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Associative Algebraic Approach to Logarithmic CFT in the Bulk: The Continuum Limit of the $\mathfrak{gl}(1|1)$ Periodic Spin Chain, Howe Duality and the Interchiral Algebra

A. M. Gainutdinov[1,2]**, N. Read**[3]**, H. Saleur**[4,5]

[1] DESY, Theory Group, Notkestrasse 85, Bldg. 2a, 22603 Hamburg, Germany.
   E-mail: azat.gainutdinov@uni-hamburg.de
[2] Laboratoire de Mathématiques et Physique Théorique, Université de Tours, Parc de Grandmont, 37200
   Tours, France
[3] Department of Physics, Yale University, P.O. Box 208120, New Haven, CT 06520-8120, USA
[4] Institut de Physique Théorique, CEA Saclay, 91191 Gif Sur Yvette, France
[5] Department of Physics and Astronomy, University of Southern California, Los Angeles, CA 90089, USA

Abstract: We develop in this paper the principles of an associative algebraic approach to *bulk* logarithmic conformal field theories (LCFTs). We concentrate on the closed $\mathfrak{gl}(1|1)$ spin-chain and its continuum limit—the $c = -2$ symplectic fermions theory—and rely on two technical companion papers, Gainutdinov et al. (Nucl Phys B 871:245–288, 2013) and Gainutdinov et al. (Nucl Phys B 871:289–329, 2013). Our main result is that the algebra of local Hamiltonians, the Jones–Temperley–Lieb algebra JTL_N, goes over in the continuum limit to a bigger algebra than $\mathcal{V}$, the product of the left and right Virasoro algebras. This algebra, $\mathcal{S}$—which we call *interchiral*, mixes the left and right moving sectors, and is generated, in the symplectic fermions case, by the additional field $S(z, \bar{z}) \equiv S_{\alpha\beta} \psi^\alpha(z) \bar{\psi}^\beta(\bar{z})$, with a symmetric form $S_{\alpha\beta}$ and conformal weights $(1, 1)$. We discuss in detail how the space of states of the LCFT (technically, a Krein space) decomposes onto representations of this algebra, and how this decomposition is related with properties of the finite spin-chain. We show that there is a complete correspondence between algebraic properties of finite periodic spin chains and the continuum limit. An important technical aspect of our analysis involves the fundamental new observation that the action of JTL_N in the $\mathfrak{gl}(1|1)$ spin chain is in fact isomorphic to an enveloping algebra of a certain Lie algebra, itself a non semi-simple version of $\mathfrak{sp}_{N-2}$. The semi-simple part of JTL_N is represented by $U\mathfrak{sp}_{N-2}$, providing a beautiful example of a classical Howe duality, for which we have a non semi-simple version in the full JTL_N image represented in the spin-chain. On the continuum side, simple modules over $\mathcal{S}$ are identified with "fundamental" representations of $\mathfrak{sp}_\infty$.

1. Introduction

Our understanding of logarithmic conformal field theory (LCFT) has greatly improved recently thanks to a renewed focus on purely algebraic features and in particular a systematic study of Virasoro indecomposable modules.

Most of the developments have concerned boundary LCFTs, where two lines of attack have been pursued. The first one is rather abstract, and follows the pioneering work of Rohsiepe [1] and Gaberdiel [2], where, in particular, fusion is interpreted as the tensor product of the symmetry algebra [3–5]. The other [6,7] is somehow more concrete, in that it relies on lattice regularizations, and exploits the (not entirely understood) similarity [8] between the properties of lattice models and their conformally invariant continuum limits. The models involved in this second approach provide [9] representations of associative algebras such as the Temperley–Lieb algebra. It turns out that the representation theory of these algebras is, in a certain (categorical, see Sect. 2.4) sense [10], similar to the one of chiral algebras in LCFT. The structure of indecomposable modules and fusion rules can then be predicted from the analysis of the scaling limit of the lattice models [6,7,11]. The results are in perfect agreement (so far) with the conclusions of the first approach based on (considerably more involved) calculations in the Virasoro algebra [4,12].

To be now a little more precise, the general philosophy of the lattice approach in the boundary case relies on the analysis of the microscopic model as a bi-module over two algebras. In physical terms, one of these algebras is generated by the local hamiltonian density, and the other is the 'symmetry' commuting with the hamiltonian densities. More precisely, the types of models we are interested in carry alternating fundamental representations of a super Lie algebra, and admit a single invariant nearest neighbor coupling, the 'super equivalent' of the Heisenberg interaction $H_i \approx \vec{S}_i \cdot \vec{S}_{i+1}$. The different H_i's turn out to obey some non-trivial algebraic relations, so that the models provide a representation of a certain abstract algebra A—the ordinary and boundary Temperley–Lieb algebras for instance.

Meanwhile, the local Hamiltonians are invariant under the super Lie algebra. In fact, they typically are invariant under a bigger algebra which is, in technical terms, the centralizer of A. We recall here that the centralizer $\mathfrak{Z}_A$ of an associative algebra A acting on a representation space V is the algebra of all intertwining operators $\mathrm{End}_A(V)$, i.e., $\mathfrak{Z}_A$ is the algebra of the maximum dimension such that $[\mathfrak{Z}_A, A] = 0$. For the open case of finite $\mathfrak{gl}(m|n)$-symmetric spin chains, this centralizer was dubbed $\mathcal{A}_{m|n}$ in [10], where it was shown to be Morita equivalent to $U_{\mathfrak{q}}s\ell(2)$ with $\mathfrak{q} + \mathfrak{q}^{-1} = m - n$, actually to the corresponding finite-dimensional $\mathfrak{q}$-Schur algebra. Usually, the representation theory of this centralizer—typically a quantum group or an algebra Morita equivalent to one—is easier to study than the representation theory of A or its scaling limit, a chiral algebra. In general, all these algebras are not semi-simple, and give rise to complicated indecomposable modules. The point is, in part, that these modules have, even for finite spin chains, properties that are closely related with those of the associated LCFT.

While the key observation in [6,10] about the similarity between the algebraic properties of the lattice models and their logarithmic continuum limit is not entirely understood, it can be then interpreted in some cases at least in terms of quantum groups. Recent studies of centralizers of chiral algebras (Virasoro and W-algebras) in continuum logarithmic models have indeed unraveled a remarkable equivalence [13–15] between the representation theory and the fusion rules of the chiral algebras and certain quantum groups. It is exactly the same quantum groups that appear as centralizers in the lattice models, hence providing the link between finite size and scaling limit properties.

Turning now to bulk LCFTs, progress has been more modest. The problem is that, on the more abstract side, one now expects indecomposability under the left and right actions of the Virasoro algebras, leading to potentially very complicated modules, which have proven too hard to study so far, except in some special cases. These include bulk

logarithmic theories [16,17] with W-algebra symmetries [18,19], and WZW models on supergroups which, albeit very simple as far as LCFTs go, provide interesting lessons on the coupling of left and right sectors [20,21]. On the more concrete side, while it is possible to define and study lattice models whose continuum limit is a (bulk) LCFT, the underlying structures are also very difficult to get: symmetries are smaller, and the lattice algebras have much more complicated representation theory. Nevertheless, it looks possible to generalize the approach in [7,10] thanks to recent results about the centralizer $\mathfrak{Z}_{JTL}$ of the Jones–Temperley–Lieb (JTL) algebra JTL_N acting on periodic super-symmetric spin chains with N tensorands/sites. This paper presents our first results in this direction for the $\mathfrak{gl}(1|1)$ case. It is based on two technical companion papers. In the first one [22], we focussed on the symmetries of the spin chain, that is, the centralizer, in the alternating product of the $\mathfrak{gl}(1|1)$ fundamental representation and its dual, of the JTL algebra. We proved that this centralizer is only a subalgebra of $U_{\mathfrak{q}}s\ell(2)$ at $\mathfrak{q} = i$ that we dubbed $U_{\mathfrak{q}}^{\mathrm{odd}}s\ell(2)$. We then analyzed the continuum limit $N \to \infty$ of the JTL algebra: using general arguments about the regularization of the stress-energy tensor, we identified families of JTL elements going over to the Virasoro generators L_n and $\bar{L}_n$. We also discussed the well known $s\ell(2)$ symmetry of symplectic fermions from the lattice point of view, and showed that this symmetry, albeit present in the continuum limit, does not have a simple, useful analog on the lattice. In our second paper [23], we analyzed the decomposition of the spin chain over the JTL algebra JTL_N, and obtained the full decomposition as a bimodule over $U_{\mathfrak{q}}^{\mathrm{odd}}s\ell(2)$ and JTL_N.

Equipped with these results, we can now explore to what extent the remarkable properties observed in the open case [10] carry over to the bulk case. Our conclusion is that the algebraic properties of the finite periodic spin chain and the bulk LCFT are again very similar. The crucial new ingredient is that the JTL algebra goes over, in the continuum limit, to a bigger operator algebra than $\mathcal{V}$, the product of the left and right Virasoro algebras. This algebra—which we call *interchiral*, mixes the left and right moving sectors, and is generated, in the symplectic fermions case, by the additional field

$$S(z,\bar{z}) \equiv S_{\alpha\beta}\psi^{\alpha}(z)\bar{\psi}^{\beta}(\bar{z}), \quad S_{12} = S_{21} = 1, \quad S_{11} = S_{22} = 0, \quad (1.1)$$

with conformal weights $(1, 1)$. More formally, we construct an inductive system of JTL_N algebras, together with their spin-chain representations, and identify the inductive limit with an infinite-dimensional operator algebra $\mathcal{S}$ generated by the modes of the field $S(z,\bar{z})$. Most of the present work is devoted to explicitly identifying this interchiral algebra $\mathcal{S}$, studying its properties, and using it to provide a new analysis of the symplectic fermions LCFT. We believe that the concept of interchiral algebra will prove fundamental in the analysis of more complicated cases—in particular those at central charge $c = 0$— as will be discussed in forthcoming work.

The paper is organized as follows. We start in Sect. 2 by a reminder of the main algebraic features, both on the lattice and in the scaling limit, for the open $\mathfrak{gl}(1|1)$ spin chain (and its $\mathfrak{gl}(n|n)$ generalizations). In Sect. 3 we similarly remind the reader of the main algebraic properties of the closed $\mathfrak{gl}(1|1)$ spin chain. The latter involves now a 'periodic' version of the Temperley–Lieb algebra which we call, following [10], the Jones–Temperley–Lieb algebra JTL_N, and the symmetry algebra $U_{\mathfrak{q}}^{\mathrm{odd}}s\ell(2)$ which is, up to some trivial elements, its centralizer $\mathfrak{Z}_{JTL}$. The structure of the corresponding bimodule is recalled in Fig. 3. We also mention briefly the antiperiodic spin chain [where the $\mathfrak{gl}(1|1)$ symmetry is broken], where one now deals with a twisted version JTL_N^{tw} of the Jones–Temperley–Lieb algebra, while the centralizer is just $Us\ell(2)$ algebra. Section 4 is the first containing new results. We discuss there how, remarkably, the image of the

JTL_N in the $\mathfrak{gl}(1|1)$ spin chains is in fact isomorphic to (the image of) the enveloping algebra of a certain non-semisimple Lie algebra containing $\mathfrak{sp}_{N-2}$ as its maximal semi-simple subalgebra. The semi-simple part of JTL_N (that is, after quotienting by the radical) is represented by $U\mathfrak{sp}_{N-2}$, providing a beautiful example of a classical Howe duality [24], for which we have a non semi-simple version in the full JTL_N case. The main goal of the rest of this paper is to understand the scaling limit of JTL_N defined as a particular inductive limit $N \to \infty$ of algebras. To do this, we start in Sect. 5 by discussing the scaling limit of the $\mathfrak{gl}(1|1)$ spin chain, in particular the inductive limit of JTL_N as algebras of *bilinears in fermion modes*—Sects. 5.1, 5.2 and, of the centralizer and the bimodule structure in Sect. 5.3. In Sect. 5.4, we describe the Virasoro algebra content of the inductive limits of simple JTL_N-modules that appear in the periodic $\mathfrak{gl}(1|1)$ model. While on the finite lattice with N sites the simples are just fundamental representations of $\mathfrak{sp}_{N-2}$, in the limit simple modules over the scaling limit of JTL_N are identified with appropriate simple $\mathfrak{sp}_\infty$-modules. In Sect. 5.5, we also describe the scaling limit of the anti-periodic $\mathfrak{gl}(1|1)$ spin chains.

We then turn to Sect. 6 where we identify the scaling limit of the JTL_N algebras as the *interchiral algebra*. This identification requires making contact with several physics concepts and introduction of completions of the inductive-limit algebras in Sect. 6.2. While the Sects. 4 and 5 are mathematically the most important of this paper—they contain our main theorems equipped with proofs—the Sect. 6 is conceptually the most important from a physics point of view and relies on results provided with only ideas of a proof or conjectures which we are not proving in the present paper. Accordingly, Sect. 6 is less rigorous, and can be considered as the "physics part" of this paper.

The idea of the interchiral algebra is that, for general models, the scaling limit of JTL_N will contain, in addition to the chiral and anti-chiral Virasoro algebras, the modes of non chiral fields, such as the degenerate conformal field $\Phi_{2,1} \times \bar{\Phi}_{2,1}$. In the particular case of symplectic fermions, this field becomes the *interchiral field* $S(z, \bar{z})$ from (1.1), and the scaling limit of JTL_N, which we discuss in Sect. 5 from the point of view of *bilinears in fermion modes*, can also be identified using bilinears in fermionic fields, which are discussed in Sect. 6.1. The interchiral algebra proper is introduced in Sect. 6.3 where it is denoted by $\mathcal{S}$. Sect. 6.5 extends the discussion to the antiperiodic model or the so-called "twisted sector". In Sect. 6.6 we discuss modules over the interchiral algebra in the LCFT and their relation with JTL_N modules on the lattice. Of particular importance is our discussion of the vacuum module over the interchiral algebra in Sects. 6.6.1 and 6.6.2. In Sect. 6.7 we discuss indecomposable modules over $\mathcal{S}$ and their relation with the symplectic fermion theory. This involves an analysis of the space of states of this theory as a module over the left–right Virasoro algebra $\mathcal{V}(2)$ (which we believe is new) in Sect. 6.7.1. We then show that this analysis agrees with the scaling limit of the bimodule over JTL_N and $\mathfrak{Z}_{JTL}$. A few conclusions—and pointers to subsequent developments—are given in the conclusion. Finally, several technical aspects are addressed in five appendices. In particular, our inductive limits constructions are in Appendix C.

1.1. Notations. To help the reader navigate through this long paper, we provide a partial list of notations (consistent with all other papers in the series)

TL_N	the (ordinary) Temperley–Lieb algebra,
T_N^a	the periodic Temperley–Lieb algebra with the translation u, or the algebra of affine diagrams,

JTL_N	the Jones–Temperley–Lieb algebra,
$\mathfrak{Z}_{JTL}$	the centralizer of JTL_N,
$\pi_{\mathfrak{gl}}$	the spin-chain representation of JTL_N,
$\rho_{\mathfrak{gl}}$	the spin-chain representation of the quantum group $U_q s\ell(2)$,
e, f	the Lusztig's divided powers in $U_q s\ell(2)$,
$\mathcal{E}_{i,j}$	the elementary matrix with exactly one nonzero entry, which is 1 in the (i, j) position,
$\mathsf{X}_{1,n}$	the simple $U_i s\ell(2)$-modules,
$\mathsf{P}_{1,n}$	the projective $U_i s\ell(2)$-modules,
X_n	the simple $U_i^{\mathrm{odd}} s\ell(2)$- and $\mathfrak{Z}_{JTL}$-modules,
T_n	the indecomposable summands in spin-chain decomposition over the centralizer $\mathfrak{Z}_{JTL}$,
$\mathcal{W}_j$	the standard modules over TL_N,
$\mathcal{P}_j$	the projective modules over TL_N,
$\mathcal{L}_{j,(-1)^{j+1}}$	the simple modules over JTL_N for which we also use the notation (d_j^0),
$\mathcal{W}_{j,(-1)^{j+1}}$	the standard modules over JTL_N,
$\widehat{\mathcal{P}}_j$	the indecomposable summands in spin-chain decomposition over JTL_N,
$\mathfrak{S}_N$	a Lie algebra introduced in Sect. 4.3.1,
$\mathfrak{S}'_N = \mathfrak{S}_N \oplus \mathbb{C}\mathbf{1}$	a central extension of $\mathfrak{S}_N$,
$\mathfrak{gl}_\infty$	the Lie algebra of infinite matrices with finite number of non-zero elements, see Sect. 5.1.2,
$\mathfrak{gl}'_\infty$	a central extension of the Lie algebra $\mathfrak{gl}_\infty$,
$\mathfrak{sp}_\infty$	the symplectic Lie algebra of infinite matrices,
$\mathfrak{S}_\infty$	a Lie algebra of infinite matrices—the scaling limit of $\mathfrak{S}'_N$—introduced in Sect. 5.1.3,
$\mathfrak{S}$	a Lie algebra of local operators introduced in Sect. 6.1.3,
$\mathcal{S}$	the interchiral algebra, see Sect. 6.3,
$\mathcal{V}(2)$	the left Virasoro algebra with $c = -2$,
$\boldsymbol{\mathcal{V}}(2)$	the product of the left and right Virasoro algebras,
$\mathcal{X}_{j,1}$	the simple Virasoro modules,
$\mathcal{P}_{n,1}$	the staggered Virasoro modules,
$\mathcal{X}_j$	a module over $U\mathfrak{S}_\infty$ which is obtained in the scaling limit of the JTL modules $\mathcal{L}_{j,(-1)^{j+1}}$,
$\mathcal{Y}_j$	a module over $U\mathfrak{S}_\infty$ which is obtained in the scaling limit of the JTL modules $\widehat{\mathcal{P}}_j$,
$\mathcal{H}^+$	the bosonic space of states in symplectic fermion theory,
$\mathcal{H}^-$	the fermionic space of states in symplectic fermion theory.

All the algebras in this paper are associative and defined over the field $\mathbb{C}$ of complex numbers.

2. A Reminder of the Open Case

2.1. The open $\mathfrak{gl}(1|1)$ super-spin chain. The open $\mathfrak{gl}(1|1)$ super-spin chain [10,22] is a tensor product representation $\mathcal{H}_N = \otimes_{j=1}^N \mathbb{C}^2$ of the Temperley–Lieb (TL) algebra of zero fugacity parameter m. We recall that the ordinary TL algebra denoted by $TL_N(m)$

is generated by e_j, with $1 \le j \le N - 1$, and has the defining relations

$$
\begin{aligned}
e_j^2 &= m e_j, \\
e_j e_{j\pm 1} e_j &= e_j, \\
e_j e_k &= e_k e_j \quad (j \ne k, \ k \pm 1),
\end{aligned}
\tag{2.1}
$$

where m is a real parameter.

The representation space $\mathcal{H}_N$ consists of $N = 2L$ tensorands labelled $j = 1, \dots, 2L$ with the fundamental representation of $\mathfrak{gl}(1|1)$ on even sites and its dual on odd sites. The representation of each TL generator e_j is given by projectors on the $\mathfrak{gl}(1|1)$-invariant in the product of two neighbour tensorands

$$
e_j^{\mathfrak{gl}} = (f_j + f_{j+1})(f_j^\dagger + f_{j+1}^\dagger), \quad 1 \le j \le 2L - 1,
\tag{2.2}
$$

where we use a free fermion representation based on operators f_j and $f_j^\dagger$ acting nontrivially only on jth tensorand and obeying

$$
\{f_j, f_{j'}\} = 0, \quad \{f_j^\dagger, f_{j'}^\dagger\} = 0, \quad \{f_j, f_{j'}^\dagger\} = (-1)^j \delta_{jj'},
\tag{2.3}
$$

where the minus sign for an odd j is due to presence of the dual representations of $\mathfrak{gl}(1|1)$. The generators $e_j^{\mathfrak{gl}}$ provide then a representation of $TL_{2L}(m = 0)$ which is known to be faithful.

The representation space $\mathcal{H}_{2L}$ is equipped with an inner product $\langle \cdot, \cdot \rangle$ such that $\langle f_j x, y \rangle = \langle x, f_j^\dagger y \rangle$ for any $x, y \in \mathcal{H}_{2L}$. We stress that the inner product is indefinite because of the sign factors in the relations (2.3). The Hamiltonian operator

$$
H_{\text{op.}} = -\sum_{j=1}^{2L-1} e_j^{\mathfrak{gl}},
$$

with the 'hamiltonian densities' $e_j^{\mathfrak{gl}}$ defined in (2.2), is self-adjoint $H_{\text{op.}} = H_{\text{op.}}^\dagger$ with respect to this inner product (actually, each $e_j^{\mathfrak{gl}}$ is a self-adjoint operator). Its eigenvalues are real and the eigenvectors can easily be computed. Because of the indefinite inner product, the self-adjoint Hamiltonian can have non-trivial Jordan cells and here it is indeed the case—the Jordan cells are of rank two [7].

The open $\mathfrak{gl}(1|1)$ spin-chain exhibits a large symmetry algebra dubbed $\mathcal{A}_{1|1}$ in [10]. This algebra is the centralizer $\mathfrak{Z}_{\text{TL}}$ of $TL_N(0)$ and is generated by the identity and the five generators

$$
F_{(1)} = \sum_{1 \le j \le N} f_j, \quad F_{(1)}^\dagger = \sum_{1 \le j \le N} f_j^\dagger,
\tag{2.4}
$$

$$
F_{(2)} = \sum_{1 \le j < j' \le N} f_j f_{j'}, \quad F_{(2)}^\dagger = \sum_{1 \le j < j' \le N} f_{j'}^\dagger f_j^\dagger, \quad \mathsf{N} = \sum_{1 \le j \le N} (-1)^j f_j^\dagger f_j.
\tag{2.5}
$$

We note that the formulas give, after trivial redefinitions, a representation of the (Lusztig) quantum group $U_q s\ell(2)$ at $\mathfrak{q} = i$. The fermionic generators above, those with the subscript '(1)', are from the nilpotent part and the bosonic ones form the
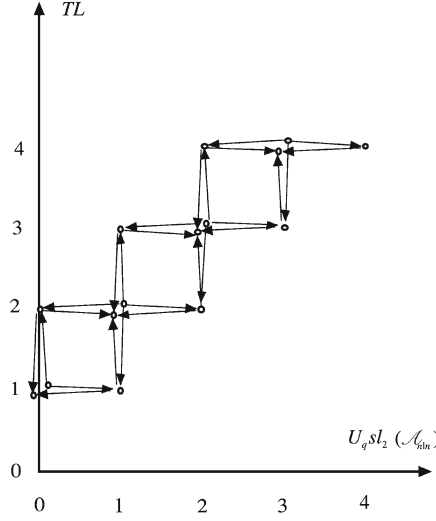
Fig. 1. The structure of the open $\mathfrak{gl}(1|1)$ spin-chain (i.e., $n = 1$) for $N = 8$ sites, as a representation of the product $TL_N \boxtimes U_i s\ell(2)$. Some nodes occur twice and have been separated slightly for clarity

$s\ell(2)$ subalgebra in $U_q s\ell(2)$. It will be convenient in what follows to introduce slightly modified generators[1] of $U_q s\ell(2)$

$$h = \frac{1}{2} \sum_{j=1}^{N} (-1)^j f_j^\dagger f_j - \frac{L}{2}, \qquad e = q^{-1} \sum_{1 \leq j_1 < j_2 \leq N} f_{j_1}^\dagger f_{j_2}^\dagger, \qquad f = q \sum_{1 \leq j_1 < j_2 \leq N} f_{j_1} f_{j_2},$$

(2.6)

$$K = (-1)^{2h}, \qquad\qquad E = \sum_{j=1}^{N} f_j^\dagger K, \qquad\qquad F = q^{-1} \sum_{j=1}^{N} f_j \qquad (2.7)$$

obeying in particular

$$KEK^{-1} = -E, \qquad KFK^{-1} = -F, \qquad E^2 = F^2 = 0, \qquad (2.8)$$
$$[h, e] = e, \qquad\quad [h, f] = -f, \qquad\quad [e, f] = 2h, \qquad (2.9)$$

see more details in [22]. The $s\ell(2)$ Cartan generator h is related to the total-spin operator S^z in the XX language, or the Fermion number N as

$$2h = S^z = N - L. \qquad (2.10)$$

For our alternating chain, the values of S^z are integer since the number of sites is even, and thus h takes integer or half integer values.

2.2. Bimodule on a finite lattice.

The decomposition of the open spin-chain as a bimodule over the pair $(TL_N, \mathcal{A}_{1|1})$ of mutual centralizers is shown on Fig. 1 for the $N = 8$

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[1] We dispense with the more correct notation $\rho_{\mathfrak{gl}}$ used in [22].
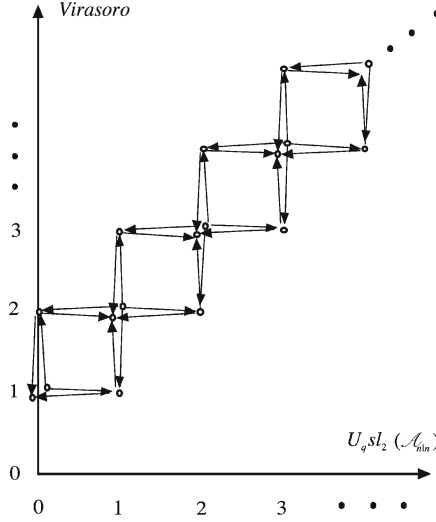
Fig. 2. Space of states of the continuum theory as a representation of $\mathcal{V}(2) \boxtimes U_i s\ell(2)$. The *vertical labels* $n = 1, 2, \ldots$ are for the Virasoro $\mathcal{V}(2)$ irreducible representations of the conformal weights $\Delta_{n,1} = n(n-1)/2$

case and borrowed from [7]. The label j in the horizontal direction corresponds to the double value of the $s\ell(2)$ spin n—the highest weight of an $s\ell(2)$-module. Each node with a Cartesian coordinate (j, k) in the bimodule diagram corresponds to a simple subquotient over the tensor product $TL_N \boxtimes U_i s\ell(2)$ of the commuting algebras and arrows show the action of both the algebras—the Temperley–Lieb TL_N acts in the vertical direction (preserving the coordinate j), while $U_i s\ell(2)$ acts in the horizontal way. Indecomposable projective TL_N-modules $\mathcal{P}_k$ can be recovered by ignoring all the horizontal arrows, while tilting $U_i s\ell(2)$-modules $\mathsf{P}_{1,j}$ (these are also projective [15]) are obtained by ignoring all the vertical arrows. For more details, see [7,23].

2.3. Bimodule in the continuum limit. The crucial observation of [7] is that an identical bimodule structure, extending to arbitrarily high values of the $s\ell(2)$ spin $n = j/2$, is present in the continuum limit. This is illustrated on Fig. 2, where the same comments as in the finite chain apply exactly, with the replacement of TL_N by the Virasoro algebra at central charge $c = -2$ denoted by $\mathcal{V}(2)$.

It is useful here to comment Fig. 2 further. In the boundary symplectic fermion theory, the space of states decomposes as a direct sum of bosonic $\mathcal{H}^+$ and fermionic $\mathcal{H}^-$ sectors mixed by the fermionic part of the $\mathfrak{gl}(1|1)$ algebra. We remind that the symplectic fermions form an $s\ell(2)$ doublet [25] and the fermionic part of the $\mathfrak{gl}(1|1)$ is formed by the two zero modes of the fermions. Each of the sectors is further decomposed as a direct sum of modules over the product $s\ell(2) \boxtimes \mathcal{V}(2)$ of the two commuting algebras, $s\ell(2)$ and Virasoro, as (see also [26])

$$\mathcal{H}^+ = \bigoplus_{n \in \mathbb{N}_0} C_n \boxtimes \mathcal{P}_{2n+1,1}, \quad \mathcal{H}^- = \bigoplus_{n \in \mathbb{N}-\frac{1}{2}} C_n \boxtimes \mathcal{P}_{2n+1,1}, \qquad (2.11)$$

where C_n denotes a $(2n+1)$-dimensional $s\ell(2)$-module of the (iso)spin n, and Virasoro modules $\mathcal{P}_{n,1}$ are the so-called staggered modules introduced in [1]. Note that we use

the term 'isospin' for $s\ell(2)$-modules on the CFT side in order to distinguish them from ones on the lattice, where the highest weight of an $s\ell(2)$-module we call in general by 'spin'. In the open case, the $Us\ell(2)$ symmetry of the symplectic fermion theory [25] and the $Us\ell(2)$ part of the full quantum group $U_{\mathfrak{q}}s\ell(2)$ are actually coincident but this will not be true in the non-chiral case. Recall again that the value of S^z for the highest weight at (isospin) n is $j = 2n$, the horizontal label used in Fig. 2.

The staggered $\mathcal{V}(2)$-modules are indecomposable and have the following subquotient structure

$$
\mathcal{P}_{2n+1,1} =
\begin{array}{ccc}
 & \mathcal{X}_{2n+1,1} & \\
 & \swarrow \qquad \searrow & \\
\mathcal{X}_{2n,1} & & \mathcal{X}_{2n+2,1} \\
 & \searrow \qquad \swarrow & \\
 & \mathcal{X}_{2n+1,1} &
\end{array}
\tag{2.12}
$$

where $\mathcal{X}_{2n+1,1}$ is the irreducible $\mathcal{V}(2)$-module with the conformal dimension $\Delta_{2n+1,1} = n(2n + 1)$, with a non-negative integer or half-integer n. We note that a south-east arrow represents an action of negative Virasoro modes while a south-west arrow represent positive modes action. In the diagram (2.12), the staggered module is a 'glueing'/extension of two indecomposable Kac modules which are highest-weight modules. The one in the top composed of $\mathcal{X}_{2n+1,1}$ and $\mathcal{X}_{2n+2,1}$ is the quotient of the Verma module with the weight $\Delta_{2n+1,1}$ by the singular vector at the level $2n + 1$, and the second Kac module in the bottom composed of $\mathcal{X}_{2n,1}$ and $\mathcal{X}_{2n+1,1}$ is a similar quotient (at the level $2n$) of the Verma module with the weight $\Delta_{2n,1}$.

The different terms in $\mathcal{H}^+$ and $\mathcal{H}^-$ from (2.11) are in turn connected by the action of $\mathfrak{gl}(1|1)$, resulting into the diagram on Fig. 2, where each node is a product of an irreducible $\mathcal{V}(2)$-module and an irreducible $s\ell(2)$-module of dimension $j + 1$. We see that the symmetry algebra in this CFT, the centralizer of $\mathcal{V}(2)$, is the semi-direct product of the fermionic part of $\mathfrak{gl}(1|1)$ and the enveloping algebra $Us\ell(2)$. This centralizer turns out to coincide with (a representation of) the quantum group $U_{\mathfrak{q}}s\ell(2)$ at $\mathfrak{q} = i$, which we also denote by $\mathcal{A}_{1|1}$, as on a finite spin-chain.

2.4. A note on other spin-chains. Note that while we discuss only $\mathfrak{gl}(1|1)$ spin-chains here, a diagram identical to Fig. 2 describes the continuum limit of alternating $\mathfrak{gl}(n|n)$ spin-chains. We recall that these spin-chains are defined in a similar fashion by introducing projections on $\mathfrak{gl}(n|n)$-invariants [27]. They give also faithful representations of $TL_N(0)$ where the horizontal action in the bimodule diagram is due to the symmetry algebra being now $\mathcal{A}_{n|n}$, an algebra which is Morita equivalent to $\mathcal{A}_{1|1}$ [10].

A similar analysis for other cases—representations of $TL_N(m)$ for other values of m—such as the alternating $\mathfrak{gl}(2|1)$ spin-chain or the XXZ spin-chains with $U_{\mathfrak{q}}s\ell(2)$ symmetry at different roots of unity show that the lattice bimodule structure can be used along the same lines to infer properties of all known boundary logarithmic CFTs. In particular, the staggered Virasoro modules for different central charges abstractly discussed in [3] or [4] can quickly be recovered in this fashion, at least their subquotient structure can be deduced from the bimodules. This opens in particular the way to measuring [28,29] indecomposability parameters (also called β invariants [12]) characterising Virasoro-module structure completely, or to computing fusion rules using an induction procedure [7,10,30].

These similarities in subquotient structures and fusion rules indicate an equivalence between corresponding tensor categories. Braided tensor structure or fusion data on the

lattice part is given by an induction bi-functor associated with two lattices of arbitrary sizes joined to each other. Rigorously, this bi-functor gives a braided tensor structure only in the infinite size limit, where a construction of inductive limits of the finite categories of modules over the Temperley–Lieb algebras is required. We note that a systematic way to construct these limits is based on the spin-chain bimodules. Then, the inductive limits should be compared with braided tensor categories of modules over the Virasoro algebras. We believe that a direct construction of centralizers of the Virasoro algeras in the LCFTs would give the desired equivalence.

This equivalence can be established by direct calculation in the case of symplectic fermions, where the continuum limit can be explicitly carried out (for steps in this direction, see [22]). For other cases however, such a calculation seems completely out of reach since models are only Bethe ansatz solvable, and precious little is known about their continuum limits, apart from critical exponents and some indecomposable modules. The equivalence thus remains a very reasonable conjecture, which will take considerably more work—in particular, in defining the continuum or scaling limit—before being rigorously established. It can nevertheless be easily understood if one recognizes the similar role played by the quantum group $U_{\mathfrak{q}} s\ell(2)$ both on the lattice and in the continuum [15,31].

Meanwhile, our strategy consists in postulating that similar equivalences between the lattice and continuum models are present in the bulk, or non-chiral, case as well. We shall then study the case of closed $\mathfrak{gl}(1|1)$ spin-chains in detail, to draw lessons that we will then apply to more complicated—and physically interesting—models.

3. A Reminder of the Algebraic Aspects in the Closed Case

3.1. The Jones–Temperley–Lieb algebra $JTL_N(0)$. A closed periodic spin-chain can be obtained by adding a last generator $e_N^{\mathfrak{gl}}$ defined using formula (2.2) but this time identifying the labels modulo N, in particular $f_{N+1} \equiv f_1$, $f_{N+1}^{\dagger} \equiv f_1^{\dagger}$:

$$e_N^{\mathfrak{gl}} = (f_N + f_1)(f_N^{\dagger} + f_1^{\dagger}). \tag{3.1}$$

The generators $e_j^{\mathfrak{gl}}$, with $1 \leq j \leq N$, satisfy the same relations (2.1) but now the indices are interpreted modulo N.

The relations (2.1) (defined modulo N) define an infinite dimensional associative algebra denoted in our companion papers by TL_N^a [32,33]. The algebra TL_N^a is known in the physics literature as the periodic Temperley–Lieb algebra [34,35]. In fact, the formulas for the generators in (2.2) and (3.1) lead to many more relations; as a result, the periodic $\mathfrak{gl}(1|1)$ spin-chain provides a non-faithful representation of TL_N^a. Moreover, it provides a non-faithful representation of a finite-dimensional quotient—called *the Jones–Temperley–Lieb* algebra $JTL_N(m = 0)$—of a slightly bigger algebra enlarging TL_N^a by a translation operator u^2, see e.g. [23, Sec. 2.1]. We denote this representation in what follows by $\pi_{\mathfrak{gl}} : JTL_N(0) \to \mathrm{End}_{\mathbb{C}}(\mathcal{H}_N)$.

To define the algebra $JTL_{2L}(m)$ (the following description is taken almost verbatim from [10]), we first introduce an operator u^2 (with inverse u^{-2}) which translates any state to the right by two sites (so as to be consistent with the distinction of two types of sites carrying dual representations). We thus have $u^2 e_j u^{-2} = e_{j+2}$ and we also impose the relation $u^{2L} = 1$. We then define abstractly an algebra of diagrams as is customary for the ordinary Temperley–Lieb (TL) algebra, but this time on an annulus (or finite cylinder),

in which a general basis element corresponds to a diagram of $2L$ sites on the inner, and $2L$ on the outer boundary; the sites are connected in pairs, but only configurations that can be represented using lines inside the annulus that do not cross are allowed. Multiplication is defined in a natural way on these diagrams, by joining an inner to an outer annulus, and removing the interior sites. We emphasize that whenever a closed loop is produced when diagrams are multiplied together, this loop must be replaced by a numerical factor m (as for the TL algebra), even for loops that wind around the annulus, as well as for those that are homotopic to a point. We also impose that non-isotopic (in the annulus) diagrams connecting the same sites are identified. Then, the resulting algebra is finite-dimensional and is generated by the elements e_j and u^2, and they obey the relations (2.1) defined modulo N, which however are not a complete set of defining relations. We note that the numerical factor m for winding loops is not a consequence of the stated relations, but a separate assumption.

