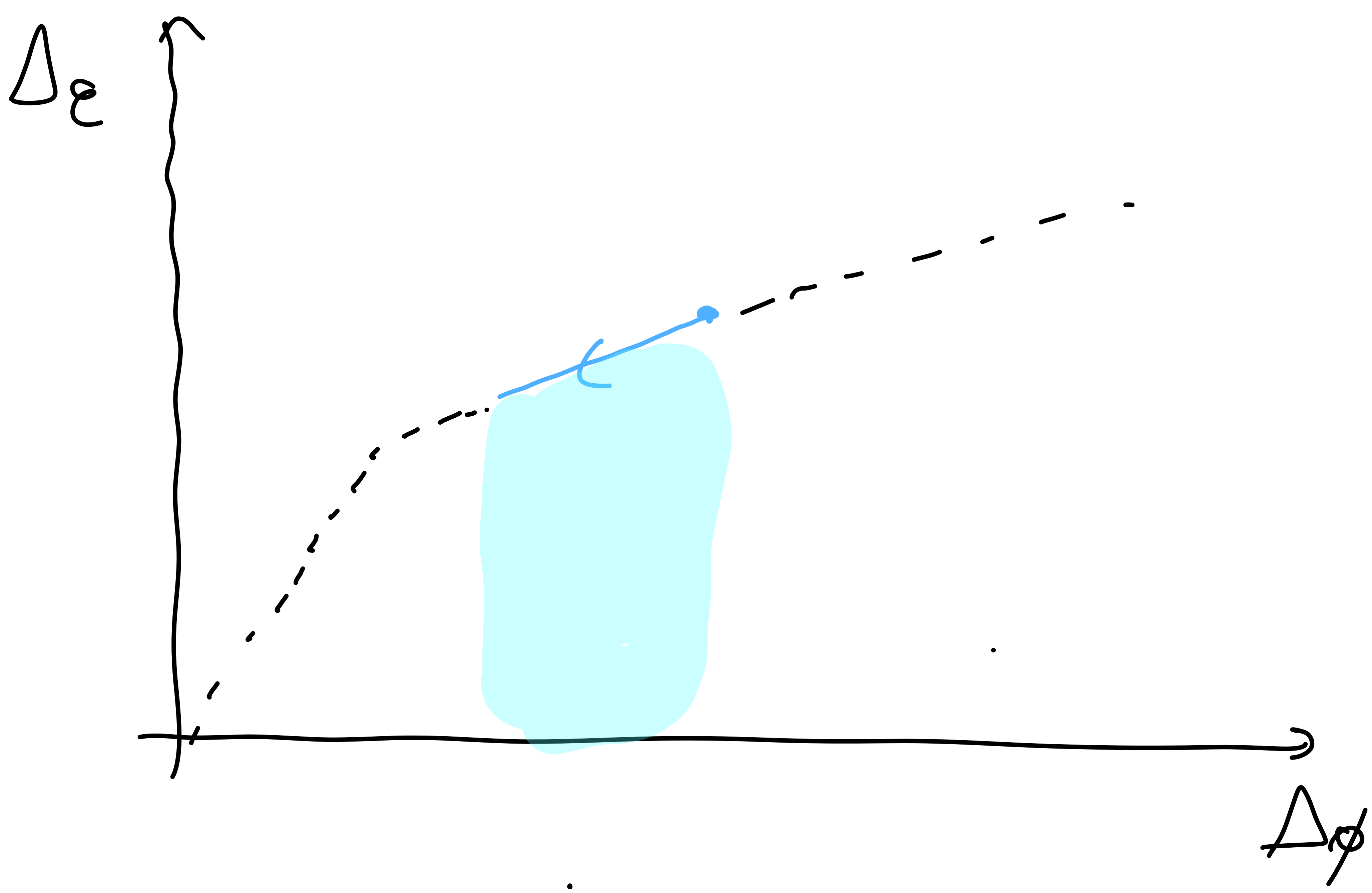
2. Flow to the exact boundary of feasibility by solving KKT w/ approximate bootstrap guess + Newton's method



