Analytically solving the Ising model in $2+\epsilon$ dimensions

Wenliang Li

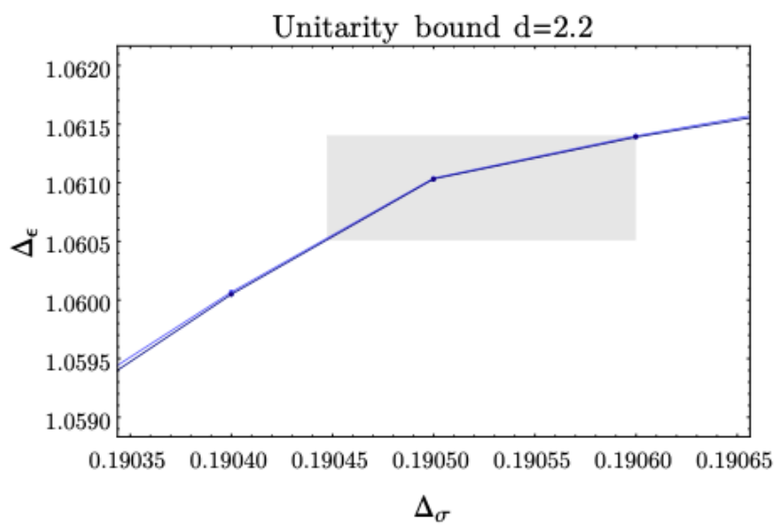
Okinawa Institute of Science and Technology Graduate University, Japan

Sun Yat-Sen University, China

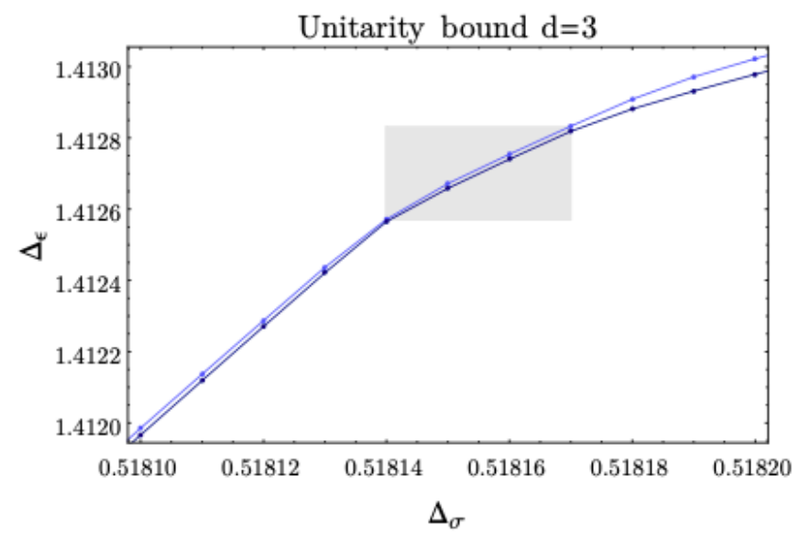
Bootstat 2021, 3-28 May

Last Friday

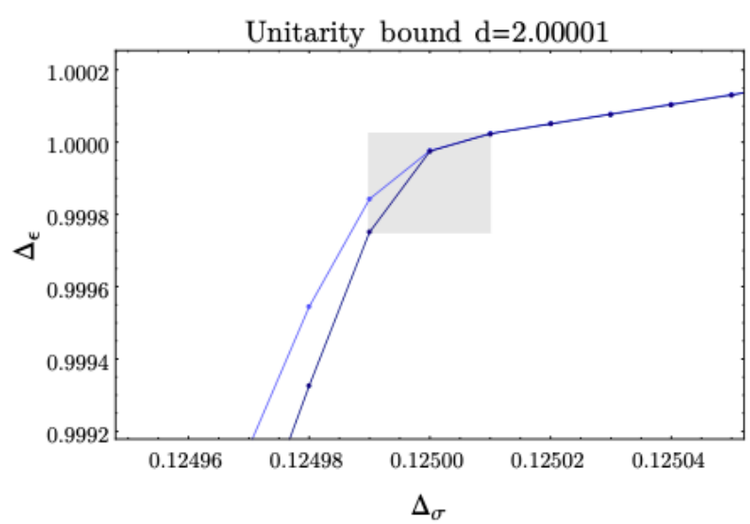
2<d<4 Ising from $\langle\sigma\sigma\sigma\sigma\rangle$ numerical bootstrap Cappelli, Maffi, Okuda ’18