In what follows, we consider only the case $m = 0$ and use the notation $JTL_N \equiv JTL_N(0)$.

The Hamiltonian operator $H_{\text{per.}}$ in the periodic case will be denoted simply by H and is given by

$$H = -\sum_{j=1}^{2L} e_j^{\mathfrak{gl}}. \tag{3.2}$$

This operator is also self-adjoint $H = H^\dagger$ with respect to the inner product, as in the open case described above. Its eigenvalues are also real and the eigenvectors can easily be computed. As a consequence of the indefinite inner product, the self-adjoint Hamiltonian has non-trivial Jordan cells of rank-two [22], see also Sect. 5.1 below.

3.2. The centralizer $\mathfrak{Z}_{JTL}$ of JTL_N. In the closed case, while the $\mathfrak{gl}(1|1)$ symmetry generated by $F_{(1)}$, $F_{(1)}^\dagger$, and N remains, the bosonic $s\ell(2)$ generators $F_{(2)}$ and $F_{(2)}^\dagger$ defined in (2.5) do not commute with the action of JTL_N [22]. What remains of $\mathcal{A}_{1|1}$ or $U_q s\ell(2)$ is only the fermionic generators

$$F_{(2n+1)} = \sum_{\substack{1 \le j_1 < j_2 < \\ \cdots < j_{2n+1} \le 2L}} f_{j_1} f_{j_2} \ldots f_{j_{2n+1}},$$
$$F_{(2n+1)}^\dagger = \sum_{\substack{1 \le j_1 < j_2 < \\ \cdots < j_{2n+1} \le 2L}} f_{j_1}^\dagger f_{j_2}^\dagger \ldots f_{j_{2n+1}}^\dagger, \qquad n \ge 0, \tag{3.3}$$

which, together with the bosonic operators N and $F_{(2L)}$ and $F_{(2L)}^\dagger$, generate the centralizer $\mathfrak{Z}_{JTL}$ of the representation of JTL_{2L}.

For further reference, we recall another and more convenient for us description of the centralizer in terms of standard $U_q s\ell(2)$ generators. We introduce the generators

$$\mathsf{E}_m \equiv \mathsf{e}^m \mathsf{E}\,\frac{\mathsf{K}^2 + \mathbf{1}}{2}, \quad \mathsf{F}_n \equiv \mathsf{f}^n \mathsf{F}\,\frac{\mathsf{K}^2 + \mathbf{1}}{2}, \quad m, n \ge 0, \tag{3.4}$$

which are represented on the spin-chain following the expressions (2.6) and (2.7). They generate a subalgebra in $U_q s\ell(2)$ which we denote as $U_q^{\text{odd}} s\ell(2)$, see [22, Dfn. 3.3.1]. This subalgebra has the basis $\mathsf{E}_n \mathsf{F}_m \mathsf{h}^k \mathsf{K}^l$, with $n, m, k \ge 0$ and $0 \le l \le 3$. The positive

Borel subalgebra is generated by h and E_n while the negative subalgebra—by h and F_n, for $n \geq 0$.

The result [22, Thm. 3.3.3] is then that the centralizer $\mathfrak{Z}_{JTL}$ of the (representation $\pi_{\mathfrak{gl}}$ of) JTL_{2L} on the alternating periodic $\mathfrak{gl}(1|1)$ spin-chain $\mathcal{H}_{2L}$ is the algebra generated by (the representation of) $U_q^{\mathrm{odd}}s\ell(2)$ and f^L, e^L. The correspondence with the fermionic expressions (3.3) is $F_n \sim F_{(2n+1)}$, $E_n \sim F_{(2n+1)}^{\dagger}$, and $N \sim h$, etc. For more details, see [22, Sec. 3.3.4].

The representation theory of $U_q^{\mathrm{odd}}s\ell(2)$ was studied in our second paper [23] of this series. The indecomposable $U_q^{\mathrm{odd}}s\ell(2)$ modules T_n, with integer $1 \leq n \leq L$, which appear in the decomposition of $\mathcal{H}_{2L}$ as direct summands are restrictions to the subalgebra $U_q^{\mathrm{odd}}s\ell(2)$ of the well known projective covers $P_{1,n}$ over $U_q s\ell(2)$. They are all indecomposable, with dimension $4n$, the same as the dimension of $P_{1,n}$. Their subquotient structure is given also in [23, Sec. 3.3].

3.3. Bimodule over JTL_N and $\mathfrak{Z}_{JTL}$. Recall that the representation $\pi_{\mathfrak{gl}}$ of JTL_N is non-faithful (faithfulness aspects will be discussed in more detail in our next paper [36]). There is thus no evident direct way to get a decomposition of the periodic $\mathfrak{gl}(1|1)$ spin-chain like in the open (faithful) case. For example, the general theory [37] of projective modules over a cellular algebra [which includes $TL_N(m)$ and $JTL_N(m)$ algebras], which can be applied for a faithful representation [36], is not directly useful here. Instead, an indirect strategy is needed, which was discussed in detail in [23, Sec. 5.2], and which we do not recall in detail here, concentrating only on the essential aspects.

We first give a diagram describing the subquotient structure of the bimodule $\mathcal{H}_{2L}$ over the pair $(JTL_N, \mathfrak{Z}_{JTL})$. The two commuting actions are presented in Fig. 3 where we show a direct sum of indecomposable spin-chain modules $\widehat{\mathcal{P}}_j$ over JTL_N, and simple subquotients over JTL_N will be denoted below by $(d_{|j|}^0)$. The direct sum is depicted as a (horizontal) sequence of diagrams for $\widehat{\mathcal{P}}_j$ from $j = -L$ on the left to $j = L$ on the right. Each node in the diagram is a simple subquotient over the product $JTL_N \boxtimes U_q^{\mathrm{odd}}s\ell(2)$. The action of JTL_N is depicted by vertical arrows while the action of $U_q^{\mathrm{odd}}s\ell(2)$ is shown by dotted horizontal lines connecting different JTL_N-modules.

In the diagram on Fig. 3, the first horizontal layer (at the bottom) contains the space of ground states and it consists of four nodes, which are simple JTL_N-modules (d_1^0), and dotted arrows mixing them. These nodes and arrows describe the indecomposable $U_q^{\mathrm{odd}}s\ell(2)$-module T_1. The second layer contains eight nodes corresponding to (d_2^0) and the dotted arrows contribute to the indecomposable module T_2, etc. We emphasize that we do not draw long-range arrows representing action of the generators $F_{>0}$ and $E_{>0}$ in modules $T_{n>1}$ in order to simplify diagrams but the arrows can be easily recovered using the subquotient structure of T_n given in [23, Sec. 3.3]—for example, the second layer of the bimodule contains in addition four long arrows going from the node $\diamond$ at $j = \mp 1$ to the node $\bullet$ at $j = \pm 2$, and from the node $\bullet$ at $j = \pm 2$ to the node $\blacklozenge$ at $j = \mp 1$. With this comment about arrows in mind, the reader can compare complexity of this bimodule with the open-case bimodule in Fig. 1.

As a JTL_N-module, the $\mathcal{H}_N$ has a decomposition

$$\mathcal{H}_N|_{JTL_N} = \bigoplus_{j=-L+1}^{L-1} \widehat{\mathcal{P}}_j \boxtimes X_j \oplus \widehat{\mathcal{P}}_L \boxtimes X_L, \tag{3.5}$$
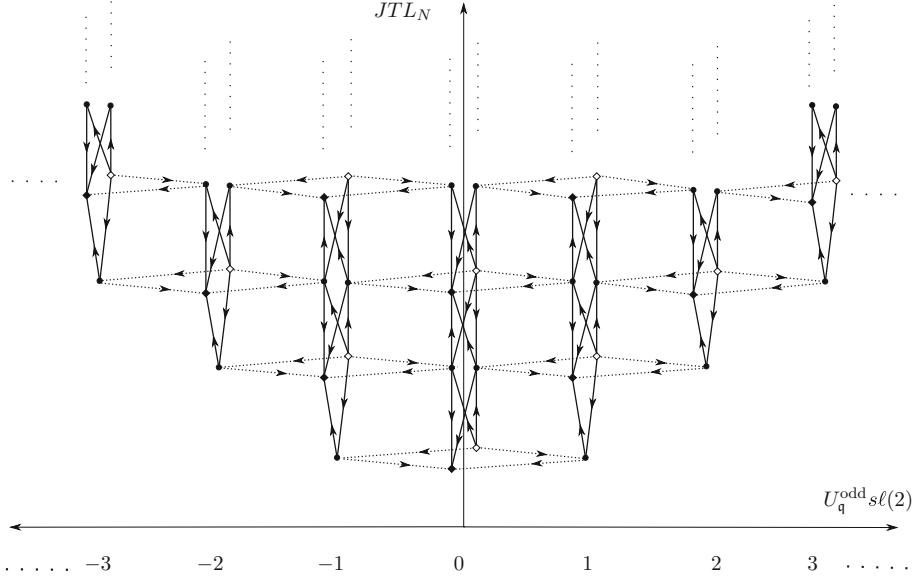
Fig. 3. Bimodule over the pair $(JTL_N, U_q^{odd}s\ell(2))$ of commuting algebras. The action of JTL_N is depicted by *vertical arrows* while the action of $U_q^{odd}s\ell(2)$ is shown by *dotted horizontal lines*. Each *label* j in the *horizontal axis* corresponds to the sector for S^z and the label runs from $-L$ on the *left* to $j = L$ on the *right*. Each *vertical tower at a label* j is the diagram for $\widehat{\mathcal{P}}_j$. The *first horizontal layer at the bottom* contains four nodes (d_1^0) and *dotted arrows* mixing them compose the $U_q^{odd}s\ell(2)$-module T_1. The second layer contains eight nodes (d_2^0) and the *dotted arrows* depict the action in the indecomposable module T_2, etc. We suppress *long-range arrows* representing action of the generators $\mathsf{F}_{>0}$ and $\mathsf{E}_{>0}$ in order to simplify diagrams. For example, the second layer of the bimodule contains in addition four *long arrows* going from the node $\diamond$ at $j = \mp 1$ to the node $\bullet$ at $j = \pm 2$, and from the node $\bullet$ at $j = \pm 2$ to the node $\blacklozenge$ at $j = \mp 1$. We also note that $\blacklozenge$'s have only *incoming arrows* and their union is the socle (maximum semisimple submodule or the bottom level) of the bimodule; further, the union of $\bullet$'s is the socle of the quotient by the socle—the *middle level*, while the $\diamond$'s have only *outgoing arrows* and their union is the *top level*

where $N = 2L$, X_j is the one-dimensional (with $S^z = j$) and X_L is the two-dimensional simple $\mathfrak{Z}_{JTL}$-modules (the representation theory of the centralizer $\mathfrak{Z}_{JTL}$ is described in [23, Sec. 3]), and $\widehat{\mathcal{P}}_L$ is the one-dimensional JTL_{2L}-module with the action of $u^2 = (-1)^{L-1}\mathbf{1}$. For any sector with non-zero $j = S^z$, the subquotient structure for $\widehat{\mathcal{P}}_j$ is given in Fig. 4, on the right, while the tower for $\widehat{\mathcal{P}}_0$ is presented on the left. All the towers are ended by the pair of simple subquotients (d_L^0). We note that the two simple subquotients at each level of the ladders are isomorphic. The Hamiltonian H from (3.2) acts by Jordan blocks of rank 2 on each pair of isomorphic simple subquotients with one at the top (having only outgoing arrows) and the second subquotient in the socle of the module (having only ingoing arrows). The Jordan block structure is due to presence of zero fermionic modes in the Hamiltonian as it is observed in [22], see also Sect. 4.1.2.

We note that the JTL_N-modules $\widehat{\mathcal{P}}_j$ in Fig. 3 are drawn in opposite direction 'from bottom to top' comparing to diagrams in Fig. 4, in order to the space of ground states to be in the bottom of the diagram.

Finally, we recall some information about the simple JTL_N modules, which we denote in [23] by $\mathcal{L}_{j,(-1)^{j+1}}$ or more conveniently by (d_j^0) above. The number d_j^0 is their
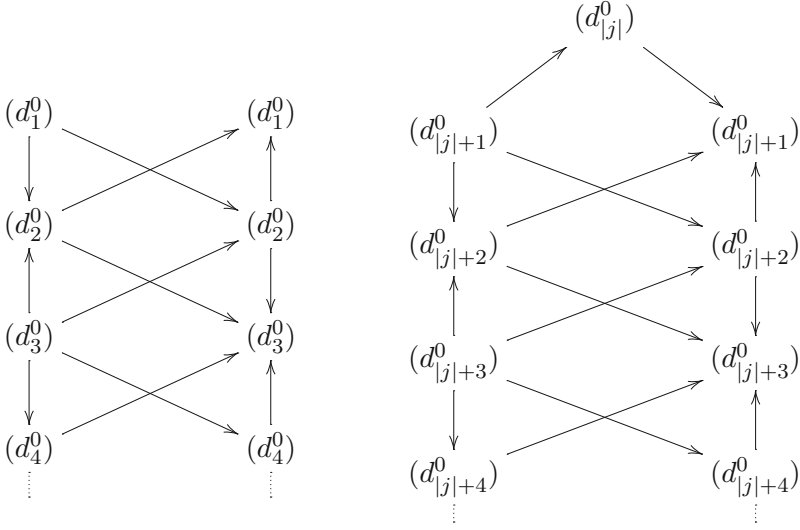
Fig. 4. The two-strands structure of the spin-chain JTL_N-modules $\widehat{\mathcal{P}}_j$ for $j = 0$ on the *left* and $j \neq 0$ on the *right side*. The *towers* are ended by the pair of (d_L^0)

dimension, obtained from the dimension of the spin-chain JTL_N-modules $\widehat{\mathcal{P}}_j$ given by

$$\dim \widehat{\mathcal{P}}_j \equiv \widehat{d}_j = \binom{2L}{L+j} \tag{3.6}$$

which is just the dimension of the sector with $S^z = j$ in the related XX spin chain with $2L$ sites. We then have from the subquotient structure for $\widehat{\mathcal{P}}_j$ in Fig. 4 that

$$\dim \mathcal{L}_{j,(-1)^{j+1}} \equiv d_j^0 = \sum_{j' \geq j} (-1)^{j'-j} d_{j'} \quad \text{with} \quad d_j = \binom{2L}{L+j} - \binom{2L}{L+j+1},$$

which can be written as

$$d_j^0 = \binom{2L-2}{L-j} - \binom{2L-2}{L-j-2}, \tag{3.7}$$

where $0 \leq j \leq L$. Note that $d_0^0 = 0$ and we thus have that all simple modules over JTL_N that appear in this spin-chain are those $\mathcal{L}_{j,(-1)^{j+1}}$ with $1 \leq j \leq L$, see also more details in our second paper [23].

We note also the obvious identities

$$\sum_{j=-L}^{L} \widehat{d}_j = \sum_{j=1}^{L} 4 d_j^0 = 2^{2L}$$

which give the full dimension of the $\mathfrak{gl}(1|1)$ spin-chain indeed.

It turns out that in this very degenerate case of JTL_N representations the $\mathcal{L}_{j,(-1)^{j+1}}$ modules are also simple for the subalgebra TL_N. Of course, the difference with the open case is in the structure of indecomposable modules—the complexity of towers in Fig. 3 can be compared with the simpler 'diamond'-type structure of TL modules in Fig. 1.

3.4. *The antiperiodic spin-chain.* We can also consider the alternating $\mathfrak{gl}(1|1)$ spin-chain with antiperiodic conditions for the fermions, obtained by setting $f_{2L+1}^{(\dagger)} = -f_1^{(\dagger)}$. The generators e_j, for $1 \leq j \leq 2L-1$, have the same representation (2.2) while the last generator is then given by

$$e_{2L} = (f_{2L} - f_1)(f_{2L}^\dagger - f_1^\dagger), \tag{3.8}$$

to be compared with (3.1). This expression provides a representation of another quotient (different from JTL_N) of the affine TL algebra: in the diagram language, non-contractible loops now should be replaced by the weight 2 (the dimension of the fundamental or its dual, instead of the superdimension in the periodic case); the relation $u^N = (-1)^j$ is also imposed in the sector with $2j$ through-lines. We will call the corresponding algebra JTL_N^{tw}, see a precise definition in our second paper [23, Sec. 6.2].

We emphasize that this antiperiodic $\mathfrak{gl}(1|1)$ spin chain does not have $\mathfrak{gl}(1|1)$ symmetry any longer. Instead, we have [22, Thm.3.4.1] that the centralizer of JTL_N^{tw} is generated by the $s\ell(2)$ generators e and f, or $F_{(2)}^\dagger$ and $F_{(2)}$ defined in (2.5).

We then recall the bimodule structure [23, Sec. 6.2] over the pair $(JTL_N^{tw}, Us\ell(2))$:

$$\mathcal{H}_N|_{JTL_N^{tw} \boxtimes Us\ell(2)} = \bigoplus_{j=0}^{L} \mathcal{L}_{j,(-1)^j} \boxtimes \mathsf{X}_{1,j+1} \tag{3.9}$$

where $\mathsf{X}_{1,j}$ is the j-dimensional simple $Us\ell(2)$-module and $\mathcal{L}_{j,(-1)^j}$ is a simple module over JTL_N^{tw}. The dimension of $\mathcal{L}_{j,(-1)^j}$ is $2d_{j+1}^0 + d_j^0 + d_{j+2}^0$ and is

$$\dim \mathcal{L}_{j,(-1)^j} = \widehat{d}_j - \widehat{d}_{j+2} = \binom{2L}{L+j} - \binom{2L}{L+j+2}. \tag{3.10}$$

Recall that the dimension of each $S^z = j$ sector $\mathcal{H}_{[j]}$ is $\binom{2L}{L+j}$.

4. JTL Algebra and Howe Duality

We now give an interpretation of the simple modules over JTL_N and JTL_N^{tw} with the dimensions (3.7) and (3.10), respectively, from the point of view of representation theory of symplectic Lie algebras. We first recall a convenient notation for lattice fermions and then describe the representation theory of JTL_N in the periodic and anti-periodic spin chains in the context of (a non-semisimple version of) Howe duality.

4.1. *A Lie algebra of fermion bilinears.* Recall that the e_j generators of the JTL_N (or JTL_N^{tw}) algebras in the (anti-)periodic spin-chains are linear combinations of bilinears in the fermions f_j and $f_j^\dagger$. For the periodic case, we have the representation

$$e_j^{\mathfrak{gl}} = (f_j + f_{j+1})(f_j^\dagger + f_{j+1}^\dagger), \quad 1 \leq j \leq 2L,$$

identifying the labels modulo N, in particular $f_{N+1} \equiv f_1$, and $f_{N+1}^\dagger \equiv f_1^\dagger$ (see also expressions in (2.2) and (3.1)). For the anti-periodic model, we have a similar expression (3.8), where the fermions are anti-periodic. It turns out that the commutators of these

combinations of bilinears can be expressed again in fermionic bilinears, and of course belong to the JTL algebra. Therefore, they generate a finite-dimensional Lie algebra. On the other hand, the spin-chain images of the JTL algebra contain many operators which are not bilinears. These non-bilinear operators are generated by the Lie algebra elements[2] because the e_j generators belong to the Lie algebra. To say things differently, the full image of the JTL algebra is isomorphic to an enveloping algebra of the Lie algebra of the fermionic bilinears (of course, it is also true for the open case discussed in Sect. 2 that the TL algebra, which is a subalgebra in the JTL algebra, is generated by a Lie algebra of fermionic bilinears). We begin our analysis with the anti-periodic spin-chain which is much simpler than the periodic one because of its semi-simplicity.

The centralizer of JTL_N^{tw} is generated as well by special bilinears in the fermions and these are the $s\ell(2)$ generators e and f, or $F_{(2)}^\dagger$ and $F_{(2)}$ defined in (2.5). It is a well-known fact due to Howe [24] that, in the Clifford algebra $\mathcal{C}_{4L}$, the centralizing algebra of this image of $Us\ell(2)$—which is the enveloping algebra of the symplectic Lie algebra $\mathfrak{sp}_2$ as well—is the enveloping algebra of $\mathfrak{sp}_{2L}$. Note that the dimensions (3.10) of the simple JTL_N^{tw}-modules coincide indeed with dimensions of fundamental representations of $\mathfrak{sp}_{2L}$. We can thus easily get the following theorem.

Theorem 4.1.1. *The image* $\pi_{\mathfrak{gl}}\big(JTL_N^{tw}\big)$ *of the twisted Jones–Temperley–Lieb algebra in the anti-periodic* $\mathfrak{gl}(1|1)$ *spin-chain is isomorphic to the image of a representation of the enveloping algebra* $U\mathfrak{sp}_{2L}$ *with all simple* $\mathfrak{sp}_{2L}$-*modules corresponding to one-column Young diagrams.*

Recall that the anti-periodic spin-chain is completely reducible as a JTL_N^{tw}-module—see the corresponding semisimple $(JTL_N^{tw}, U\mathfrak{sp}_2)$-bimodule in Sect. 3.4—and we thus have the double-centralizing property [38, Thm. 4.1.13]. Then, the proof of Theorem 4.1.1 is a simple consequence of the classical $(\mathfrak{sp}(n), \mathfrak{sp}(m))$-type Howe duality, where $n = 2$ and $m = 2L$ in our case.

We next describe explicitly the representation of $\mathfrak{sp}_{2L}$ from Theorem 4.1.1, and introduce a more convenient fermion basis which we will use below in studying scaling limits.

4.1.2. The Clifford algebra. The periodic spin chain admits the natural action of a Clifford algebra $\mathcal{C}_{4L}$ with $4L$ generators: χ_p, η_p and their adjoints $\chi_p^\dagger$, $\eta_p^\dagger$, where we set $p = m\pi/L$ and $0 \le m \le L - 1$. As operators, they are defined in Appendix A in (A3) and satisfy the following anti-commutation relations

$$\left\{\chi_p^\dagger, \chi_{p'}\right\} = \left\{\eta_p^\dagger, \eta_{p'}\right\} = \delta_{p,p'}, \quad \left\{\chi_p, \eta_{p'}\right\} = \left\{\chi_p^\dagger, \eta_{p'}^{(\dagger)}\right\} = \left\{\eta_p^\dagger, \chi_{p'}^{(\dagger)}\right\} = 0. \quad (4.1)$$

The Hamiltonian from (3.2) can then be written [22, Sec. 4.1] in this Clifford algebra as

$$H = 2 \sum_{\substack{p=\epsilon \\ \text{step}=\epsilon}}^{\pi-\epsilon} \sin p\left(\chi_p^\dagger \chi_p - \eta_p^\dagger \eta_p\right) + 4\chi_0^\dagger \eta_0, \quad \epsilon = \frac{\pi}{L}, \quad (4.2)$$

i.e., the $\chi_p^{(\dagger)}$ and $\eta_p^{(\dagger)}$ fermions are creation and annihilation operators that generate an eigenvector of H of the energy $2\sin(p)$ and momentum p. We note that H is non-diagonalizable and its off-diagonal part $\chi_0^\dagger \eta_0$ generates Jordan cells of rank 2. The

---

[2] Note that the translation generator u^2 action in our spin-chains can be essentially expressed by products of e_j's, see more concrete statements in [22,23].

creation operators generate all root- and eigen-vectors of H from the space of ground states. The latter has the following structure:

$$
\begin{array}{ccc}
 & \boldsymbol{\omega} & \\
\chi_0^\dagger \swarrow & \Big\downarrow H & \searrow -\eta_0 \\
\phi^2 & & \phi^1 \\
\eta_0 \searrow & \Big\downarrow & \swarrow \chi_0^\dagger \\
 & \boldsymbol{\Omega} &
\end{array}
\tag{4.3}
$$

where the two bosonic states—the vacuum $\boldsymbol{\Omega}$ and the state $\boldsymbol{\omega}$—have $S^z = 0$ and form a two-dimensional Jordan cell for the lowest eigenvalue for H, while the two fermionic states ϕ^2 and ϕ^1 belong to the sectors with $S^z = +1$ and $S^z = -1$, respectively. We also showed in (4.3) the action of the quantum-group generators $\mathsf{F} \sim \eta_0$ and $\mathsf{E} \sim \chi_0^\dagger$.

All the excitations over the ground states $\boldsymbol{\omega}$ and $\boldsymbol{\Omega}$ are thus generated by the free action of the fermionic creation modes which are $\chi_p^\dagger$ and η_p, for $p \in (0, \pi)$, and $\eta_0^\dagger$ and χ_0 (see precise definitions of these operators in (A3)). The annihilation modes acts as

$$
\chi_p \boldsymbol{\Omega} = \eta_p^\dagger \boldsymbol{\Omega} = \chi_0^\dagger \boldsymbol{\Omega} = \eta_0 \boldsymbol{\Omega} = 0, \quad p \in (0, \pi). \tag{4.4}
$$

For the antiperiodic spin-chains, we have also the action of a Clifford algebra with $4L$ generators χ_p, η_p and adjoint ones $\chi_p^\dagger$, $\eta_p^\dagger$, where now the momenta run over a different set: $p = (m - \frac{1}{2})\pi/L$ and $1 \leq m \leq L$. In this case, there are no zero modes, and in particular the ground state of the Hamiltonian with anti-periodic conditions is non-degenerate [22].

4.1.3. A special basis in (twisted) JTL algebras. We now go back to the anti-periodic model and describe explicitly the representation of $\mathfrak{sp}_{2L}$ from Theorem 4.1.1. First, recall the well known fact that the bilinears in the generators of the Clifford algebra $\mathcal{C}_{4L}$ introduced in (4.1) that commute with the operator $S^z = 2\mathsf{h}$ (or with the fermion number operator) give a basis in the Lie algebra $\mathfrak{gl}_{2L}$. Indeed, introducing the elementary matrices $\mathcal{E}_{m,n}$ with matrix elements unity at the corresponding (m, n)th position and zero otherwise (a standard basis in $\mathfrak{gl}_{2L}$), we can write them in terms of fermions as

$$
\begin{array}{ll}
\mathcal{E}_{m,n} = \chi_p^\dagger \chi_q, & \mathcal{E}_{m,L+n} = \chi_p^\dagger \bar{\eta}_q, \\
\mathcal{E}_{L+m,n} = \bar{\eta}_p^\dagger \chi_q, & \mathcal{E}_{L+m,L+n} = \bar{\eta}_p^\dagger \bar{\eta}_q,
\end{array}
\quad 1 \leq m, n \leq L \tag{4.5}
$$

where we set $p = (m - \frac{1}{2})\pi/L$ and $q = (n - \frac{1}{2})\pi/L$, and we used also the notation $\bar{\eta}_p = \eta_{\pi-p}$, $\bar{\eta}_p^\dagger = \eta_{\pi-p}^\dagger$. One can check using the relations (4.1) that the defining relations for $\mathfrak{gl}_{2L}$:

$$
[\mathcal{E}_{m,n}, \mathcal{E}_{k,l}] = \delta_{n,k}\mathcal{E}_{m,l} - \delta_{m,l}\mathcal{E}_{k,n}, \quad 1 \leq m, n, k, l \leq 2L. \tag{4.6}
$$

are satisfied. We then note that the linear combinations

$$
\begin{array}{l}
\mathcal{A}_{m,n} = \mathcal{E}_{m,n} - \mathcal{E}_{L+n,L+m}, \quad \mathcal{B}_{m,n} = \mathcal{E}_{m,L+n} + \mathcal{E}_{n,L+m}, \\
\mathcal{C}_{m,n} = \mathcal{E}_{L+m,n} + \mathcal{E}_{L+n,m},
\end{array}
\quad 1 \leq m, n \leq L, \tag{4.7}
$$

span a Lie subalgebra in $\mathfrak{gl}_{2L}$ isomorphic to $\mathfrak{sp}_{2L}$ (note that $\mathcal{B}_{n,m} = \mathcal{B}_{m,n}$ and $\mathcal{C}_{n,m} = \mathcal{C}_{m,n}$), and these combinations commute with the action of $\mathfrak{sp}_2$ spanned by e, f, and h.

Moreover, these combinations span the maximum Lie subalgebra in $\mathfrak{gl}_{2L}$ commuting with the generators of $\mathfrak{sp}_2$: it is straightforward to check using (2.6) and transformations in Appendix A that the complement of the subspace $\mathfrak{sp}_{2L}$ in the $\mathfrak{gl}_{2L}$ does not commute with the action of e and f. We then recall [24] that the generators of the Howe-dual for the $\mathfrak{sp}_2$ are also bilinears in the fermionic operators. We thus obtain that the Lie algebra $\mathfrak{sp}_{2L}$ from Theorem 4.1.1 acts as in (4.7) and its centralizer is the $U s\ell(2)$ acting by (2.6). Because of the double-centralizing property the basis elements of $\mathfrak{sp}_{2L}$ therefore generate the action of the centralizer of $U s\ell(2)$ which is $\pi_{\mathfrak{gl}}(JTL_N^{tw})$, as was stated in Theorem 4.1.1. We do not give here explicit expressions of e_j's in terms of these basis elements of $\mathfrak{sp}_{2L}$ but will give some formulas below. What is important to note now is that taking products of the $\mathfrak{sp}_{2L}$ basis elements we obtain a special basis in the image of JTL_N^{tw} which is used later for taking the scaling limit of the JTL algebras.

Note finally that the $\mathcal{A}_{n,n}$ generators are basis elements in the Cartan subalgebra of $\mathfrak{sp}_{2L}$. Then, the simple modules $\mathcal{L}_{j,(-1)^j}$ over JTL_N^{tw} are highest-weight representations of $\mathfrak{sp}_{2L}$ with the weights[3] $(1, 1, \ldots, 1, 0, \ldots, 0)$, which are sequences of length L of consecutive 1's and then 0's, with j of 0's and $0 \leq j \leq L$. We thus obtain a decomposition (3.9) of the anti-periodic spin-chain with respect to the two commuting Lie algebras, $\mathfrak{sp}_2$ and $\mathfrak{sp}_{2L}$, where the $\mathfrak{sp}_2$ representation with j boxes corresponds to the jth fundamental (one-column) representation for $\mathfrak{sp}_{2L}$.

We then extend this classical Howe duality to the non-semisimple case which corresponds to the periodic model. The final result is given in Theorem 4.4 and Corollary 4.5 but first we discuss the semi-simple part of the action of the JTL_N algebra which parallels the previous discussion.

4.2. Semisimple part of JTL_{2L} and Lie algebra $\mathfrak{sp}_{2L-2}$.

For the periodic model, we introduce the standard basis for $\mathfrak{gl}_{2L}$ as in (4.5), where now $0 \leq m, n \leq L - 1$ and $p = m\pi/L$, and $q = n\pi/L$. Note once again that we have zero modes $\chi_0^{(\dagger)}$ and $\eta_0^{(\dagger)}$ in this case, with the identification $\bar{\eta}_0^{(\dagger)} = \eta_0^{(\dagger)}$. Then, the linear combinations from (4.7), with $1 \leq m, n \leq L - 1$, span a Lie subalgebra $\mathfrak{sp}_{2L-2}$. Note that we do not include zero modes because half of them do not commute with the symmetry algebra $U_q^{odd}s\ell(2)$. Recall that the $U_q^{odd}s\ell(2)$ generators E_m and F_n are represented in the spin-chain by $\mathsf{e}^m\mathsf{E}$ and $\mathsf{f}^n\mathsf{F}$, see (3.4) and note that $\mathsf{K}^2 = \mathbf{1}$ for even N. We can write [22] fermionic expression for these generators in the $\chi-\eta$ notation as

$$\mathsf{E}_m = (-1)^{S^z}\sqrt{N}\left(\sum_{p=\epsilon}^{\pi-\epsilon}\eta_p^\dagger\chi_{\pi-p}^\dagger\right)^m\chi_0^\dagger, \quad \mathsf{F}_n = -i\sqrt{N}\left(\sum_{p=\epsilon}^{\pi-\epsilon}\eta_p\chi_{\pi-p}\right)^n\eta_0. \quad (4.8)$$

It is then easy to check that the combinations of bilinears that give a basis in the $\mathfrak{sp}_{2L-2}$ commute with this action of $U_q^{odd}s\ell(2)$. Moreover, we show below that the bilinears in fermions (4.7) are the only bilinears that do not contain zero modes and belong to the image of JTL_N under the representation $\pi_{\mathfrak{gl}}$. Put a bit differently, we show that the operators $\mathcal{A}_{m,n}$, $\mathcal{B}_{m,n}$, and $\mathcal{C}_{m,n}$ generate the semisimple part of the associative algebra $\pi_{\mathfrak{gl}}(JTL_N)$, while we will see later that similar combinations containing zero modes $\chi_0^\dagger$ and η_0 generate its Jacobson radical. Indeed, the dimensions (3.7) of the simple

---

[3] Actually, for our choice of Cartan elements one should replace 1 by -1 in order to obtain correct weights with respect to $\mathcal{A}_{n,n}$.

JTL_{2L}-modules coincide with dimensions of fundamental representations of $\mathfrak{sp}_{2L-2}$, as in the anti-periodic case but now for a smaller symplectic Lie algebra. This observation suggests the following lemma.

Lemma 4.2.1. *The image of the representation* (4.7) *of the enveloping algebra* $U\mathfrak{sp}_{2L-2}$ *is isomorphic to the quotient* $\pi_{\mathfrak{gl}}(JTL_N)/\mathrm{Rad}$, *where* Rad *is the Jacobson radical of the image of the Jones–Temperley–Lieb algebra in the periodic* $\mathfrak{gl}(1|1)$ *spin-chain.*

Proof. We note that all simple modules $\mathcal{L}_{j,(-1)^{j+1}}$ over JTL_N that appear in the spin-chain module $\mathcal{H}_N$ appear also in the socle (maximum semisimple submodule) of $\mathcal{H}_N$, see Fig. 4. Then, in order to study the semisimple part of the action of JTL_N, it is enough to consider the action restricted on the socle of $\mathcal{H}_N$. Recall that simple modules over JTL_N are also simple modules over its subalgebra TL_N, see Sect. 3.3, and the centralizer of the TL_N action is given the Lusztig's (restricted specialization of) $U_q s\ell(2)$ at $q = i$ [39,40]—the one with the divided powers e and f. Therefore, the centralizer of the action restricted on the socle —the intersection of the kernels of F and E in $\mathcal{H}_N$—is given by a representation of $U\mathfrak{sp}_2$ realized by the divided powers of $U_i s\ell(2)$. Since the socle is freely generated from the vacuum state $\boldsymbol{\Omega}$ by the action of a Clifford algebra $\mathcal{C}_{4L-4}$ with $4L-4$ generators $\chi_p^{(\dagger)}, \eta_p^{(\dagger)}$, where $p = m\pi/L$ and $1 \leq m \leq L-1$, we have a classical symplectic Howe duality [24] between $\mathfrak{sp}_2$ and $\mathfrak{sp}_{2L-2}$. Indeed, the socle is a multiplicity-free semisimple bimodule over JTL_N and $U\mathfrak{sp}_2$, where both algebras are represented by appropriate bilinears in the generators of $\mathcal{C}_{4L-4}$. By Howe duality, we obtain that the action of JTL_N on the socle of $\mathcal{H}_N$, and thus its semisimple part on $\mathcal{H}_N$, is generated by the $\mathfrak{sp}_{2L-2}$ action. ☐

In the open case, the simple modules are the same as for JTL_N in the periodic model. Therefore, we conclude that the semisimple part of TL_N, or the quotient TL_N/Rad by its Jacobson radical, is also generated by an $\mathfrak{sp}_{N-2}$ action.