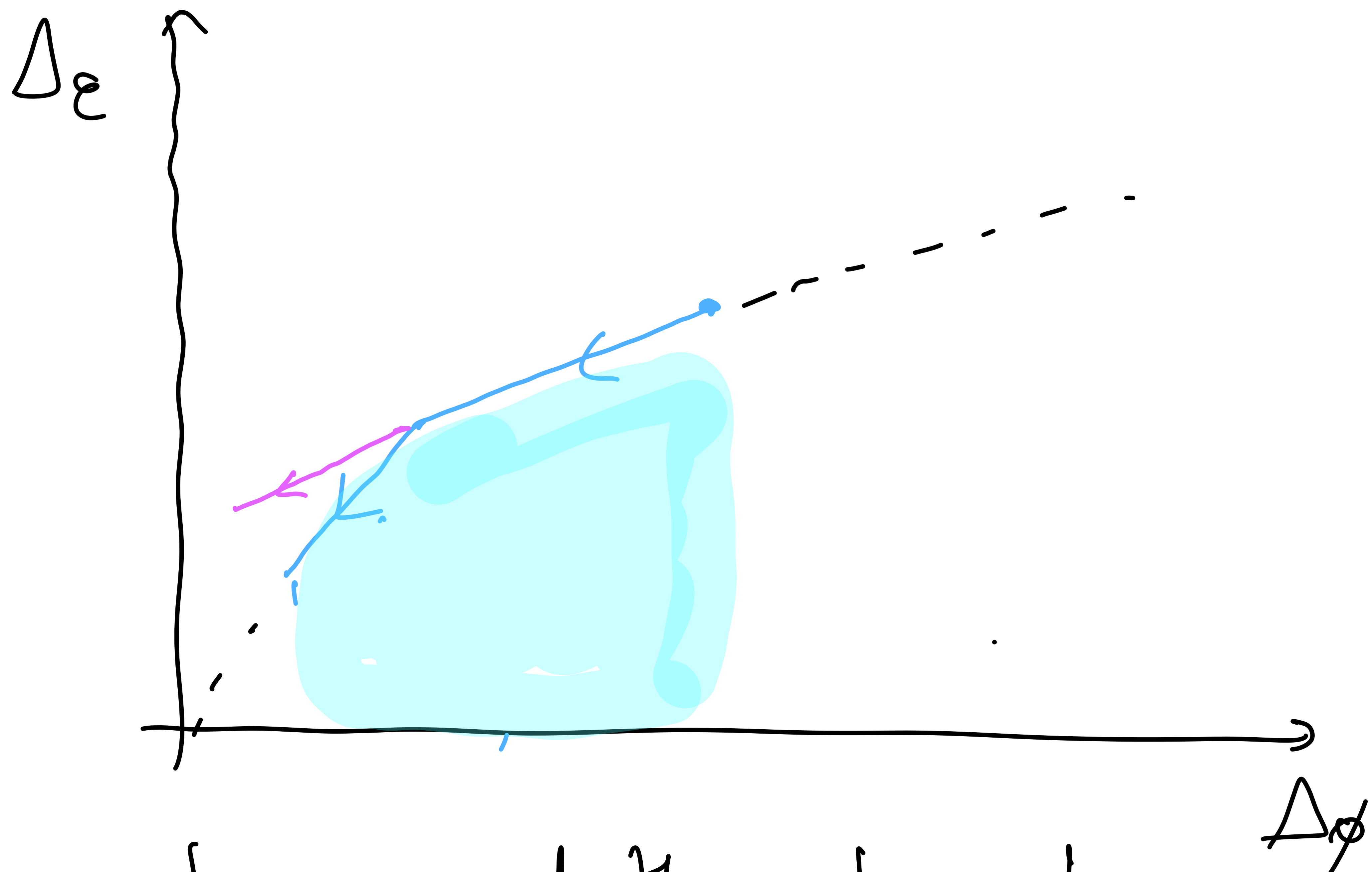
e.g. from 10^{-3} accuracy we obtain 10^{-30}