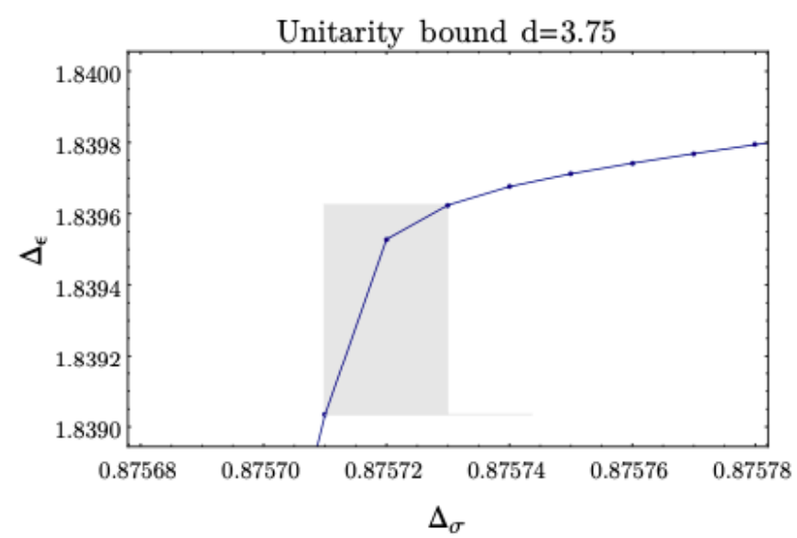


(a)



(a)





“the conformal states rearrange themselves around d=2.2”

ϵ - expansion

- Slightly below 4d
  weakly coupled Wilson-Fisher fixed point

$$G^{\text{Free}}_{\sigma\sigma\sigma\sigma}(u,v) = 1 + u^{\Delta_\sigma} + \frac{u^{\Delta_\sigma}}{v^{\Delta_\sigma}}$$

$$\langle \phi(x_1) \cdots \phi(x_4) \rangle = \frac{g(u,v)}{|x_{12}|^{2\Delta_\phi}\, |x_{34}|^{2\Delta_\phi}}$$

$$u = \frac{|x_{12}|^2\, |x_{34}|^2}{|x_{13}|^2\, |x_{24}|^2} \qquad v = \frac{|x_{14}|^2 |x_{23}|^2}{|x_{13}|^2 |x_{24}|^2}$$

- 2d Ising model is exactly solvable

see Vichi's note

$$G^{(d=2)}_{\sigma\sigma\sigma\sigma}(u,v) = \frac{(1 + u^{1/2} + v^{1/2})^{1/2}}{\sqrt{2}\, v^{1/8}}$$

It is tempting to deform the strongly coupled solution to $2+\epsilon$ d
using the analytic conformal bootstrap
(Weakly coupled in fermionic representation, but nonlocal?)

ϵ - expansion

- Consider $\epsilon = d - 2$ is small

- To first order

$$G^{(d=2+\epsilon)}_{\sigma\sigma\sigma\sigma}(u,v) = G^{(0)}_{\sigma\sigma\sigma\sigma}(u,v) + \epsilon\, G^{(1)}_{\sigma\sigma\sigma\sigma}(u,v) + \mathcal{O}(\epsilon^2)$$

- Assume
  1. conformal symmetry
  2. leading corrections are linear in ϵ

scaling dimension $$\Delta_i = \Delta_i^{(0)} + \epsilon\, \Delta_i^{(1)} + O(\epsilon^2)$$

OPE coefficient $$\lambda_i = \lambda_i^{(0)} + \epsilon\, \lambda_i^{(1)} + O(\epsilon^2)$$