Using Lemma 4.2.1 and a correspondence between weights in the symplectic Howe duality [24,41], we obtain the following corollary.

Corollary 4.2.2. *The simple modules* $\mathcal{L}_{j,(-1)^{j+1}}$, *with* $1 \leq j \leq L$, *over* JTL_{2L} (*or* TL_{2L}) *are simple modules over the enveloping algebra for* $\mathfrak{sp}_{2L-2}$. *The dimensions* (3.7) *of all these simple modules correspond to all highest-weight representations of* $\mathfrak{sp}_{2L-2}$ *of the weights* $(1, 1, \ldots, 1, 0, \ldots, 0)$, *which are sequences of length* $L-1$ *of consecutive* 1*'s and then* 0*'s, with* $j-1$ *of* 0*'s.*

We note once again that the generators $\mathcal{A}_{n,n}$, now with $1 \leq n \leq L-1$, span a basis in the Cartan subalgebra of the $\mathfrak{sp}_{2L-2}$. The diagonal part of the Hamiltonian H has a very simple expression in terms of these generators:

$$H = 2 \sum_{m=1}^{L-1} \sin(p) \, \mathcal{A}_{m,m} + 4 \, \mathcal{E}_{0,L}. \tag{4.9}$$

This allows us to describe JTL_N simple modules as highest-weight representations of $\mathfrak{sp}_{2L-2}$ where highest-weight vectors play the role of charged vacua for the Hamiltonian. The vacuum state $\boldsymbol{\Omega}$ in the space of ground states coincides with the highest-weight vector of the unique $\mathfrak{sp}_{2L-2}$ module of the weight $(1, 1, \ldots, 1, 1)$, the next one corresponds to the weight $(1, 1, \ldots, 1, 0)$, etc.

4.3. The image of JTL_N as a Lie algebra representation. We introduce now special elements in JTL_N which span a subspace of all elements in JTL_N that are bilinear in the χ-η fermions. We begin with the definition of the Lie algebra $\mathfrak{S}_N$.

Definition 4.3.1. We define the Lie algebra $\mathfrak{S}_N$ to be generated by $\mathcal{A}_{m,n}$, $\mathcal{B}_{m,n}$, $\mathcal{C}_{m,n}$ from (4.7) and the elementary matrices $\mathcal{E}_{0,n}$, $\mathcal{E}_{0,L}$, $\mathcal{E}_{0,L+n}$, $\mathcal{E}_{m,L}$, and $\mathcal{E}_{L+m,L}$, where $1 \leq m, n \leq L - 1$. This Lie algebra can be schematically depicted by matrices of the form (in the standard basis of $\mathfrak{gl}_{2L}$)

$$
\mathfrak{S}_N : \begin{pmatrix}
0 & \times & \ldots & \times & \times & \times & \ldots & \times \\
0 & & & & \times & & & \\
0 & & \mathcal{A}_{m,n} & & \times & & \mathcal{B}_{m,n} & \\
0 & & & & \times & & & \\
0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\
0 & & & & \times & & & \\
0 & & \mathcal{C}_{m,n} & & \times & & & \\
0 & & & & \times & & &
\end{pmatrix}, \tag{4.10}
$$

where the crosses $\times$ stand for the corresponding elements $\mathcal{E}_{0,n}$, $\mathcal{E}_{0,L}$, $\mathcal{E}_{0,L+n}$, $\mathcal{E}_{m,L}$, and $\mathcal{E}_{L+m,L}$.

Note that $\mathfrak{S}_N$ is a non-semisimple Lie algebra and admits $\mathfrak{sp}_{2L-2}$ spanned by $\mathcal{A}_{m,n}$, $\mathcal{B}_{m,n}$, and $\mathcal{C}_{m,n}$ as a Lie subalgebra. The dimension of $\mathfrak{S}_N$ is

$$
\dim \mathfrak{S}_N = \frac{(N-1)(N+2)}{2} - 1, \tag{4.11}
$$

where we recall that $\mathcal{B}$- and $\mathcal{C}$-blocks are symmetric matrices. The Lie radical of $\mathfrak{S}_N$ is generated by $\mathcal{E}_{0,n}, \mathcal{E}_{0,L}, \mathcal{E}_{0,L+n}, \mathcal{E}_{m,L}$, and $\mathcal{E}_{L+m,L}$: these are the generators corresponding the the the crosses $\times$ in (4.10).

We next show that the representation of this Lie algebra of fermionic bilinears in the periodic spin-chain generates the image of JTL_N, proving the following theorem.

Theorem 4.4. *The image of the representation* (4.5)–(4.7) *of the enveloping algebra* $U\mathfrak{S}_N$ *is equal to the image* $\pi_{\mathfrak{gl}}(JTL_N)$ *of the Jones–Temperley–Lieb algebra in the periodic* $\mathfrak{gl}(1|1)$ *spin-chain.*

Proof. We first prove that the centralizer of the $U\mathfrak{S}_N$ action is given by $\mathfrak{Z}_{JTL}$—the centralizer of $\pi_{\mathfrak{gl}}(JTL_N)$. Recall that $\mathfrak{Z}_{JTL}$ is described in Sect. 3.2. By a direct computation, we see that the image of the representation (4.5)–(4.7) of the enveloping algebra $U\mathfrak{S}_N$ commutes with the $U_q^{\mathrm{odd}} s\ell(2)$ action given in (4.8). We then show that the generators of $\mathfrak{S}_N$ are the only bilinears in the χ-η fermions that commute with $U_q^{\mathrm{odd}} s\ell(2)$. Let us for this introduce bilinears as in (4.7) but with the opposite sign and denote them by $\mathcal{A}'_{m,n}$, $\mathcal{B}'_{m,n}$, and $\mathcal{C}'_{m,n}$, respectively. Together with the bilinears $\mathcal{E}_{n,0}$ and $\mathcal{E}_{L,n}$, they belong to the complement of the vector space $\mathfrak{S}_N$. We then compute commutators of all these bilinears which are not in $\mathfrak{S}_N$ with the generator E_1. It turns out that all these commutators are non-zero and linearly independent bilinears. So, there are no linear combinations among $\mathcal{A}'_{m,n}$, $\mathcal{B}'_{m,n}$, $\mathcal{C}'_{m,n}$, $\mathcal{E}_{n,0}$, and $\mathcal{E}_{L,n}$ that would commute with E_1. This proves the statement that $\mathfrak{S}_N$ is the maximum Lie subalgebra of bilinears (in the χ-η fermionic operators) which commute with $U_q^{\mathrm{odd}} s\ell(2)$. We then note that the image of JTL_N is generated by e_j's which are also bilinears in the χ-η and they commute with $U_q^{\mathrm{odd}} s\ell(2)$. This image is thus contained in the image of $U\mathfrak{S}_N$ and both algebras

have the common centralizer $\mathfrak{Z}_{JTL}$ (it is easy to check that $\mathfrak{S}_N$ also commutes with the $\mathfrak{f}^L$ and $\mathfrak{e}^L$).

To finish our proof, we need to show that the image of $U\mathfrak{S}_N$ is not bigger than the image of JTL_N. To show this, we use a representation-theoretic approach. We recall the subquotient structure for each $S^z = \pm j$ sector $\widehat{\mathcal{P}}_j$ considered as a module over the centralizer of $\mathfrak{Z}_{JTL}$ which is isomorphic by the definition to the algebra $\mathrm{End}_{\mathfrak{Z}_{JTL}}(\mathcal{H}_N)$. The centralizer obviously contains $\pi_{\mathfrak{gl}}(JTL_N)$ and, following the previous paragraph, the image of $U\mathfrak{S}_N$ as well. The opposite inclusion is not true as was shown in our second paper [23, Sec. 5] and we thus can not rest on a double-centralizing argument.

The subquotient structure can be obtained using intertwining operators respecting $\mathfrak{Z}_{JTL}$ action. These are described in Thm. 3.4.4 in [23]. The only difference from the diagrams for JTL_N in Fig. 4 is that there are additional ('long') arrows mapping a top subquotient (d_j^0) (having only outgoing arrows) to (d_k^0) in the socle (having only ingoing arrows) whenver $|j-k| \geq 4$ is an even number. We note these long arrows are not composites of any short arrows mapping from the top to the middle level,[4] and from the middle to the socle. It turns out that $\mathfrak{S}_N$ generators correspond only to these short arrows and not to the long ones, and therefore there is no element from $\mathfrak{S}_N$ represented by a long arrow. This can be shown using a direct calculation with the fermionic expressions (4.5)–(4.7). Due to Lemma 4.2.1, we need to analyze only the radical of $\mathfrak{S}_N$ generated by $\mathcal{E}_{0,n}$, $\mathcal{E}_{0,L}$, $\mathcal{E}_{0,L+n}$, $\mathcal{E}_{m,L}$, and $\mathcal{E}_{L+m,L}$. These are represented by bilinears in the Clifford algebra $\mathcal{C}_{4L}$ generators. As was noted before, $N-2$ creation modes $\chi_{p>0}^{\dagger}$ and $\eta_{p>0}$ generate the bottom level—the intersection of the kernels of F and E in $\mathcal{H}_N$—from the vacuum state $\boldsymbol{\Omega}$, and also the top level from one cyclic vector $\boldsymbol{\omega}$ which is involved with $\boldsymbol{\Omega}$ into a Jordan cell for the Hamiltonian H. Among the Clifford algebra generators, there are two—zero modes η_0 and $\chi_0^{\dagger}$—proportional to F and EK^{-1}, respectively. These are the only generators mapping vectors from the top level to the middle level, and from the middle to the bottom level. Among the generators in the radical of $\mathfrak{S}_N$, there is only one, $\mathcal{E}_{0,L}$, which is the product of the two zero modes. The product maps the top to the bottom but commutes with the JTL_N action and thus maps a top subquotient (d_j^0) only to the bottom (d_j^0). All other generators of the radical are bilinears in fermions containing only one of the zero modes and they thus map only by one level down. Therefore, they correspond to the short arrows in the diagram in Fig. 4. We conclude that the action of $U\mathfrak{S}_N$ can not correspond to long arrows connecting the top and the bottom and which are not composites of any short arrows because any element of the radical in the image of $U\mathfrak{S}_N$ is a linear combination of monomials in its generators.

We can thus conclude that the modules over the enveloping algebra of $\mathfrak{S}_N$ in the spin-chain representation have the same subquotient structure as for the $\pi_{\mathfrak{gl}}(JTL_N)$ and therefore their Jacobson radicals are isomorphic. In addition to the analysis on their semisimple parts given in Lemma 4.2.1—both algebras have the same equivalence classes of simple modules—we also conclude that they are isomorphic as associative algebras. This finally proves the theorem. □

As a consequence of Theorem 4.4, we get the following corollary which we consider as a non-semisimple version of the (symplectic) Howe duality.

Corollary 4.5. *The centralizer $\mathfrak{Z}_{U\mathfrak{S}_N}$ of the universal enveloping algebra $U\mathfrak{S}_N$ in the periodic $\mathfrak{gl}(1|1)$ spin-chain representation in (4.5)–(4.7) is given by $\mathfrak{Z}_{JTL}$, which is generated by the image of $U_{\mathfrak{q}}^{odd}s\ell(2)$ in (4.8) and $\mathfrak{f}^L$ and $\mathfrak{e}^L$.*

---

[4] The level consisting of all nodes having both ingoing and outgoing arrows.

Note that in this case the "Howe-dual" algebra is not described as the enveloping of a Lie algebra of bilinears, in contrast to the anti-periodic case. It is rather a $\mathbb{Z}_2$-graded algebra due to presence of the $\mathfrak{gl}(1|1)$ subalgebra. It is easier to show for the open case, where the TL_N is the enveloping algebra of a non-semisimple Lie algebra (having maximum semisimple Lie subalgebra $\mathfrak{sp}_{2L-2}$ and its radical is smaller than in the JTL_N case) and the action of its centralizer $U_i s\ell(2)$ is generated by the semidirect product $\mathfrak{sp}_2 \ltimes \mathfrak{gl}(1|1)$ of the Lie algebras $\mathfrak{sp}_2$ and $\mathfrak{gl}(1|1)$.

4.6. Normal ordered basis. We introduce here a few formalities which will be useful to analyze the $N \to \infty$ limits of our algebras. Recall that the excitations over the space of ground states are generated by the creation modes $\chi_p^\dagger$ and η_p, with $p \in (0, \pi)$, while the annihilation modes are χ_p and $\eta_p^\dagger$. We can thus introduce normal ordering prescriptions like $:\chi_p\chi_{p'}^\dagger: \equiv -\chi_{p'}^\dagger\chi_p$ and $:\eta_p^\dagger\eta_{p'}: \equiv -\eta_{p'}\eta_p^\dagger$, etc. Then, a normal ordered basis in $\mathfrak{gl}_{2L} \oplus \mathbb{C}\mathbf{1}$ (a trivial central extension) is given by $\mathcal{E}'_{m,n} = :\mathcal{E}_{m,n}:$, where the elementary matrices $\mathcal{E}_{m,n}$, for $0 \leq m, n \leq 2L-1$, are introduced above in (4.5). The normal ordering affects only one half of the Cartan elements: $\mathcal{E}'_{n,n} = \mathcal{E}_{n,n}$ while $\mathcal{E}'_{L+n,L+n} = \mathcal{E}_{L+n,L+n} - \mathbf{1}$, where now $0 \leq n \leq L - 1$ and $\mathbf{1}$ is the identity in End $\mathcal{H}_{2L}$. The defining relations in the normally ordered basis are slightly changed—they have the central element term:

$$[\mathcal{E}'_{m,L+n}, \mathcal{E}'_{L+k,l}] = \delta_{n,k}\mathcal{E}'_{m,l} - \delta_{m,l}\mathcal{E}'_{L+k,L+n} - \delta_{n,k}\delta_{m,l}\mathbf{1}, \tag{4.12}$$

while all the other commutators have the same form as in the standard basis (4.6).

We also consider central extensions of the Lie subalgebras $\mathfrak{sp}_{2L-2} \subset \mathfrak{gl}_{2L}$ and $\mathfrak{S}_{2L} \subset \mathfrak{gl}_{2L}$ by the identity $\mathbf{1}$. So, we introduce the Lie algebras

$$\mathfrak{sp}'_{2L-2} = \mathfrak{sp}_{2L-2} \oplus \mathbb{C}\mathbf{1} \quad \text{and} \quad \mathfrak{S}'_N = \mathfrak{S}_N \oplus \mathbb{C}\mathbf{1}. \tag{4.13}$$

Then, we choose the normal ordered basis in their Cartan subalgebras as

$$\mathcal{A}'_{n,n} \equiv :\mathcal{A}_{n,n}: = \mathcal{E}'_{n,n} - \mathcal{E}'_{L+n,L+n} = \mathcal{A}_{n,n} + \mathbf{1}, \tag{4.14}$$

see notations in (4.7). Similarly to (4.12), the commutators $[\mathcal{B}_{m,n}, \mathcal{C}_{k,l}]$ have the central element term in the normally ordered basis in $\mathfrak{sp}'_{2L-2}$ while other relations are not changed.

Note finally that the weights of highest-weight $\mathfrak{sp}'_{2L}$-modules are now different due to the different choice of Cartan elements (or simple roots). For example, the vacuum irreducible module $\mathcal{L}_{1,1}$ has now the weight $(0, 0, \ldots, 0)$, the next one $\mathcal{L}_{2,-1}$ corresponds to the weight $(1, 0, \ldots, 0)$, etc.

5. Scaling Limit

We are now interested in studying how the algebraic properties of the finite spin chain relate with those of its continuum—or scaling—limit. Our ultimate purpose is to extract from the $\mathfrak{gl}(1|1)$ case lessons that can be used in other, more complicated situations. We shall for this purpose, introduce and discuss in the next section the new concept of interchiral algebra and its relationship with fields in a LCFT setting. For now however, we do not discuss field theory *per se*, and define and discuss the scaling limit of the spin chains using fermionic modes.

In this section, we consider mostly the periodic model and discuss briefly the anti-periodic case only at the end in Sect. 5.5. The limit of JTL_N algebras is constructed in Sects. 5.1 and 5.2. We discuss in Sect. 5.3 the structure of the bimodule over the two commuting algebras, $U_q^{odd} s\ell(2)$ and the scaling limit of JTL_N or $U\mathfrak{S}_N$ algebras denoted by $U\mathfrak{S}_\infty$. In Sect. 5.4, we also describe simple modules over $U\mathfrak{S}_\infty$ that appear in the space $\mathcal{H}$ of scaling states. Finally, the full generating functions of energy levels are computed in Sect. 5.6.

5.1. The full scaling limit of the closed $\mathfrak{gl}(1|1)$ spin-chains.

In this section, we recall the construction of the scaling limit [22] of the (anti)periodic spin-chain that gives a conformal field theory model. We then use the Lie algebraic reformulation of the (images of) JTL_N and JTL_N^{tw} to construct the full scaling limit of these algebras. An essential ingredient in the general definition of the scaling limit is the low-lying eigenstates of the Hamiltonian H. In order to study the action of JTL elements on these eigenstates in the limit $L \to \infty$ (recall $N = 2L$) we first truncate each $\mathcal{H}_{2L}$, keeping only eigenspaces up to an energy level M, for each positive number M. Each such truncated space turns out to be finite-dimensional in the limit, i.e., it depends on M but not L. Then, keeping matrix elements of JTL elements that correspond to the action only within these truncated spaces of "scaling" states, we obtain well-defined operators in the limit $L \to \infty$. The corresponding operators acting on all scaling states of the CFT can be finally obtained (if they exist) in the second limit $M \to \infty$. A bit more formally, *the scaling limit* denoted simply by '$\mapsto$' is defined as a limit over graded spaces of coinvariants with respect to smaller and smaller subalgebras in the creation modes algebra, see details in Sec. 4.3 from [22]. Meanwhile, in the case of our $\mathfrak{gl}(1|1)$ spin-chains we can actually give a clearer definition of the scaling limit (of algebras and their modules) by means of inductive systems, see Appendix C.

5.1.1. The Clifford algebra scaling limit.

We first recall [22] the scaling limit of the Clifford algebra $\mathcal{C}_{4L}$ generators introduced in Sect. 4.1.2—the χ and η fermions—and then define a representation of $\mathfrak{gl}_\infty$ which contains the full scaling limit of JTL. The lattice fermions appropriately rescaled coincides in the scaling limit with the symplectic-fermions modes:

$$\sqrt{m}\,\chi_p \mapsto \psi_m^1, \quad \sqrt{m}\,\bar\eta_p^\dagger \mapsto \psi_m^2, \quad \sqrt{m}\,\bar\chi_p \mapsto \bar\psi_m^1, \quad -\sqrt{m}\,\eta_p^\dagger \mapsto \bar\psi_m^2,$$
$$-\sqrt{m}\,\bar\eta_p \mapsto \psi_{-m}^1, \quad \sqrt{m}\,\chi_p^\dagger \mapsto \psi_{-m}^2, \quad \sqrt{m}\,\eta_p \mapsto \bar\psi_{-m}^1, \quad \sqrt{m}\,\bar\chi_p^\dagger \mapsto \bar\psi_{-m}^2, \tag{5.1}$$

where we consider any finite integer $m > 0$ and set $p = m\pi/L$ and $\bar\eta_p^{(\dagger)} = \eta_{\pi-p}^{(\dagger)}$, and $\bar\chi_p^{(\dagger)} = \chi_{\pi-p}^{(\dagger)}$, while the scaling limit for zero modes is

$$\sqrt{L/\pi}\,\eta_0 \mapsto \psi_0^1 = \bar\psi_0^1, \quad \sqrt{L/\pi}\,\chi_0^\dagger \mapsto \psi_0^2 = \bar\psi_0^2,$$
$$\sqrt{\pi/L}\,\eta_0^\dagger \mapsto i\phi_0^2, \quad \sqrt{\pi/L}\,\chi_0 \mapsto -i\phi_0^1. \tag{5.2}$$

Indeed, having this limit we obtain that the scaling limit of the Hamiltonian $H = \sum_{j=1}^N e_j$ gives the zero mode of the stress energy tensor $T(z) + \bar T(\bar z)$ in the symplectic fermions theory:

$$\frac{L}{2\pi}\left(H + \frac{4L}{\pi}\right) \mapsto \sum_{m\in\mathbb{Z}} :\psi_{-m}^2\psi_m^1: + (\psi \to \bar\psi) - \frac{c}{12} = L_0 + \bar L_0 - \frac{c}{12}, \tag{5.3}$$

where we use the fermionic normal ordering prescription introduced in Sect. 4.6 which takes the form [25]

$$:\psi_m^\alpha \psi_n^\beta: = \begin{cases} \psi_m^\alpha \psi_n^\beta & for\ m < 0, \\ -\psi_n^\beta \psi_m^\alpha & for\ m > 0. \end{cases} \tag{5.4}$$

We note that the fermionic operators act on the space $\mathcal{H}$ of scaling states, which are limits of low-energy eigenvectors[5] of the Hamiltonian H, in the sense of construction in Appendix C. The ground subspace in $\mathcal{H}$ has the same structure (4.3) as in the finite chain $\mathcal{H}_N$, where one should replace $\chi_0^\dagger$ and η_0 by the zero modes ψ_0^2 and ψ_0^1, respectively. The module structure (over the infinite-dimensional Clifford algebra) on $\mathcal{H}$ containing all the excitations over the ground states $\boldsymbol{\omega}$, $\psi_0^2\boldsymbol{\omega}$, $\psi_0^1\boldsymbol{\omega}$ and $\psi_0^1\psi_0^2\boldsymbol{\omega} = \boldsymbol{\Omega}$ is thus generated by the free action of the creation modes $\psi_n^{1,2}$ and $\bar{\psi}_m^{1,2}$, for integers $n, m < 0$. The annihilation modes, or positive modes, act on the vacuum states in the usual way.

We have also shown in our first paper [22] that the limit (5.1) allows one to obtain all left and right Virasoro modes L_n and $\bar{L}_n$ in the symplectic fermions representation

$$L_n = \sum_{m\in\mathbb{Z}} :\psi_{n-m}^2 \psi_m^1:, \quad \bar{L}_n = \sum_{m\in\mathbb{Z}} :\bar{\psi}_{n-m}^2 \bar{\psi}_m^1:, \quad n \in \mathbb{Z}, \tag{5.5}$$

as the scaling limit of particular JTL elements, see also more details below in Sect. 6.1. A comment is necessary about the infinite sums in the definition of L_n's and $\bar{L}_n$'s. Note that the Hilbert space $\mathcal{H}$ has bi-grading by the pair $(L_0, \bar{L}_0)$, where each homogeneous or root subspace $\mathcal{H}^{n,m}$ for positive n and m is spanned by states of the form $\psi_{n_1}^{1,2} \ldots \psi_{n_k}^{1,2} \bar{\psi}_{m_1}^{1,2} \ldots \bar{\psi}_{m_l}^{1,2}$ acting on any ground state such that $\sum_{i=1}^k n_i = -n$ and $\sum_{j=1}^l m_l = -m$. The space $\mathcal{H}$ then can be described as the direct sum $\oplus_{n,m}\mathcal{H}^{n,m}$ which means that any state $v \in \mathcal{H}$ can be written as $v = \sum_{n,m} v^{(n,m)}$, with $v^{(n,m)} \in \mathcal{H}^{n,m}$, and there exist positive integers N and M such that $v^{(n,m)} = 0$ for any $n > N$ or $m > M$. This is what we call *the finite-energy and finite-spin* (or simply *scaling*) *states space* and it is a dense[6] subspace in the the direct product of the root subspaces $\mathcal{H}^{n,m}$. Therefore, the formally infinite sums in (5.5) actually reduce to finite sums in the scaling states space $\mathcal{H}$. This is standard, and discussed for instance in [45].

5.1.2. The symplectic fermion representation of $\mathfrak{gl}_\infty$. We recall that the bilinears in the χ-η fermions give a standard basis $\mathcal{E}_{m,n}$, the usual elementary matrices, in the Lie algebra $\mathfrak{gl}_{2L}$, see Sect. 4.2 for details. Due to the fact that low-lying eigenstates are generated by the Clifford generators with their momenta p close to 0 or π (in the large L limit), we keep in the scaling limit all bilinears in the fermions with any of these momenta. Note that basis elements (4.5) of $\mathfrak{gl}_{2L}$ can be divided into 4 blocks with the (m, n) indices running from 0 to $L - 1$. Because in our limit we keep only those indices close to 0 or to L, then each block is further divided into four infinite blocks in the scaling limit. This limit of $\mathfrak{gl}_{2L}$ can be schematically described in terms of its standard generators as

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[5] It is better actually to say 'root-vectors' because the Hamiltonian is not diagonalizable on $\mathcal{H}_N$ and we have a basis where relations such as $(H - \lambda\mathbf{1})^2 v = 0$, for $v \in \mathcal{H}_N$, are satisfied.

[6] See the discussion of completions and topology in Sect. 6.2 after (6.41).

$$\left[\, \mathcal{E}_{m,n} = \chi_p^\dagger \chi_q \,\right]_{0 \le p,q \le L-1} \xrightarrow{\;L \to \infty\;}
\begin{matrix}
\chi_0^\dagger \chi_0 & \chi_0^\dagger \chi_\epsilon & \cdots & \chi_0^\dagger \bar{\chi}_{2\epsilon} & \chi_0^\dagger \bar{\chi}_\epsilon \\
\chi_\epsilon^\dagger \chi_0 & \chi_\epsilon^\dagger \chi_\epsilon & \cdots & \chi_\epsilon^\dagger \bar{\chi}_{2\epsilon} & \chi_\epsilon^\dagger \bar{\chi}_\epsilon \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\bar{\chi}_{2\epsilon}^\dagger \chi_0 & \bar{\chi}_{2\epsilon}^\dagger \chi_\epsilon & \cdots & \bar{\chi}_{2\epsilon}^\dagger \bar{\chi}_{2\epsilon} & \bar{\chi}_{2\epsilon}^\dagger \bar{\chi}_\epsilon \\
\bar{\chi}_\epsilon^\dagger \chi_0 & \bar{\chi}_\epsilon^\dagger \chi_\epsilon & \cdots & \bar{\chi}_\epsilon^\dagger \bar{\chi}_{2\epsilon} & \bar{\chi}_\epsilon^\dagger \bar{\chi}_\epsilon
\end{matrix} \tag{5.6}$$

for the left-top block in the matrix algebra, and similarly for other three blocks. Here, we set for the momenta variables $p = m\epsilon$, $q = n\epsilon$, with $\epsilon = \pi/L$, and these momenta on the right-hand side are formal variables—one should of course make substitutions (5.1) and (5.2). We emphasize that each item of the block on the right is an infinite elementary matrix having identity at the corresponding position and zeros otherwise. These elementary matrices (for all four blocks) define a representation of an infinite-dimensional Lie algebra $\mathfrak{gl}_\infty$ on the space $\mathcal{H}$ which we call the symplectic-fermion representation of $\mathfrak{gl}_\infty$.

A comment is necessary about the exact definition of the algebra $\mathfrak{gl}_\infty$. Any element of $\mathfrak{gl}_\infty$ is an infinite matrix with a *finite* number of non-zero elements, i.e., it is represented on $\mathcal{H}$ by a finite sum of the generators—the elementary matrices. The commutation relations in this algebra are given by corresponding limits of (4.6) and they correspond to usual basis and relations [46] in $\mathfrak{gl}_\infty$ after appropriate rearranging rows and columns, see more precise statements in Appendix B.

We note that formally the Virasoro generators L_n's and $\bar{L}_n$'s from (5.5) do not belong to such defined $\mathfrak{gl}_\infty$ (recall there exist many versions of $\mathfrak{gl}_\infty$ algebras [46]) but belong, for $n \ne 0$, to its completed version $\overline{\mathfrak{gl}}_\infty$, where there is now a possibly infinite number of non-zero elements, but the matrix still has a finite number of non zero diagonals, see details in Appendix B. The only problem is with L_0 and $\bar{L}_0$, because of the well-known central anomaly that appears in CFT. In order to get a convenient algebra containing the L_0 and $\bar{L}_0$ operators we should actually take the scaling limit of the family of algebras $\{\mathfrak{gl}_{2L}\}$ in their normally ordered basis $\mathcal{E}'_{m,n}$ introduced in Sect. 4.6. Using then (5.1)–(5.2), we obtain finally the generators (after rescaling by $\pm\sqrt{mn}$) of another infinite-dimensional Lie algebra we call $\mathfrak{gl}'_\infty$ given by the following list of bilinears

$$\mathfrak{gl}'_\infty = \langle\, {:}\psi_m^2 \psi_n^1{:}, \quad {:}\bar{\psi}_m^2 \psi_n^1{:}, \quad {:}\psi_m^2 \bar{\psi}_n^1{:}, \quad {:}\bar{\psi}_m^2 \bar{\psi}_n^1{:}, \quad {:}\psi_m^2 \phi_0^1{:}, \quad {:}\bar{\psi}_m^2 \phi_0^1{:},$$
$$\quad {:}\phi_0^2 \psi_n^1{:}, \quad {:}\phi_0^2 \bar{\psi}_n^1{:}, \quad {:}\phi_0^2 \phi_0^1{:}, \quad m, n \in \mathbb{Z}\rangle, \tag{5.7}$$

which are now all normally ordered (note that some of these bilinears, $\psi_{-n}^1 \psi_n^2$, are not elementary matrices). This Lie algebra turns out to be a central extension of the $\mathfrak{gl}_\infty$, i.e., it is $\mathfrak{gl}'_\infty = \mathfrak{gl}_\infty \oplus \mathbb{C}\mathbf{1}$, as a vector space. The central element $\mathbf{1}$ is obtained by the scaling limit of the identity in the (trivial) central extension $\mathfrak{gl}_{2L} \oplus \mathbb{C}\mathbf{1}$ considered in Sect. 4.6, see also a formal inductive limit construction in Appendix C. Commutation relations in this algebra are obtained as limits of the relation (4.12) which fixes the central charge of $\mathfrak{gl}'_\infty$. The contribution of the central term in commutators of the new generators can be also obtained using the two-cocycle [46] $c(\mathcal{E}'_{m,n}, \mathcal{E}'_{m',n'})$ and it coincides with the vacuum expectation of the commutator $[\mathcal{E}'_{m,n}, \mathcal{E}'_{m',n'}]$ as it should. Now, all the left–right Virasoro operators on $\mathcal{H}$ are embedded into the completed Lie algebra $\overline{\mathfrak{gl}}'_\infty$ including L_0 and $\bar{L}_0$. We will actually work below with this Lie algebra $\overline{\mathfrak{gl}}'_\infty$ containing the Virasoro generators and not with $\mathfrak{gl}_\infty$.

5.1.3. The scaling limit of the JTL algebras, $\mathfrak{sp}_\infty$, and $\mathfrak{S}_\infty$. The scaling limit of the (spin-chain representations of) JTL_N algebras can then be taken using the Lie algebraic description given in Theorem 4.4. It was shown there that the spin-chain representation of JTL_N is given by a representation of the enveloping algebra $U\mathfrak{S}_N$. We thus need only to take the scaling limit of the central extension $\mathfrak{S}'_N = \mathfrak{S}_N \oplus \mathbb{C}\mathbf{1}$ introduced in Sect. 4.6 which is straightforward since there is an embedding of $\mathfrak{S}_N$ into the $\mathfrak{gl}_{2L}$, see Definition 4.10. In particular, the Lie subalgebra $\mathfrak{sp}'_{2L-2} \subset \mathfrak{S}'_N$ from (4.13) generated by normally ordered elements in (4.14) and (4.7) has the limit generated by the following quadratic expressions

$$J_{\alpha\beta}{:}\psi_m^\alpha \psi_n^\beta{:}, \quad J_{\alpha\beta}{:}\bar\psi_m^\alpha \bar\psi_n^\beta{:}, \quad S_{\alpha\beta}\psi_m^\alpha \bar\psi_n^\beta, \quad m, n \in \mathbb{Z}/\{0\}, \tag{5.8}$$

where $J_{\alpha\beta}$ is such that $J_{\alpha\beta}J^{\beta\gamma} = \delta_\alpha^\gamma$ and we use the symplectic form $J^{12} = -J^{21} = 1$, and $S_{\alpha\beta}$ is the symmetric form $S_{12} = S_{21} = 1$, $S_{11} = S_{22} = 0$. We show below that the vector space spanned by finite linear combinations of these operators is closed under taking commutators. The infinite-dimensional Lie algebra they generate coincides with the central extension of the Lie algebra $\mathfrak{sp}_\infty \subset \mathfrak{gl}_\infty$.

Further, we introduce $S_{m,n}$, $T_{m,n}$, and $\bar T_{m,n}$ as the operators

$$S_{m,n} = S_{\alpha\beta}\psi_m^\alpha \bar\psi_n^\beta, \quad T_{m,n} = J_{\alpha\beta}\psi_m^\alpha \psi_n^\beta, \quad \bar T_{m,n} = J_{\alpha\beta}\bar\psi_m^\alpha \bar\psi_n^\beta, \quad \text{with} \quad n, m \in \mathbb{Z}, \tag{5.9}$$

i.e., as in (5.8) but without normal ordering and including combinations involving the zero fermionic modes ψ_0^1 and ψ_0^2 (note that there are no bilinears here involving conjugate modes ϕ_0^2 and ϕ_0^1). Actually, we have the symmetry $T_{m,n} = T_{n,m}$ and $\bar T_{m,n} = \bar T_{n,m}$. The operators in (5.9) together with the central term $\mathbf{1}$ give now generators of the scaling limit of $\mathfrak{S}'_N$ which we denote simply by $\mathfrak{S}_\infty \subset \mathfrak{gl}'_\infty$. We conclude that the corresponding action of $U\mathfrak{S}_\infty$ is *the scaling limit* of the (spin-chain images of the) JTL algebras. We give in Appendix C a more formal construction of this scaling-limit algebra as a direct/inductive limit $\varinjlim_L \pi_{\mathfrak{gl}}(JTL_{2L})$ of the finite-dimensional spin-chain representations $\pi_{\mathfrak{gl}}$ of the JTL algebras.

5.2. Commutation relations in $\mathfrak{S}_\infty$. We next obtain commutation relations between the generators $S_{n,m}$, $T_{m,n}$ and $\bar T_{m,n}$ of $\mathfrak{S}_\infty$. Using a straightforward calculation and identities like $J_{\alpha\beta}J_{\gamma\delta}J^{\beta\gamma} = J_{\alpha\delta}$, we first obtain

$$[T_{m,n}, T_{k,l}] = [J_{\alpha\beta}\psi_m^\alpha \psi_n^\beta, J_{\gamma\delta}\psi_k^\gamma \psi_l^\delta] = J_{\alpha\beta}n\big(\delta_{n+k,0}\,\psi_m^\alpha \psi_l^\beta + \delta_{n+l,0}\,\psi_m^\alpha \psi_k^\beta\big)$$
$$+ J_{\alpha\beta}m\big(\delta_{m+k,0}\,\psi_l^\alpha \psi_n^\beta + \delta_{m+l,0}\,\psi_k^\alpha \psi_n^\beta\big) \tag{5.10}$$

and a similar expression with $\psi_n^\alpha \to \bar\psi_n^\alpha$. We then have

$$[S_{m,n}, T_{k,l}] = S_{\alpha\beta}\,m\big(\delta_{k+m,0}\,\psi_l^\alpha \bar\psi_n^\beta + \delta_{m+l,0}\,\psi_k^\alpha \bar\psi_n^\beta\big), \tag{5.11}$$

$$[S_{m,n}, \bar T_{k,l}] = S_{\alpha\beta}\,n\big(\delta_{k+n,0}\,\psi_m^\alpha \bar\psi_l^\beta + \delta_{n+l,0}\,\psi_m^\alpha \bar\psi_k^\beta\big), \tag{5.12}$$

and the commutators

$$[S_{n,m}, S_{k,l}] = J_{\alpha\beta}\big(l\,\delta_{l+m,0}\,\psi_n^\alpha \psi_k^\beta + k\,\delta_{k+n,0}\,\bar\psi_m^\alpha \bar\psi_l^\beta\big) - 2kl\,\delta_{k+n,0}\delta_{l+m,0}, \tag{5.13}$$

where we use repeatedly the identities $S_{\alpha\beta}J_{\gamma\delta}J^{\alpha\gamma} = S_{\beta\delta}$ and $S_{\alpha\beta}J_{\gamma\delta}J^{\alpha\delta} = -S_{\beta\gamma}$.