$\to$ no optimisation required

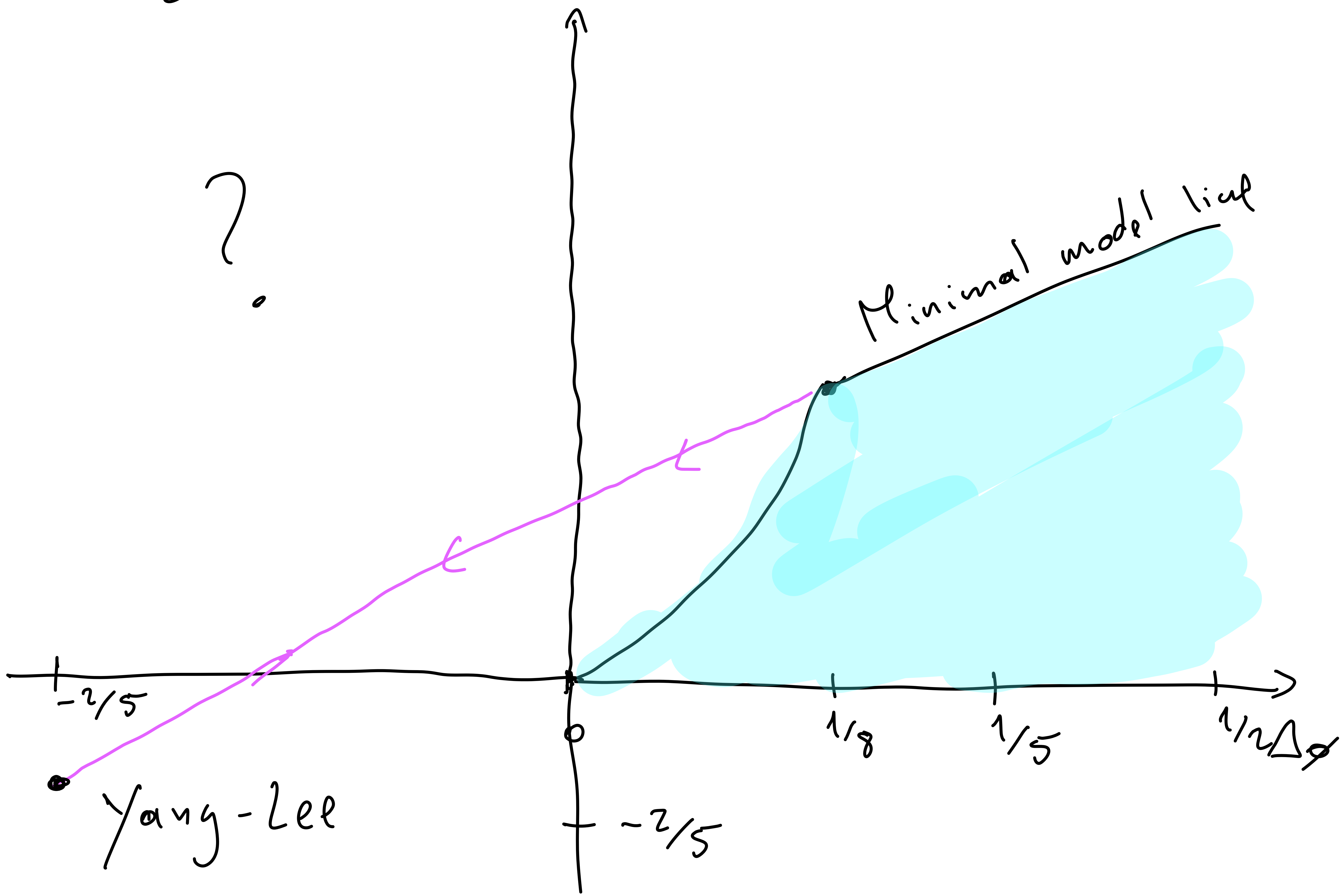
3. Deform solution by varying parameter say Δ_ϕ. We linearize KKT, + Newton's method. At any step along flow we have solution + functional $\Rightarrow$ bounds