Conformal bootstrap

- Crossing equation

$$\sum_{\Delta,\ell} \lambda^2_{\Delta,\ell} \left(u^{-\Delta\phi} g_{\Delta,\ell}(u,v) - v^{-\Delta\phi} g_{\Delta,\ell}(v,u) \right) = 0$$

$$F_{\Delta,\ell}(u,v)$$

see Vichi's note

- To first order,

$$\begin{aligned} 0 &= \sum \lambda_i^2 F_{\Delta_i,\ell_i} \\ &= \epsilon \sum \left(2\lambda_i^{(0)}\lambda_i^{(1)} + (\lambda_i^{(0)})^2 \Delta_i^{(1)} \partial_\Delta\right) F_{\Delta,\ell}\Big|_{d\to 2, \Delta\to\Delta_i^{(0)}} \\ &\quad + \epsilon \sum (\lambda_i^{(0)})^2 \left(\Delta_\sigma^{(1)} \partial_{\Delta_\sigma} + \partial_d\right) F_{\Delta,\ell}\Big|_{d\to 2, \Delta\to\Delta_i^{(0)}} \\ &\quad + \mathcal{O}(\epsilon^2) \end{aligned}$$

sum of 2d blocks

fixed by 2d data

the intermediate states are the same as 2d

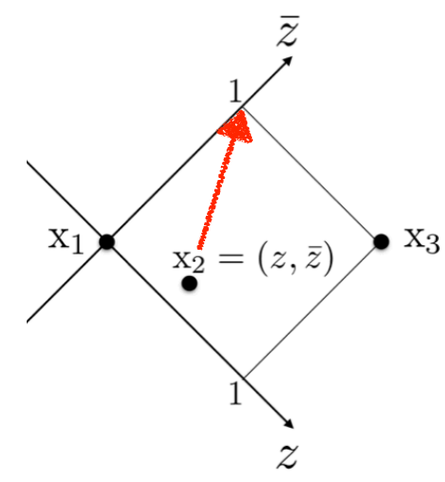
Lightcone bootstrap

- In small u,v expansion, the crossing solution is symmetric in u,v (convergent series?)

$$v^{\Delta_\sigma}\, G(u,v) = \sum_{m,n} c_{m,n}\, u^{\tau_m/2} v^{\tau_n/2}\,, \quad c_{m,n} = c_{n,m}$$

where $\tau = \Delta - \ell$ is twist

Alday, Zhiboedov '15



- 2d Ising $$G^{(d=2)}_{\sigma\sigma\sigma\sigma}(u,v) = \frac{(1+u^{1/2}+v^{1/2})^{1/2}}{\sqrt{2}\, v^{1/8}}$$

$$\sigma \times \sigma = I + \epsilon \qquad \tau_I = 4n\,, \quad \tau_\epsilon = 4n+1$$

$$u = z\bar{z}$$
$$v = (1-z)(1-\bar{z})$$

(sparse, highly degenerate)

See Henriksson's talk for more advanced analytical methods

Lightcone bootstrap

- In s-channel OPE, the power laws in v are associated with twist accumulation points with large-spin asymptotics

$$u^{\tau/2}\, v^{p} \quad \Longleftrightarrow \quad \lambda_i^2 \sim \ell^{-2p-1} \frac{\Gamma(\ell)^2}{\Gamma(2\ell)} + \dots$$

- In 3d, the vacuum contribution is dual to $u^{\Delta_\sigma} v^{0-\Delta_\sigma}$
  At large spin, mean fields $\tau_{[\sigma\sigma]_n} \sim 2\Delta_\sigma + 2n$ dominate

vacuum state $\Longleftrightarrow$ double-twist trajectories

- But 2d has no twist gap.
  LHS involves infinitely many states: I, T, T^2, … (self-dual)

Lightcone bootstrap

$$u = z\bar{z}$$
$$v = (1-z)(1-\bar{z})$$

- ϵ-expansion + lightcone expansion

$$\epsilon \sum \left(2\lambda_i^{(0)}\lambda_i^{(1)} + (\lambda_i^{(0)})^2\, \Delta_i^{(1)}\partial_\Delta\right) F_{\Delta,\ell}\Big|_{d\to 2,\Delta\to\Delta_i^{(0)}}$$

sum of 2d blocks

$$= -\epsilon \sum (\lambda_i^{(0)})^2\, (\Delta_\sigma^{(1)}\partial_{\Delta_\sigma} + \partial_d) F_{\Delta,\ell}\Big|_{d\to 2,\Delta\to\Delta_i^{(0)}}$$