Using the definition of $S_{n,m}$, $T_{m,n}$ and $\bar T_{m,n}$ given in (5.9), we see that the relations (5.10) and (5.13) together with (5.11) and (5.12) prove the following proposition which can be considered as an alternative definition of the $\mathfrak{S}_\infty$ Lie algebra.

Proposition 5.2.1. *The Lie algebra $\mathfrak{S}_\infty$ has $S_{n,m}$, $T_{k,l}$ and $\bar{T}_{r,s}$, with $n, m, k, l, r, s \in \mathbb{Z}$, as its basis elements, with the defining relations*

$$[S_{n,m}, S_{k,l}] = l\delta_{l+m,0}\,T_{n,k} + k\,\delta_{k+n,0}\bar{T}_{m,l} - 2kl\,\delta_{k+n,0}\delta_{l+m,0}, \tag{5.14}$$

$$[S_{m,n}, T_{k,l}] = m\big(\delta_{k+m,0}\,S_{l,n} + \delta_{m+l,0}\,S_{k,n}\big), \tag{5.15}$$

$$[S_{m,n}, \bar{T}_{k,l}] = n\big(\delta_{k+n,0}\,S_{m,l} + \delta_{n+l,0}\,S_{m,k}\big), \tag{5.16}$$

$$[T_{m,n}, T_{k,l}] = n\big(\delta_{n+k,0}\,T_{m,l} + \delta_{n+l,0}\,T_{m,k}\big) + m\big(\delta_{m+k,0}\,T_{l,n} + \delta_{m+l,0}\,T_{k,n}\big), \tag{5.17}$$

$$[\bar{T}_{m,n}, \bar{T}_{k,l}] = n\big(\delta_{n+k,0}\,\bar{T}_{m,l} + \delta_{n+l,0}\,\bar{T}_{m,k}\big) + m\big(\delta_{m+k,0}\,\bar{T}_{l,n} + \delta_{m+l,0}\,\bar{T}_{k,n}\big), \tag{5.18}$$

$$[T_{m,n}, \bar{T}_{k,l}] = 0. \tag{5.19}$$

We note that it is straightforward to check the Jacobi identitites.

5.3. Bimodule in the scaling limit and the centralizer of $\mathfrak{S}_\infty$. Using a formal construction of inverse/projective limits, we define *the scaling limit of the JTL centralizers $\mathfrak{Z}_{JTL}$* in Appendix C. The limit is an infinite dimensional associative algebra which we identify with a quotient of $U_q^{odd}s\ell(2)$ and we denote this quotient by $\rho\big(U_q^{odd}s\ell(2)\big)$, see (C13). Fermionic expressions for the generators in the scaling limit of the centralizers were computed in our first paper [22]:

$$\mathsf{E}_n = \left[\sum_{m>0}\Big(\frac{\psi_m^2\psi_{-m}^2}{m} - \frac{\bar{\psi}_m^2\bar{\psi}_{-m}^2}{m}\Big)\right]^n \psi_0^2, \quad \mathsf{F}_n = \left[\sum_{m>0}\Big(\frac{\psi_m^1\psi_{-m}^1}{m} - \frac{\bar{\psi}_m^1\bar{\psi}_{-m}^1}{m}\Big)\right]^n \psi_0^1, \tag{5.20}$$

with the representation of the Cartan element h as

$$\mathsf{h} = -i/2\big(\psi_0^2\phi_0^1 + \psi_0^1\phi_0^2\big) + \sum_{m>0}\frac{1}{m}\big(\psi_{-m}^2\psi_m^1 + \psi_{-m}^1\psi_m^2 + \bar{\psi}_{-m}^2\bar{\psi}_m^1 + \bar{\psi}_{-m}^1\bar{\psi}_m^2\big) \tag{5.21}$$

while the generator $\mathsf{K} = (-1)^{2\mathsf{h}}$.

It is obvious that both $\mathfrak{gl}'_\infty$ and its subalgebra $\mathfrak{S}_\infty$ act on the space $\mathcal{H}$ of scaling states. To study the symmetry algebra of the Lie algebra $\mathfrak{S}_\infty$ on this space, we first note that using identities like

$$\Big[\psi_m^2, \psi_{|m|}^1\psi_{-|m|}^1\Big] = |m|\psi_m^1, \quad \Big[\psi_m^1, \psi_{|m|}^2\psi_{-|m|}^2\Big] = -|m|\psi_m^2, \quad m \in \mathbb{Z}, \tag{5.22}$$

it is straightforward to check that the generators (5.9) commute with the $U_q^{odd}s\ell(2)$ action given in (5.20) and (5.21). Moreover, we prove in Theorem C.7 that the centralizer of the direct-limit algebra $\mathfrak{S}_\infty$ equals the inverse limit of the centralizers for each term $\mathfrak{S}_N$. In view of the importance of this fact, we repeat it here: *the centralizer $\mathfrak{Z}_{U\mathfrak{S}_\infty}$ of the enveloping algebra $U\mathfrak{S}_\infty$ action (5.9) on $\mathcal{H}$ is given by the representation of $U_q^{odd}s\ell(2)$ in (5.20) and (5.21).*

We next describe bimodule structure of $\mathcal{H}$ over $U\mathfrak{S}_\infty$ and its centralizer $\rho\big(U_q^{odd}s\ell(2)\big)$.

As a consequence of the direct limit construction of $\mathfrak{S}_\infty$ and of its module $\mathcal{H}$ in Sect. C.1, we obtain that the (direct) limit of irreducible representations over $\mathfrak{S}_N$ is also irreducible with respect to the action of $\mathfrak{S}_\infty$. This is proved in Proposition C.3 and it in particular means that the only simple $\mathfrak{S}_\infty$-modules that appear in the direct limit space $\mathcal{H}$ are the limits $\mathfrak{X}_j$. Further, see a note after Proposition C.3, the indecomposable but

reducible JTL_N-modules $\widehat{\mathcal{P}}_j$ that appear in $\mathcal{H}_N$ as direct summands go in the scaling (or direct) limit to indecomposable but reducible $U\mathfrak{S}_\infty$-modules (so, $\widehat{\mathcal{P}}_j$ do not split on direct sums in the limit) and their subquotient structure has the same pattern as in Fig. 4, or more precisely in its infinite analogue in Fig. 7 in Sect. 6.7.2. All this is not surprising as we have essentially the same centralizer $U_q^{\mathrm{odd}}s\ell(2)$ in CFT as for any JTL_N algebra in the periodic spin-chain, see Theorem C.7 about the centralizing algebra for $U\mathfrak{S}_\infty$. The only difference from finite chains bimodules described in Sect. 3.3 is that the representations of $U_q^{\mathrm{odd}}s\ell(2)$ now admit the "total spin" (the h eigenvalues) of any integer value: we have a decomposition of $\mathcal{H}$ as a module over $U_q^{\mathrm{odd}}s\ell(2)$ onto the indecomposable direct summands T_j, for any $j > 1$, and with multiplicities given now by graded vector spaces which are the simple $\mathcal{X}_j$ modules over $U\mathfrak{S}_\infty$. Then, in our space of scaling states where each state is a finite linear combination of basis ones, we can easily extend our analysis from [23] and obtain the bimodule structure over the pair of commuting algebras $\left(U\mathfrak{S}_\infty, U_q^{\mathrm{odd}}s\ell(2)\right)$ as in Fig. 3. In the figure for the bimodule in the scaling limit, each node is now a simple $\mathfrak{S}_\infty$-module $\mathcal{X}_j$ and the towers have infinite length. The first question is of course about the left–right Virasoro content of these scaling limits, which we discuss below.

5.4. The content of simple $\mathfrak{S}_\infty$-modules. We start our analysis by discussing the left–right Virasoro $\mathcal{V}(2) = \mathcal{V}(2) \boxtimes \overline{\mathcal{V}}(2)$ content of the simple JTL_N modules in the scaling limit. By this, we mean more precisely the Virasoro representation content of the states that contribute to the direct limit $\mathcal{X}_j$ of the simple JTL_N-modules $\mathcal{L}_{j,(-1)^{j+1}}$ (recall that this limit is simple as well, as a $\mathfrak{S}_\infty$-module). It is convenient for this purpose to first evaluate the generating functions (E1) of energy-momentum levels in the JTL_N modules $\widehat{\mathcal{P}}_j$ (the direct summands of the spin-chain) we have encountered previously. This can be done by reorganizing the fermions such that $\psi^{1,2}$'s correspond to even modes of new fermions and $\bar{\psi}^{1,2}$'s give odd modes of the fermions, see Appendix B. Then, we can produce a generating function $F_{j,(-1)^{j+1}}$ for each charged sector $\widehat{\mathcal{P}}_j$ with $S^z = j$ using the new fermions. Repeating an exercise similar to what can be found in [46], we obtain the character formula for $F_{j,(-1)^{j+1}}$ in the limit which reads

$$F_{j,(-1)^{j+1}} = \frac{q^{1/12}\bar{q}^{1/12}}{P(q)P(\bar{q})} \sum_{n\in\mathbb{Z}} q^{\frac{(j+n)^2+j+n}{2}} \bar{q}^{\frac{n^2+n}{2}} \tag{5.23}$$

with

$$P(q) = \prod_{n=1}^{\infty}(1 - q^n) = q^{-1/24}\eta(q). \tag{5.24}$$

We present a more general derivation of this result in Appendix E which is based on well-know scaling properties of the spectrum for twisted XXZ models, and will be useful in our future studies [36]. From the structure of the spin chain modules (see Sect. 3.3; Fig. 4), we deduce then the traces $F^{(0)}_{j,(-1)^{j+1}}$ for the simple $\mathfrak{S}_\infty$-modules $\mathcal{X}_j$ which read, in terms of Virasoro characters, (see a calculation in Appendix E)

$$F^{(0)}_{j,(-1)^{j+1}} = \sum_{j_1,j_2>0}^{*} \chi_{j_1,1}\overline{\chi}_{j_2,1} \tag{5.25}$$

where the sum is done with the following constraints:

$$|j_1 - j_2| + 1 \leq j, \quad j_1 + j_2 - 1 \geq j, \quad j_1 + j_2 - j = 1 \bmod 2 \qquad (5.26)$$

(note this is equivalent to treating j not as a spin but as a degeneracy, i.e., setting $j = 2s + 1$, $j_i = 2s_i + 1$ and combining s_1 and s_2 to obtain spin s), and we recall the characters of the simple modules with Kac labels $(j, 1)$ over Virasoro at $c = -2$:

$$\chi_{j,1} = \frac{q^{(2j-1)^2/8} - q^{(2j+1)^2/8}}{\eta(q)}. \qquad (5.27)$$

For instance, we have

$$
\begin{aligned}
F_{0,-1}^{(0)} &= 0, \\
F_{1,1}^{(0)} &= \sum_{r=1}^{\infty} \chi_{r1} \bar{\chi}_{r1}, \\
F_{2,-1}^{(0)} &= \sum_{r=1}^{\infty} \chi_{r1} \left(\bar{\chi}_{r-1,1} + \bar{\chi}_{r+1,1} \right), \\
F_{3,1}^{(0)} &= \chi_{11} \bar{\chi}_{31} + \chi_{21} \left(\bar{\chi}_{21} + \bar{\chi}_{41} \right) + \sum_{r=3}^{\infty} \chi_{r1} \left(\bar{\chi}_{r-2,1} + \bar{\chi}_{r1} + \bar{\chi}_{r+2,1} \right).
\end{aligned}
\qquad (5.28)
$$

The result (5.25) implies that, in contrast with the open case, simple modules of the JTL_N algebra do not become simple modules over $\mathcal{V}(2)$ in the scaling limit. While in general one might expect that this scaling limit gives rise to non-fully-reducible modules over $\mathcal{V}(2)$, it turns out that one gets in fact a direct sum of simple modules over the full Virasoro algebra $\mathcal{V}(2)$. This can be checked either by explicitly working out this scaling limit as was done in [22], or by using a direct argument. We first note that the left Virasoro generators are particular endomorphisms of the module structure over the right Virasoro $\bar{\mathcal{V}}(2)$ action. Then, taking into account possible extensions/'glueings' between simple $\mathcal{V}(2)$-modules—a module $\mathcal{X}_{j,1}$ can be extended only by $\mathcal{X}_{j\pm1,1}$ into an indecomposable module,—the left Virasoro algebra can map vectors from subquotients $\mathcal{X}_{j,1} \boxtimes \bar{\mathcal{X}}_{k,1}$ only to $\mathcal{X}_{j\pm1,1} \boxtimes \bar{\mathcal{X}}_{k,1}$, and similarly for the right Virasoro algebra. Therefore, the character of a reducible but indecomposable $\mathcal{V}(2)$-module has to have at least one pair of the indices (j, k), say (j_1, k_1) and (j_2, k_2), such that $|j_1 - j_2| = 1$ and $k_1 = k_2$ or $|k_1 - k_2| = 1$ and $j_1 = j_2$. None of these conditions is true for the functions $F_{n,(-1)^{n-1}}^{(0)}$ given above. We thus conclude that $\mathcal{V}(2)$-modules with characters in (5.25), for any fixed j, are semisimple. The foregoing analysis gives thus the left–right Virasoro content for the direct limits of simple JTL_N-modules[7]:

$$\mathcal{L}_{j,(-1)^{j+1}} \mapsto \mathcal{X}_j |_{\mathcal{V}(2)} = \bigoplus_{j_1,j_2>0}^{*} \mathcal{X}_{j_1,1} \boxtimes \bar{\mathcal{X}}_{j_2,1}, \qquad (5.29)$$

with the conditions (5.26) on the sum.

---

[7] Having the identification of the Hamiltonians $H = H_0(0)$ with the Cartan-subalgebra elements of $\mathfrak{sp}_{2L-2}$, see (6.12) below, it should be possible using the Weyl character formula for fundamental representations of $\mathfrak{sp}_{2L-2}$ to study the character asymptotics and extract the $\mathcal{V}(2)$ content of the limits as in (5.29). We leave this exercise for future work.

Recall now Proposition C.3 where we found that direct limits of simples are again simple modules, and thus the only simple $\mathfrak{S}_\infty$-modules that appear in the direct limit space $\mathcal{H}$ are the limits $\mathfrak{X}_j$. We thus conclude this subsection with an important statement: *the $U\mathfrak{S}_\infty$ simple modules that appear in $\mathcal{H}$ are the direct limits $\mathfrak{X}_j$ of the JTL_N simples $\mathcal{L}_{j,(-1)^{j+1}}$ and considered as modules over $\mathcal{V}(2)$ they are the direct sums (5.29) over $\mathcal{V}(2)$ simples.*

5.5. Scaling limit of the anti-periodic chains. Of course, we could also consider the antiperiodic model (or the twisted sector) for the symplectic fermions where the modes are half-integer [25]. This corresponds to the scaling limit of the anti-periodic $\mathfrak{gl}(1|1)$ spin-chain from Sect. 3.4. The JTL_N algebra is then replaced by JTL_N^{tw} while $U_q^{odd}s\ell(2)$ is replaced by $Us\ell(2)$. The corresponding bimodule is semisimple and is given in (3.9).

We take the scaling limit of JTL_N^{tw} using Theorem 4.1.1 where it was shown that the spin-chain representation of JTL_N^{tw} is isomorphic to a representation of the enveloping algebra $U\mathfrak{sp}_N$. We thus need only to take the scaling limit of the Lie algebra $\mathfrak{sp}_N$ which is quite straightforward. Repeating the analysis given in Sect. 5.1 or following lines in Sect. C.1, we obtain in this case that $\mathfrak{sp}_N$ has the direct limit generated by the same bilinears (5.8) as in the periodic case but now with $n, m \in \mathbb{Z} - \frac{1}{2}$. This representation of $\mathfrak{sp}_\infty$ gives a symplectic-fermion representation of the universal enveloping algebra $U\mathfrak{S}_\infty$ in the twisted model as well and the generators $S_{m,n}$, $T_{m,n}$, and $\bar{T}_{m,n}$ have similar expressions as in (5.9) but with fermionic modes shifted by one-half:

$$S_{m,n} = S_{\alpha\beta}\psi^\alpha_{m\mp\frac{1}{2}}\bar{\psi}^\beta_{n\mp\frac{1}{2}}, \quad T_{m,n} = J_{\alpha\beta}\psi^\alpha_{m\mp\frac{1}{2}}\psi^\beta_{n\mp\frac{1}{2}}, \quad \bar{T}_{m,n} = J_{\alpha\beta}\bar{\psi}^\alpha_{m\mp\frac{1}{2}}\bar{\psi}^\beta_{n\mp\frac{1}{2}},$$
$$n, m \in \mathbb{Z}/\{0\}, \tag{5.30}$$

where $m \mp \frac{1}{2}$ is by definition $m - \frac{1}{2}$ for positive m and $m + \frac{1}{2}$ for negative values of m. In particular, we see that the radical of $\mathfrak{S}_\infty$ which is a subalgebra generated by $S_{0,n}$, $T_{m,0}$, etc., is trivially represented in this model.

There are no zero modes in the anti-periodic model and the continuum limit of the $Us\ell(2)$ generators reads as [22]

$$\tilde{Q}^a = d^a_{\alpha\beta} \sum_{n=0}^\infty \left(\frac{\psi^\alpha_{-n-1/2}\psi^\beta_{n+1/2}}{n+1/2} - \frac{\bar{\psi}^\alpha_{-n-1/2}\bar{\psi}^\beta_{n+1/2}}{n+1/2} \right), \tag{5.31}$$

with the matrices

$$d^0_{\alpha\beta} = \frac{1}{2}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad d^1_{\alpha\beta} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad d^2_{\alpha\beta} = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \tag{5.32}$$

with $[\tilde{Q}^a, \tilde{Q}^b] = f^{ab}_c \tilde{Q}^c$ and $f^{01}_2 = -1$. We check then that the action of the Lie algebra $\mathfrak{S}_\infty$ of fermionic bilinears commutes with this action of $Us\ell(2)$. Moreover, we repeat the construction in Sect. C.4 and prove Lemma C.5 in this context. This allows then to state an analogue of Theorem C.7 about the centralizer in the half-integer-mode sector.

Theorem 5.5.1. *The centralizer $\mathfrak{Z}_{U\mathfrak{S}_\infty}$ of the enveloping algebra $U\mathfrak{S}_\infty$ action in the half-integer-mode (twisted) sector, which factorizes to the action of $U\mathfrak{sp}_\infty$ in this model, is given by the representation of $Us\ell(2)$ from (5.31). The bimodule over $(\mathfrak{Z}_{U\mathfrak{S}_\infty}, U\mathfrak{sp}_\infty)$ is semisimple and given by the direct sum (3.9) over all $j \geq 0$.*

This is confirmed by the decomposition of the full partition function to which we now turn.

5.6. The full generating functions. The generating function of levels for the periodic $\mathfrak{gl}(1|1)$ model—that is, the left hand side of (E1) reads, from the analysis in [27],

$$Z = F_{0,-1} + 2 \sum_{j \text{ even}} F_{j,-1} + 2 \sum_{j \text{ odd}} F_{j,1} \tag{5.33}$$

(in the reinterpretation as the partition function of the XX model with appropriate twists, the factors 2 arise from the $\mathbb{Z}_2$ spin-flip symmetry, so the summation index j is positive). Elementary algebra using

$$F_{2n,-1}^{(0)} + F_{2n+1,1}^{(0)} = F_{2n,-1} - F_{2n+1,1} \tag{5.34}$$

leads to

$$Z = 4 \sum_{j=0}^{\infty} j F_{j,(-1)^{j+1}}^{(0)}. \tag{5.35}$$

This formula is a direct translation of the bimodule structure discussed in Sect. 3.3: the same decomposition holds in fact for a finite system, with $F^{(0)}$ the trace over the simple modules over JTL_N. Note that it is similar to the formula $Z_{\text{op}} = 4 \sum_{j=0}^{\infty} j \chi_{j,1}$ obtained in [7] for the open $\mathfrak{gl}(1|1)$ spin chains. In the open case, the degeneracy $4j$ arose as dimensions of irreducible representations of the full quantum group, which coincides with the algebra $\mathcal{A}_{1|1}$. In the periodic case, we lose the full quantum group symmetry and retain only $U_q^{\text{odd}} s\ell(2)$ on the lattice. This leads, nevertheless, to the same $4j$ for the simple modules: $\mathcal{L}_{j,(-1)^{j+1}}$ in the periodic case, $\mathcal{X}_{j,1}$ in the open case.

Finally, we rewrite the full generating function as

$$Z = 4 \sum_{j=0}^{\infty} j F_{j,(-1)^{j+1}}^{(0)} = 4 \left| \sum_{j=0}^{\infty} j \chi_{j,1} \right|^2 = 4 \left| q^{1/12} \prod_{n=1}^{\infty} (1+q^n)^2 \right|^2. \tag{5.36}$$

The right hand side is now seen to coincide with the generating function for the level of symplectic fermions with periodic boundary conditions indeed [25]. Of course, the corresponding partition function of doubly periodic symplectic fermions on the torus vanishes exactly due to the $\mathfrak{gl}(1|1)$ symmetry; it is thus trivially modular invariant. The generating function Z in (5.33) is not modular invariant, nor does it have to be.

In the anti-periodic case or twisted model, the generating function of levels is now

$$Z = F_{0,1} + 2 \sum_{\text{odd } j>0} F_{j,-1} + 2 \sum_{\text{even } j>0} F_{j,1} \tag{5.37}$$

corresponding to

$$Z = \sum_{j=0}^{\infty} (j+1) F_{j,(-1)^j}^{(0)}, \tag{5.38}$$

in accordance with the decomposition (3.9). The $(j+1)'s$ are now dimensions of the $Us\ell(2)$-modules; the $\mathfrak{gl}(1|1)$ symmetry is broken. As before, we can rewrite

$$Z = \left| \sum_{j=1}^{\infty} j \chi_{j,2} \right|^2 = \left| q^{-1/24} \prod_{n=1}^{\infty} \left(1 + q^{n+1/2}\right)^2 \right|^2 \tag{5.39}$$

which is a well known expression in this sector as well.

6. Interchiral Algebra and Local Fields

We are now discussing a reinterpretation of our previous results in a way that, we expect, can be generalized to other LCFTs. This requires us to perform a few manipulations whose validity, however likely, we cannot prove at this stage. The present section, while more physical, is thus of a less mathematical nature, and, ultimately, based on some conjectures we discuss below.

While the structure of the symplectic fermion LCFT is fully consistent with the one of the $\mathfrak{gl}(1|1)$ spin chain, there is an obvious difference due to the presence of the $s\ell(2)$ symmetry in the continuum theory, which, in the periodic case, does not have a natural analog on the lattice. Investigation of more complicated models [36] suggests that the most productive way to think of this difference is to essentially forget the $s\ell(2)$ symmetry, which seems to be an artifact from the point of view of the $\mathfrak{gl}(1|1)$ theory. Rather, we believe that the bimodule structure of the lattice model does suggest to us the proper symmetries, and the proper way to analyze the scaling limit. Put otherwise, the good algebraic object that will organize the spectrum of most general logarithmic lattice models should be *the scaling limit of the Jones–Temperley–Lieb algebra.* Switching our point of view in this way leads to profound and maybe not so surprising conclusions, in particular that the organizing algebra of bulk LCFTs should contain non-chiral objects. Indeed, while the focus in the past [42] had been mostly to extract the stress-energy tensor modes from the Temperley–Lieb or Jones–Temperley–Lieb algebras, it is well known [27] that the scaling limit of some elements in JTL_N can lead to other physical observables corresponding to different bulk scaling fields. A very important example of such a field is what we will call the energy operator, which is associated with the staggered sum

$$\sum_{i=1}^{N}(-1)^i e_i \; \mapsto \; \int dx\, \Phi_{2,1} \times \overline{\Phi}_{21}(x, \tau = 0), \tag{6.1}$$

where the integral is taken over the circumference of a cylinder at constant imaginary-time $\tau = 0$. In the Potts model of statistical mechanics, where the Temperley–Lieb algebra with positive, integer values of m appears, this field is canonically coupled to the temperature. The field in (6.1) is the non chiral degenerate field with conformal weights $h = \bar{h} = h_{2,1}$; later, we will denote it by $S_{\text{cyl.}}(x, \tau)$. Of course, the introduction of such fields in the organizing algebra of a LCFT requires discussion of objects which mix chiral and anti-chiral sectors. We shall in this section introduce the new concept of an *interchiral algebra*, and discuss its structure and role in the case of the closed $\mathfrak{gl}(1|1)$ chains and the bulk symplectic fermions.

It is important to note that in the open case, the continuum limit of the Temperley–Lieb algebra TL_N only leads to the enveloping algebra of the Virasoro algebra, and does not involve other conformal fields. A related fact is, for instance, that the boundary energy operator at the ordinary transition in the Potts model coincides with the stress energy tensor [43]. There are probably cases for open spin chains where the algebra of hamiltonian densities will lead, in the continuum limit, to a bigger algebra than the Virasoro algebra—for instance, the super Virasoro algebra, W-algebras [44], etc.

From another, more formal point of view, recall we showed in [22, Sec. 5.2] that the scaling limit of the JTL centralizer $\mathfrak{Z}_{JTL}$—an infinite dimensional representation of the $U_q^{\text{odd}} s\ell(2)$—gives an algebra of intertwining operators respecting the left and right Virasoro algebras. On the other hand, we also showed [22] that the centralizer of $\mathcal{V}(2)$ contains $U_q^{\text{odd}} s\ell(2)$ but is a bigger algebra. It is thus reasonable to expect that the scaling

limit of the Jones-Temperley-Lieb algebra is bigger than the non-chiral Virasoro algebra $\mathcal{V}(2)$, and that the latter is only a proper subalgebra.

To make progress, we turn back to the scaling limit analysis of the $\mathfrak{gl}(1|1)$ spin-chains discussed in [22] and extract an additional field that generates the full scaling limit of the JTL algebras. It turns out that the limit gives modes of a field of conformal dimension $(1, 1)$. We call this field the *interchiral field $S(z, \bar{z})$*; it is expressed in terms of derivatives of the symplectic fermions as

$$S(z, \bar{z}) = S_{\alpha\beta}\psi^\alpha(z)\bar\psi^\beta(\bar{z}) \quad \text{with} \quad \psi^\alpha(z) = \partial\Phi^\alpha(z, \bar{z}), \quad \bar\psi^\alpha(\bar{z}) = \bar\partial\Phi^\alpha(z, \bar{z}), \quad (6.2)$$

where we introduced the symmetric form

$$S_{12} = S_{21} = 1, \quad S_{11} = S_{22} = 0. \quad (6.3)$$

We recall that the derivatives of the symplectic fermions have the mode decomposition [25]

$$\psi^\alpha(z) = \sum_{n\in\mathbb{Z}}\psi_n^\alpha z^{-n-1}, \quad \bar\psi^\alpha(\bar{z}) = \sum_{n\in\mathbb{Z}}\bar\psi_n^\alpha \bar{z}^{-n-1}, \quad (6.4)$$

and are primary fields of conformal dimensions $(1, 0)$ and $(0, 1)$, respectively. The fermionic modes satisfy the anti-commutation relations

$$\{\psi_m^\alpha, \psi_{m'}^\beta\} = mJ^{\alpha\beta}\delta_{m+m',0}, \quad \alpha, \beta \in \{1, 2\}, \quad m, m' \in \mathbb{Z}, \quad (6.5)$$

with the symplectic form $J^{12} = -J^{21} = 1$, and the same formulas for the antichiral modes $\bar\psi_n^\alpha$, and $\{\psi_m^\alpha, \bar\psi_{m'}^\beta\} = 0$. We also introduce 'constant' modes ϕ_0^2 and ϕ_0^1 which are conjugate to the zero modes

$$\left\{\phi_0^1, \psi_0^2\right\} = i, \quad \left\{\phi_0^2, \psi_0^1\right\} = -i. \quad (6.6)$$

6.1. Modes of local fields and the algebra $\mathfrak{S}$. While dealing with bilinears in fermionic modes is quite convenient and provides the most obvious approach to the scaling limit of JTL algebras, it is also important to understand what happens in more physical terms. From a (L)CFT point of view indeed, the natural objects are, for instance, not so much bilinears such as $:\psi_{n-m}^2\psi_m^1:$, but the modes of the stress energy tresor, which are (infinite) sums such as $L_n = \sum_{m\in\mathbb{Z}}:\psi_{n-m}^2\psi_m^1:$. Also, while Howe duality (in its non semi-simple version) gave us a powerful and new angle on JTL_N and its scaling limit, it is not likely that this will generalize to other models. It is thus crucial to be able to come up with an alternative, if less pleasant, approach to the scaling limit of JTL algebras.

6.1.1. Higher Hamiltonians. We begin by recalling the lattice analysis given in [22] where it was shown how to proceed from the JTL_N generators to get Virasoro modes in the non-chiral LCFT of symplectic fermions: the combinations

$$H(n) = -\sum_{j=1}^N e^{-iqj}e_j^{\mathfrak{gl}}, \quad P(n) = \frac{i}{2}\sum_{j=1}^N e^{-iqj}[e_j^{\mathfrak{gl}}, e_{j+1}^{\mathfrak{gl}}], \quad q = \frac{n\pi}{L}, \quad (6.7)$$

of the JTL_N generators (recall that $e_j^{\mathfrak{gl}}$ denotes the representation of e_j given in (2.2) and (3.1)) converge in the scaling limit as $L \to \infty$ to the well-known symplectic fermions representation of the left and right Virasoro generators

$$\frac{L}{2\pi} H(n) \mapsto L_n + \bar{L}_{-n}, \quad \frac{L}{2\pi} P(n) \mapsto L_n - \bar{L}_{-n}. \tag{6.8}$$

We note that the limit (6.8) of $H(n)$ and $P(n)$ is taken for finite n.

Here, we go further and consider also limits with n close to $N/2$ (corresponding, physically, to alternating sum of JTL generators, related with the energy operator), or equivalently we consider $H(L - k)$ and $P(L - k)$ with $L \to \infty$ and finite k. It turns out that this limit gives modes of a field of conformal dimension (1, 1). To show this, we introduce a family of operators $H_l(n)$ which are Fourier images of all the higher Hamiltonians:

$$H_l(n) = -\frac{1}{2} e^{-il\frac{\pi}{2}} \sum_{j=1}^{N} e^{-iqj} E_{j,l}, \quad q = \frac{n\pi}{L} \quad \text{and} \quad l \in \mathbb{N}, \quad n \in \mathbb{Z}, \tag{6.9}$$

where we used the following notation for multiple commutators of the JTL_N-generators e_j

$$E_{j,l} = \Big[e_j, \big[e_{j+1}, \ldots e_{j+l-2}, [e_{j+l-1}, e_{j+l}] \ldots \big] \Big], \quad 1 \le j \le N. \tag{6.10}$$

We call these Fourier images generalized higher Hamiltonians. Note that $P(n) = H_1(n)$ and we set $H_0(n) = H(n)$.

The fermionic expressions for the $H_l(n)$'s were obtained in our first paper [22, Sec. 4.5.1] and it turns out that the $H_l(n)$'s can be written in the basis (4.10) of the Lie algebra $\mathfrak{S}_N$, for $l \ge 0$ and $0 \le n \le L - 1$, as

$$
\begin{aligned}
H_l(n) = 2e^{iq\frac{l+1}{2}} \bigg(& \sum_{m=1}^{L-n-1} \cos l \Big(p + \frac{q}{2} \Big) \sqrt{\sin(p)\sin(q+p)} \, \mathcal{A}_{m,m+n} \\
& + \frac{1}{2} \sum_{m=1}^{n-1} \cos l \Big(p - \frac{q}{2} \Big) \sqrt{\sin(p)\sin(q-p)} \big(\mathcal{C}_{m,n-m} + (-1)^l \mathcal{B}_{L-m,L+m-n} \big) \\
& + \cos \frac{lq}{2} \sqrt{\sin q} \big(\mathcal{E}_{0,n} + (-1)^l \mathcal{E}_{0,2L-n} + \mathcal{E}_{L+n,L} + (-1)^l \mathcal{E}_{L-n,L} \big) \\
& + 2\delta_{n,0} \big(1 + (-1)^l \big) \mathcal{E}_{0,L} \bigg),
\end{aligned} \tag{6.11}
$$

where we set $p = m\pi/L$ and $q = n\pi/L$ as usually. The expressions for negative values of n are given in (D4). Note that all zero modes $H_l(0)$ and the lattice translation generator u^2 have very simple expressions:

$$H_l(0) = 2 \sum_{m=1}^{L-1} \cos(lp) \sin(p) \, \mathcal{A}_{m,m} + 4\big(1 + (-1)^l\big) \mathcal{E}_{0,L} \tag{6.12}$$

and

$$u^2 = \exp\left(-\frac{2i\pi}{L} \sum_{n=1}^{L-1} n \mathcal{A}_{n,n} \right) \tag{6.13}$$

and they belong, for odd values of l, to the Cartan subalgebra of $\mathfrak{sp}_{2L-2}$.

We check then that the element $H_0(L)$ is the energy operator indeed $\sum_{j=1}^{N}(-1)^j e_j$ on the lattice (which differs from the Hamiltonian H), which reads

$$\sum_{j=1}^{N}(-1)^j e_j = 2i \sum_{\substack{p=\epsilon \\ \text{step}=\epsilon}}^{\pi-\epsilon} \sin p \left(\chi_p^\dagger \eta_p + \eta_p^\dagger \chi_p\right). \tag{6.14}$$

The energy operator rescaled by the factor $\frac{L}{2\pi i}$ has then the limit

$$\frac{L}{2\pi i} \sum_{j=1}^{N}(-1)^j e_j \mapsto \sum_{m\in\mathbb{Z}} \left(\psi_m^2 \bar{\psi}_m^1 + \psi_m^1 \bar{\psi}_m^2\right) \equiv S_0.$$

We denote this limit by S_0 and show below that it corresponds to the zero mode of a primary field of dimension $(1, 1)$. First however, we discuss what will correspond to higher modes S_n of this primary field. We refer to our calculations in Appendix D where we obtain in the scaling limit for $H_0(L-k)$ (keeping only the leading order in $1/N$), with finite integer k,

$$\frac{L}{2\pi i} H_0(L-k) \mapsto S_{\alpha\beta} \sum_{m\in\mathbb{Z}} \psi_m^\alpha \bar{\psi}_{m+k}^\beta \equiv S_k^{(0)}, \quad k \in \mathbb{Z}, \tag{6.15}$$

and the scaling limit for $H_1(L-k)$ is

$$-\frac{L^2}{\pi^2} H_1(L-k) \mapsto S_{\alpha\beta} \sum_{m\in\mathbb{Z}} (2m+k)\psi_m^\alpha \bar{\psi}_{m+k}^\beta \equiv S_k^{(1)}, \quad k \in \mathbb{Z}, \tag{6.16}$$

where we introduce, for $l \geq 0$ and $k \in \mathbb{Z}$, the operators

$$S_k^{(l)} = S_{\alpha\beta} \sum_{m\in\mathbb{Z}} (2m+k)^l \psi_m^\alpha \bar{\psi}_{m+k}^\beta. \tag{6.17}$$

The whole family of operators $S_k^{(l)}$, with $l \geq 0$ and $k \in \mathbb{Z}$, can be extracted from the scaling limit of the operators $H_{l'}(L-k)$ keeping not only the leading terms in their formal expansion in $1/L$ but also all sub-leading terms in the expansion. Put differently, we can formally expand $H_{l'}(L-k)$ as Laurent series in N where coefficients are fermionic bilinears of the form $S_k^{(l)}$. For simplicity, we refrain from discussing this any further here.

