Random Field Ising model and dimensional reduction

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The random–field Ising model (RFIM)

Generalization of the standard ferromagnetic Ising model, $J > 0$ and $S_x \pm 1$:

$$\mathcal{H}^{(\mathrm{RFIM})} = -J \sum_{\langle x,y \rangle} S_x S_y - \sum_x h_x S_x$$

with $\{h_x\}$ a random variable.

Equivalent to the experimentally relevant Diluted AntiFerromagnetic model in a Field (Fishman and Aharony, 1979).

$$\mathcal{H}^{(\mathrm{DAFF})} = -J \sum_{\langle x,y \rangle} S_x S_y \eta_x \eta_y - H_0 \sum_x (-1)^x S_x$$

with η_x a dilution variable and H_0 an external homogeneous field.

The random–field Ising model (RFIM)

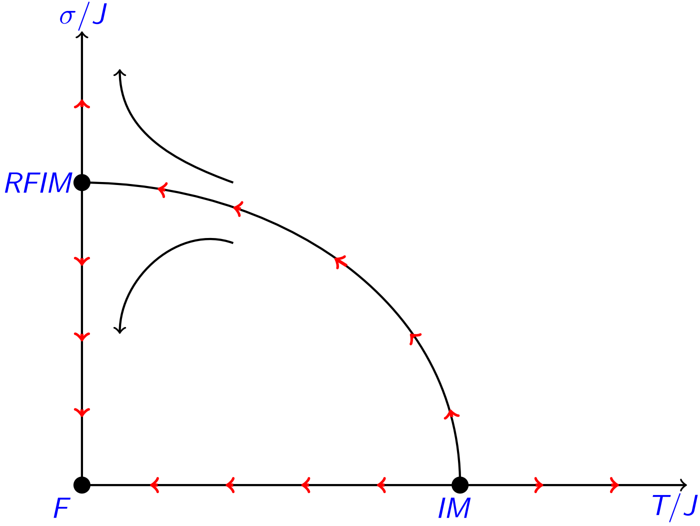
$$\mathcal{H}^{(\mathrm{RFIM})} = -J \sum_{\langle x,y \rangle} S_x S_y - \sum_x h_x S_x$$

- $\{h_x\}$ independent random magnetic fields with zero mean and dispersion σ.
- ferromagnetic/paramagnetic transition from small σ to large σ.
- Ferromagnetic state is stable only for $D > 2$ (Imry & Ma, 1975).

Relevant dimensions : $3 \leq D \leq 6$

with $D = 6$ the upper critical dimension for RFIM.

RG fixed point & phase diagram



Mean Field for the RFIM

- MF Hamiltonian, after averaging the random fields in a replicated system :

$$\mathcal{H}^{MF} = \int d^D r \left[\sum_a \left((\nabla S_a(r))^2 + t S_a^2(r) + \lambda S_a^4(r) \right) - \sigma \sum_{a,b} S_a(r) S_b(r) \right]$$

- Propagator : $(k^2 \delta_{a,b} - \sigma M_{a,b})^{-1} \to \frac{\delta_{a,b}}{k^2} - \frac{\sigma M_{a,b}}{k^2(k^2 - n\sigma)}$
- Then, two propagators :
  - $G_{xy}^{(\mathrm{dis})} = \overline{\langle S_x S_y \rangle}$ and $\simeq 1/k^4$.
  - $G_{xy}^{(\mathrm{con})} = \overline{\langle S_x S_y \rangle} - \overline{\langle S_x \rangle \langle S_y \rangle}$ and $\simeq 1/k^2$.
- Bellow the upper critical dimension, each propagator will have an anomalous dimension.