fixed by 2d data

- 2d global conformal blocks also factorize

$$g_{\Delta,\ell}(z,\bar{z}) = K_{\Delta+\ell}(z)\, K_{\Delta-\ell}(\bar{z}) + (z \leftrightarrow \bar{z})$$

from Vichi's note

$$K_\beta(x) = x^{\beta/2}\, {}_2F_1(\beta/2, \beta/2, \beta; x)$$

small $z, 1-\bar{z}$

1st line $$\epsilon \sum_\tau u^{-\Delta_\sigma^{(0)}} (f_{1,\tau}(1-\bar{z}) + f_{2,\tau}(1-\bar{z})\partial_\tau) k_\tau(z) - (z \leftrightarrow 1-\bar{z})$$

Crossing involves $\log(z),\ \log(1-\bar{z})$

Lightcone bootstrap

- $\sum (\lambda_i^{(0)})^2 \, \partial_d F_{\Delta,\ell}\Big|_{d\to 2, \Delta\to\Delta_i^{(0)}}$ is more technical

1. derive the large spin expansion of 2d OPE coefficients

2. use general d conformal blocks
   to derive the lightcone expansion of resummed results

3. take the d derivative and set d to 2

- 2012.09710

$$g_{\Delta,\ell}^{(d,a,b)}(z,\bar{z}) \sim \sum_{k,p,q} \frac{D_k^{p,q}}{J^{2p}} \left(\frac{z}{1-z}\right)^{\frac{d-2}{2}+p} k_{\gamma+2q}(z)\, k_\beta(\bar{z})$$

$$\gamma = \tau - d + 2$$

$$\beta = \tau + 2\ell$$

$$J^2 = \beta(\beta-2)/4$$

$$\sum_\ell \lambda_{\Delta,\ell}^2 \, g_{\Delta,\ell}(z,\bar{z}) \sim \left(\frac{z}{1-z}\right)^{\frac{d-2}{2}} k_\gamma(z)\,(1-\bar{z})^p$$

$$+ \left(\frac{z}{1-z}\right)^{\frac{d-2}{2}+1} \frac{\left(D_0^{1,0} + D_1^{1,0}\right) k_\gamma(z) \; + D_0^{1,1} k_{\gamma+2}(z)}{(p+1)(p+1-a+b)} (1-\bar{z})^{p+1} + \ldots$$

Results

- NO solution beyond 1st order in $z, 1-\bar{z}$
- d=2 is a singularity !
- At least one assumption is wrong

Relax the assumptions

- 1. Conformal symmetry

  scale vs conformal invariance ?

  strong evidence for conformal invariance in d = 2, 3, 4-ϵ

- 2. CFT data corrections are linear in ϵ

  Otherwise, the leading corrections should be ϵ^a with $a < 1$

  To match the d-derivative terms, a=1/k, k=2,3,4,5,…

- Other branch cut possibility $(-\epsilon)^a$

Relax the assumptions

- The simplest possibility is $\epsilon^{1/2}$
- Infinite higher-spin towers to generate lightcone singularities
- a) deform the 2d data $\tau_I = 4n\,, \quad \tau_\epsilon = 4n+1$

$$\Delta_i = \Delta_i^{(0)} + \epsilon^{1/2}\,\Delta_i^{(1)} + \ldots \qquad \lambda_i = \lambda_i^{(0)} + \epsilon^{1/2}\,\lambda_i^{(1)} + \ldots$$

b) add new states/trajectories to reduce twist spacing

$$\lambda_i = 0 + \epsilon^{1/2}\,\lambda_i^{(1)} + \ldots$$

$$\tau_{I'} = 4n+2\,, \quad \tau_{\epsilon'} = 4n+3$$

2d twist spacing

- Global conformal blocks are special in 2d

$$g_{\Delta,\ell}(z,\bar{z}) = K_{\Delta+\ell}(z)\,K_{\Delta-\ell}(\bar{z}) + (z \leftrightarrow \bar{z})$$

invariant under the "spin shadow" transform

$$\ell \leftrightarrow 2-d-\ell$$

- So the 2d data should be invariant under

twist $\Delta - \ell \leftrightarrow \Delta + \ell$ **conformal spin**

$$\tau \leftrightarrow \tau + 2\ell$$

identical correlator: cs spacing is 4, so twist spacing is also 4
(decompose Virasoro blocks into global blocks)

$$\sigma \times \sigma = I + \epsilon \qquad \tau_I = 4n\,, \quad \tau_\epsilon = 4n+1$$

Do some quasi-primary states decouple from $\langle\sigma\sigma\sigma\sigma\rangle$?