6.1.2. Fields. The foregoing family of operators $S_k^{(l)}$, with $l \geq 0$ and $k \in \mathbb{Z}$, can be obtained as modes of a primary field of conformal dimension $(1, 1)$, and of its descendants. We denote this primary field defined on the complex plane as

$$S_{\text{pl.}}(z, \bar{z}) = S_{\alpha\beta} \psi^\alpha(z) \bar{\psi}^\beta(\bar{z}), \tag{6.18}$$

where $S_{\alpha\beta}$ is the symmetric form, i.e., $S_{12} = S_{21} = 1$, $S_{11} = S_{22} = 0$. We call this field the *inter*chiral field; note that it is of course neither chiral nor anti-chiral.

Recall first the mode decomposition of the fermion fields

$$\psi^\alpha(z) = \sum_{n\in\mathbb{Z}} \psi_n^\alpha z^{-n-1}, \quad \bar{\psi}^\alpha(\bar{z}) = \sum_{n\in\mathbb{Z}} \bar{\psi}_n^\alpha \bar{z}^{-n-1}. \tag{6.19}$$

Then, $S_{\text{pl.}}(z, \bar{z})$ has a formal expansion

$$S_{\text{pl.}}(z, \bar{z}) = S_{\alpha\beta} \sum_{n,m\in\mathbb{Z}} \psi_n^\alpha \bar{\psi}_m^\beta z^{-n-1} \bar{z}^{-m-1}. \qquad (6.20)$$

Interpreting z and $\bar{z}$ as independent complex coordinates allows one to extract $\psi_n^\alpha \bar{\psi}_m^\beta$ by a double integration over $z, \bar{z}$. This is not too pleasant however, since we rather would like to deal with modes obtained by integration in the physical theory, where $z, \bar{z}$ are not independent. Note also that in the Virasoro case, there is no $\bar{z}$ we can integrate over, so there is no integration that can give rise to the single product $\psi_n^\alpha \bar{\psi}_m^\beta$ starting from the stress energy tensor and its descendants.

To make contact with the hamiltonian formulation we used so far, it is now convenient to perform a conformal transformation $z \to w(z)$, $\bar{z} \to w^*(\bar{z})$ onto a cylinder of circumference L', with $w(z) = \frac{L'}{2\pi} \log(z)$. We set $w = \tau - ix$ and now $w^* = \tau + ix$, we then obtain

$$S_{\text{cyl.}}(x, \tau) = \left(\frac{2\pi}{L'}\right)^2 z\bar{z} S_{\text{pl.}}(z, \bar{z}) = \left(\frac{2\pi}{L'}\right)^2 S_{\alpha\beta} \sum_{n,m\in\mathbb{Z}} \psi_n^\alpha \bar{\psi}_m^\beta \exp\left[-\frac{2\pi}{L'}\left(\tau(m+n)+ix(m-n)\right)\right].$$
$$(6.21)$$

Introducing the mode expansion of the field $S_{\text{cyl.}}(x, \tau)$ on the cylinder in the Heisenberg presentation, where the modes depend on the imaginary time τ,

$$S_{\text{cyl.}}(x, \tau) = \left(\frac{2\pi}{L'}\right)^2 \sum_{k\in\mathbb{Z}} S_k(\tau) e^{2\pi ikx/L'}, \qquad (6.22)$$

we then obtain

$$S_k(\tau) = \frac{L'}{(2\pi)^2} \int_0^{L'} dx\, e^{-2\pi ikx/L'} S_{\text{cyl.}}(x, \tau) \qquad (6.23)$$

which is evaluated as

$$S_k(\tau) = S_{\alpha\beta} \sum_{n\in\mathbb{Z}} \psi_{n+k}^\alpha \bar{\psi}_n^\beta e^{-2\pi\tau(2n+k)/L'}. \qquad (6.24)$$

Finally, we get the expansion

$$S_k(\tau) = \sum_{l\geq 0} \frac{(-2\pi\tau)^l}{l!(L')^l} S_{-k}^{(l)} \qquad (6.25)$$

and in particular

$$S_k(0) = S_{-k}^{(0)}, \quad k \in \mathbb{Z}, \qquad (6.26)$$

where $S_k^{(l)}$ were introduced above in (6.17).

It is important to note that applying the zero mode S_0 of the local field $S_{\text{cyl.}}(x, \tau)$ on the vacuum state, we get then an infinite sum $\sum_{m<0}(\psi_m^2 \bar{\psi}_m^1 + \psi_m^1 \bar{\psi}_m^2)\Omega$. All other modes S_k have similar formally divergent action. These formal divergences are not very surprising as a local operator in general is able to excite states of arbitrarily high energy (the spin is fixed in the action of our local operators). Note nevertheless that the modes $S_k(\tau)$ in the Heisenberg picture are well-defined as they have the convergence factor $e^{-2\pi\tau(2n+k)/L'}$ which eliminates high excitations. We could thus regularize the action of

our local operators by introducing convergence factors in the formally divergent sums but this is not very convenient for us. In order to describe representations of the algebra generated by these local operators, we prefer to choose another regularization in which a locally defined field at a point $w = \tau - ix$ becomes a bilocal field at points z and $\bar{z}$ (hence formally "point splitting" by going to $\mathbb{C} \times \mathbb{C}$). So, we interpret the field $S(z, \bar{z})$ as a regularization of the local interchiral field on the cylinder where the variables z and $\bar{z}$ might be considered independent. After such a regularization the modes of $S(z, \bar{z})$ given in (6.20) now indeed generate states of finite energy only.

Recall now that the modes of the regularized field, $S(z, \bar{z})$, are elements of the Lie algebra $\mathfrak{S}_\infty$ which is a subalgebra in the algebra $\mathfrak{gl}_\infty$ of infinite matrices with finite number of non-zero elements. It turns out that the modes of the local, non-regularized, field $S_{\mathrm{cyl.}}(x, \tau)$ belong to the algebraic completion $\overline{\mathfrak{gl}}_\infty$ of this $\mathfrak{gl}_\infty$. The Lie algebra $\overline{\mathfrak{gl}}_\infty$ was already mentioned above in Sect. 5.1.2 and it consists of infinite matrices with finite number of diagonals having (possibly infinite number of) non-zero elements, see also Appendix B. Working now in the completion $\overline{\mathfrak{gl}}_\infty$, we can ask about the Lie algebra generated by the local operators. We show below that all the other operators $S_k^{(l)}$ that appear in the scaling limit of the $H_{l'}(n)$ operators are produced from $S_k^{(0)}$ by the conformal generators L_n and $\bar{L}_n$. It is of course interesting to study, more generally, commutation relations involving these operators, that is, to study the Lie algebraic structure generated from the $S_k^{(l)}$.

6.1.3. Commutation relations and the Lie algebra $\mathfrak{S}$.

We first find commutation relations among the modes $S_k^{(0)}$ with the result

$$[S_r^{(0)}, S_s^{(0)}] = J_{\alpha\beta} \sum_{n \in \mathbb{Z}} \left((n - s) \bar{\psi}_n^\alpha \bar{\psi}_{r+s-n}^\beta - (n + r) \psi_n^\alpha \psi_{-(r+s)-n}^\beta \right). \tag{6.27}$$

Recall that the modes of the chiral stress-energy tensor can be written as $L_k = \frac{1}{2} J_{\alpha\beta} \sum_{n \in \mathbb{Z}} :\psi_n^\alpha \psi_{k-n}^\beta:$, and similarly for the anti-chiral modes. We introduce also

$$L_k^{(l)} = \frac{1}{2} J_{\alpha\beta} \sum_{n \in \mathbb{Z}} n^l :\psi_n^\alpha \psi_{k-n}^\beta:, \quad \bar{L}_k^{(l)} = \frac{1}{2} J_{\alpha\beta} \sum_{n \in \mathbb{Z}} n^l :\bar{\psi}_n^\alpha \bar{\psi}_{k-n}^\beta:, \quad l \geq 0, \ k \in \mathbb{Z}, \tag{6.28}$$

with $L_k^{(0)} = L_k$ and $\bar{L}_k^{(0)} = \bar{L}_k$. Then, (6.27) has the form, for $r \neq -s$,

$$[S_r^{(0)}, S_s^{(0)}] = 2\bar{L}_{r+s}^{(1)} - 2L_{-(r+s)}^{(1)} - 2s\bar{L}_{r+s}^{(0)} - 2r L_{-(r+s)}^{(0)}. \tag{6.29}$$

Note that the new modes $L_k^{(1)}$ and $\bar{L}_k^{(1)}$ together with $L_n^{(0)}$ and $\bar{L}_n^{(0)}$ generate a subalgebra:

$$2[L_n^{(1)}, L_m^{(1)}] = m(n - 2m)L_{n+m}^{(1)} + m^3 L_{n+m}^{(0)}, \tag{6.30}$$

$$2[\bar{L}_n^{(1)}, \bar{L}_m^{(1)}] = m(n - 2m)\bar{L}_{n+m}^{(1)} + m^3 \bar{L}_{n+m}^{(0)}, \tag{6.31}$$

$$[L_n^{(1)}, \bar{L}_m^{(1)}] = 0, \quad [L_n^{(1)}, \bar{L}_m^{(0)}] = 0, \quad [\bar{L}_n^{(1)}, L_m^{(0)}] = 0, \tag{6.32}$$

$$[L_n^{(1)}, L_m^{(0)}] = (n - 2m)L_{n+m}^{(1)} + m^2 L_{n+m}^{(0)}, \tag{6.33}$$

$$[\bar{L}_n^{(1)}, \bar{L}_m^{(0)}] = (n - 2m)\bar{L}_{n+m}^{(1)} + m^2 \bar{L}_{n+m}^{(0)}, \tag{6.34}$$

where we give relations for $n \neq -m$ (at the case $n = m$ there is a central charge term in all these commutators, which is not important for our purposes). Note also that $2[L_n^{(1)}, L_m^{(1)}] = m[L_n^{(1)}, L_m^{(0)}]$.

In general, we have

$$[S_r^{(l)}, S_s^{(l')}] = \sum_{k=0}^{l+l'+1} \left(a_k L_{-r-s}^{(k)} + \bar{a}_k \bar{L}_{r+s}^{(k)} \right), \tag{6.35}$$

where a_k and $\bar{a}_k$ are coefficients in the expansions

$$\sum_{k=0}^{l+l'+1} a_k x^k = (-1)^{l'+1}(2x + r)^l (x + r)(2x + 2r + s)^{l'} \tag{6.36}$$

$$\sum_{k=0}^{l+l'+1} \bar{a}_k x^k = (-1)^{l+1}(2x - s)^{l'} (-x + s)(2x - 2s - r)^l, \tag{6.37}$$

and we again suppose that $r \neq -s$.

We compute next for all higher modes the action of the left and right Virasoro, i.e., commutation relations between $S_r^{(l)}$ and $L_s^{(0)}, \bar{L}_s^{(0)}$:

$$2[S_r^{(l)}, L_s^{(0)}] = S_{r-s}^{(l+1)} - \sum_{k=0}^{l}(-s)^{l-k} \left[s\binom{l+1}{k} + r\binom{l}{k} \right] S_{r-s}^{(k)},$$
$$\hspace{6cm} r, s \in \mathbb{Z}. \tag{6.38}$$
$$2[S_r^{(l)}, \bar{L}_s^{(0)}] = S_{r+s}^{(l+1)} - \sum_{k=0}^{l}(-s)^{l-k} \left[s\binom{l+1}{k} - r\binom{l}{k} \right] S_{r+s}^{(k)},$$

In general, we have

$$[S_r^{(l)}, L_s^{(l')}] = \frac{-1}{2^{l'+2}} \sum_{k=0}^{l+1} \sum_{k'=0}^{l'+1} (-s)^{l-k} \left[s(s-r)^{l'-k'} \binom{l+1}{k}\binom{l'}{k'} + r(s-r)^{l'-k'} \binom{l}{k}\binom{l'}{k'} \right.$$
$$\left. + (-1)^{l'}(r+s)(-s-r)^{l'-k'} \binom{l}{k}\binom{l'+1}{k'} \right] S_{r-s}^{(k+k')} \tag{6.39}$$

and a similar formula for $[S_r^{(l)}, \bar{L}_s^{(l')}]$, with $r, s \in \mathbb{Z}$.

Although it is obvious that the commutator $[L_n^{(l)}, L_m^{(l')}]$ is expressed as a linear combination of $L_{n+m}^{(l'')}$ with $l'' \leq l + l'$, its precise expression looks more complicated and we do not give it for simplicity. We note also that the commutators from (6.29)–(6.35) have central charge terms which are the vacuum expectations of these commutators (we do not give explicit computations for brevity). Finally, we conclude that the vector space with the basis $S_n^{(l)}$, $L_n^{(l)}$ and $\bar{L}_n^{(l)}$, with $l \geq 0$ and $n \in \mathbb{Z}$, has a Lie algebra structure. We will denote *this Lie algebra as* $\mathfrak{S}$. It is noteworthy to mention that all operators that appear in repeated commutators of $S_n^{(0)}$, including all $S_n^{(l)}$, do commute with the action of $U_q^{\mathrm{odd}} s\ell(2)$. We thus have that the action of the enveloping algebra $U\mathfrak{S}$ commutes with $U_q^{\mathrm{odd}} s\ell(2)$.

6.2. *The relation between the $\mathfrak{S}$ and $\mathfrak{S}_\infty$ algebras and completions.* We have so far obtained two different Lie algebras. One, $\mathfrak{S}_\infty$, is generated by bilinears in fermion modes, while the other, $\mathfrak{S}$, is generated by the modes of bilinears in fermionic fields, which are local operators. The corresponding objects such as $S_n^{(l)}$ and so on expand on an infinite sum of bilinears in fermion modes, and we now have to face the question of the equivalence of these two descriptions. We note first that the generators of the second algebra, $\mathfrak{S}$, are formally infinite sums of the generators of $\mathfrak{S}_\infty$. On the other hand, any element in the first algebra $\mathfrak{S}_\infty$, is expressed (by definition) as a *finite* linear combination of fermion bilinears. Therefore, in order to compare we should first take an algebraic completion $\overline{\mathfrak{S}}_\infty$ of the Lie algebra $\mathfrak{S}_\infty$, that now admits infinite sums like those for $S_n^{(l)}$. The completion $\overline{\mathfrak{S}}_\infty$ is defined as a Lie subalgebra in $\overline{\mathfrak{gl}}'_\infty$ (the completion of $\mathfrak{gl}_\infty$ introduced in Sect. 5.1.2 and Appendix B) that contains any (possibly infinite) linear combinations of the bilinears

$$\overline{\mathfrak{S}}_\infty: X_{\alpha,\beta}{:}\boldsymbol{\psi}_m^\alpha\boldsymbol{\psi}_{-n}^\beta{:}\quad\text{for a fixed value of }(m-n),\tag{6.40}$$

where the fermions $\boldsymbol{\psi}^\alpha$ are defined in (B4) and $X_{\alpha,\beta} = S_{\alpha,\beta}$ for m and n of different parity and $X_{\alpha,\beta} = J_{\alpha,\beta}$ for m and n of same parity. Then, recalling the expressions (6.17) and (6.28) we obviously have inclusions

$$\mathfrak{S}\subset\overline{\mathfrak{S}}_\infty\subset\overline{\mathfrak{gl}}'_\infty.$$

So, considering the completion of the Lie algebra appearing in the scaling limit of our spin-chains is crucial if we want to describe local operators.

Strictly speaking, the algebra $\overline{\mathfrak{S}}_\infty$, as well as its subalgebra $\mathfrak{S}$, does not act on the space $\mathcal{H}$ of the scaling states. It acts on the completion

$$\overline{\mathcal{H}} = \prod_{n\geq 0}\mathcal{H}^{(n)},\quad\text{where }\mathcal{H}^{(n)}\text{ are root/eigen-spaces for }E = L_0 + \bar{L}_0,\tag{6.41}$$

constructed initially as the projective limit of our spin-chains in Sect. C.4 and described in details in the paragraph preceding Corollary C.6. This is the completion in the so-called formal topology [46]: all subspaces $\bigoplus_{\mathbb{N}/M}\mathcal{H}^{(n)}\subset\mathcal{H}$, for a finite subset $M\subset\mathbb{N}$, are declared to be the fundamental system of neighbourhoods of zero. Then, the completion of $\mathcal{H}$ in this topology is the direct product $\prod_{\mathbb{N}}\mathcal{H}^{(n)}$ of its homogeneous, or fixed energy, subspaces $\mathcal{H}^{(n)}$, and $\mathcal{H}\subset\overline{\mathcal{H}}$ is a dense subspace. We will denote the representation of the universal enveloping algebra of the Lie algebra $\overline{\mathfrak{S}}_\infty$ on the completed space by $\pi: U\overline{\mathfrak{S}}_\infty\to\text{End}\,\overline{\mathcal{H}}$ and for simplicity denote the image $\pi(\overline{\mathfrak{S}}_\infty)$ of the abstract Lie algebra just by $\overline{\mathfrak{S}}_\infty$ in what follows.

It is interesting to go back for a while to the finite lattice and to mention that there is an isomorphism between the Lie algebra generated by the generalized higher Hamiltonians $H_l(n)$, from which we extracted operators like $S_n^{(l)}$, and the Lie algebra $\mathfrak{S}_N$ introduced in Sect. 4 in order to give special generators of $\pi_{\mathfrak{gl}}(JTL_N)$ as bilinear monoms in the fermionic modes. The existence of such an isomorphism is discussed in Sect. D.1. The $\mathfrak{S}_N$ algebras in the scaling limit give the $\mathfrak{S}_\infty$, while the Lie algebra of $H_l(n)$'s gives in the limit the Lie algebra $\mathfrak{S}$. It is natural to expect that the isomorphism between the two Lie algebras for finite spin chains carries over to their scaling limit, after taking a proper completion. The completion $\overline{\mathfrak{S}}$ (of the image of the representation) of $\mathfrak{S}$ acting

in $\overline{\mathcal{H}}$ is defined as the Lie algebra with the basis of $\mathfrak{S}$ but containing also infinite linear combinations of $S_n^{(l)}$ or $L_n^{(l)}$, or $\bar{L}_n^{(l)}$'s for any fixed n such that

$$\overline{\mathfrak{S}}: \quad \sum_{l \geq 0} C_l S_n^{(l)}, \quad \sum_{l \geq 0} C_l L_n^{(l)}, \quad \sum_{l \geq 0} C_l \bar{L}_n^{(l)}, \quad \text{with} \quad \lim_{l \to \infty} C_l = 0. \tag{6.42}$$

Actually, we will require a stronger condition: for any (infinite) sequence of the constants C_l, the series $\sum_{l=0}^{\infty} C_l$ has to be convergent. Note that this way defined completion $\overline{\mathfrak{S}}$ is a subalgebra in $\overline{\mathfrak{gl}}_{\infty}'$. We can show that the infinite combinations in (6.42) with the coefficients series $\sum_{l=0}^{\infty} C_l$ equal 0 or 1 converge on $\overline{\mathcal{H}}$ to the generators of $\mathfrak{S}_{\infty}$. We first note that the $\mathfrak{S}$ generators are obtained from those of $\mathfrak{S}_{\infty}$ by infinite Vandermonde-type matrices:

$$2L_k^{(l)} = \sum_{n \in \mathbb{Z}} n^l {:}T_{n,k-n}{:}, \quad 2\bar{L}_k^{(l)} = \sum_{n \in \mathbb{Z}} n^l {:}\bar{T}_{n,k-n}{:}, \quad S_k^{(l)} = \sum_{n \in \mathbb{Z}} (2n+k)^l S_{n,k+n}, \quad l \geq 0 \tag{6.43}$$

where we set $n^0 = 1$, for any n. Let us consider for simplicity only the case $k = 0$ in details. We start with $L_0^{(l)}$, which are non-zero only for even values of l (recall that $T_{n,-n} = T_{-n,n}$). Then, we can write

$$L_0^{(2l)} = \tfrac{1}{2}\delta_{l,0}T_{0,0} + \sum_{n > 0} n^{2l} T_{n,-n}, \tag{6.44}$$

or introducing infinite vectors $\mathsf{L} = (L_0^{(0)}, L_0^{(2)}, L_0^{(4)}, \dots)^T$ and $\mathsf{T} = (\tfrac{1}{2}T_{0,0}, T_{1,-1}, T_{2,-2}, \dots)^T$, we have

$$\mathsf{L} = \mathcal{V} \cdot \mathsf{T}, \tag{6.45}$$

where $\mathcal{V}$ is the transposed of the classical infinite Vandermonde matrix

$$\mathcal{V}^T = \begin{pmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n & \cdots \\ 1 & x_1 & x_1^2 & \cdots & x_1^n & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & x_m & x_m^2 & \cdots & x_m^n & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \end{pmatrix}, \quad \text{with} \quad x_j = j^2. \tag{6.46}$$

Now, we face a problem of finding the inverse $\mathcal{V}^{-1} = (C_{k,n})_{k,n \geq 0}$. In general, it is a non-trivial problem but our Vandermonde matrix is special: it has one row and one column of units 1's ($x_1 = 1$) and this property allows at least to prove that the series $\sum_{n \geq 0} C_{k,n}$ associated with each row of $\mathcal{V}^{-1}$ converge. Indeed, we have by definition of the inversed matrix

$$\sum_{n \geq 0} C_{k,n} x_m^n = \delta_{m,k}. \tag{6.47}$$

Introduce then formal power series in x as

$$P_k(x) = \sum_{n \geq 0} C_{k,n} x^n \tag{6.48}$$

and using (6.47) they take the following values at points in the sequence $\{x_0, x_1, x_2, \dots\}$:

$$P_k(x_m) = \delta_{m,k}. \tag{6.49}$$

It is now obvious that

$$\sum_{n\geq 0} C_{k,n} = P_k(1) = P_k(x_1) = \delta_{k,1} \tag{6.50}$$

and in particular $\lim_{n\to\infty} C_{k,n} = 0$. We have thus shown that

$$T_{0,0} = 2\sum_{l\geq 0} C_{0,l} L_0^{(2l)}, \quad T_{k,-k} = \sum_{l\geq 0} C_{k,l} L_0^{(2l)} \tag{6.51}$$

are well defined and belong to our completion $\overline{\mathfrak{S}}$. Now, we turn to the zero modes $S_0^{(l)}$ and introduce an infinite vector $\mathsf{L} = \left(\frac{1}{2^l} S_0^{(l)}\right)_{l\geq 0}$. It can be expressed again as a product of a new Vandermonde matrix $\mathcal{V} = (x_m^n)_{n\geq 0, m\in\mathbb{Z}}$, where $x_m = m$, with the vector $\mathsf{T} = (S_{k,k})_{k\in\mathbb{Z}}$. Note that $\mathcal{V}$ has again one column and one row of units. Proceeding as above, we have again that the inverse $\mathcal{V}^{-1} = (C_{k,n})_{k\in\mathbb{Z}, n\geq 0}$ has rows with the properties that the series $\sum_{n\geq 0} C_{k,n}$ converge to 0 or 1. Therefore, the elements $S_{k,k} = \sum_{l\geq 0} C_{k,l} S_0^{(l)}$ are in the completion $\overline{\mathfrak{S}}$. For the non-zero modes k, we divide the generators of $\mathfrak{S}$ by appropriate polynomials in k of order l and proceed similarly to establish the desired properties for the coefficients $C_{k,l}$.

Summarizing the previous arguments, we have thus established an existence of an isomorphism

$$\overline{\mathfrak{S}} \cong \overline{\mathfrak{S}}_\infty \tag{6.52}$$

of Lie algebras. Though the explicit transformation requires computation of the coefficients $C_{k,n}$ in the power series (6.48), it is not necessary for our purposes.

We can justify the isomorphism in (6.52) by another, heuristic but more physical, argument. Recall that the modes $S_k = S_k^{(0)}$ were obtained by expanding the field $S_{\text{pl.}}(z, \bar{z})$ defined in (6.18) on the unit circle in the complex plane (or at time $\tau = 0$ in the cylinder geometry). The higher modes $S_k^{(l)}$ can be obtained by expansion at different moments τ. Indeed, we see from (6.38) that the dilatation operator $L_0 + \bar{L}_0$ generates $S_k^{(1)}$ from $S_k^{(0)}$, and so on. This action can be interpreted as an expansion of $S_{\text{pl.}}(z, \bar{z})$ on a different (non-unit) circle with the same centre. Transforming the unit-circle expansion by the conformal generators L_n and $\bar{L}_n$ we generate all higher modes $S_n^{(l)}$ and cover an expansion of the interchiral field $S_{\text{pl.}}(z, \bar{z})$ on the whole complex plane. On the other hand, this expansion is given by the quadratic monoms $S_{n,m}$ in the fermions modes. This suggest strongly that both Lie algebras—the one generated by $S_n^{(l)}$ and the other, $\mathfrak{S}_\infty$, generated by $S_{n,m}$—are isomorphic, after taking proper completions.

6.3. The interchiral algebra $\mathcal{S}$. We define *the interchiral algebra* for the $\mathfrak{gl}(1|1)$ models, denoted by $\mathcal{S}$, as follows. Consider the completion $\overline{\mathfrak{S}}_\infty$ of the Lie algebra $\mathfrak{S}_\infty$ generated by the modes of the regularized interchiral field $S(z, \bar{z})$ in its double mode expansion. The interchiral algebra $\mathcal{S}$ is then defined as the associative algebra generated by $\overline{\mathfrak{S}}_\infty$, i.e., it is the homomorphic image of $U\overline{\mathfrak{S}}_\infty$:

$$U\overline{\mathfrak{S}}_\infty \longrightarrow \mathcal{S}. \tag{6.53}$$

The defining relations in $\mathfrak{S}_\infty$ and thus in the universal enveloping algebra $U\overline{\mathfrak{S}}_\infty$ are computed in Sect. 5.2. The point is, there might be additional relations in the fermionic representation of the abstract algebra $U\overline{\mathfrak{S}}_\infty$, and the particular modules realized in $\overline{\mathcal{H}}$,

as also happens say for the Virasoro Lie algebra and its enveloping algebra $\mathcal{V}(2)$. This is why the map from (6.53) is not an isomorphism but a covering homomorphism of associative algebras.

We also recall the discussion in the previous subsection about relations between both Lie algebras $\mathfrak{S}$ and $\mathfrak{S}_\infty$. In view of existence of an isomorphism (6.52) of their completions, we can give another, more physical, definition of the interchiral algebra. The interchiral algebra can be considered as the associative algebra generated by the modes of the following *local* fields: the stress-energy tensor fields $T(x, \tau)$ and $\bar{T}(x, \tau)$ and the interchiral field $S_{\mathrm{cyl.}}(x, \tau)$, see the mode expansion in (6.22) and in (6.25), where x and τ are the coordinates on the cylinder. The Lie algebra generated by these modes requires of course the completion, $\overline{\mathfrak{S}}$ in the sense of (6.42), in order to get indeed a definition of $\mathcal{S}$ equivalent to the first one. In the symplectic-fermion theory, the interchiral algebra is then the homomorphic image of the (representation of the) enveloping algebra $U\overline{\mathfrak{S}}$. The Lie algebra $\mathfrak{S}$, and its completion as well, has $S_n^{(l)}$, $L_n^{(l)}$ and $\bar{L}_n^{(l)}$, with $l \geq 0$ and $n \in \mathbb{Z}$, as its basis. The commutation relations in this basis were computed also in Sect. 6.1.3, which give relations in $\mathcal{S}$, but probably not all defining relations, because there should be more in the representation of $U\mathfrak{S}$. We write therefore

$$U\overline{\mathfrak{S}}_\infty \cong U\overline{\mathfrak{S}} \longrightarrow \mathcal{S} \qquad (6.54)$$

where the arrow is again a covering homomorphism of associative algebras and $\mathcal{S}$ is the image of $U\overline{\mathfrak{S}}$ under this homomorphism.

Note that while the second definition of the interchiral algebra is probably better for further generalizations of the concept of interchiral algebras for other models, we will use the first, more technical (and less physically pleasant), definition of the interchiral algebra in studying its simple modules below. We note further that for more complicated logarithmic theories like those describing scaling limit of $\mathfrak{gl}(n|n)$ ($n > 1$) periodic spin-chains, we will have a different and now faithful representation of JTL_N, and a description in terms of Lie algebras will probably not be available. Nevertheless, a description in terms of a (properly defined) interchiral algebra should still exist. This will be discussed in further work.

6.4. OPEs. An important advantage in using the interchiral algebra is that it gives a convenient "vertex-operator algebra" framework[8] where one introduces operator-valued generating functions of formal variables z and $\bar{z}$ and their OPE in order to define an algebraic structure. It is worth spending some time discussing this approach in the context of the bulk theory. We recall first the OPE of derivatives $\psi^\alpha(z)$ and $\bar\psi^\alpha(\bar{z})$ of the symplectic fermions

$$\psi^\alpha(z)\psi^\beta(w) = J^{\alpha\beta}\left(\frac{1}{(z-w)^2} - T(w) + \mathrm{reg.}\right), \quad \bar\psi^\alpha(\bar{z})\bar\psi^\beta(\bar{w}) = J^{\alpha\beta}\left(\frac{1}{(\bar{z}-\bar{w})^2} - \bar{T}(\bar{w}) + \mathrm{reg.}\right),$$
(6.55)

where we use the symplectic form $J^{\alpha\beta}$, with $J^{12} = -J^{21} = 1$, and computed the coefficient in front of the stress tensor $T(w)$ using usual conformal-invariance arguments. The stress tensor $T(z)$ is given in fermions as $T(z) = \frac{1}{2}J_{\alpha\beta} : \psi^\alpha(z)\psi^\beta(z) :$ with $J_{\alpha\beta}$ such that $J_{\alpha\beta}J^{\beta\gamma} = \delta_\alpha^\gamma$.

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[8] Our interchiral algebra strictly speaking is not a vertex-operator algebra because of the neither-chiral-nor-antichiral fields. A proper generalization of the vertex-operator algebras for our context is still required.

Then, OPE of two $S(z, \bar{z})$ fields can be written using (6.55) as

$$S(z, \bar{z})S(w, \bar{w}) = S_{\alpha\beta}\psi^{\alpha}(z)\bar{\psi}^{\beta}(\bar{z})S_{\gamma\delta}\psi^{\gamma}(w)\bar{\psi}^{\delta}(\bar{w})$$

$$= -S_{\alpha\beta}S_{\gamma\delta}\left(J^{\alpha\gamma}\left(\frac{1}{(z-w)^2} - T(w) + [\mathbf{1}]_{>2}\right)J^{\beta\delta}\left(\frac{1}{(\bar{z}-\bar{w})^2} - \bar{T}(\bar{w}) + [\bar{\mathbf{1}}]_{>2}\right)\right.$$

$$+ J^{\alpha\gamma}\frac{1}{(z-w)^2}\left(: \bar{\psi}^{\beta}(\bar{w})\bar{\psi}^{\delta}(\bar{w}) : +(\bar{z}-\bar{w}) : \bar{\partial}\bar{\psi}^{\beta}(\bar{w})\bar{\psi}^{\delta}(\bar{w}) : +\cdots\right)$$

$$\left. + J^{\beta\delta}\frac{1}{(\bar{z}-\bar{w})^2}\left(: \psi^{\alpha}(w)\psi^{\gamma}(w) : +(z-w) : \partial\psi^{\alpha}(w)\psi^{\gamma}(w) : +\cdots\right) + \text{reg.}\right)$$

(6.56)

where we take into account contributions of all possible single and double contractions, and denote by $[\mathbf{1}]_{>2}$ and $[\bar{\mathbf{1}}]_{>2}$ contributions of descendants from levels $(n, 0)$ and $(0, n)$ with $n > 2$, respectively. We note that $S_{\alpha\beta}S_{\gamma\delta}J^{\alpha\gamma}J^{\beta\delta} = -2$ and $S_{\alpha\beta}S_{\gamma\delta}J^{\alpha\gamma} = J_{\beta\delta}$. Therefore, the OPE is

$$S(z, \bar{z})S(w, \bar{w}) = 2\left(\frac{1}{(z-w)^2(\bar{z}-\bar{w})^2} - \frac{2T(w)}{(\bar{z}-\bar{w})^2} - \frac{2\bar{T}(\bar{w})}{(z-w)^2} + \frac{[\mathbf{1}]_{>2}}{(\bar{z}-\bar{w})^2} + \frac{[\bar{\mathbf{1}}]_{>2}}{(z-w)^2}\right)$$

$$- \frac{(z-w)\partial T(w) + \ldots}{(\bar{z}-\bar{w})^2} - \frac{(\bar{z}-\bar{w})\bar{\partial}\bar{T}(\bar{w}) + \ldots}{(z-w)^2} + \text{reg.}$$

(6.57)

We note that the right-hand side in the relation (6.29) is consistent with the OPE (6.57) in a sense that commuting two $S(z, \bar{z})$ fields we can get only the identity and a linear combination of modes of $T(z)$ and $\bar{T}(\bar{z})$ and of their descendants, due to presence of the symplectic form $J_{\alpha\beta}$ in (5.13).

The OPE of $S(z, \bar{z})$ with $T(w)$ and $\bar{T}(\bar{w})$ is quite obvious because the field $S(z, \bar{z})$ is primary for both chiral and anti-chiral stress tensors. We describe below in Sect. 6.6.2 the vacuum module over the interchiral algebra $\mathcal{S}$ and some other its simple modules explicitly using this "vertex-operator" formulation.

6.5. The interchiral algebra in the twisted model. It turns out that we have also a representation of the interchiral algebra in the twisted sector of the symplectic fermions, where the fermionic modes are half-integer [25]. As we have seen in Sect. 5.5, this sector corresponds to the scaling limit of the anti-periodic $\mathfrak{gl}(1|1)$ spin-chain from Sect. 3.4. The symplectic-fermion representation of the universal enveloping algebra $U\mathfrak{S}_\infty$ in the twisted model is given by the expressions (5.30) for the generators $S_{m,n}$, $T_{m,n}$, and $\bar{T}_{m,n}$. Using the local operators or modes of the local fields $S_{\text{cyl.}}(x, \tau)$ and the energy-momentum tensors in the half-integer sector, we obtain the corresponding representation of the interchiral algebra in the anti-periodic model. And once again, taking then the completion $\overline{\mathfrak{sp}}_\infty$ of $\mathfrak{sp}_\infty$ as in (6.40), in this half-integer sector, we obtain the representation of $\mathcal{S}$ as the representation of $U\overline{\mathfrak{sp}}_\infty$, which is also a quotient of $U\overline{\mathfrak{S}}_\infty$.

6.6. Modules over the interchiral algebra $\mathcal{S}$. In this new section, we describe the structure of simple and indecomposable modules over the interchiral algebra $\mathcal{S}$ that is defined as the representation of the completion $\overline{\mathfrak{S}}_\infty$. Recall that we already described all simple modules $\mathcal{X}_j$ over $U\mathfrak{S}_\infty$ that appear in the whole space $\mathcal{H}$ of the scaling states in Sects. 5.3 and 5.4, together with the bimodule structure of $\mathcal{H}$ over $U\mathfrak{S}_\infty$ and the centralizer $U_{\mathfrak{q}}^{\text{odd}}s\ell(2)$. Here, we give in Sect. 6.6.1 an explicit construction of the module $\mathcal{X}_1$ and

then description of the vacuum module over the interchiral algebra $\mathcal{S}$, and the twisted vacuum module in Sect. 6.6.4. Finally, indecomposable modules over the interchiral algebra are analyzed in Sect. 6.7.

6.6.1. The module $\mathcal{X}_1$. Recall that $\mathcal{X}_j$ denotes the module over $U\mathfrak{S}_\infty$ which is obtained in the scaling (direct) limit of the JTL simple modules $\mathcal{L}_{j,(-1)^{j+1}}$. Suppose we wish to check directly that $\mathcal{X}_j$ are simple modules over the algebra $U\mathfrak{S}_\infty$ and that their completions $\overline{\mathcal{X}}_j$ in the sense of the formal topology, see (6.41), are simple modules over $\mathcal{S}$. Our strategy consists in a few steps: (i) we state that the $T_{m,n}$ and $\bar{T}_{m,n}$ basis elements of $\mathfrak{S}_\infty$ generate the algebras of endomorphims of simple modules (as graded vector spaces) over the left and right Virasoro algebras, respectively; (ii) we then recall that the generators $S_{m,n}$ are modes of the interchiral field $S(z,\bar{z})$ of dimension $(1,1)$ and (iii) finally we use the operator-state correspondence in CFT and identify composite fields in $S(z,\bar{z})$ and its derivatives that produce the direct summands in (5.29) for a few explicit examples.