Mean Field for the RFIM

- The IM bellow the upper critical dimension is characterized by two quantities, ν and the anomalous dimension η of the (single) propagator.
- The RFIM bellow the upper critical dimension is characterized by three quantities, ν and the anomalous dimensions η and $\bar{\eta}$ of the two propagators.
- Dimensional reduction : $\epsilon = 6 - D$ Perturbative computation gives, for all critical exponents and at each order

$$\alpha^{RFIM,D} = \alpha^{IM,D-2} \tag{1}$$

  (Aharony, Imry, and Ma, 1976 and Young, 1977)
- Then $\eta = \bar{\eta}$

Dimensional reduction versus sharp reality

- The dimensional reduction is explained by a hidden supersymmetry in the Random Field Ising model (Parisi & Sourlas, 1979)
  Supersymmetry $\rightarrow$ Dimensional reduction

- Failure: The 3D RFIM orders while the 1D Ising model (IM) does not!

- Then $\eta \neq \bar{\eta} \rightarrow$ 3 independent critical exponents !!!

- 2 or 3 independent exponents ? $\bar{\eta} = 2\eta$ (Schwartz et al., 1985)

Dimensional reduction versus sharp reality

- Many reason have been put forward to explain breaking of dimensional reduction :
  - Non perturbative effect due to bound states in replica theory
  - the breakdown of perturbation theory is due to a large number local minimum in the energy landscape.
  - Existences of large scale excitations
- different scenarios are possible :
  1. Nonperturbative effects could destroy supersymmetry at a finite order in the ϵ expansion or, even worse, at $D = 6$.
  2. Violations of supersymmetry might be exponentially small $\sim \exp(-A/\epsilon)$.
  3. Supersymmetry has been suggested to be exact but only for $D > D_{\mathrm{int}} \approx 5.1$ (Tarjus et al.). For $D < D_{\mathrm{int}}$ the supersymmetric fixed point becomes unstable with respect to non-supersymmetric perturbations.

$D_c \simeq 5$ also appeared in recent works by S. Hikami (2018), Kaviraj, Rychkov and Trevisani (2020)

Recent numerical works

Large scale simulations in $D = 3, 4$ and 5 with the goal of :

1. Examine universality in terms of different distributions of the random fields $\{h_x\}$.
2. Check the puzzle with the number of independent exponents.
3. Revisit dimensional reduction $\mathrm{RFIM}^{(\mathrm{D})} \to \mathrm{IM}^{(\mathrm{D}-2)}$ at higher dimensions to check the above mentioned scenarios.

Simulation

- **Optimization methods**: Graph theoretical algorithms that calculate exact ground states of the model in polynomial time, avoiding equilibration problems & simulating much larger system sizes: $L_{\rm max}^{D} = \{192^3, 60^4, 28^5\}$.

- We consider a D dimensional hyper-cubic lattice with periodic boundary conditions and energy units $J = 1$.

- h_x are independent quenched random fields with a distribution $\mathcal{P}(h, \sigma)$. We considered the following distributions, with σ as the single parameter :

  1. Gaussian distribution : $\mathcal{P}^{(\rm G)}(h_x, \sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(h_x)^2}{2\sigma^2}}$

  2. Poissonian distribution : $\mathcal{P}^{(\rm P)}(h_x, \sigma) = \frac{1}{2|\sigma|} e^{-|h_x|/\sigma}$

  (test of universality !!!!)

- Extensive averaging over **10 million samples**.

Observables

- Binder cumulant: $m = \frac{1}{L^D} \sum_x S_x \rightarrow U_4 = \frac{\overline{\langle m^4 \rangle}}{\overline{\langle m^2 \rangle}^2}$
- Disconnected and connected correlation lenght :

$$G^{(\mathrm{dis})}_{xy} = \overline{\langle S_x S_y \rangle} \sim \frac{1}{r^{D-4+\bar{\eta}}} \; ; \; G^{(\mathrm{con})}_{xy} = \overline{\frac{\partial \langle S_x \rangle}{\partial h_y}} \sim \frac{1}{r^{D-2+\eta}}$$

$$\xi^{\#} = \frac{1}{2\sin(\pi/L)} \sqrt{\frac{\chi^{\#}_{(0,\cdots)}}{\chi^{\#}_{(2\pi/L,0,\cdots)}} - 1} \, . \qquad (2)$$

with $\chi^{\#}_{\vec{k}}$ the Fourrier transform of $G^{\#}_{xy}$

- Dimensionless quantities : $U_4(L,\sigma); \xi^{(\mathrm{dis})}(L,\sigma)/L$ and $\xi^{(\mathrm{con})}(L,\sigma)/L$.
- For a dimensionless quantity, we have, close to a critical point

$$g(L,\sigma) = F_g(L^{1/\nu}(\sigma - \sigma_c)) + \mathcal{O}(L^{-\omega}) \cdots \qquad (3)$$

with $F_g(x)$ some universal function and ω the leading irrelevant correction.