Count quasi-primary

- Based on (4.9) and (4.10) in Cappelli, Maffi, Okuda ’18

```
{{Δ → 0, l[0]}, {Δ → 1, l[0]}, {Δ → 2, l[2]}, {Δ → 3, 0}, {Δ → 4, l[0] + l[4]},
 {Δ → 5, l[4]}, {Δ → 6, l[2] + l[6]}, {Δ → 7, l[6]}, {Δ → 8, l[0] + l[4] + l[7] + 2 l[8]},
 {Δ → 9, l[0] + l[8]}, {Δ → 10, l[2] + 2 l[6] + l[9] + 2 l[10]}, {Δ → 11, l[2] + 2 l[10] + l[11]},
 {Δ → 12, l[0] + l[3] + 2 l[4] + 2 l[8] + l[11] + 3 l[12]},
 {Δ → 13, l[0] + l[4] + l[9] + 3 l[12] + l[13]},
 {Δ → 14, l[1] + 2 l[2] + l[5] + 2 l[6] + 3 l[10] + 2 l[13] + 4 l[14]},
 {Δ → 15, l[0] + l[2] + 2 l[6] + l[7] + l[11] + 3 l[14] + 2 l[15]},
 {Δ → 16, 4 l[0] + l[1] + l[3] + 2 l[4] + l[7] + 3 l[8] + 4 l[12] + 3 l[15] + 5 l[16]},
 {Δ → 17, l[0] + l[2] + 2 l[4] + l[5] + 3 l[8] + l[9] + 2 l[13] + 5 l[16] + 3 l[17]},
 {Δ → 18, l[1] + 4 l[2] + 2 l[3] + l[5] + 3 l[6] + 2 l[9] + 4 l[10] + 5 l[14] + 4 l[17] + 7 l[18]},
 {Δ → 19, l[0] + 2 l[2] + 2 l[3] + l[4] + 3 l[6] + l[7] + 3 l[10] + 2 l[11] + 3 l[15] + 6 l[18] + 4 l[19]},
 {Δ → 20, 4 l[0] + 2 l[1] + l[3] + 6 l[4] + 3 l[5] + 2 l[7] + 4 l[8] + 3 l[11] + 5 l[12] +
   7 l[16] + 6 l[19] + 9 l[20]}}
```

multiplicities l[spin]

$$\mathcal{N}_m(q,\bar{q}) = |\widehat{\chi}_{11}(q)|^2 + |q^{h_{13}}\widehat{\chi}_{13}(q)|^2$$
$$= \sum_{\Delta=0}^{\infty} \sum_{|\ell|\leq\Delta} d^{QP}(\Delta,\ell)\, x^{\Delta}\, y^{\ell}, \qquad q = xy, \quad \bar{q} = xy^{-1}.$$