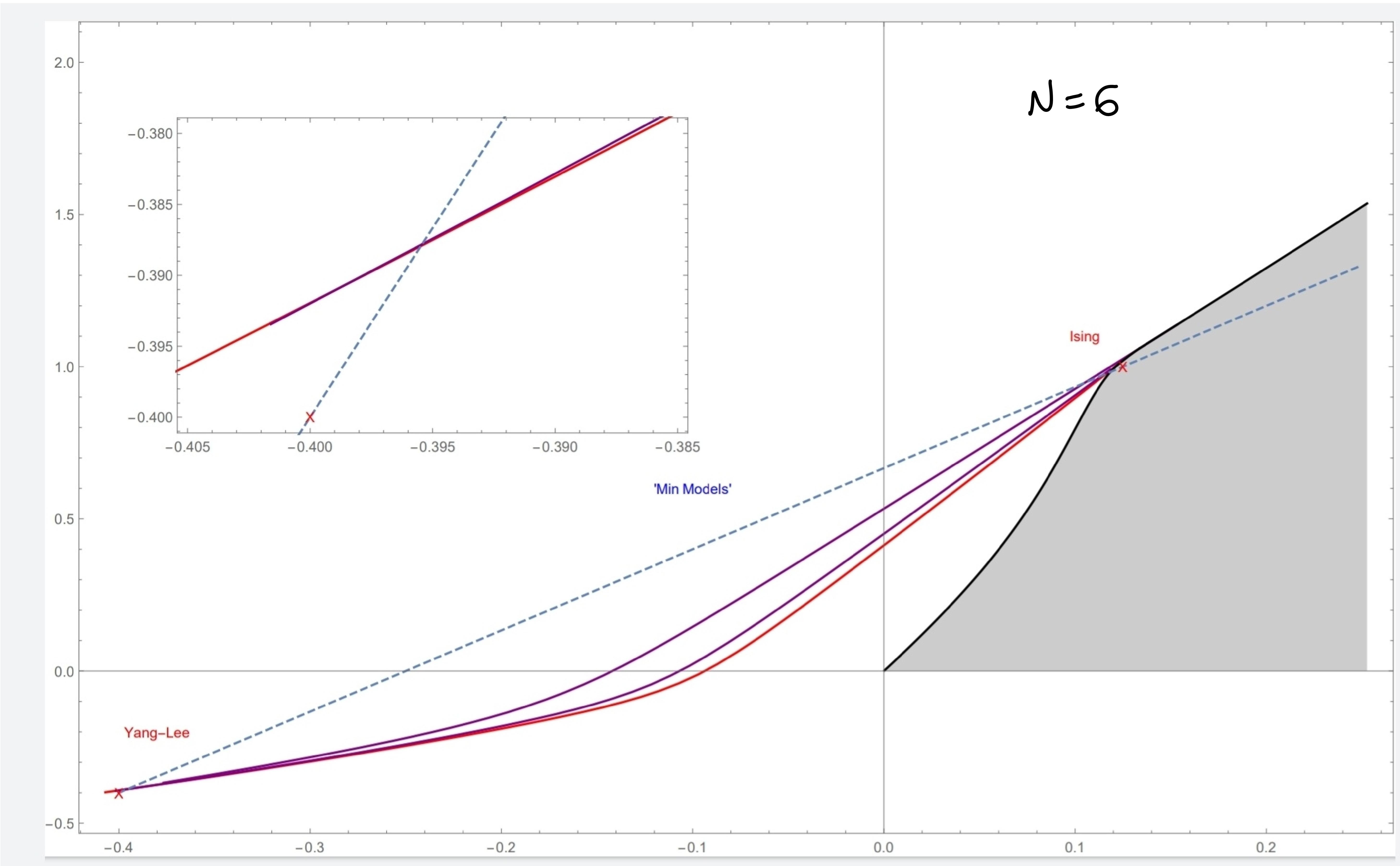


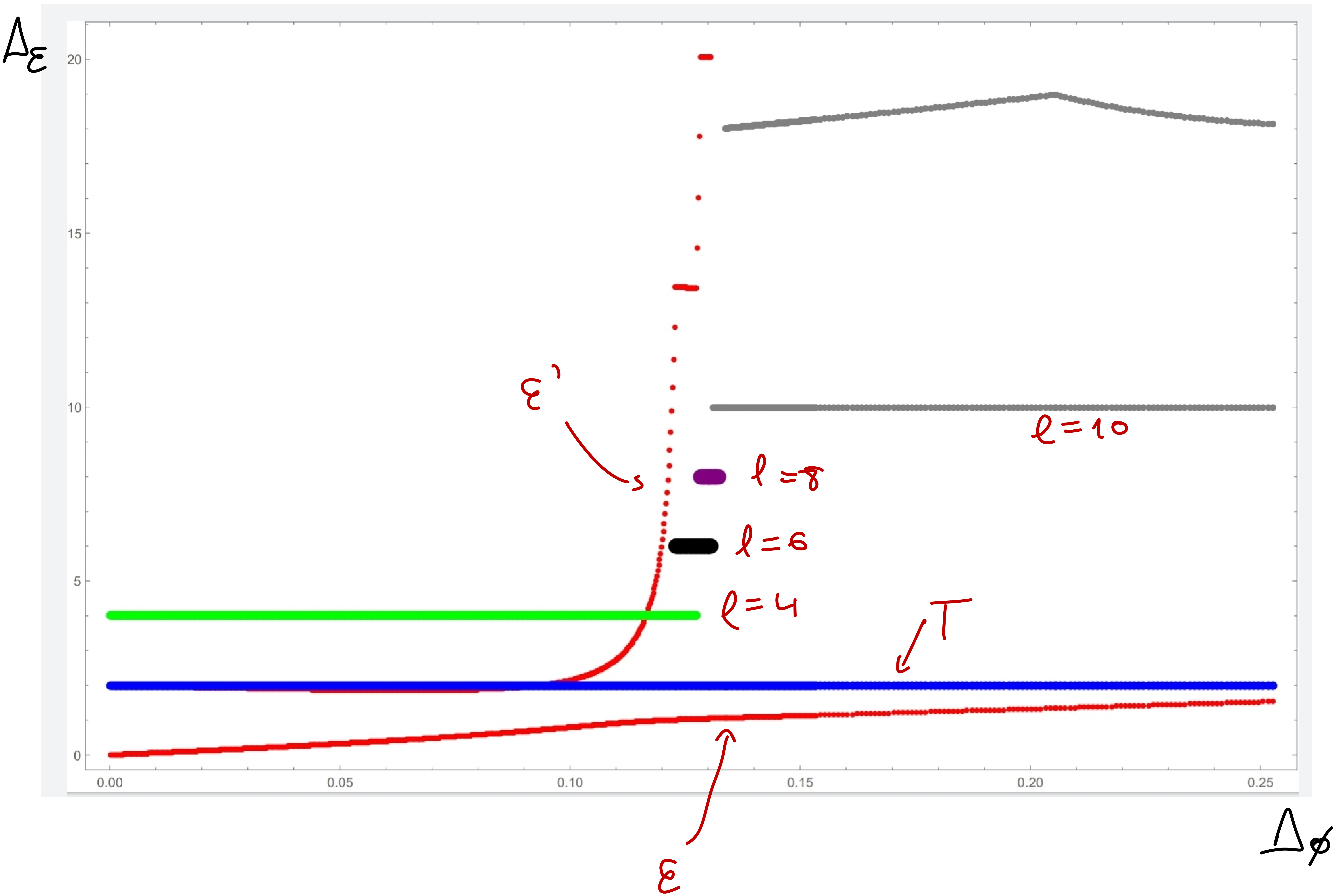
4. Singularities may occur, or $\lambda_i^2 \to 0$



Can choose whether to stay in unitary branch or move on to non-unitary one ex $D=2$




N=6
Ising
Yang–Lee
'Min Models'



Δ_ε
Δ_ϕ
0
5
10
15
20
0.00
0.05
0.10
0.15
0.20
0.25
ε'
$\ell=8$
$\ell=6$
$\ell=4$
$\ell=10$
T
ε

• Outlook

- Implement flows for multiple correlator + interface SDPB
- Direct construction:
  - Ambiguities?
  - Multidim. search?

  } BCFT good starting point
- Flows
  - In Δ_ϕ, N, D, ...
  - Ambiguities?
- Better functional bases, more correlators instead of more N, ...