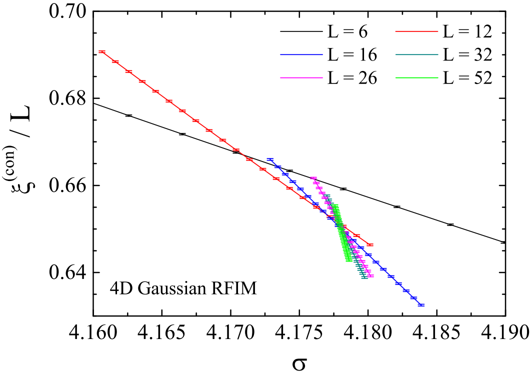
Finite–size scaling using quotients

- We solve numerically $g(L, \sigma_c(L)) = g(2L, \sigma_c(L))$.
- At the lowest order $\sigma_c(L) = \sigma_c + \alpha L^{-\omega-1/\nu} \to \omega + 1/\nu$
- We measure at the points $\sigma_c(L)$.

$$g(L, \sigma_c(L)) = F_g(L^{1/\nu} \alpha L^{-\omega-1/\nu}) + \mathcal{O}(L^{-\omega}) \cdots = g(\sigma_c) + \beta L^{-\omega} + \cdots \to \omega$$

- We fit **simultaneously** several data sets: 2 field distributions and up to 3 crossing points: $\mathrm{Z}^{(\mathrm{x})}$, where Z = G, or P and x = (con), (dis), or U_4.
- Estimation of ω using joint fits for several magnitudes.
- Individual extrapolation of all other observables fixing ω.

Finite–size scaling using quotients

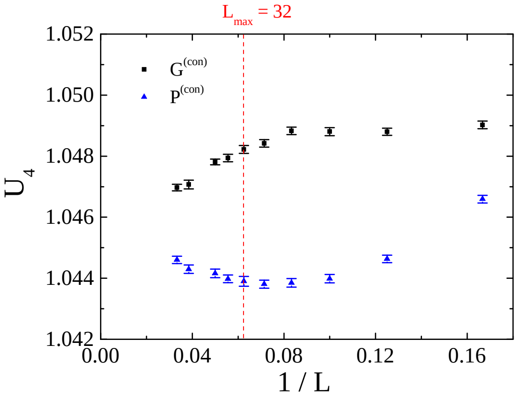


with $\sigma_c(6) = 4.17091(22)$; $\sigma_c(16) = 4.17813(7)$; $\sigma_c(26) = 4.17790(5)$.

Not monotonic !!!!

Non-monotonic behavior (4D RFIM)

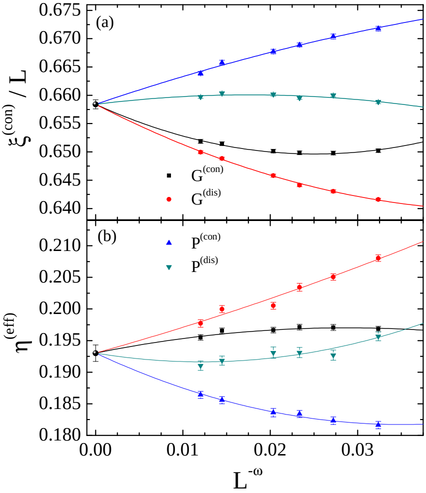
Possible explanation of previously reported universality violations



Higher-order corrections are necessary: $g_L = g^* + a_1 L^{-\omega} + a_2 L^{-2\omega} + \cdots$