m=3

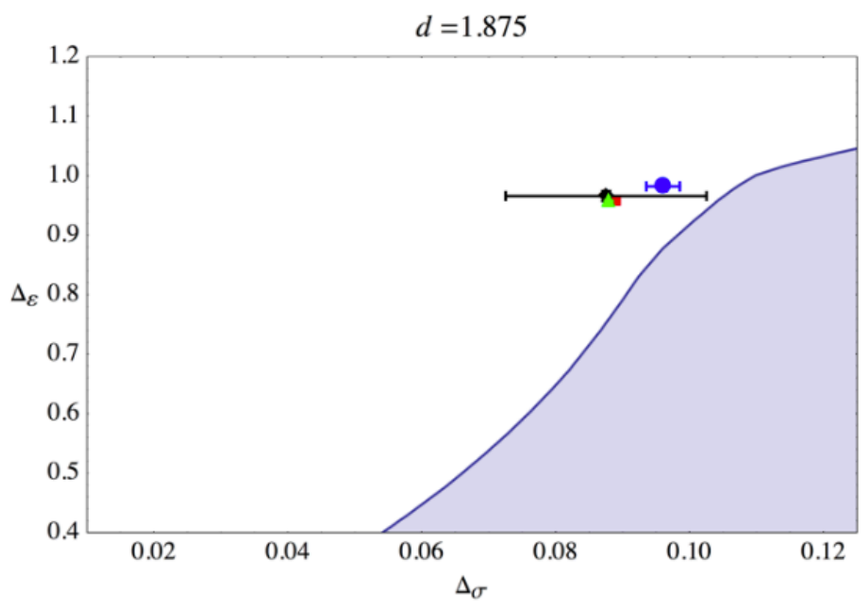
Count quasi-primary

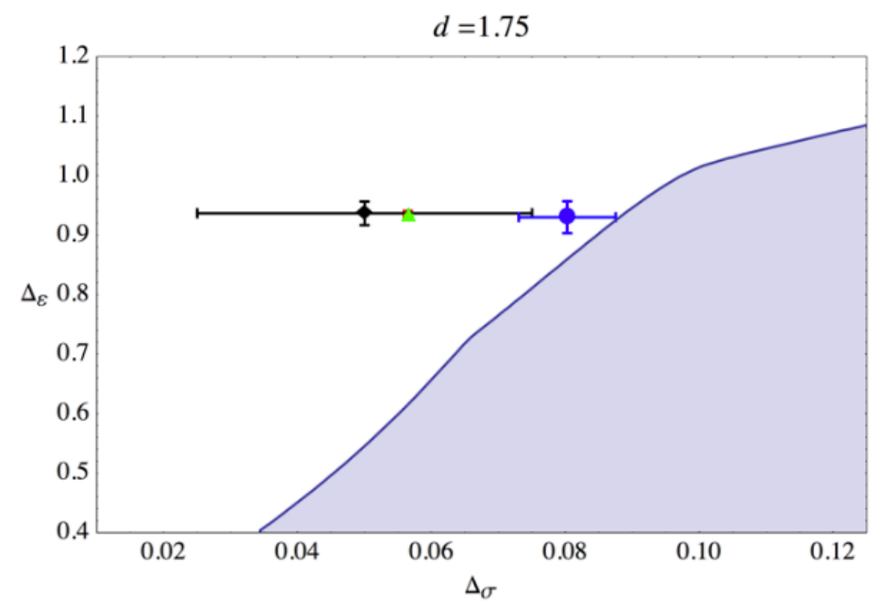
```
{Δ → 21, 4 + 2 l[1] + l[2] + 3 l[4] + 2 l[5] + 2 l[6] + 3 l[8] + 2 l[9] + 5 l[12] +
  3 l[13] + 4 l[17] + 8 l[20] + 6 l[21]},
{Δ → 22, l[0] + 2 l[1] + 6 l[2] + 3 l[3] + 2 l[5] + 8 l[6] + 3 l[7] + 3 l[9] +
  5 l[10] + 4 l[13] + 7 l[14] + 9 l[18] + 8 l[21] + 11 l[22]},
{Δ → 23, l[0] + 3 l[1] + 6 l[2] + 2 l[3] + 2 l[4] + 3 l[6] + 4 l[7] + 3 l[8] +
  5 l[10] + 3 l[11] + 6 l[14] + 4 l[15] + 6 l[19] + 11 l[22] + 9 l[23]},
{Δ → 24, 9 l[0] + 3 l[1] + l[2] + 4 l[3] + 8 l[4] + 3 l[5] + 3 l[7] + 10 l[8] +
  5 l[9] + 4 l[11] + 7 l[12] + 6 l[15] + 9 l[16] + 11 l[20] + 10 l[23] + 15 l[24]},
{Δ → 25, 9 l[0] + 3 l[1] + 2 l[2] + 4 l[3] + 6 l[4] + 4 l[5] + 3 l[6] + 5 l[8] +
  6 l[9] + 4 l[10] + 6 l[12] + 4 l[13] + 8 l[16] + 6 l[17] + 9 l[21] + 15 l[24] + 11 l[25]}
{Δ → 26, l[0] + 6 l[1] + 12 l[2] + 3 l[3] + 2 l[4] + 6 l[5] + 10 l[6] + 5 l[7] +
  4 l[9] + 14 l[10] + 6 l[11] + 6 l[13] + 9 l[14] + 8 l[17] + 11 l[18] + 15 l[22] +
  14 l[25] + 19 l[26]},
```