In our analysis, we use the $s\ell(2)$ action or "global $SU(2)$ symmetry" of the symplectic fermions commonly known as Kausch's $s\ell(2)$ action [25] which we denote by $\boldsymbol{s\ell}(2)$. The generators of the $\boldsymbol{s\ell}(2)$ in the non-chiral symplectic-fermion theory are

$$Q^a = d^a_{\alpha\beta}\left\{i\phi_0^\alpha\psi_0^\beta + \sum_{n=1}^\infty\left(\frac{\psi_{-n}^\alpha\psi_n^\beta}{n} + \frac{\bar{\psi}_{-n}^\alpha\bar{\psi}_n^\beta}{n}\right)\right\} \tag{6.58}$$

where $d^a_{\alpha\beta}$ are defined in (5.32), and with $[Q^a, Q^b] = f^{ab}_c Q^c$ and $f^{01}_2 = -1$. We note that the generators $T_{m,n}$ and $\bar{T}_{m,n}$ commute with this $\boldsymbol{s\ell}(2)$ action. Meanwhile, it is important to note that the generators $S_{m,n}$ *do not* commute with the $\boldsymbol{s\ell}(2)$ and belong to the zero isospin-projection in the triplet $\boldsymbol{s\ell}(2)$-module.

First, we can consider chiral and anti-chiral sectors separately to show that simples $\mathcal{X}_{j,1}$ over the Virasoro Lie algebra with $c = -2$ are also simples over the Lie algebra of $T_{m,n}$'s, and similarly for the anti-chiral part. To analyze the chiral sector, it is technically easier to work in a smaller space—in the chiral symplectic-fermions theory. Recall that the scaling limit of the open $\mathfrak{gl}(1|1)$ spin-chains described in [7] and in Sect. 2.3 gives this chiral LCFT. The symmetry algebra of this theory, which is now the centralizer of the Virasoro algebra $\mathcal{V}(2)$, is a representation of the full quantum group $U_\mathfrak{q}s\ell(2)$ at $\mathfrak{q} = i$. We recall that this centralizer can be equivalently described as the semi-direct product of the $\mathfrak{gl}(1|1)$ and the enveloping algebra $Us\ell(2)$. The generators of the $Us\ell(2)$ are the divided powers of the quantum-group generators and their action coincides with the action of Kausch's global $s\ell(2)$, while the generators $\psi_0^{1,2}$ of $\mathfrak{gl}(1|1)$ correspond to horizontal arrows in the bimodule diagram in Fig. 2. On the other hand, the centralizer of $U_is\ell(2)$ is an associative algebra of endomorphisms of graded vector spaces $\text{End}(\mathcal{H}) = \bigoplus_{n,m\in\mathbb{N}}\text{Hom}(\mathcal{H}^{(n)}, \mathcal{H}^{(m)})$, where $\mathcal{H}^{(n)}$ are finite-dimensional homogeneous subspaces of the nth energy level, and such that they commute with $U_is\ell(2)$. This centralizer obviously contains Virasoro $\mathcal{V}(2)$ but it is a bigger algebra—a completion[9] of $\mathcal{V}(2)$.

The chiral space $\mathcal{H}$ is the direct sum $\mathcal{H}^+ \oplus \mathcal{H}^-$ of bosonic and fermionic degrees of freedom, and it is decomposed onto modules over the product $Us\ell(2) \boxtimes \mathcal{V}(2)$ as

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[9] The completion of the Virasoro contains in particular projectors onto a fixed, kth, energy subspace. Such operators are obviously in the centralizer of $U_is\ell(2)$ but they are not in the universal enveloping of the Virasoro Lie algebra. Of course, there is no double-centralizing property in infinite-dimensional spaces and this is why an appropriate completion is necessary for describing centralizers of quantum groups in the CFTs.

in (2.11). The full bimodule structure is given in Fig. 2. We note that the generators $T_{m,n}$ are well-defined operators from End($\mathcal{H}$) just introduced and they commute with the $U_i s\ell(2)$ action. They are therefore in the centralizer of $U_i s\ell(2)$. The socle of $\mathcal{H}$ is the intersection of the kernels of the fermionic generators of the $\mathfrak{gl}(1|1)$ part of $U_i s\ell(2)$. We can thus restrict the action of all $T_{m,n}$'s onto the socle of the modules in (2.11). Using then the $U s\ell(2)$ symmetry, we obtain that simple modules over the Lie algebra generated by $T_{m,n}$'s are the same, as graded vector spaces, as the simple modules over the Virasoro. Indeed, the vacuum modules $\mathcal{X}_{1,1}$ appears with multiplicity one and it is the $U s\ell(2)$ invariant but all $T_{m,n}$'s are also $U s\ell(2)$ invariants and they thus generate the same $\mathcal{X}_{1,1}$ from the vacuum state. Further, we take the highest-weight state of each higher multiplet for the $U s\ell(2)$ and see that all $T_{m,n}$'s generate from it a module identified with (isomorphic as a graded vector space to) the simple Virasoro module $\mathcal{X}_{j,1}$. This proves our claim that all $T_{m,n}$, for $m, n \in \mathbb{Z}$, are graded-vector spaces endomorphisms of the simple Virasoro modules $\mathcal{X}_{j,1}$, for $j \geq 1$. Moreover, we also check that the fermion bilinears $T_{m,n}$ are the only bilinears that commute with $U_i s\ell(2)$. They are therefore generators of the centralizer of $U_i s\ell(2)$. This means that simple modules over Virasoro are also simple modules over the Lie algebra with the basis given by $T_{m,n}$. We use the same arguments to show that the Lie algebra with the basis given by $\bar{T}_{m,n}$ has simple modules $\bar{\mathcal{X}}_{j,1}$.

In the full non-chiral theory, the left (or right) Virasoro algebra has the same isomorphism classes of simple and indecomposable modules as in the chiral theory, see a discussion in Sect. 6.7.1 below. We thus have essentially the same representation of $T_{m,n}$—the same expression for the generators as in the chiral theory but acting in a much bigger space, so a difference is only in 'multiplicities' which are infinite now and correspond to modules over the right Virasoro $\bar{\mathcal{V}}(2)$. Summarizing, we give the following statement. The simple modules $\mathcal{X}_{j,1} \boxtimes \bar{\mathcal{X}}_{k,1}$ over the left–right Virasoro algebra $\mathcal{V}(2)$ are also simple modules over the Lie algebra with the basis given by $T_{m,n}$ and $\bar{T}_{m,n}$, with $m, n \in \mathbb{Z}$.

We then continue by analyzing the vacuum module over $\mathcal{S}$ which contains the identity field. By definition, the vacuum module is a vector space generated from one state—the vacuum $\boldsymbol{\Omega}$—by those $S_{n,m}$, $T_{n,m}$ and $\bar{T}_{n,m}$ that have negative indexes n and m (note that we used here our first definition of the interchiral algebra from Sect. 6.3). It turns out that this vector space as a $\mathfrak{S}_\infty$-module is decomposed over the left–right Virasoro exactly like $\mathcal{X}_1$. We stated just above that the $T_{n,m}$ and $\bar{T}_{n,m}$ generators of $\mathfrak{S}_\infty$ generate the endomorphism algebras of a simple Virasoro module $\mathcal{X}_{j,1}$ or $\bar{\mathcal{X}}_{j,1}$, respectively. Therefore, to extract the Virasoro content of the vacuum module and compare it with $\mathcal{X}_1$ we only need to identify highest-weight vectors for $\mathcal{V}(2)$ which are generated from the vacuum state by the $S_{n,m}$ basis elements. Equivalently, using the operator-state correspondence, we are going to identify left–right Virasoro primary fields generated from the identity by a field having these $S_{n,m}$ as its modes. Such a field was identified in Sect. 6.1.2 with the interchiral field $S(z, \bar{z}) = S_{\alpha\beta} \psi^\alpha(z) \bar{\psi}^\beta(\bar{z})$. The $S(z, \bar{z})$ is a primary field and corresponds to the state

$$|S\rangle = \lim_{z, \bar{z} \to 0} S(z, \bar{z}) \boldsymbol{\Omega}$$

which belongs to the subspace $\mathcal{X}_{2,1} \boxtimes \bar{\mathcal{X}}_{2,1}$. Note that this subspace is in the triplet sector with respect to $s\ell(2)$. Then, $T_{n,m}$ and $\bar{T}_{n,m}$ generate this $\mathcal{V}(2)$-module from the highest-weight state $|S\rangle$. We thus obtained first two terms in the decomposition onto left–right Virasoro modules

$$\mathcal{X}_1|_{\mathcal{V}(2)} = \bigoplus_{j \geq 1} \mathcal{X}_{j,1} \boxtimes \bar{\mathcal{X}}_{j,1}, \tag{6.59}$$

which is the direct sum in (5.29) for $j = 1$. Highest-weight vectors in other direct summands in (6.59) can be constructed by taking appropriate composite fields in $S(z, \bar{z})$ and in its derivatives $\partial^j \bar{\partial}^j S(z, \bar{z})$ applied to the vacuum $\mathbf{\Omega}$ in the limit $z, \bar{z} \to 0$. So, the next primary field from the decomposition (6.59) should have conformal dimensions $(3, 3)$ and it might be identified with the composite field $:\partial\bar{\partial}S(z, \bar{z})S(z, \bar{z}):$, up to descendants of $S(z, \bar{z})$ and of the identity field on the level $(3, 3)$. We note that this highest-weight state belongs to a 5-dimensional $s\ell(2)$-module. This result can be obtained by a direct calculation using the double mode expansion (6.20) together with normal-ordering prescriptions or using OPE formulas given in (6.57).

The analysis can be continued to construct in a similar way states belonging to higher $s\ell(2)$-multiplets and contributing thus to new primary fields. Using (6.58), we obtain that the composite field

$$:\prod_{j=0}^{n} \partial^j \bar{\partial}^j S(z, \bar{z}): \tag{6.60}$$

has the conformal weight $(\Delta_{n+2,1}, \Delta_{n+2,1})$ and it is in a $(n + 1)$-dimensional $s\ell(2)$-module, up to contributions from lower $s\ell(2)$-multiplets[10]. This state therefore belongs to the direct sum $\oplus_{j=1}^{n+2} \mathcal{X}_{j,1} \boxtimes \bar{\mathcal{X}}_{j,1}$. By induction, this finally gives the decomposition (6.59).

We next observe that the state

$$|S^2\rangle = \lim_{z, \bar{z} \to 0} :S^2(z, \bar{z}):\mathbf{\Omega}$$

is on the level $(2, 2)$ of the direct sum $\mathcal{X}_{2,1} \boxtimes \bar{\mathcal{X}}_{2,1} \oplus \mathcal{X}_{1,1} \boxtimes \bar{\mathcal{X}}_{1,1}$. This means that the square of the interchiral field maps back to the identity module over $\mathcal{V}(2)$. A similar analysis for states (6.60) living in higher $s\ell(2)$ multiplets shows that our vector space $\mathcal{X}_1$ generated by composite fields in $S(z, \bar{z})$ and its derivatives is indeed a simple module over $\mathfrak{S}_\infty$.

6.6.2. Vacuum module over $\mathcal{S}$. Recall then our discussion about the isomorphism of two different Lie algebras, $\overline{\mathfrak{S}}_\infty$ and $\overline{\mathfrak{S}}$, given in Sect. 6.2. The completed algebras, as well as the interchiral algebra $\mathcal{S}$, act on the completed space $\overline{\mathcal{H}}$ defined in (6.41). It is natural then to take the completion of $\mathcal{X}_1 = \oplus_n \mathcal{X}_1^{(n)}$ as the direct product $\overline{\mathcal{X}}_1 = \prod_n \mathcal{X}_1^{(n)}$ of the eigenspaces of $E = L_0 + \bar{L}_0$ and declare it as *the vacuum module of the interchiral algebra*. Note that this module has the basis of $\mathcal{X}_1$ and the latter is its dense subspace in the formal topology, see Sect. 6.2. The first question is now whether the vacuum module $\overline{\mathcal{X}}_1$ is simple or not: making the completion we add many new vectors which could generate invariant subspaces. However, it turns out that having $v \in \overline{\mathcal{X}}_1$ written as an infinite combination of the basis elements, i.e., if v is not in the subspace $\mathcal{X}_1$ we can find a word in the generators of $\mathcal{S}$ such that the image of v under the action of the word belongs again to $\mathcal{X}_1$. For example, for

$$v = S_0^{(0)}\mathbf{\Omega} = S_{\alpha,\beta} \sum_{m<0} \psi_m^\alpha \bar{\psi}_m^\beta \mathbf{\Omega} \in \overline{\mathcal{H}}$$

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[10] Strictly speaking, the composite field constructed belongs actually to a direct sum $\oplus_{j=0}^{n/2} C_j$ where C_j denotes the $(2j + 1)$-dimensional $s\ell(2)$-module.

we can write its image under $(L_0 - 2)(\bar{L}_0 - 2) - L_{-2}\bar{L}_{-2}$ as

$$\left((L_0 - 2)(\bar{L}_0 - 2) - L_{-2}\bar{L}_{-2}\right)v = S_{\alpha,\beta}\psi_{-1}^{\alpha}\bar{\psi}_{-1}^{\beta}\mathbf{\Omega} \in \mathcal{H}$$

and the image is just the Virasoro highest-weight state corresponding to the primary field $S(z, \bar{z})$.

In general, we formulate the following conjecture based on many explicit checks.

Conjecture 6.6.3. *Taking the completions $\overline{\mathcal{X}}_j$ (those in the formal topology) of the simple $U\mathfrak{S}_{\infty}$-modules $\mathcal{X}_j$ gives simple modules over the interchiral algebra $\mathcal{S}$.*

6.6.4. The twisted vacuum module. We finally discuss an example of a simple module over the interchiral algebra $\mathcal{S}$ in the sector with half-integer fermionic modes. Recall that we have found in Sect. 6.5 an action of the interchiral algebra $\mathcal{S}$ in the twisted model of symplectic fermions. The full symmetry algebra in this case is the $U s\ell(2)$ as it was stated in Proposition 5.5.1. We define then the twisted vacuum module over the interchiral algebra as the space of $s\ell(2)$ invariants. The structure of the twisted vacuum module can be described in a way parallel to Sects. 5.4 and 6.6.1. The generating function of levels obtained as in [27] now involves the characters

$$\chi_{j,2} = \frac{q^{(j-1)^2/2} - q^{(j+1)^2/2}}{\eta(q)} \tag{6.61}$$

and we have the identity for the Kac character, see Appendix E,

$$K_{r,2j+2} = \sum_{s=0}^{j} \chi_{r-j+2s,2}. \tag{6.62}$$

This time, the simple modules of JTL_N^{tw} are obtained with a single subtraction according to (3.10), so their left–right Virasoro $\mathcal{V}(2)$ content is

$$F_{j,(-1)^j}^{(0)} = F_{j,(-1)^j} - F_{j+2,(-1)^j} = \sum_{r=1}^{\infty} \chi_{r,2}\left(\overline{\chi}_{r-j,2} + \overline{\chi}_{r-j+2,2} + \ldots + \overline{\chi}_{r+j,2}\right). \tag{6.63}$$

The JTL_N^{tw} modules are all semi-simple in this model, and so are the $\mathcal{V}(2)$ modules, according to the decomposition (3.9). We thus obtain the left–right Virasoro content in the scaling limit

$$\mathcal{L}_{j,(-1)^j} \mapsto \bigoplus_{r\geq 1} \mathcal{X}_{r,2} \boxtimes \left(\bar{\mathcal{X}}_{r-j,2} \oplus \bar{\mathcal{X}}_{r-j+2,2} \oplus \ldots \oplus \bar{\mathcal{X}}_{r+j,2}\right), \tag{6.64}$$

which is again a direct sum of infinite number of simple modules over $\mathcal{V}(2)$.

Let $\mathcal{X}_j^{tw}$ denotes the module over $U\mathfrak{S}_{\infty}$ (or $U\mathfrak{sp}_{\infty}$) which is obtained in the scaling limit of the JTL^{tw} simple modules $\mathcal{L}_{j,(-1)^j}$. With an analysis similar to one in Sect. 6.6.1, we generate the vacuum module applying by composite fields in $S(z, \bar{z})$ and its derivatives on the vacuum state, the highest-weight state in $\mathcal{X}_{1,2} \boxtimes \bar{\mathcal{X}}_{1,2}$. The vacuum module $\mathcal{X}_0^{tw}$ decomposition onto $\mathcal{V}(2)$-modules is

$$\mathcal{X}_0^{tw}|_{\mathcal{V}(2)} = \bigoplus_{j\geq 1} \mathcal{X}_{j,2} \boxtimes \bar{\mathcal{X}}_{j,2} \tag{6.65}$$

and it is an irreducible module over $U\mathfrak{sp}_{\infty}$. We do not give details of calculations as their essentially repeat the previous. Correspondingly, its completion, see a discussion in Sect. 6.6.2, describes the vacuum module over $\mathcal{S}$ in the twisted sector.

6.7. Indecomposable modules over $\mathcal{S}$. To proceed, we must now discuss the scaling limit of the indecomposable JTL_N-modules $\widehat{\mathcal{P}}_j$, which are indecomposable $U\mathfrak{S}_\infty$-modules denoted by $\mathcal{Y}_j$, and compare it with potentially similar structure in the symplectic fermion theory. This will be facilitated by a preliminary discussion of the latter.

6.7.1. The structure of the Virasoro representations in symplectic fermions. We already reminded the reader of the known results about the chiral conformal field theory of symplectic fermions in Sect. 2.3. We now turn to a similar analysis of the non-chiral theory and its decomposition over the left–right Virasoro $\mathcal{V}(2)$. In this section, we will denote our space of scaling states in the bulk theory by $\mathcal{H}$ in order to not be confused with the decomposition (2.11) of the chiral theory.

In the integer-mode sector, the space of states $\mathcal{H}$ for the non chiral theory is decomposed into a bosonic sector $\mathcal{H}^+$ and the fermionic one $\mathcal{H}^-$ with the $\boldsymbol{s\ell}(2)$ and left–right Virasoro $\mathcal{V}(2)$ content [47]

$$\mathcal{H}^+|_{\boldsymbol{s\ell}(2)\boxtimes\mathcal{V}(2)} = \bigoplus_{k\in\mathbb{N}_0} C_k \boxtimes \mathcal{H}_k^+, \quad \mathcal{H}^-|_{\boldsymbol{s\ell}(2)\boxtimes\mathcal{V}(2)} = \bigoplus_{k\in\mathbb{N}-\frac{1}{2}} C_k \boxtimes \mathcal{H}_k^-, \quad (6.66)$$

where C_n denotes now a $(2n+1)$-dimensional $\boldsymbol{s\ell}(2)$-module of the isospin n, and $\mathcal{H}_k^\pm$ are $\mathcal{V}(2)$-modules which we describe below.

With the left and right $\mathcal{V}(2)$-modules $\mathcal{H}_k^{\pm,(l)}$ and $\mathcal{H}_k^{\pm,(r)}$ denoting restriction of $\mathcal{H}_k^\pm$ on $\mathcal{V}(2)$ and $\overline{\mathcal{V}}(2)$, respectively, the bosonic components $\mathcal{H}_k^+$ have the decompositions, with non-negative integer k,

$$\mathcal{H}_k^{+,(l)} = \bigoplus_{a=-k}^{k} \bigoplus_{n>k-a} \mathcal{P}_{n,1} \boxtimes \bar{\mathcal{X}}_{n+2a,1}, \quad \mathcal{H}_k^{+,(r)} = \bigoplus_{a=-k}^{k} \bigoplus_{n>k-a} \mathcal{X}_{n,1} \boxtimes \bar{\mathcal{P}}_{n+2a,1}, \quad k\in\mathbb{N}_0,$$

$$(6.67)$$

and the fermionic components $\mathcal{H}_k^-$ decompose as, with a positive half-integer k,

$$\mathcal{H}_k^{-,(l)} = \bigoplus_{a=-k}^{k} \bigoplus_{n>k-a} \mathcal{P}_{n,1} \boxtimes \bar{\mathcal{X}}_{n+2a,1}, \quad \mathcal{H}_k^{-,(r)} = \bigoplus_{a=-k}^{k} \bigoplus_{n>k-a} \mathcal{X}_{n,1} \boxtimes \bar{\mathcal{P}}_{n+2a,1}, \quad k\in\mathbb{N}-\frac{1}{2},$$

$$(6.68)$$

where we use the tensor product $\boxtimes$ for two Virasoro modules in order to show that 'multiplicities' of left- or right-Virasoro staggered modules in the decompositions are simple subquotients over right or left Virasoro, respectively. We stress that direct summands in (6.67) and (6.68) are not $\mathcal{V}(2)$-modules and the rest of this section is devoted to describing a subquotient structure for the product of the two Virasoro algebras.

The sector with the trivial $\boldsymbol{s\ell}(2)$ action ($k=0$) has a particular interest because it contains the vacuum state $\boldsymbol{\Omega}$ and its logarithmic partner $\boldsymbol{\omega}$ ($L_0\boldsymbol{\omega} = \bar{L}_0\boldsymbol{\omega} = \boldsymbol{\Omega}$),

$$\mathcal{H}_0^{+,(l)} = \bigoplus_{n\geq 1} \mathcal{P}_{n,1} \boxtimes \bar{\mathcal{X}}_{n,1}, \quad \mathcal{H}_0^{+,(r)} = \bigoplus_{n\geq 1} \mathcal{X}_{n,1} \boxtimes \bar{\mathcal{P}}_{n,1}, \quad \boldsymbol{\Omega}, \boldsymbol{\omega} \in \mathcal{P}_{1,1} \boxtimes \bar{\mathcal{X}}_{1,1}. \quad (6.69)$$

These two states are the only bosonic states with conformal dimension $(0,0)$. There are two other states of the same conformal dimension in the fermionic sector $\mathcal{H}_{1/2}^-$ in agreement with [48].

We use the decompositions (6.67) and (6.68) over the left and right Virasoro in studying the subquotient structure over their product $\mathcal{V}(2)$. We first note an obvious fact that
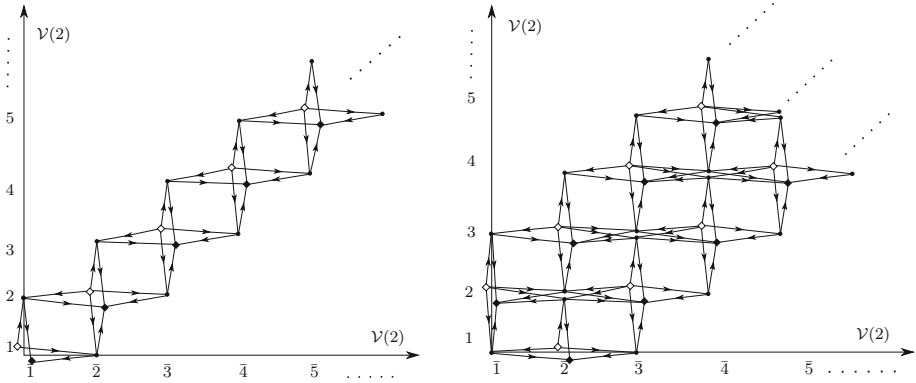
Fig. 5. Module structure over $\boldsymbol{\mathcal{V}}(2) = \mathcal{V}(2) \boxtimes \overline{\mathcal{V}}(2)$ for the vacuum sector $\mathcal{H}_0^+$ [with zero $s\ell(2)$-isospin] on the *left diagram* while the *right one* is for the doublet-sector $\mathcal{H}_{1/2}^-$. Each node with a coordinate $(\bar{n}, n')$ is a simple subquotient over $\boldsymbol{\mathcal{V}}(2)$ with the conformal weight $(\Delta_{n',1}, \bar{\Delta}_{n,1})$. *Vertical arrows* represent the action of the *left* Virasoro $\mathcal{V}(2)$ and *horizontal arrows of the right* Virasoro $\overline{\mathcal{V}}(2)$

right-Virasoro algebra elements are represented on each $\mathcal{H}_k^\pm$ as intertwining operators for the left-Virasoro action, i.e., there is a homomorphism from the universal enveloping algebra $U\big(\overline{\mathcal{V}}(2)\big)$ to $\mathrm{End}_{\mathcal{V}(2)}(\mathcal{H}_k^\pm)$. Therefore, we should begin with a description of the space $\mathrm{Hom}_{\mathcal{V}(2)}(\mathcal{P}_{n,1}, \mathcal{P}_{n',1})$. From the subquotient structure (2.12) of the staggered modules, we deduce that the Hom-space is one-dimensional only for $n = n'$ or $n = n' \pm 1$ and zero-dimensional otherwise. In the case $n = n'$, the image of a basis element in the Hom-space is the simple submodule $\mathcal{X}_{n,1}$ while the image in the case $n = n' + 1$ is the indecomposable Kac module $\mathcal{X}_{n,1} \rightarrow \mathcal{X}_{n+1,1}$ and the image in the case $n = n' - 1$ is the contragredient Kac module $\mathcal{X}_{n,1} \rightarrow \mathcal{X}_{n-1,1}$. Ananlysing then all possible endomorphisms on $\mathcal{H}_k^{\pm,(l)}$ respecting the left Virasoro and combining with the decomposition of $\mathcal{H}_k^{\pm,(r)}$ over the right Virasoro, we end up in diagrams depicting a subquotient structure over the product $\boldsymbol{\mathcal{V}}(2)$ of two Virasoro algebras for each $\mathcal{H}_k^\pm$.

The simplest is for $k = 0$, where we obtain the structure represented on the left in Fig. 5 using the decompositions (6.69). On the right in Fig. 5, we show the subquotient structure for $k = 1/2$ as well, which is in the fermionic sector $\mathcal{H}^-$. For $k = 1$ meanwhile we have the structure represented on Fig. 6. More detailed analysis of the $\boldsymbol{\mathcal{V}}(2)$-module structure on each $\mathcal{H}_k^\pm$ can be also found in a companion paper [47].

Each node with a coordinate $(\bar{n}, n')$ in the diagrams on Figs. 5 and 6 corresponds to a simple subquotient over $\boldsymbol{\mathcal{V}}(2) = \mathcal{V}(2) \boxtimes \overline{\mathcal{V}}(2)$ with the lowest conformal weight $(\Delta_{n',1}, \bar{\Delta}_{n,1})$ and arrows show the action of both Virasoro algebras—the Virasoro $\mathcal{V}(2)$ acts in the vertical direction (preserving the coordinate $\bar{n}$), while the right Virasoro $\overline{\mathcal{V}}(2)$ acts in the horizontal way. Some values $(\bar{n}, n')$ occur twice and those nodes/subquotients are separated slightly for clarity; we denote top subquotients by $\diamond$, bottom ones $\blacklozenge$, and subquotients in the middle level that have incoming as well as outgoing arrows are denoted by $\bullet$, in order to make reading the diagrams easier. We also note that some horizontal arrows (of the same direction) connecting a $\diamond$ with a pair of $\bullet$'s at the same coordinate or a pair of $\bullet$'s with a $\blacklozenge$, say connecting $(\bar{1}, 3)$ with two nodes at $(\bar{2}, 3)$ in Fig. 6, are actually doubled which means that right-Virasoro elements corresponding to the pair of arrows with the same source/sink map to/from a fixed linear combination
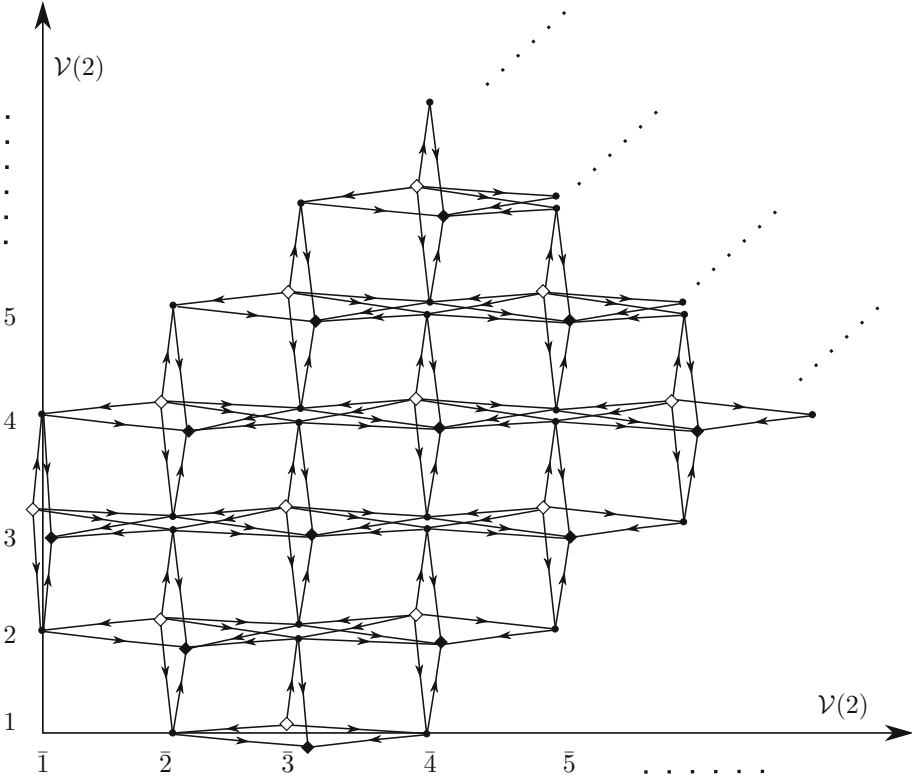
Fig. 6. $\mathcal{V}(2)$-module subquotient structure for the triplet-sector $\mathcal{H}_1^+$. Each node with a coordinate $(\bar{n}, n')$ is a simple subquotient over $\boldsymbol{\mathcal{V}}(2) = \mathcal{V}(2) \boxtimes \overline{\mathcal{V}}(2)$ with the conformal weight $(\Delta_{n',1}, \bar{\Delta}_{n,1})$. *Vertical arrows* represent the action of the *left* Virasoro $\mathcal{V}(2)$ and *horizontal arrows of the right* Virasoro $\overline{\mathcal{V}}(2)$

of the pair of subquotients depicted by $\bullet$. We use such a linear combination for each pair of $\bullet$'s that corresponds to a basis adapted for the left-Virasoro action[11] that has no (vertical) doubled arrows and is explicitly decomposed onto diagrams from (2.12). We finally assume in our diagrams that right-Virasoro elements map states from two neighbour $\diamond$'s, say $(\bar{1}, 3)$ and $(\bar{3}, 3)$ in Fig. 6, to linearly independent subquotients of the same coordinate, say $(\bar{2}, 3)$, in the middle level.

The indecomposable staggered $\mathcal{V}(2)$-modules $\mathcal{P}_{n',1}$ introduced in (2.12) and appearing in the decompositions (6.67) and (6.68) can be recovered by ignoring all the horizontal arrows, while staggered $\overline{\mathcal{V}}(2)$-modules $\bar{\mathcal{P}}_{n,1}$ are obtained by ignoring all the vertical arrows in the diagrams and taking an appropriate linear combination of two simple subquotients for each pair of middle-level nodes denoted by $\bullet$ and sharing the same coordinate $(\bar{n}, n')$.

We note that the interchiral field $S(z, \bar{z})$ from (6.18) generating our interchiral algebra $\mathcal{S}$ belongs to the submodule identified in the diagram for the triplet sector in Fig. 6 with the node $\blacklozenge$ at the position $(\bar{2}, 2)$.

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[11] Of course, it is a matter of a convention, and we could choose a basis where horizontal arrows are not doubled but the vertical ones would be doubled. It is important to note that there exists no a basis without the doubled arrows in the corresponding diagram.
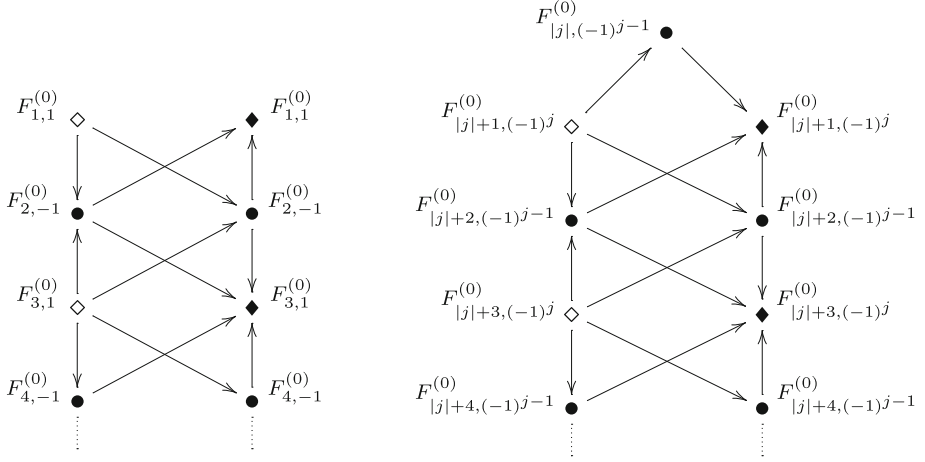
Fig. 7. Scaling limit of indecomposable spin-chain modules $\widehat{\mathcal{P}}_j$ for $j = 0$ on the *left* and $j \neq 0$ on the *right side*. These are indecomposable $U\mathfrak{S}_\infty$-modules $\mathcal{Y}_j$ with simple subquotients and the Virasoro character $F^{(0)}_{|j|,(-1)^{j+1}}$ of each subquotient is given by (5.25)

We will comment on the similarity and consistency of these figures with our results of the (limits of) JTL_N-modules in the next subsection. Notice that the way the left and right indecomposables are glued together in (the vacuum sector of) the non-chiral theory is similar that what is observed for super WZW models on $\mathfrak{gl}(1|1)$ and $\mathfrak{su}(2|1)$ [20,21].

Finally, we stress that the symplectic fermion theory also admits action of $\mathfrak{gl}(1|1)$ which connects in particular the bosonic $\mathcal{H}^+$ and fermionic sectors $\mathcal{H}^-$ of the theory. Its action is quite straightforward, and we refrain from discussing this for simplicity.

6.7.2. Indecomposable $U\mathfrak{S}_\infty$ and $\mathcal{V}(2)$-modules. Recall that we denote the scaling limit of indecomposable spin-chain modules $\widehat{\mathcal{P}}_j$ by $\mathcal{Y}_j$. These are indecomposable but reducible modules over $U\mathfrak{S}_\infty$, recall the discussion in Sect. 5.3, with subquotient structure given by diagrams in Fig. 7 which is an infinite analogue of the finite ladders in Fig. 4. As we discussed in previous sections, the interchiral algebra $\mathcal{S}$ acts on the completed space $\overline{\mathcal{H}}$. From its definition in (6.41), we obviously have a decomposition over $\mathcal{S}$:

$$\overline{\mathcal{H}} = \bigoplus_{j \in \mathbb{Z}} \overline{\mathcal{Y}}_j, \tag{6.70}$$

where each direct summand is the completion (as defined in Sect. 6.2) of the indecomposable $U\mathfrak{S}_\infty$-modules $\mathcal{Y}_j$. The main objective of this section is to show that the structure of $\mathcal{V}(2)$-modules described above is consistent with the diagrams in Fig. 7. Comparing the structure of modules over $U\mathfrak{S}_\infty$ and $\mathcal{V}(2)$, which is a subalgebra in $U\overline{\mathfrak{S}}_\infty$, we then make conclusions on the structure of indecomposable modules $\overline{\mathcal{Y}}_j$ over the interchiral algebra $\mathcal{S}$.