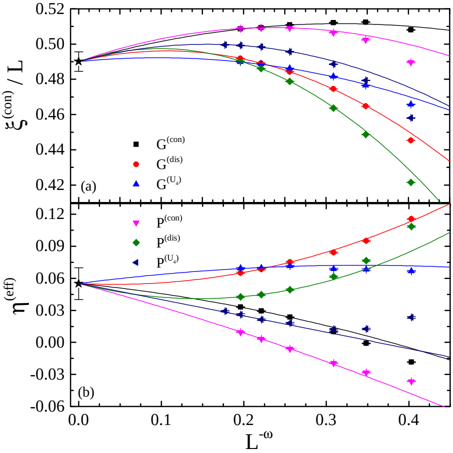
Universality in the 4D RFIM

$\omega = 1.30(9)$
$\xi^{(\mathrm{con})}/L = 0.6584(8)$
$\eta = 0.1930(13) \neq 0.25 = \eta^{(\mathrm{2D\ IM})}$



Universality in the 5D RFIM

$\omega = 0.66(15) \sim 0.82966(9) = \omega^{(\text{3D IM})}$
$\xi^{(\text{con})}/L = 0.4901(55)$
$\eta = 0.055(15) \sim 0.036298(2) = \eta^{(\text{3D IM})}$



A summary of results for the RFIM at $3 \leq D < 6$

	3D RFIM	4D RFIM	5D RFIM	2D IM	3D IM	MF
ν	1.38(10)	0.8718(58)	0.626(15)	1	0.629971 (4)	1/2
η	0.5153(9)	0.1930(13)	0.055(15)	0.25	0.036298(2)	0
$\bar{\eta}$	1.028(2)	0.3538(35)	0.052(30)	0.25	0.036298(2)	0
$\Delta_{\eta,\bar{\eta}} = 2\eta - \bar{\eta}$	0.0026(9)	0.0322(23)	0.058(7)	0.25	0.036298(2)	0
β	0.019(4)	0.154(2)	0.329(12)	0.125	0.326419(3)	1/2
γ	2.05(15)	1.575(11)	1.217(31)	1.875	1.237075(10)	1
θ	1.487(1)	1.839(3)	2.00(2)	2	2	2
α	-0.16(35)	0.12(1)	-	-	-	-
α (from hyperscaling)	-0.09(15)	0.12(1)	0.12(5)	0	0.110087 (12)	0
$\alpha + 2\beta + \gamma$	2.00(31)	2.00(3)	2.00(11)	2	2.000000 (28)	2
$\sigma_c(G)$	2.27205(18)	4.17749(6)	6.02395(7)	-	-	-
$\sigma_c(P)$	1.7583(2)	3.62052(11)	5.59038(16)	-	-	-
U_4	1.0011(18)	1.04471(46)	1.103(16)			
$\xi^{(\mathrm{con})}/L$	1.90(12)	0.6584(8)	0.4901(55)			
$\xi^{(\mathrm{dis})}/L$	8.4(8)	2.4276(70)	1.787(8)			
ω	0.52(11)	1.30 (9)	0.66(+15/-13)		0.82966(9)	0

Within our numerical resolution: 5D RFIM $\rightarrow$ 3D IM

Supersymmetry ?

N.G. Fytas, V. Martín-Mayor, G. Parisi, M. Picco, and N. Sourlas, PRL **122**, 240603 (2019).

- So far, we have checked about dimensional reduction which seems to exists between $D = 5$ RFIM and $D = 3$ IM.
- What about supersymmetry predicted by Parisi and Sourlas (1979) ? Remember that dimensional reduction is a consequence of supersymmetry, not the other way around !!!
- We consider measurements in $5D$ with the geometry :

$$L_x = L_y = L_z = L \; ; \; L_t = L_u = RL \; ; \; R \geq 1 \tag{4}$$

and look for the limit $R \to \infty$

- The correction limit is to take $R \to \infty$ **before** the thermodynamic limit, $L \to \infty$.

$$O(D, 2) \to O(2, 2).$$

Supersymmetry ?