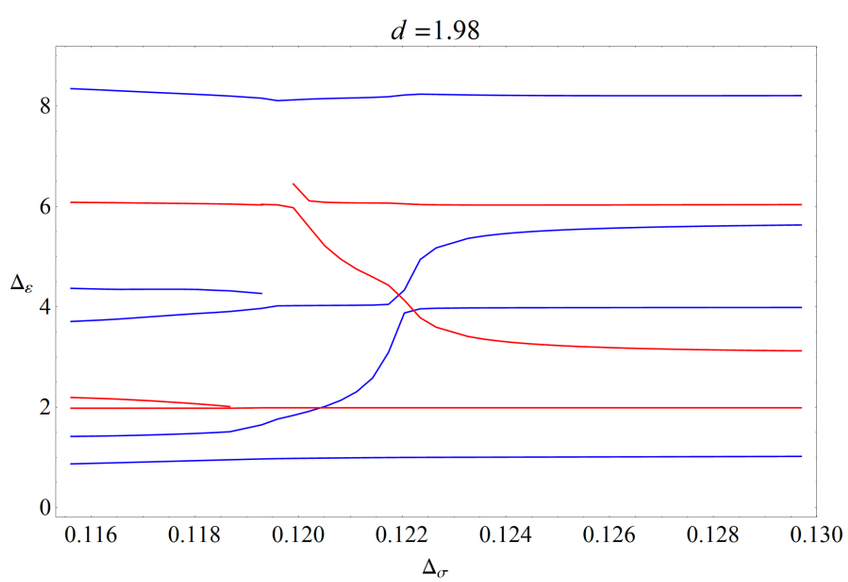
1<d<2

- Double kinks

Golden, Paulos ’15







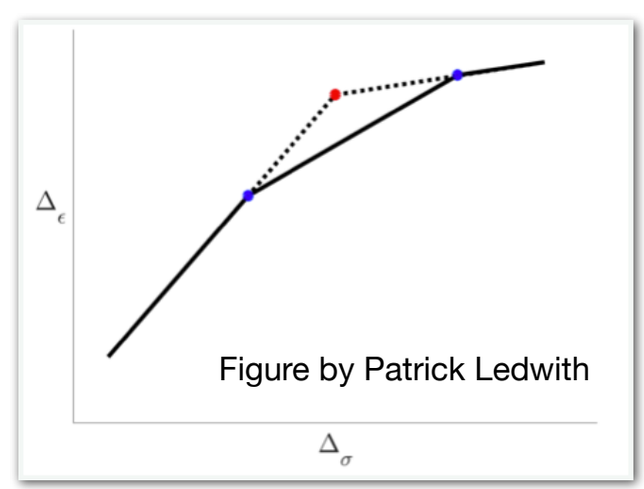


Figure by Patrick Ledwith

More constraints

- Introduce new intermediate trajectories with $\tau = 4n + 2\,, 4n + 3$

  $\lambda_i = 0 + \epsilon^{1/2}\,\lambda_i^{(1)} + \ldots$

  Then there are many solutions.

- s-t crossing is just the 1st constraint

- s-u crossing
  can't be studied using the above lightcone expansion (Mellin space?)

- Ising decoupling
  In 2d, [twist-1, spin-2] and [twist-5, all spin] decouple at the Ising point

  Do they persist in $2 + \epsilon$ dimensions ?
  [twist-1, spin-2] seems to persist, [twist-5, spin=0] partly does.
  (Cappelli, Maffi, Okuda '18)

- Consistency in Lorentzian signature, such as the Regge limit

Open questions

- If 2d data is deformed by fractional power of ϵ, do we still need new states?
- Numerically solving the crossing equation based on a tentative spectrum with new states?
- More no-go results due to 2d peculiarity? Other models?
- Other analytic methods?
  Analytic functional, dispersive sum rules, Tauberian theorems, ...

Thank you!