Having obtained in Sect. 5.4 the content under $\mathcal{V}(2)$ of the simple subquotients, we study a filtration[12] of $\mathcal{Y}_j$ by $\mathcal{V}(2)$-modules described in Sect. 6.7.1. Each $\mathcal{Y}_j$ as a $\mathcal{V}(2)$-module has the same as in Fig. 7 subquotient structure, where a crucial point is that a

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[12] Recall that a filtration of A-module M by its submodules M_i, with $0 \leq i \leq n$, is called a sequence of embeddings $0 = M_0 \subset M_1 \subset \cdots \subset M_i \subset \cdots \subset M_{n-1} \subset M_n = M$.

node stands now for an infinite direct sum of simples over $\boldsymbol{\mathcal{V}}(2)$, *i.e*, it is a decomposable $\boldsymbol{\mathcal{V}}(2)$-module.

This subquotient structure agrees with the final results of Sect. 6.7.1. Indeed, the $\boldsymbol{\mathcal{V}}(2)$-modules (6.66) $\mathcal{H}|_{\mathsf{h}=j} \equiv \mathcal{H}_{(j)} = \bigoplus_{k \geq |j|} \mathcal{H}_k^+$ in each sector $\mathsf{h} = j$ have a filtration

$$\mathrm{Soc}\big(\mathcal{H}_{(j)}\big) \subset \mathrm{Mid}\big(\mathcal{H}_{(j)}\big) \subset \mathcal{H}_{(j)} \to \mathrm{Top}\big(\mathcal{H}_{(j)}\big) \to 0 \qquad (6.71)$$

by $\boldsymbol{\mathcal{V}}(2)$-submodules (which are described below) consistent with the structure of $\mathcal{Y}_j$ presented on Fig. 7, where each node is described by formulas in (5.25) and (5.28). In particular, for $j = 0$, we obtain from decompositions (6.67) and (6.68) the left Virasoro structure on $\mathcal{H}$, forgetting about the $\boldsymbol{s\ell}(2)$ content for a while,

$$\mathcal{H}_{(0)}^{(l)} = \bigoplus_{n,m \geq 0} \big(2\min(n,m)+1\big)\mathcal{P}_{2n+1,1} \boxtimes \bar{\mathcal{X}}_{2m+1,1} \oplus \bigoplus_{n,m \geq 1} 2\min(n,m)\mathcal{P}_{2n,1} \boxtimes \bar{\mathcal{X}}_{2m,1},$$
$$(6.72)$$

and the right structure $\mathcal{H}^{(r)}$ at the grade $\mathsf{h} = 0$ is given by the substitutions $\mathcal{P} \to \mathcal{X}$ and $\bar{\mathcal{X}} \to \bar{\mathcal{P}}$. This decomposition allows us immediately to compare particular submodules/quotients—the terms in the filtration (6.71)—with the ones in $\mathcal{Y}_0$ on Fig. 7. First, the socle (the maximal semisimple submodule) $\mathrm{Soc}(\mathcal{Y}_0)$ has the Virasoro character

$$\mathrm{char}\,[\mathrm{Soc}(\mathcal{Y}_0)] = \sum_{j-\mathrm{odd}} F_{j,1}^{(0)} = \sum_{j_1,j_2 \geq 0} \big(2\min(j_1,j_2)+1\big)\chi_{2j_1+1,1}\bar{\chi}_{2j_2+1,1}$$
$$+ \sum_{j_1,j_2 \geq 1} 2\min(j_1,j_2)\chi_{2j_1,1}\bar{\chi}_{2j_2,1}$$

which coincides with the character of $\mathrm{Soc}(\mathcal{H}_{(0)})$ easily extracted from (6.72). The same is true for the top parts (the maximal semisimple quotient) Top of $\mathcal{H}_{(0)}$ and $\mathcal{Y}_0$, which are isomorphic to the socle in our case. Second, we compare the middle-level subquotient Mid/Soc in the filtration (6.71) consisting of all subquotients/nodes having in-arrows from the top and out-arrows directed to the bottom/socle. The middle level of $\mathcal{Y}_0$, constituting of two copies for each $F_{2j,-1}^{(0)}$ (see Fig. 7), has the same $\boldsymbol{\mathcal{V}}(2) \boxtimes \overline{\boldsymbol{\mathcal{V}}}(2)$ content as the $\boldsymbol{\mathcal{V}}(2)$-module $\mathcal{H}$ has at the $\mathsf{h} = 0$ grade, where we use (6.72) and (2.12). Similarly one can proceed for any $\mathsf{h} = j$ grade.

The Kausch's $\boldsymbol{s\ell}(2)$ action discussed in Sec. 5 of the first paper [22], see also Sect. 6.6.1, 'splits' the sum (5.29) into sectors (direct summands) of different isospins[13]—e.g., with $k \geq \frac{j}{2}$ and $j \geq 1$, in the bosonic case. For each sector, the formulas (6.67) and (6.68) give the indecomposable structure under $\boldsymbol{\mathcal{V}}(2)$. The simplest is for $k = 0$, where we obtain the structure represented on the left in Fig. 5. By a slight change of 'geometry' the diagram for this $\boldsymbol{\mathcal{V}}(2)$-module can be represented as well as on Fig. 8, with a pattern of arrows coincident with the one in Fig. 7. The case $k = 1$ in Fig. 6 appears more complicated, but it is only because of additional 'gaps' in the arrows when compared to $k = 0$. The decomposition is still fully compatible with Fig. 7. Here, additional checks of consistency between structures of $\mathcal{Y}_j$ and $\mathcal{H}_{(j)}$ involve finer filtrations. Consider for example the following filtration of $U\mathfrak{S}_\infty$-submodules in $\mathcal{Y}_j$

$$0 = \mathcal{F}_{j-1} \subset \mathcal{F}_j \subset \mathcal{F}_{j+1} \subset \mathcal{F}_{j+2} \subset \cdots \subset \mathcal{Y}_j, \qquad (6.73)$$

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[13] In principle, it is possible to find the $\boldsymbol{s\ell}(2)$ content of the $\mathcal{Y}_j$ modules in Fig. 7 using the lattice realization of the Kausch's $\boldsymbol{s\ell}(2)$ given in Sec. 5.3 in [22] but we do not do it in this paper.
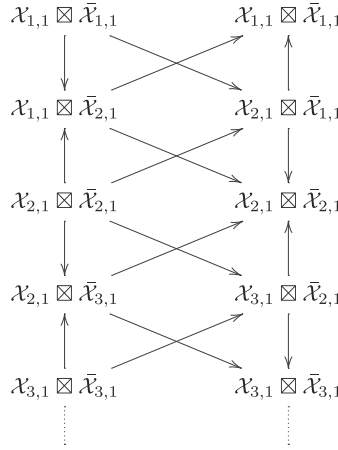
Fig. 8. Two-strands structure of the vacuum sector for non-chiral symplectic fermions

where $\mathcal{F}_j$ is the submodule generated from the top subquotients $F^{(0)}_{k,(-1)^{k+1}}$, with $k \leq j$, in Fig. 7. It turns out that each term of this filtration is a direct sum of indecomposable $\mathcal{V}(2)$ submodules, belonging in general to different isospin-sectors. This follows from the fact that the algebra $U_q^{\mathrm{odd}} s\ell(2)$ of all intertwining operators between $U\mathfrak{S}_\infty$-modules gives also intertwining operators for $\mathcal{V}(2)$-modules. To show an explicit example of a correspondence between the terms of the filtration (6.73) and $\mathcal{V}(2)$ submodules, we consider for simplicity again the case $j = 0$, where the first non-trivial term in (6.73) is $\mathcal{F}_1$ with the subquotient structure

$$(6.74)$$

It is decomposed over $\mathcal{V}(2)$ into the direct sum (over all integer isospins)

where the first and second summands are particular $\mathcal{V}(2)$-submodules of $\mathcal{H}_0^+$ in Fig. 8 and $\mathcal{H}_1^+$ in Fig. 6, respectively (we just rearranged nodes introducing red arrows for the right or anti-chiral Virasoro action). The other direct summands of higher isospins appear

in the same pattern (6.74) with arrows and nodes in the $U\mathfrak{S}_\infty$-picture split according to (5.25) in order to get the left–right Virasoro diagrams. All the other terms $\mathcal{F}_j$ in the filtration (6.73) can in principle be analyzed in a similar way but pictures are more complicated and we do not give them.

To conclude, we checked that all arrows present in the diagrams for the subquotient structure of the scaling limit of the spin-chain modules $\widehat{\mathcal{P}}_j$ or indecomposables $\mathcal{Y}_j$ over $U\mathfrak{S}_\infty$ are also in the corresponding infinite diagrams for the subquotient structure of modules over $U\overline{\mathfrak{S}}_\infty$, which contains the Virasoro algebra $\mathcal{V}(2)$, and vice versa, confirming our earlier results. Our analysis was based on a decomposition onto direct summands over the left–right Virasoro algebra. Then, due to an isomorphism found around (6.54) and Conjecture 6.6.3 about simple modules over $\mathcal{S}$, we formulate our final conjecture.

Conjecture 6.7.3. *Taking the completions $\overline{\mathcal{Y}}_j$ (in the formal topology as defined in Sect. 6.2) of the indecomposable $U\mathfrak{S}_\infty$-modules $\mathcal{Y}_j$ gives indecomposable modules over the interchiral algebra $\mathcal{S}$ and their subquotient structure is given by the same infinite towers as for $\mathcal{Y}_j$ in Fig. 7.*

7. Conclusion

This paper terminates our series on the scaling limit of the $\mathfrak{gl}(1|1)$ spin chain. The conclusions of the analysis of [7] in the open case do carry over to the periodic case, but at the price of several complications. The bimodule structure of the spin chain over JTL_N and its centralizer $U_i^{\mathrm{odd}}s\ell(2)$ does turn out to be compatible with the known symplectic fermion continuum limit. However, simple modules over JTL_N correspond to direct sums of modules over $\mathcal{V}(2)$, the left–right Virasoro algebra, while the known $s\ell(2)$ symmetry of Kausch is not present on the discrete spin chain in the bulk case (in contrast with the boundary case). The lesson drawn is that the good organizing object of the LCFT is truly the scaling limit of the JTL_N algebra, and that it contains more than $\mathcal{V}(2)$, which must be extended by non chiral fields [not commuting with $s\ell(2)$], giving rise to the *interchiral algebra $\mathcal{S}$.* The centralizer of this interchiral algebra remains $U_i^{\mathrm{odd}}s\ell(2)$ in the scaling limit, and the bimodule structure of the $\mathfrak{gl}(1|1)$ discrete spin chain carries over identically now to the scaling limit, exactly as in the open case. Interestingly, we note that the bimodule over the pair of algebras $\mathcal{S}$ and $U_i^{\mathrm{odd}}s\ell(2)$ produces acyclic complexes with the differentials E_n and F_n ($\mathsf{E}_n^2 = \mathsf{F}_n^2 = 0$). These can be interpreted as non-chiral analogues of the well-known Felder resolutions for chiral Virasoro representations.

The jaded reader might argue of course that the complex algebraic analysis presented here and in [22,23] is not really necessary nor useful to understand most physical properties of symplectic fermions. The point however is that we are in the process of developing a strategy to tackle interacting models for which very little is known, and a direct solution based on a (free) action is simply unavailable. In this case, we believe that the algebraic analysis is an essential tool to make progress, and answer such simple questions as which conformal fields are degenerate and which are not, which fields have logarithmic partners, what are the logarithmic couplings, etc. It is also probably possible to obtain information about fusion rules using this approach.

Sadly, things are bound to be more complicated in most cases than they were for $\mathfrak{gl}(1|1)$. The reason is, that for $\mathfrak{gl}(1|1)$, the representation of JTL_N is in fact non-faithful—the structure of the bulk and boundary theories are, as a result, quite similar.

General $\mathfrak{gl}(n|m)$ or $\mathfrak{osp}(n|2m)$ spin chains, such as those necessary to study the cases $c = 0$, $c = 1$, will provide faithful representations, for which the structure of the indecomposable JTL_N modules will be considerably more involved. Their analysis will, in fact, require use of more sophisticated algebraic techniques, and will be started in [36]. Note that even for $c = -2$, the $\mathfrak{gl}(1|1)$ spin chain is non generic and the $\mathfrak{gl}(2|2)$ spin chain for instance will lead, in the scaling limit, to a $c = -2$ LCFT that bears little resemblance with symplectic fermions.

Appendix A: Fourier Transforms

It is convenient to introduce Fourier transforms of the f_j and $f_j^\dagger$ fermions used in the definition of our JTL_N representation in (2.2) and (3.1). We set, for $1 \leq m \leq N$ (recall that we set $N = 2L$),

$$\theta_{p_m} = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} e^{-ikp_m} i^{-k} f_k, \quad \theta_{p_m}^\dagger = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} e^{ikp_m} i^{-k} f_k^\dagger \tag{A1}$$

with the set of allowed momenta

$$p_m = \begin{cases} \frac{2\pi m}{N}, & L\text{-even,} \\ \frac{(2m-1)\pi}{N}, & L\text{-odd,} \end{cases} \quad 1 \leq m \leq N, \tag{A2}$$

and with the usual anti-commutation relations

$$\{\theta_{p_1}, \theta_{p_2}^\dagger\} = \delta_{p_1,p_2}, \quad \{\theta_{p_1}, \theta_{p_2}\} = \{\theta_{p_1}^\dagger, \theta_{p_2}^\dagger\} = 0.$$

We then introduce the following linear combinations of fermions

$$\chi_p^\dagger = \frac{1}{\sqrt{2}} \left(\sqrt{\tan \frac{p}{2}}\, \theta_{p-\frac{\pi}{2}}^\dagger + \sqrt{\cot \frac{p}{2}}\, \theta_{p+\frac{\pi}{2}}^\dagger \right), \quad \chi_p = \frac{1}{\sqrt{2}} \left(\sqrt{\cot \frac{p}{2}}\, \theta_{p-\frac{\pi}{2}} + \sqrt{\tan \frac{p}{2}}\, \theta_{p+\frac{\pi}{2}} \right),$$

$$\eta_p^\dagger = \frac{1}{\sqrt{2}} \left(\sqrt{\tan \frac{p}{2}}\, \theta_{p-\frac{\pi}{2}}^\dagger - \sqrt{\cot \frac{p}{2}}\, \theta_{p+\frac{\pi}{2}}^\dagger \right), \quad \eta_p = \frac{1}{\sqrt{2}} \left(\sqrt{\cot \frac{p}{2}}\, \theta_{p-\frac{\pi}{2}} - \sqrt{\tan \frac{p}{2}}\, \theta_{p+\frac{\pi}{2}} \right),$$

$$\tag{A3}$$

$$\chi_0^\dagger = \theta_{\frac{\pi}{2}}^\dagger, \quad \chi_0 = \theta_{\frac{\pi}{2}}, \qquad\qquad\qquad \eta_0^\dagger = \theta_{\frac{3\pi}{2}}^\dagger, \quad \eta_0 = \theta_{\frac{3\pi}{2}},$$

with momenta p shifted by $\pi/2$ and taking thus values $p = p_n = \epsilon n$, where $\epsilon = \frac{\pi}{L}$ and $1 \leq n \leq L - 1$, for even and odd L. The χ and η fermions satisfy the anti-commutation relations

$$\left\{ \chi_p^\dagger, \chi_{p'} \right\} = \left\{ \eta_p^\dagger, \eta_{p'} \right\} = \delta_{p,p'}, \quad \{\chi_p, \eta_{p'}\} = \left\{ \chi_p^\dagger, \eta_{p'}^{(\dagger)} \right\} = \left\{ \eta_p^\dagger, \chi_{p'}^{(\dagger)} \right\} = 0. \tag{A4}$$

These χ and η fermions are creation and annihilation operators for the Hamiltonian H from (3.2) and they were found in [22].

For the anti-periodic model, we introduce θ_p and $\theta_p^\dagger$ with the same formal expression (A1) but now the momenta p_m take values $\frac{2\pi m}{N}$ for L odd and $\frac{(2m-1)\pi}{N}$ for L even, with $1 \leq m \leq N$. As a result, the values $p = \frac{\pi}{2}, \frac{3\pi}{2}$ are not allowed, and there are no zero modes. Finally, we introduce $\chi_p^{(\dagger)}$ and $\eta_p^{(\dagger)}$ fermions generating Hamiltonian eigenstates from the vacuum by the same formal definition (A3) but now momenta take values $\epsilon/2 \leq p \leq \pi - \epsilon/2$ with the step $\epsilon = \pi/L$.

Appendix B: Lie Algebras $\mathfrak{gl}_\infty$, $\overline{\mathfrak{gl}}_\infty$, and $\mathfrak{S}$

Recall [46] that $\mathfrak{gl}_\infty$ is a Lie algebra of infinite matrices with a *finite* number of non-zero elements. This algebra admits a well-known representation (so-called basic representation) in the free fermion models. Having fermion modes $\boldsymbol{\psi}_n^2$ with their conjugates $\boldsymbol{\psi}_{-n}^1$, the basis elements $\mathcal{E}_{n,m}$ of $\mathfrak{gl}_\infty$—the elementary matrices having identity at the position (n, m) and zeros otherwise—can be written as

$$\mathcal{E}_{n,m} = \boldsymbol{\psi}_n^2 \boldsymbol{\psi}_{-m}^1, \quad n, m \in \mathbb{Z}. \tag{B1}$$

The commutation relations in this algebra are then the usual ones as in a finite $\mathfrak{gl}_N$ but now indices are in $\mathbb{Z}$. The algebra $\mathfrak{gl}_\infty$ admits a central extension $\mathfrak{gl}_\infty \oplus \mathbb{C}\mathbf{1}$ which we denote by $\mathfrak{gl}'_\infty$. This extension is defined using the two-cocylce $c\colon \mathfrak{gl}_\infty \times \mathfrak{gl}_\infty \to \mathbb{C}$ given by (see [46])

$$c(\mathcal{E}_{n,m}, \mathcal{E}_{m,n}) = -c(\mathcal{E}_{m,n}, \mathcal{E}_{n,m}) = 1, \quad \text{if } n \leq 0, m \geq 1, \quad \text{and} \quad c(\mathcal{E}_{n,m}, \mathcal{E}_{k,l}) = 0 \quad \text{otherwise.} \tag{B2}$$

Note that c satisfies the cocycle condition that allows to modify the commutation relations in $\mathfrak{gl}_\infty$ as

$$[a + x\mathbf{1}, b + y\mathbf{1}] = [a, b] + c(a, b)\mathbf{1}, \quad a, b \in \mathfrak{gl}_\infty, \ x, y \in \mathbb{C}. \tag{B3}$$

The completed algebra $\overline{\mathfrak{gl}}_\infty$ is then defined as a Lie algebra of infinite matrices with a possibly infinite number of non-zero elements but the matrix still has a finite number of non-zero diagonals, those which are along the main diagonal containing the Cartan elements $\mathcal{E}_{n,n}$. This completion can be also described using a formal algebraic completion of the Cartan subalgebra in $\mathfrak{gl}_\infty$, see [46].

Recall that we introduced the Lie algebra $\mathfrak{S}$ in the end of Sect. 6.1.3—it is a Lie algebra of our local operators. We now define an embedding of $\mathfrak{S}$ into $\overline{\mathfrak{gl}}_\infty$, or more precisely to its central extension $\overline{\mathfrak{gl}}'_\infty$ due to the presence of L_0 and $\bar{L}_0$ operators (see Sect. 5.1.2), in the following way. First, we introduce new fermionic modes $\boldsymbol{\psi}_m^\alpha$, for $\alpha = 1, 2$ and $m \in \mathbb{Z}$, and reorganize the modes of the chiral and anti-chiral symplectic fermions, ψ_n^α and $\bar{\psi}_n^\alpha$, such that the chiral modes are identified with even modes of $\boldsymbol{\psi}$'s and the anti-chiral ones correspond to odd modes of $\boldsymbol{\psi}$'s:

$$\psi_n^\alpha = \boldsymbol{\psi}_{2n}^\alpha, \quad \bar{\psi}_n^\alpha = \boldsymbol{\psi}_{-2n+1}^\alpha, \quad n \in \mathbb{Z}, \quad \alpha = 1, 2. \tag{B4}$$

Recall that we also have a pair of modes $\phi_0^{1,2}$ conjugate to $\boldsymbol{\psi}_0^{2,1}$—we do not use new notations for them. Under the identification (B4), it is then easy to see that the scaling limit of our $\mathfrak{gl}_N$ algebras described in Sect. 5.1.2, and in more precise terms using direct limits in Appendix C, is given by the basic representation (B1) of $\mathfrak{gl}_\infty$ where one also includes bilinears like $\boldsymbol{\psi}_n^2 \phi_0^1$, etc. The Lie algebra $\mathfrak{S}$ which has basis elements as infinite

sums of $\mathcal{E}_{n,m}$'s can be now embedded into the $\overline{\mathfrak{gl}}'_\infty$ algebra. Indeed, the Virasoro-like generators $L_n^{(l)}$ are the sums of bilinears in the ψ modes that have infinite number of terms like $:\psi_{2k}^2 \psi_{-2k+2n}^1:$ and all these terms belong to the same diagonal along the main one containing the Cartan elements $\psi_k^2 \psi_{-k}^1$. Therefore, any finite linear combination of $L_n^{(l)}$ belongs to $\overline{\mathfrak{gl}}'_\infty$, and similarly for $\bar{L}_n^{(l)}$. Note then that the basis elements $S_n^{(l)}$ are given by the sums containing infinite number of terms like $\psi_{2k}^2 \psi_{-2k-2n+1}^1$ and they also belong to $\overline{\mathfrak{gl}}'_\infty$. Finally, we obtain that any linear and finite combination of basis elements in $\mathfrak{S}$ belongs to $\overline{\mathfrak{gl}}'_\infty$.

Appendix C: Direct and Inverse Limits

In order to satisfy the more formal reader, we give here precise definition of scaling limits of spin chains and define the Lie algebra $\mathfrak{S}_\infty$ using direct-limit constructions. We first remind a formal definition of the direct limit. A direct (or inductive) system is a pair $\{A_i, \phi_{ij}\}$ of a family of algebraic objects A_i (vector spaces, algebras, etc.) indexed by an ordered set I and a family of homomorphisms $\phi_{ij}: A_i \rightarrow A_j$ for all $i \leq j$ satisfying the following properties: (i) ϕ_{ii} is the identity on A_i, and (ii) $\phi_{ik} = \phi_{jk}\phi_{ij}$ for all $i \leq j \leq k$. The *direct limit* $A_\infty \equiv \varinjlim A_i$ of the direct system $\{A_i, \phi_{ij}\}$ is defined as the disjoint union $\coprod_i A_i / \sim$ modulo an equivalence relation: two elements $a_i \in A_i$ and $a_j \in A_j$ in the disjoint union are equivalent if and only if they eventually become equal in the direct system, i.e., $a_i \sim a_j$ if there is $k \in I$ such that $\phi_{ik}(a_i) = \phi_{jk}(a_j)$. We obtain from this definition canonical homomorphisms $\phi_i: A_i \rightarrow A_\infty$ mapping each element to its equivalence class. The algebraic operations on A_∞ are then defined using these maps. Recall also that an inverse or projective system of algebraic objects can be defined in a similar way reverting order and arrows everywhere. So, we have a family of homomorphisms $\phi_{ij}: A_j \rightarrow A_i$, for all $i \leq j$, with similar to (i) and (ii) properties. The *inverse limit* is then defined as a particular subobject (subalgebra/subspace) in the direct product of the A_i's:

$$A \equiv \varprojlim A_i = \left\{ a \in \prod_i A_i \mid a_i = \phi_{ij}(a_j) \text{ for all } i \leq j \right\}. \tag{C1}$$

The inverse limit A is equipped with canonical projections $P_i: A \rightarrow A_i$ defined by taking the ith component of the direct product.

C.1. The scaling limit of JTL algebras as a direct limit. Consider a case of odd $L = N/2$ for simplicity and an embedding ϕ_L of the Clifford algebra $\mathcal{C}_{4L}$ into a bigger one $\mathcal{C}_{4(L+2)}$ defined by

$$\phi_L\left(\chi_p^{(\dagger)}\right) = \chi_{p'}^{(\dagger)}, \quad \phi_L\left(\eta_p^{(\dagger)}\right) = \eta_{p'}^{(\dagger)},$$
$$\phi_L\left(\chi_{\pi-p}^{(\dagger)}\right) = \chi_{\pi-p'}^{(\dagger)}, \quad \phi_L\left(\eta_{\pi-p}^{(\dagger)}\right) = \eta_{\pi-p'}^{(\dagger)}, \quad \text{for} \quad 1 \leq m \leq \frac{L-1}{2}. \tag{C2}$$

where we set $p \equiv p(m) = \frac{m\pi}{L}$, with $1 \leq m \leq L-1$, and for the indexing set in the bigger Clifford algebra we set $p' \equiv p'(m) = \frac{m\pi}{L+2}$, with $1 \leq m \leq L+1$, and for the zero modes we set

$$\phi_L(\chi_0) = \sqrt{\frac{L}{L+2}}\,\chi_0, \quad \phi_L(\chi_0^\dagger) = \sqrt{\frac{L+2}{L}}\,\chi_0^\dagger, \quad \phi_L(\eta_0) = \sqrt{\frac{L+2}{L}}\,\eta_0, \quad \phi_L(\eta_0^\dagger) = \sqrt{\frac{L}{L+2}}\,\eta_0^\dagger. \tag{C3}$$

We note that $\mathcal{C}_{4(L+2)}$ is generated by its subalgebra $\phi_L(\mathcal{C}_{4L})$ and eight additional generators $\chi^{(\dagger)}_{\frac{\pi}{2}\pm\frac{\epsilon'}{2}}$ and $\eta^{(\dagger)}_{\frac{\pi}{2}\pm\frac{\epsilon'}{2}}$, with $\epsilon' = \frac{\pi}{L+2}$. The additional fermionic operators generate in $\mathcal{H}_{2(L+2)}$ eigenstates of highest eigenvalue of the Hamiltonian H in the one-particle subspace.

The embeddings ϕ_L of algebras induce the corresponding embedding of the $\mathcal{C}_{4L}$-module $\mathcal{H}_{2L}$ into the $\mathcal{C}_{4(L+2)}$-module $\mathcal{H}_{2(L+2)}$ by $\phi_L\left(\chi^{\dagger}_p\boldsymbol{\Omega}\right) = \chi^{\dagger}_{p'}\boldsymbol{\Omega}$, etc., and $\phi_L(ab\boldsymbol{\Omega}) = \phi_L(a)\phi_L(b\boldsymbol{\Omega})$, for any $a, b \in \mathcal{C}_{4L}$. Here, we denote the embeddings of the representation spaces of the Clifford algebras by the same letter $\phi_L : \mathcal{H}_{2L} \to \mathcal{H}_{2(L+2)}$. We thus construct two direct systems: the one for the Clifford algebras

$$\mathcal{C}_4 \xrightarrow{\phi_1} \mathcal{C}_{12} \xrightarrow{\phi_3} \cdots \xrightarrow{\phi_{L-2}} \mathcal{C}_{4L} \xrightarrow{\phi_L} \mathcal{C}_{4(L+2)} \xrightarrow{\phi_{L+2}} \cdots$$

with the direct limit an infinite-dimensional Clifford algebra $\mathcal{C} \equiv \varinjlim_L \mathcal{C}_{4L}$ generated by the rescaled generators or the symplectic fermions modes $\psi_n^{1,2}$, $\bar{\psi}_m^{1,2}$, and $\phi_0^{1,2}$ introduced in (5.1) and (5.2) and satisfying (6.5) and (6.6); the second direct system is for modules over the Clifford algebras or the spin-chains

$$\mathcal{H}_2 \xrightarrow{\phi_1} \mathcal{H}_6 \xrightarrow{\phi_3} \cdots \xrightarrow{\phi_{L-2}} \mathcal{H}_{2L} \xrightarrow{\phi_L} \mathcal{H}_{2(L+2)} \xrightarrow{\phi_{L+2}} \cdots \qquad (\text{C4})$$

Using the canonical homomorphisms in the definition of direct systems, the direct limit $\varinjlim_L \mathcal{H}_{2L}$ is then a $\mathcal{C}$-module. This module coincides with the space $\mathcal{H}$ of scaling states or finite-energy and finite-spin states described in Sect. 5.1.1.

Notice then that the direct system of Clifford algebras defines a direct system of the matrix algebras $\mathfrak{gl}_{2L}$ in the following way[14]

$$\cdots \xrightarrow{\phi_{L-2}} \begin{pmatrix} \mathcal{E}_{m,n} & \mathcal{E}_{m,L+n} \\ \mathcal{E}_{L+m,n} & \mathcal{E}_{L+m,L+n} \end{pmatrix}_{2L\times 2L} \xrightarrow{\phi_L} \begin{pmatrix} 0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & \mathcal{E}_{m,n} & 0 & 0 & \mathcal{E}_{m,L+n} & 0 \\ 0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & \mathcal{E}_{L+m,n} & 0 & 0 & \mathcal{E}_{L+m,L+n} & 0 \\ 0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \end{pmatrix}_{2(L+2)\times 2(L+2)} \xrightarrow{\phi_{L+2}} \cdots$$

where the elements $\mathcal{E}_{i,j}$ denote usual elementary matrices, with $0 \leq n, m \leq L - 1$, defined in (4.5) and we abuse notations denoting the embeddings of matrix algebras by the same ϕ_L as for the Clifford algebras. The direct limit $\varinjlim_L \mathfrak{gl}_{2L}$ then gives the infinite-dimensional Lie algebra that we call $\mathfrak{gl}_\infty$—an algebra of infinite matrices with a finite number of non-zero elements. An identification of this limit with the standard representation (B1) of $\mathfrak{gl}_\infty$ is given via the identification (B4) of the fermionic generators.

In order to get the central extension $\mathfrak{gl}'_\infty$ algebra having the proper normally ordered basis, necessary for physics, see (5.7), we should map the normally ordered generators of $\mathfrak{gl}'_{2L} = \mathfrak{gl}_{2L} \oplus \mathbb{C}\mathbf{1}$, i.e., we define

$$\phi'_L : \mathfrak{gl}'_{2L} \to \mathfrak{gl}'_{2(L+2)}, \quad \text{with} \quad \phi'_L(\mathcal{E}'_{m,n}) = \mathcal{E}'_{m,n} \qquad (\text{C5})$$

and set $\phi'_L(\mathbf{1}) = \mathbf{1}$, where $\mathbf{1}$ is the identity or the central element. In particular, mapping the normally ordered generators of $\mathfrak{S}'_N \subset \mathfrak{gl}'_{2L}$ subalgebra (defined in (4.13) and

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[14] After actually rearranging four rows and four columns of zeros for a simpler presentation which is just a matter of conventions.

Sect. 4.3.1) as $\phi'_L(\mathcal{A}'_{m,n}) = \mathcal{A}'_{m,n}$, etc., we get in the direct limit $\varinjlim_L \mathfrak{S}'_{2L}$ what we call the $\mathfrak{S}_\infty$ Lie algebra and the identity is mapped under the canonical homomorphisms $\mathfrak{S}'_N \to \mathfrak{S}_\infty$ to the central element in $\mathfrak{S}_\infty$.

It turns out that the Lie algebra $\mathfrak{S}_\infty$ has $T_{m,n}$, $\bar{T}_{m,n}$ and $S_{m,n}$ with $m, n \in \mathbb{Z}$ as its basis or equivalently the universal enveloping algebra $U\mathfrak{S}_\infty$ has the symplectic-fermion representation on $\mathcal{H}$ generated by $\mathbf{1}$ and

$$S_{n,m} = S_{\alpha\beta}\psi^\alpha_n \bar{\psi}^\beta_m, \quad T_{k,l} = J_{\alpha\beta}\psi^\alpha_k \psi^\beta_l, \quad \bar{T}_{r,s} = J_{\alpha\beta}\bar{\psi}^\alpha_r \bar{\psi}^\beta_s, \quad \text{with} \quad n, m, k, l, r, s \in \mathbb{Z},$$
(C6)

as was also pointed out in Sect. 5.1.3.

Using Theorem 4.4, we can define the scaling limit of JTL algebras using the direct system for $\mathfrak{S}'_N$ algebras. Indeed, the generators of $\mathfrak{S}'_N$ are also generators of the image $\pi_{\mathfrak{gl}}(JTL_N)$. Therefore, the direct system for the enveloping algebras of $\mathfrak{S}'_N$ defines the direct system for the images of the JTL algebras

$$\pi_{\mathfrak{gl}}(JTL_2) \xrightarrow{\phi'_1} \pi_{\mathfrak{gl}}(JTL_6) \xrightarrow{\phi'_3} \cdots \xrightarrow{\phi'_{L-2}} \pi_{\mathfrak{gl}}(JTL_{2L}) \xrightarrow{\phi'_L} \pi_{\mathfrak{gl}}(JTL_{2(L+2)}) \xrightarrow{\phi'_{L+2}} \cdots$$
(C7)

where ϕ'_i are defined in (C5) and are embeddings of the finite-dimensional associative algebras. Note that an existence of embeddings for JTL algebras is a non-trivial (comparing to the open case) result and it is very hard to express these embeddings in terms of diagrams or the initial generators e_j's and u^2. Nevertheless, we found a special basis in which a system of embeddings is easily constructed and the direct system (C7) gives the limit—an enveloping algebra of $\mathfrak{S}_\infty$—which we call *the scaling limit of JTL* algebras.

Now, we can consider the direct system (C4) of spin-chains as the direct system of JTL modules (by restriction of the action from $\mathcal{C}_{4L}$ to JTL_{2L}) and therefore the direct limit space $\mathcal{H} = \varinjlim_L \mathcal{H}_{2L}$ has canonically the $U\mathfrak{S}_\infty$-module structure: an element a of $\mathfrak{S}_\infty$ has its representative in the subalgebra $\mathfrak{S}_N$, for some N, or simply saying $a \in \mathfrak{S}_N \subset \mathfrak{S}_\infty$ for large enough N, and then the action of a is given by the action of this representative on $\mathcal{H}$, with the latter defined by the direct limit construction of the $\mathfrak{S}_N$-modules. The result obviously does not depend on the choice of N.

We then show that the monomorphisms ϕ_L in (C2)–(C4) commute also with the action of the centralizer $\mathfrak{Z}_{JTL}$ or the image of $U^{\text{odd}}_{\mathfrak{q}} s\ell(2)$ in each term $\mathcal{H}_{2L}$.

Theorem C.2. *The embeddings ϕ_L in the direct system (C4) commute with the actions of the JTL_{2L} and $\mathfrak{Z}_{JTL}$ algebras, or equivalently (C4) defines a direct system of bimodules over $(JTL_{2L}, \mathfrak{Z}_{JTL})$.*

Proof. The statement about the JTL_{2L} action was proved by the previous discussion around (C7) noting that we have the equality $\phi'_L = \phi_L$ for homomorphisms of associative algebras. Recall then the expressions (4.8) for the $U^{\text{odd}}_{\mathfrak{q}} s\ell(2)$ generators. We first note that for all $L \geq 1$ the embeddings ϕ_L defined in (C2)–(C3) commute with the action of h or the S^z operator because ϕ_L do not change the number of fermionic operators in a homogeneous $v \in \mathcal{H}_{2L}$. Using (C3), we see that the ϕ_L's commute also with $\mathsf{E} = \mathsf{E}_0 = (-1)^{S^z}\sqrt{N}\chi^\dagger_0$ and $\mathsf{F} = \mathsf{F}_0 = -i\sqrt{N}\eta_0$ generators. It is thus enough to show that ϕ_L commutes with the operators

$$A_L = \sum_{p=\epsilon}^{\pi-\epsilon} \eta_p \chi_{\pi-p}, \quad B_L = \sum_{p=\epsilon}^{\pi-\epsilon} \chi^\dagger_{\pi-p}\eta^\dagger_p,$$

for any $L \geq 1$ and the sum is with the step $\epsilon = \frac{\pi}{L}$ as usual. For any $v \in \mathcal{H}_{2L}$ and odd L we have

$$\phi_L(A_L v) = \sum_{\substack{p'=\epsilon' \\ p' \neq \frac{\pi}{2} \pm \frac{\epsilon'}{2}}}^{\pi-\epsilon'} \eta_{p'} \chi_{\pi-p'} \phi_L(v) = A_{L+2}\phi_L(v),$$

with $\epsilon' = \frac{\pi}{L+2}$ and in the last equality we used the fact that χ_p is an annihilation operator and thus the image $\phi_L(v)$ is in the kernel of the operators $\chi_{\frac{\pi}{2} \pm \frac{\epsilon'}{2}}$. Similar calculation for even L and the B_L, f^L and e^L operators finishes our proof. $\quad\square$

We can of course construct direct limits of any submodules over JTL_N in $\mathcal{H}_N$. In particular, for any simple JTL_N-subquotient $\mathcal{L}_{j,(-1)^{j+1}}$ that appear in $\mathcal{H}_N$, as a fundamental representation of $\mathfrak{sp}_{N-2}$ due to Corollary 4.2.2, we have an isomorphic submodule in (the socle of) $\mathcal{H}_N$. This fact follows from the explicit analysis of the module structure in [23] or see Fig. 4. Fixing such an irreducible submodule, say in the fermionic basis, the direct system (C4) thus gives a direct system of the simple modules $\mathcal{L}_{j,(-1)^{j+1}}$ for different values of N. We denote their direct limit by $\mathfrak{X}_j$, for any integer $j \geq 1$, and these infinite-dimensional spaces have canonically the action of $U\mathfrak{S}_\infty$ as explained above.