- We consider the disconnected correlation function $G^{(\mathrm{dis})}_{(x_1,u);(x_2,u)} = \overline{\langle S_{x_1,u} S_{x_2,u} \rangle}$, with x_1 or x_2 the 3 dimensional part and u the 2 dimensional part.
- Supersymmetry prediction

$$G^{(\mathrm{dis})}_{(x_1,u);(x_2,u)} = \mathcal{Z}\, G^{\mathrm{3d\ Ising}}_{x_1;x_2} \tag{5}$$

with $\mathcal{Z}$ a position-independent normalization constant.

- In practice, we first define a Fourier transform as :

$$\chi^{(\mathrm{dis})}_k = \frac{1}{L^{D-2}} \sum_{x_1,x_2} \mathrm{e}^{\mathrm{i}k\cdot(x_1-x_2)}\, \overline{G^{(\mathrm{dis})}_{(x_1,u);(x_2,u)}} \tag{6}$$

Note that the average over the disorder corresponds to an average over u.

Supersymmetry ?

- Compute a correlation length ($\mathcal{Z}$ disappeared !!!)

$$\xi^{(\mathrm{dis})} = \frac{1}{2\sin(\pi/L)}\sqrt{\frac{\chi^{(\mathrm{dis})}_{(0,0,0)}}{\chi^{(\mathrm{dis})}_{(2\pi/L,0,0)}} - 1}\,. \qquad (7)$$

- Similar argument also for the Binder ratio :

$$U_4(L) = \frac{\overline{\langle m_u^4 \rangle}}{\overline{\langle m_u^2 \rangle}^2}\,. \qquad (8)$$

Again, the average over the disorder corresponds to an average over u.

Supersymmetry ?

- We can also make a direct check of the supersymmetry. It predicts the following Ward identity

$$G_r^{(\mathrm{con})} = -\mathcal{Z}_2 \frac{\mathrm{d}}{\mathrm{d}r^2} G_r^{(\mathrm{dis})} \,, \qquad (9)$$

which relates the connected and disconnected correlation functions, with $r^2 = (u_1 - u_2)^2$.

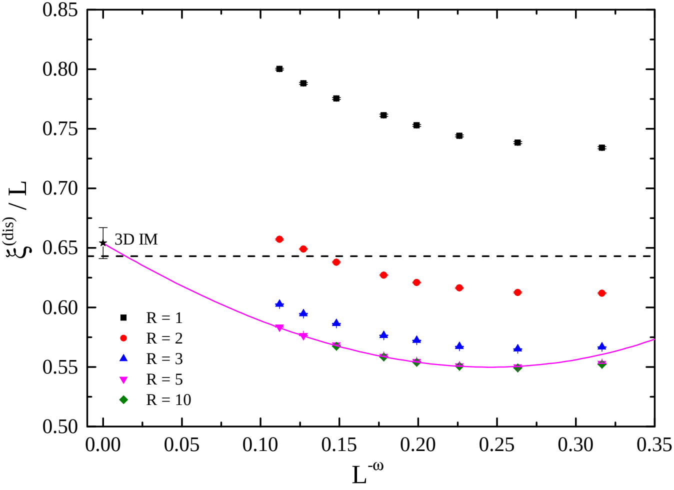
- As simple integration gives

$$\int_0^\infty \mathrm{d}\rho^2 \, G^{(\mathrm{con})}_{\mathbf{x}_1,0,0;\mathbf{x}_2,\rho,0} \simeq \mathcal{Z}_2 \, G^{(\mathrm{dis})}_{\mathbf{x}_1,0,0;\mathbf{x}_2,0,0} \qquad (10)$$

- We can then compare the correlation length obtained from the integrated connected correlation function with the correlation length from the disconnected correlation function (which is related to the $D-2$ correlation length !).

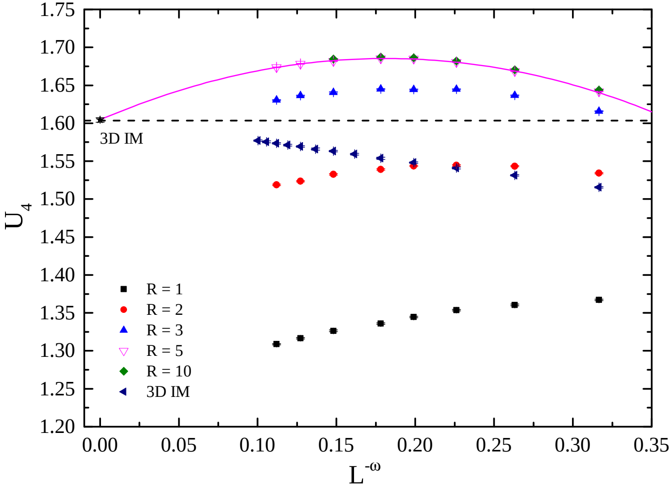
Check of Supersymmetry (1)

$\xi^{(\mathrm{dis})}(L,R)/L$ vs. $L^{-\omega}$ for various R values, as computed in the $D=5$ RFIM with $3D$ IM ω.



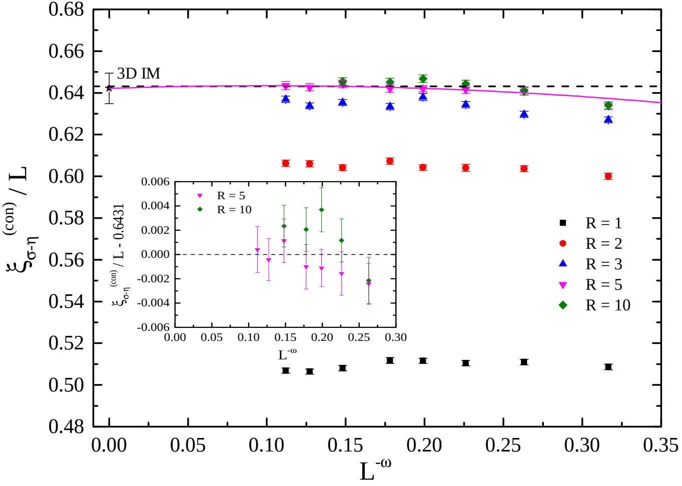
Check of Supersymmetry (2)

$U_4(L, R)$ vs. $L^{-\omega}$ for various R values, as computed in the $D = 5$ RFIM.



Check of Supersymmetry (3)

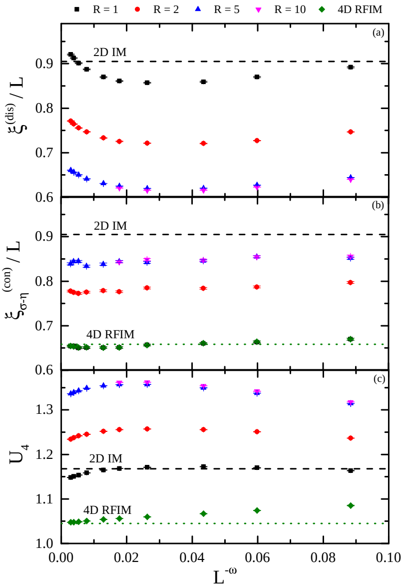
$\xi^{(\mathrm{con})}_{\sigma-\eta}(L,R)/L$ vs. $L^{-\omega}$ for various R values, as computed in the $D=5$ RFIM.
$\xi^{(\mathrm{con})}_{\sigma-\eta}(L,R)/L \simeq$ connected correlation length.



Check of Supersymmetry (4)

$4D$ RFIM $\rightarrow$ $2D$ IM ?

NO !!



Conclusions

- Universality in the RFIM in finite D.
- High-accuracy estimates for various universal ratios and the whole set of critical exponents and all relevant dimensions $D = \{4, 5\}$ with 3 independent exponents for $D = 4$.
- Our estimates for the critical exponents indicate that dimensional reduction seems to be at play at, or close to, $D = 5$.
- The checked predictions of supersymmetry are satisfied between the $D = 5$ RFIM and the $D = 3$ Ising model with a good precision.