Proposition C.3. *The direct limit modules $\mathfrak{X}_j$ over $U\mathfrak{S}_\infty$ are irreducible for any integer $j \geq 1$.*

Proof. Assume that for some j the $\mathfrak{S}_\infty$-module $V \equiv \mathfrak{X}_j$ is reducible and has thus an invariant subspace $W \subset V$. Take any element $v \in V$ such that it belongs to the complement of W. Then, by definition of the direct limit, there exist a positive number N such that the element v has its representative in the subspace $\mathcal{L}_{j,(-1)^{j+1}}[N] \subset V$, where $\mathcal{L}_{j,(-1)^{j+1}}[N]$ is just notation for the simple module over $\mathfrak{S}_N$, or over $\mathfrak{sp}_{N-2}$. Simply saying v is in the subspace $\mathcal{L}_{j,(-1)^{j+1}}[N]$ for large enough N. Take then the intersection W' of the $\mathfrak{S}_N$-module $\mathcal{L}_{j,(-1)^{j+1}}[N]$ with W. The $\mathfrak{S}_N$ acts on W', otherwise it would contradict to the fact that W is the invariant subspace for $\mathfrak{S}_\infty$ and thus for $\mathfrak{S}_N \subset \mathfrak{S}_\infty$. We also have that W' is a proper subspace of $\mathcal{L}_{j,(-1)^{j+1}}[N]$, otherwise v would belong to W' and it contradicts to the assumption that v is in the complement to W. We thus obtain a contradiction to the fact that $\mathcal{L}_{j,(-1)^{j+1}}[N]$ is irreducible for $\mathfrak{S}_N$. $\quad\square$

Using the same idea of a reduction from $\mathcal{H}$ to its subspace $\mathcal{H}_N \subset \mathcal{H}$ for large enough N and transporting thus an assumption in $\mathcal{H}$ to the corresponding assumption in $\mathcal{H}_N$, one can similarly prove that the direct summands $\widehat{\mathcal{P}}_j$ in $\mathcal{H}_N$ (these are indecomposable but reducible JTL_N-modules from Fig. 4) have the direct limit denoted by $\mathcal{Y}_j$ as an indecomposable but reducible $U\mathfrak{S}_\infty$-module and its subquotient structure (with simple subquotients) is described by the infinite tower in Fig. 4.

C.4. Inverse limit of centralizers. By Theorem C.2, we have a direct system of bimodules over $\left(JTL_{2L}, U_q^{odd}s\ell(2)\right)$ and the direct limit space $\mathcal{H}$ is canonically a bimodule over $(U\mathfrak{S}_\infty, U_q^{odd}s\ell(2))$. The action of $U\mathfrak{S}_\infty$ on $\mathcal{H}$ is described in (C6) and the $U_q^{odd}s\ell(2)$ action has the fermionic expressions computed in our first paper [22]:

$$\mathsf{E}_n = \left[\sum_{m>0}\left(\frac{\psi_m^2\psi_{-m}^2}{m} - \frac{\bar{\psi}_m^2\bar{\psi}_{-m}^2}{m}\right)\right]^n \psi_0^2, \quad \mathsf{F}_n = \left[\sum_{m>0}\left(\frac{\psi_m^1\psi_{-m}^1}{m} - \frac{\bar{\psi}_m^1\bar{\psi}_{-m}^1}{m}\right)\right]^n \psi_0^1,$$

$$\tag{C8}$$

with the representation of the Cartan element h as

$$\mathsf{h} = -i/2\big(\psi_0^2\phi_0^1 + \psi_0^1\phi_0^2\big) + \sum_{m>0}\frac{1}{m}\big(\psi_{-m}^2\psi_m^1 + \psi_{-m}^1\psi_m^2 + \bar\psi_{-m}^2\bar\psi_m^1 + \bar\psi_{-m}^1\bar\psi_m^2\big) \quad \text{(C9)}$$

while the generator $\mathsf{K} = (-1)^{2\mathsf{h}}$. This action can be obtained by constructing an *inverse system* for the spin-chain representations ρ_N of $U_{\mathsf{q}}^{\mathrm{odd}}s\ell(2)$ as[15]

$$\rho_2\big(U_{\mathsf{q}}^{\mathrm{odd}}s\ell(2)\big) \xleftarrow{p_1} \rho_6\big(U_{\mathsf{q}}^{\mathrm{odd}}s\ell(2)\big) \xleftarrow{p_3} \cdots \xleftarrow{p_{L-2}} \rho_{2L}\big(U_{\mathsf{q}}^{\mathrm{odd}}s\ell(2)\big)$$
$$\xleftarrow{p_L} \rho_{2(L+2)}\big(U_{\mathsf{q}}^{\mathrm{odd}}s\ell(2)\big) \xleftarrow{p_{L+2}} \cdots \qquad\qquad \text{(C10)}$$

where ρ_N is defined in (4.8) and p_n are surjective homomorphisms defined in terms of the fermionic generators as

$$p_L\big(\chi_{p'}^{(\dagger)}\big) = \Big(1 - \delta_{p',\frac{\pi}{2}\pm\frac{\epsilon'}{2}}\Big)\chi_p^{(\dagger)}, \quad p_L\big(\eta_{p'}^{(\dagger)}\big) = \Big(1 - \delta_{p',\frac{\pi}{2}\pm\frac{\epsilon'}{2}}\Big)\eta_p^{(\dagger)}, \quad 1 \le m \le L+1,$$
$$\text{(C11)}$$

where as usual $p \equiv p(m) = \frac{m\pi}{L}$ and $p' \equiv p'(m) = \frac{m\pi}{L+2}$, and for the zero modes as

$$p_L(\chi_0) = \sqrt{\frac{L+2}{L}}\,\chi_0, \quad p_L(\chi_0^\dagger) = \sqrt{\frac{L}{L+2}}\,\chi_0^\dagger, \quad p_L(\eta_0) = \sqrt{\frac{L}{L+2}}\,\eta_0, \quad p_L(\eta_0^\dagger) = \sqrt{\frac{L+2}{L}}\,\eta_0^\dagger.$$
$$\text{(C12)}$$

In other words, we define first p_L as a vector-space homomorphism $\mathcal{C}_{4(L+2)} \to \mathcal{C}_{4L}$ such that its kernel equals the cokernel of ϕ_L. Then using (4.8), it is easy to check that the surjective maps $p_L \colon \mathcal{C}_{4(L+2)} \twoheadrightarrow \mathcal{C}_{4L}$ of vector spaces induce homomorphisms of subalgebras $\rho_{2(L+2)}\big(U_{\mathsf{q}}^{\mathrm{odd}}s\ell(2)\big) \subset \mathcal{C}_{4(L+2)}$ onto $\rho_{2L}\big(U_{\mathsf{q}}^{\mathrm{odd}}s\ell(2)\big)$ and correspondingly homomorphisms $p_L \colon \mathcal{H}_{2(L+2)} \twoheadrightarrow \mathcal{H}_{2L}$ of modules over $\rho_{2(L+2)}\big(U_{\mathsf{q}}^{\mathrm{odd}}s\ell(2)\big)$. Then, the inverse limit

$$\rho\big(U_{\mathsf{q}}^{\mathrm{odd}}s\ell(2)\big) = \varprojlim \rho_{2L}\big(U_{\mathsf{q}}^{\mathrm{odd}}s\ell(2)\big) \qquad\qquad \text{(C13)}$$

defines an infinite-dimensional associative algebra which we identify with a quotient of $U_{\mathsf{q}}^{\mathrm{odd}}s\ell(2)$ that admits among irreducible representations only one-dimensional ones, see representation theory of $U_{\mathsf{q}}^{\mathrm{odd}}s\ell(2)$ in [23, Sec. 3]. We call this inverse limit as *the scaling limit of the JTL centralizers* $\mathfrak{Z}_{JTL}$. The corresponding inverse limit space $\overline{\mathcal{H}} = \varprojlim \mathcal{H}_{2L}$ of the $U_{\mathsf{q}}^{\mathrm{odd}}s\ell(2)$-modules is a module over $\rho\big(U_{\mathsf{q}}^{\mathrm{odd}}s\ell(2)\big)$ in (C13). The action is defined componentwise following the definition (C1). The explicit action (C8) and (C9) of the generators of the inverse limit on $\overline{\mathcal{H}}$ is calculated using (C11) and (C12) and the identification in (5.1) and (5.2).

The direct and inverse systems of the spaces $\mathcal{H}_N$ have a useful coherence property in the sense of the following simple lemma.

Lemma C.5. *The direct system* $\phi_L \colon \mathcal{H}_{2L} \hookrightarrow \mathcal{H}_{2(L+2)}$ *from* (C4) *and defined in* (C2) *and* (C3) *satisfies*

$$p_L \circ \phi_L = \mathrm{id}, \quad \text{for} \ \ L \ge 1, \qquad\qquad \text{(C14)}$$

where the projective system $p_L \colon \mathcal{H}_{2(L+2)} \twoheadrightarrow \mathcal{H}_{2L}$ *is defined in* (C11) *and* (C12).

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[15] This inverse system actually corresponds to the usual one [49] for the q-Schur algebras—quotients of $U_{\mathsf{q}}s\ell(2)$.

Remark C.5.1. As a consequence of Lemma C.5, the algebra $\mathfrak{S}_{2L}$ acts on the image of p_L and this action is identical to the one on the subspace $\mathcal{H}_{2L} \subset \mathcal{H}_{2(L+2)}$. Moreover, it is straightforward to check that the map p_L intertwines the $\mathfrak{S}_{2L}$ action on $\mathcal{H}_{2(L+2)}$ and $\mathcal{H}_{2L}$.

Now, we discuss a relation between the two limits, $\mathcal{H}$ and $\overline{\mathcal{H}}$, so far constructed. Recall that the Hamiltonian H in the scaling limit (5.3) of the JTL algebras gives the grading (or energy) operator $E = L_0 + \bar{L}_0$, which is a bounded below operator, while the momentum operator $P = P(0)$ in (6.7) gives the conformal spin operator $S = L_0 - \bar{L}_0$ (not bounded). Following the definition of ϕ_L and p_L and using commutation relations of the fermionic generators with E and S, we conclude that the direct limit is a bi-graded vector space with the decomposition $\mathcal{H} = \oplus_{n \geq 0, m \in \mathbb{Z}} \mathcal{H}^{(n,m)}$ onto generalized eigenspaces or root spaces of (E, S), while the projective limit $\overline{\mathcal{H}}$ is the direct product $\overline{\mathcal{H}} = \prod_{n \geq 0, m \in \mathbb{Z}} \mathcal{H}^{(n,m)}$. Therefore, the inverse limit $\overline{\mathcal{H}}$ contains the direct limit space $\mathcal{H}$ as a dense subspace (in the formal topology discussed in Sect. 6.2) and the basis in $\mathcal{H}$ is also a basis in $\overline{\mathcal{H}}$. Thus, the space $\overline{\mathcal{H}}$ can be considered as a formal completion of $\mathcal{H}$ and the canonical projections $P_N \colon \overline{\mathcal{H}} \twoheadrightarrow \mathcal{H}_N$ reduce on the dense subspace $\mathcal{H}$ to projections $P_N \colon \mathcal{H} \twoheadrightarrow \mathcal{H}_N$. Correspondingly, the action of $\rho(U_q^{\text{odd}} s\ell(2))$ reduces to the action on $\mathcal{H}$ given by the same formulas (C8) and (C9). Let $\Phi_N \colon \mathcal{H}_N \hookrightarrow \mathcal{H}$ denotes the canonical embedding defined by the direct system (C4). Using Lemma C.5 and Remark C.5.1, we formulate a corollary.

Corollary C.6. *The canonical projections $P_N \colon \mathcal{H} \twoheadrightarrow \mathcal{H}_N$ are in the coherence property with the canonical embeddings $\Phi_N \colon \mathcal{H}_N \hookrightarrow \mathcal{H}$:*

$$P_N \circ \Phi_N = \text{id}, \quad \text{for even } N \geq 2. \tag{C15}$$

Therefore, the image of the surjective map $P_N \colon \mathcal{H} \twoheadrightarrow \mathcal{H}_N$ can be identified with the subspace $\mathcal{H}_N \subset \mathcal{H}$. The map P_N intertwines the $\mathfrak{S}_N$ action on $\mathcal{H}$ and $\mathcal{H}_N$.

It turns out that the centralizer of the algebra $U\mathfrak{S}_\infty$ action on $\mathcal{H}$ is given by the representation ρ of $U_q^{\text{odd}} s\ell(2)$ defined in (C13).

Theorem C.7. *The centralizer $\mathfrak{Z}_{U\mathfrak{S}_\infty}$ of the enveloping algebra $U\mathfrak{S}_\infty$ action (C6) on $\mathcal{H}$ equals $\rho(U_q^{\text{odd}} s\ell(2))$, with the action in (C8) and (C9).*

Proof. The centralizer of $U\mathfrak{S}_\infty$ obviously commutes with the normally ordered operators $:T_{n,-n}:$ and $:\bar{T}_{n,-n}:$, see (C6) and also discussion in Sect. 5.1.3, and therefore with the energy operator

$$E = L_0 + \bar{L}_0 = -\sum_{n>0} \left(:T_{n,-n}: + :\bar{T}_{n,-n}:\right) - :T_{0,0}:,$$

which is an element of the completed algebra $\overline{\mathfrak{S}}_\infty$ but on each $v \in \mathcal{H}$ the sum reduces to a finite sum. Let $\mathcal{H}^{(n)}$ denotes the generalized eigenspace (or root space) of eigenvalue n for the E operator, i.e., we have $\mathcal{H} = \oplus_n \mathcal{H}^{(n)}$. From the partition function analysis given in Sect. 5.6, we conclude that n is non-negative integer and each $\mathcal{H}^{(n)}$ is finite-dimensional. We thus have that the centralizer is a subalgebra in the direct product $\prod_{n \geq 0} \text{End}(\mathcal{H}^{(n)})$ of finite matrix algebras.

Using the simple fact that the fermionic operators $\psi_{-m}^{1,2}$ and $\bar{\psi}_{-m}^{1,2}$ change an eigenvalue of E by m, we have by construction of the direct system (C4), with the identifications

in (5.1) and (5.2), that for a fixed positive j there exists a big enough $N > 0$ such that any $v \in \mathcal{H}^{(j)} \subset \mathcal{H}$ has its representative in the N-site spin-chain $\mathcal{H}_N \subset \mathcal{H}$. Using the canonical embedding $\Phi_N : \mathcal{H}_N \hookrightarrow \mathcal{H}$ we can consider any such v as an element in $\mathcal{H}_N$ (or in other words, we observe that the fixed energy subspaces $\mathcal{H}^{(n)}$ stabilize with the rise of N).

Assume then that there exists a non-zero operator $O \in \prod_{n\geq 0} \mathrm{End}(\mathcal{H}^{(n)})$ such that it commutes with the action of $U\mathfrak{S}_\infty$ but is not an element of the algebra $\rho(U_q^{\mathrm{odd}}s\ell(2))$. In other words, writing the operator O componentwise as $O = (O_0, O_1, O_2, \dots)$, with $O_j \in \mathrm{End}(\mathcal{H}^{(j)})$, there exists a positive number j such that the jth component O_j of O can not be written as a finite linear combination of words in the generators (C8) and (C9). Following the preceding discussion we can take a big enough N such that the $\mathcal{H}^{(j)}$ is in the subspace $\mathcal{H}_N \subset \mathcal{H}$ (under the canonical embedding Φ_N). By the coherence property (C15) in Corollary C.6, this subspace is identified with the image of the canonical projection $P_N : \mathcal{H} \twoheadrightarrow \mathcal{H}_N$ and therefore on the subspace $\mathcal{H}_N \subset \mathcal{H}$ we have the action of $\rho(U_q^{\mathrm{odd}}s\ell(2))$ that factorizes up to the action of $\rho_N(U_q^{\mathrm{odd}}s\ell(2))$. The latter also acts on $\mathcal{H}^{(j)} \subset \mathcal{H}_N$. By the assumption on the operator O, we have that the composition $P_N O P_N \in \mathrm{End}\,\mathcal{H}_N$ is not an element of $\rho_N(U_q^{\mathrm{odd}}s\ell(2))$, otherwise its action on $\mathcal{H}^{(j)} \subset \mathcal{H}_N$ could be written in terms of the $\rho(U_q^{\mathrm{odd}}s\ell(2))$ generators and it contradicts to the initial assumption. On the other hand, the composition $P_N O P_N$ commutes with the action of the subalgebra $\mathfrak{S}_N \subset \mathfrak{S}_\infty$ or the JTL_N algebra (recall Theorem 4.4) because P_N intertwines the action of $\mathfrak{S}_N$ (see Corollary C.6) and O commutes with any subalgebra of $\mathfrak{S}_\infty$ by the assumption. Therefore, the operator $P_N O P_N$ equals[16]

$$P_N O P_N = a_1 \mathsf{f}^{N/2} + a_2 \mathsf{e}^{N/2}, \quad \text{for} \quad a_1, a_2 \in U_q^{\mathrm{odd}}s\ell(2), \tag{C16}$$

because the centralizer of JTL_N on $\mathcal{H}_N$ is the algebra generated by $\rho_N(U_q^{\mathrm{odd}}s\ell(2))$ and $\mathsf{f}^{N/2}$ and $\mathsf{e}^{N/2}$ (see Sect. 3.2). It is then easy to show a contradiction. Indeed, taking similarly the projection onto $\mathcal{H}_{N+2}$, which contains the $\mathcal{H}_N$, we conclude that $P_{N+2} O P_{N+2}$ commutes with the $\mathfrak{S}_{N+2}$ or the JTL_{N+2} action and by (C16) it equals $b + \rho_{N+2}(a_1 \mathsf{f}^{N/2} + a_2 \mathsf{e}^{N/2})$, where $b \in \mathfrak{Z}_{JTL_{N+2}}$ such that $b \in \ker P_N$. By this equality we obtain that $\rho_{N+2}(a_1 \mathsf{f}^{N/2} + a_2 \mathsf{e}^{N/2})$ is in $\mathfrak{Z}_{JTL_{N+2}}$ and it is possible only if (i) the operator $\rho_{N+2}(a_1 \mathsf{f}^{N/2} + a_2 \mathsf{e}^{N/2})$ is in $\rho_{N+2}(U_q^{\mathrm{odd}}s\ell(2))$ and thus $P_N O P_N \in \rho_N(U_q^{\mathrm{odd}}s\ell(2))$ or (ii) $a_1 = a_2 = 0$ and thus $P_N O P_N = 0$. The first assertion contradicts to the assumption on O and the last assertion contradicts to the assumption that $P_N O P_N$ is a non-zero operator. Thus we obtain the statement of the theorem. $\quad\square$

Appendix D: Emergence of the Interchiral Algebra from the Lattice

We look at the Fourier transforms of e_j's introducing $q = n\pi/L$, for $n \in \mathbb{Z}$, and defining the operators

$$H(n) = -\sum_{j=1}^{N} e^{-iqj} e_j = \sum_p \left[1 + e^{iq} + ie^{-ip} - ie^{i(p+q)} \right] \theta_p^\dagger \theta_{p+q+\pi} \tag{D1}$$

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[16] For brevity, we write $\mathsf{f}^{N/2}$ instead $\rho_N(\mathsf{f}^{N/2})$, etc.

which were obtained in our first paper [22]. These can be rewritten using the more convenient χ and η fermions from (A3) as

$$H(n) = 2e^{iq/2}\left(\sqrt{\sin q}\, \chi_0^\dagger(\chi_q + \eta_q) + \sum_{\substack{p=\epsilon \\ \text{step}=\epsilon}}^{\pi-q-\epsilon} \sqrt{\sin(p)\sin(p+q)}(\chi_p^\dagger \chi_{p+q} - \eta_p^\dagger \eta_{p+q})\right.$$

$$\left. + \sum_{\substack{p=\epsilon \\ \text{step}=\epsilon}}^{q-\epsilon} \sqrt{\sin(p)\sin(q-p)}(\chi_{\pi-p}^\dagger \eta_{q-p} + \eta_{\pi-p}^\dagger \chi_{q-p}) + \sqrt{\sin q}\,(\chi_{\pi-q}^\dagger + \eta_{\pi-q}^\dagger)\eta_0\right),$$

(D2)

where $0 < n < L$, while a similar expression can be written for other values of the modes n.

In order to take the scaling limit for n close to L we define $k = L - n$ and consider operators $\tilde{H}(k) = H(L - k)$ and study their scaling limit for finite k. We thus first substitute $n \to L - k$ and $q \to \pi - q'$, with $q' = k\pi/L$, in (D2). Then, using the limit (5.1)–(5.2) to the fermions $\psi^{1,2}$ and $\bar{\psi}^{1,2}$ and linearizing the dispersion relation, we obtain in the scaling limit (keeping terms in the sums with the momentum p close to 0 or to π), with finite positive k,

$$-i\frac{L}{2\pi}\tilde{H}(k) \mapsto \psi_0^2(\bar{\psi}_k^1 - \psi_{-k}^1) + \sum_{m=1}^{k-1}(\psi_{-m}^2\bar{\psi}_{k-m}^1 - \bar{\psi}_m^2\psi_{m-k}^1)$$

$$+ \sum_{m\geq k+1}(\psi_{-m}^2\bar{\psi}_{k-m}^1 - \bar{\psi}_m^2\psi_{m-k}^1)$$

$$+ \sum_{m\geq 1}(\psi_m^2\bar{\psi}_{m+k}^1 - \bar{\psi}_{-m}^2\psi_{-m-k}^1) + (\psi_{-k}^2 - \bar{\psi}_k^2)\psi_0^1.$$

Gathering all terms and using the relations (6.5), we finally obtain the contribution corresponding to low-lying excitations over the ground state,

$$\frac{L}{2\pi i}\tilde{H}(k) \mapsto S_{\alpha\beta}\sum_{m\in\mathbb{Z}}\psi_m^\alpha\bar{\psi}_{m+k}^\beta \equiv S_k, \quad k > 0,$$

which we denote in what follows by S_k or $S_k^{(0)}$. A similar computation gives the same scaling limit of $\frac{L}{2\pi i}\tilde{H}(k)$ for negative finite k.

We then consider the Fourier transformation of the momentum operator P,

$$P(n) = \frac{i}{2}\sum_{j=1}^{N} e^{-iqj}[e_j, e_{j+1}], \quad q = \frac{n\pi}{L},$$

which gives [22] on a finite spin-chain the expression in χ-η fermions:

$$P(n) = 2e^{iq}\left(\cos\frac{q}{2}\sqrt{\sin q}\, \chi_0^\dagger(\chi_q - \eta_q) + \sum_{\substack{p=\epsilon \\ \text{step}=\epsilon}}^{\pi-q-\epsilon} \cos\left(p + \frac{q}{2}\right)\sqrt{\sin(p)\sin(p+q)}\right.$$

$$\times \left(\chi_p^\dagger \chi_{p+q} + \eta_p^\dagger \eta_{p+q}\right)$$

$$-\sum_{\substack{p=\epsilon \\ \text{step}=\epsilon}}^{q-\epsilon} \cos\left(p-\tfrac{q}{2}\right)\sqrt{\sin(p)\sin(q-p)}\left(\chi^{\dagger}_{\pi-p}\eta_{q-p}-\eta^{\dagger}_{\pi-p}\chi_{q-p}\right)$$

$$-\cos\tfrac{q}{2}\sqrt{\sin q}\left(\chi^{\dagger}_{\pi-q}-\eta^{\dagger}_{\pi-q}\right)\eta_0\Bigg).$$

Once again, to take the scaling limit for n close to L we consider operators $\tilde{P}(k) = P(L-k)$ and study their scaling limit for finite k. It turns out that the scaling limit (for finite k) is

$$-\frac{L^2}{\pi^2}\tilde{P}(k)\mapsto S_{\alpha\beta}\sum_{m\in\mathbb{Z}}(2m+k)\psi_m^{\alpha}\bar{\psi}_{m+k}^{\beta}\equiv S_k^{(1)},\quad k\in\mathbb{Z}, \tag{D3}$$

where the operators $S_k^{(l)}$ are introduced in (6.17), for $l\geq 0$ and $k\in\mathbb{Z}$.

D.1. Higher Hamiltonians and the Lie algebra $\mathfrak{S}_N$. Recall that in Sect. 4 we gave a Lie algebraic $\mathfrak{S}_N$ description of the JTL algebra in terms of fermion bilinears. We consider here a different basis in $\mathfrak{S}_N$ that is spanned by the generalized higher Hamiltonians $H_l(n)$ introduced (6.9). These Hamiltonians for positive modes n were expressed in (6.11) in the basis of $\mathfrak{S}_N$. In order to check that the linear span of all $H_l(n)$'s gives the vector space $\mathfrak{S}_N$ we need also negative-mode expressions. For $l\geq 0$ and $1\leq n\leq L$, we find a slightly different formula

$$H_l(-n) = 2e^{-iq\frac{l+1}{2}}\Bigg(\sum_{m=n+1}^{L-1}\cos l\left(p-\tfrac{q}{2}\right)\sqrt{\sin(p)\sin(p-q)}\,\mathcal{A}_{m,m-n}$$

$$-\frac{1}{2}\sum_{m=1}^{n-1}\cos l\left(p-\tfrac{q}{2}\right)\sqrt{\sin(p)\sin(q-p)}\left(\mathcal{C}_{L-m,L+m-n}+(-1)^l\mathcal{B}_{m,n-m}\right)$$

$$+\cos\frac{lq}{2}\sqrt{\sin q}\left(\mathcal{E}_{0,L-n}-(-1)^l\mathcal{E}_{0,L+n}+\mathcal{E}_{n,L}-(-1)^l\mathcal{E}_{2L-n,L}\right)\Bigg). \tag{D4}$$

We checked then up to $N=20$ that a basis in the linear span of all $H_l(n)$'s, with $l\geq 0$ and $n\in\mathbb{Z}$, resides actually in the range $-\pi+l+1\leq q=n\pi/L\leq\pi+l+2$ and l—the index for the family—runs from 0 to $L-1$. Moreover, the number of linearly independent $H_l(n)$'s in this range is given by the dimension of the Lie algebra $\mathfrak{S}_N$ introduced in Definition 4.3.1. We thus conclude that the $H_l(n)$'s should be closed under commutations (which is really hard to compute explicitly) and that they just give a different basis in the Lie algebra $\mathfrak{S}_N$.

Appendix E: The Character Formulas

We provide here an analysis of the left–right Virasoro $\mathcal{V}(2)=\mathcal{V}(2)\boxtimes\overline{\mathcal{V}}(2)$ content of the simple JTL_N modules in the scaling limit using a general strategy (via the XXZ spin chain) that will be useful for other models as well. We recall the basic fact [50] that

$$\mathrm{Tr}\,e^{-\beta_R(H-Ne_0)}e^{-i\beta_I P}\rightarrow\mathrm{Tr}\,q^{L_0-c/24}\bar{q}^{\bar{L}_0-c/24} \tag{E1}$$

where H and P are a lattice Hamiltonian (normalized such that the velocity of sound is unity) and momentum (see [22] and below), e_0 is the ground state energy per site in the scaling limit, $q(\bar{q}) = \exp\left[-\frac{2\pi}{N}(\beta_R \pm i\beta_I)\right]$ with $\beta_{R,I}$ real and $\beta_R > 0$, and N is the length of the chain. On the right-hand side, we have L_0 and $\bar{L}_0$ as zero modes of the stress-energy tensor, and c is the corresponding central charge. The trace on the left is taken over a subspace (of scaling states) of the spin chain, and the trace on the right is over the states occurring in this subspace in the scaling limit.

The calculation of an expression such as the trace on the left of (E1) is most easily done when the spin chain is the well known XXZ chain with appropriate value of the deformation parameter $\mathfrak{q}$ and appropriately twisted boundary conditions. To be more specific, we consider the Hamiltonian

$$H_{\text{XXZ}}(K) = \sum_{j=1}^{2L-1} \left(\sigma_j^+ \sigma_{j+1}^- + \sigma_j^- \sigma_{j+1}^+\right) + e^{2iK}\sigma_{2L}^+\sigma_1^- + e^{-2iK}\sigma_{2L}^-\sigma_1^+ + \sum_{j=1}^{2L} \frac{\mathfrak{q} + \mathfrak{q}^{-1}}{4}\sigma_j^z\sigma_{j+1}^z, \tag{E2}$$

where the 2×2-matrices $\sigma^\pm$, σ^z are Pauli matrices. The momentum operator $P_{\text{XXZ}}(K)$ can be chosen either by using the exact eigenvalues of the translation operator on the spin chain, or by using the general formulas in [22]. Parametrizing $\mathfrak{q} = e^{i\pi/(x+1)}$, it has been known for a long time thanks to Coulomb gas and Bethe ansatz arguments [8,51,52] that

$$\text{Tr}_{\mathcal{H}_{[j]}} e^{-\beta_R(H_{\text{XXZ}}(K) - 2Le_0)} e^{-i\beta_I P_{\text{XXZ}}(K)} \rightarrow F_{j,e^{2iK}} \tag{E3}$$

where

$$F_{j,e^{2iK}} = \frac{q^{-c/24}\bar{q}^{-c/24}}{P(q)P(\bar{q})} \sum_{n \in \mathbb{Z}} q^{\frac{(xj+(x+1)(n+K/\pi))^2-1}{4x(x+1)}} \bar{q}^{\frac{(xj-(x+1)(n+K/\pi))^2-1}{4x(x+1)}} \tag{E4}$$

and the subspace $\mathcal{H}_{[j]}$ is the subspace of spin $S^z = j$, with $-L \leq j \leq L$, of the spin-chain of length $N = 2L$ (also denoted by $\widehat{\mathcal{P}}_j$), and $P(q)$ is defined in (5.24).

On the other hand, it is also well known [8] that this XXZ Hamiltonian can be written as

$$H_{\text{XXZ}}(K) = -\sum_{i=1}^{2L} e_i, \tag{E5}$$

where the e_i are expressed in the spin-1/2 basis [8] and, together with u^2, provide a representation of the Jones–Temperley–Lieb algebra JTL_N whenever $e^{2ijK} = 1$, for $j \neq 0$, and $e^{2iK} = \mathfrak{q}^2$, for $j = 0$. This representation has for dimension the usual binomial coefficient $\binom{2L}{L+j}$, and is believed to be isomorphic to the standard module $\mathcal{W}_{|j|,e^{\pm i\phi}}$ for $\mathfrak{q}$ generic and $\phi = -2K$, where the '+' is for a positive j, and the '−' is for a negative value of j. The standard modules (defined and discussed in [23]) on opposite sectors $S^z = \pm j$ are conjugate to each other with respect to the bilinear pairing $\mathcal{W}_{|j|,z} \times \mathcal{W}_{|j|,z^{-1}} \rightarrow \mathbb{C}$ from [32] invariant with respect to the action of the affine TL algebra denoted there by T_N^a. The pairing has a trivial radical only in the generic cases.

When $\mathfrak{q}$ is a root of unity, the representation obtained from the twisted XXZ chain is not isomorphic to a standard module any longer. But it is well established [8] that the traces over the modules and their scaling limit behave smoothly across those points. This means that we can use (E4) to obtain the generating function of conformal weights (that is, eigenvalues of L_0 and $\bar{L}_0$) in the scaling limit of the modules occurring in our $\mathfrak{gl}(1|1)$ spin chain.

Let us specialize now to $q = i$ ($x = 1$) where $c = -2$ and

$$F_{j,e^{2iK}} = \frac{q^{1/12}\bar{q}^{1/12}}{P(q)P(\bar{q})} \sum_{n\in\mathbb{Z}} q^{\frac{(j+2(n+K/\pi))^2-1}{8}} \bar{q}^{-\frac{(j-2(n+K/\pi))^2-1}{8}} \tag{E6}$$

(note that $F_{0,e^{2iK}} = F_{0,e^{-2iK}}$). Recall the Kac formula for the value of the central charge $c = -2$

$$h_{r,s} = \frac{(2r-s)^2-1}{8}. \tag{E7}$$

Introduce now the character of the Kac representation

$$K_{r,s} = \frac{q^{h_{rs}} - q^{h_{r,-s}}}{q^{c/24}P(q)} = \frac{q^{(2r-s)^2/8} - q^{(2r+s)^2/8}}{\eta(q)} \tag{E8}$$

where the Dedekind's eta function is $\eta(q) = q^{1/24}P(q)$. A short calculation establishes the crucial formula

$$F_{j,q^{2j+2k}} - F_{j+k,q^{2j}} = \sum_{r=1}^{\infty} K_{r,k}\bar{K}_{r,k+2j}, \tag{E9}$$

with $q = i$ here.

From the structure of the spin chain modules (see Sect. 3.3; Fig. 4), we deduce the traces

$$F^{(0)}_{j,(-1)^{j+1}} \equiv \lim_{L\to\infty} \mathrm{Tr}_{\mathcal{L}_{j,(-1)^{j+1}}} e^{-\beta_R(H(0)-Le_0)}e^{-i\beta_I P(0)} \tag{E10}$$

in each of the simple subquotients $\mathcal{L}_{j,(-1)^{j+1}}$ appearing in the (scaling limit of) JTL-modules $\widehat{\mathcal{P}}_k$:

$$F^{(0)}_{j,(-1)^{j+1}} = \sum_{j'\geq j}(-1)^{j'-j} \sum_{r=1}^{\infty} K_{r,1}\bar{K}_{r,2j'+1}, \quad j \leq k. \tag{E11}$$

On the other hand, recall the characters of the simples of Virasoro at $c = -2$

$$\chi_{j,1} = \frac{q^{(2j-1)^2/8} - q^{(2j+1)^2/8}}{\eta(q)} \tag{E12}$$

from which it follows that

$$K_{r,2j+1} = \sum_{s=-r+1}^{r} \chi_{j+s,1}. \tag{E13}$$

Moreover, we formally define the character when r is negative by the same formula, so we have $\chi_{-j,1} = -\chi_{j,1}$. It follows that

$$K_{r,1} = \chi_{r,1}. \tag{E14}$$

Straightforward algebra then leads to the key result

$$F^{(0)}_{j,(-1)^{j+1}} = \sum_{j_1,j_2>0}^{*} \chi_{j_1,1}\overline{\chi}_{j_2,1} \tag{E15}$$

where the sum is done with the following constraints:

$$|j_1 - j_2| + 1 \leq j, \quad j_1 + j_2 - 1 \geq j, \quad j_1 + j_2 - j = 1 \bmod 2 \tag{E16}$$

(note this is equivalent to treating j not as a spin but as a degeneracy, i.e., setting $j = 2s + 1$, $j_i = 2s_i + 1$ and combining s_1 and s_2 to obtain spin s).

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