Transport of charged particles propagating in turbulent magnetic fields as a red-noise process

[ApJ 920:87 (2021)]

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i. Diffusion in isotropic magnetic turbulence

Diffusion-tensor coefficients

- Spatial diffusion tensor related to the velocity correlation function through a time integration [Kubo, 1957]:

$$D_{ij}(t) = \int_0^t \mathrm{d}t' \, \langle v_{0i} v_j(t') \rangle$$

- $\langle \cdot \rangle$: averages taken over several space and time correlation scales of the turbulent field
- Ergodic fluctuations $\rightarrow \langle \cdot \rangle =$ averaging over an ensemble of systems
- Aim: relate $\langle v_i(t) \rangle$ to the statistical properties of the turbulence
- Equation of motion: $\partial v_i(t)/\partial t = \delta\Omega \, \epsilon_{ijk} v_j(t) \delta b_k(t)$, with $\delta\Omega = c^2 Z|e|\delta B/E$ the gyrofrequency and $\delta b_k(t) \equiv \delta b_k(\mathbf{x}(t))$ the k-th component of the magnetic field, expressed in units of δB, at the spatial coordinate $\mathbf{x}(t)$ of the particle at time t

Solution as a Dyson series

- First iterative solution: $\langle v_i(t)\rangle = v_{i0} + \delta\Omega\, \epsilon_{ijk} \int_0^t \mathrm{d}t' \langle v_j(t')\delta b_k(t')\rangle$
- Dyson series:

$$\langle v_{i_0}(t)\rangle = v_{0i_0} + \sum_{n=1}^{\infty} \delta\Omega^n\, \epsilon_{i_0 i_1 j_1}\epsilon_{i_1 i_2 j_2}\ldots\epsilon_{i_{n-1} i_n j_n} v_{0i_n} \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \ldots \int_0^{t_{n-1}} \mathrm{d}t_n \langle \delta b_{j_1}(t_1)\ldots\delta b_{j_n}(t_n)\rangle$$

- Use of the Wick theorem in the Gaussian approximation to express the expectation value in the integrand in terms of all possible permutations of products of contractions of pairs of $\langle \delta b_{i_1}(t_{j_1})\delta b_{i_2}(t_{j_2})\rangle$
- Using the Ansatz $\langle \delta b_{i_1}(t_{j_1})\delta b_{i_2}(t_{j_2})\rangle = \frac{\delta_{i_1 i_2}}{3}\varphi(t_{j_1} - t_{j_2})$, expression for the "propagator" $\langle v_i(t)\rangle = u(t) v_{0i}$:

$$u(t) = 1 + \sum_{n=1}^{\infty}\left(\frac{-2\,\delta\Omega^2}{3}\right)^n \times \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \ldots \int_0^{t_{2n-1}} \mathrm{d}t_{2n} \sum_{\{i<j\}} \prod \varphi(t_i - t_j),$$

with $\sum_{\{i<j\}} \prod \varphi(t_i - t_j)$ the $(2n-1)!!$ permutations of products of contractions of pairs

Diagrammatic representation

- Convenient way to represent the various iterations contributing to $u(t)$ in the form of diagrams [e.g. Kraichnan1961,Bourret1962,Frisch1966]
- Rules:
  - $1 \to$ free propagator $u^{(0)}(t)$,
  - contraction of $\langle b_{i_1}(t_{j_1})\delta b_{i_2}(t_{j_2})\rangle = \langle \delta b_{i_1}(t_{j_1}) u^{(0)}(t) \delta b_{i_2}(t_{j_2})\rangle \to$ interaction,
  - dashed lines $\to$ integrations over the ordered times crossing the continuous line
- In these conditions, equation for $u(t)$ symbolically written as

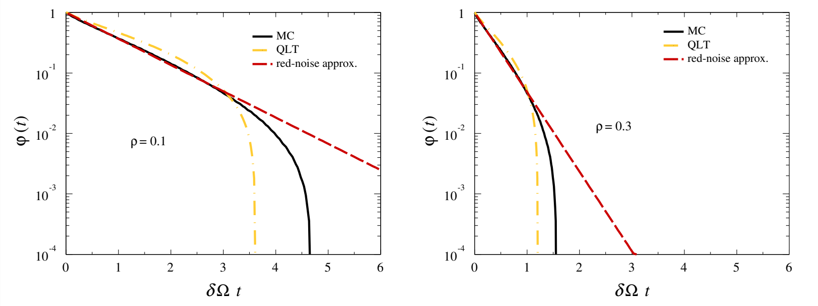
- For instance:

=

$$\left(\frac{-2\delta\Omega^2}{3}\right)^3 \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \int_0^{t_2} \mathrm{d}t_3 \int_0^{t_3} \mathrm{d}t_4 \int_0^{t_4} \mathrm{d}t_5 \int_0^{t_5} \mathrm{d}t_6 \; \varphi(t_1 - t_3)\varphi(t_2 - t_5)\varphi(t_4 - t_6)$$

ii. 2-pt correlation function of the experienced magnetic field

2-pt correlation function of the experienced magnetic field



- Formal expression:

$$\langle \delta b_i(t)\delta b_j(0)\rangle = \iint \mathrm{d}\mathbf{k}\mathrm{d}\mathbf{k}' \, \langle \delta b_i(\mathbf{k})\delta b_j(\mathbf{k}')e^{i\mathbf{k}\cdot\mathbf{x}(t)}\rangle$$

- High-energy regime: $\varphi(t) \simeq \tau\delta(t)$ [Plotnikov+2011, A&A, 532, A68]
- Red-noise approximation for $\varphi(t)$ with an exponential function, $\varphi(t) \simeq \exp\left(-t/\tau\right)$

Parameter τ

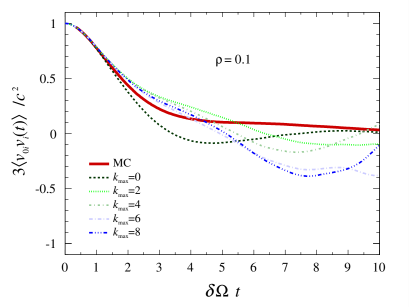
- Small-deflection regime (high rigidity, for $\rho \gtrsim \pi L_\mathrm{c}/L_\mathrm{max}$): $\tau \simeq L_\mathrm{c}/c$
- For $\rho \lesssim \pi L_\mathrm{c}/L_\mathrm{max}$, selection of modes for which particles do not experience spiral motions, modes that hence prevent decorrelations from occurring on relevant time scales
- Heuristic estimate observed to reproduce simulations (similarly to [Casse+2001, PRD 65 023002])

$$\tau \simeq \frac{1}{c} \frac{\int_{k_\star}^{k_\mathrm{max}} \mathrm{d}k \, k^{-1} \mathcal{E}(k)}{\int_{k_\star}^{k_\mathrm{max}} \mathrm{d}k \, \mathcal{E}(k)},$$

  with $\mathcal{E}(k)$ the kinetic energy spectrum of the turbulence
- ρ dependency from that of the lower boundary $k_\star(\rho) = \rho_\star k_\mathrm{min}/\rho$ with $\rho_\star = 2L_\mathrm{c}/(\pi L_\mathrm{max})$

iii. Transport in the red-noise approximation

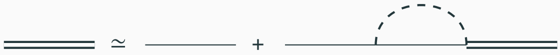
Truncation of the series?



- Summation of all terms up to some order $k_{\max}$?
- Series absolutely convergent for all t, very many terms required for $t > 3/(2\delta\Omega^2\tau)$
- $\rightarrow$ combinatorics that determines the number of occurrence of each diagram rapidly non-tractable

Partial summation

- Simplest scheme: Bourret propagator
- $\rightarrow$ substitute the “mass operator” in the Dyson equation for unconnected contributions:



- $\rightarrow$ sum of unconnected diagrams
- NB: exact solution in the case of white-noise process (cancellation of nested/crossed diagrams) [Plotnikov+2011, A&A, 532, A68]
- Decoupling in Laplace-transform space: $U(p) = p - \frac{2\delta\Omega^2}{3}\frac{U(p)}{p+\tau}$
- Inverse transform: $u(t) = \frac{A}{2B}e^{-At/6\tau}(e^{Ct/\tau} - 1)$, with $A = 3 + \sqrt{9 - 24\tau^2\delta\Omega^2}$, $B = A - 3$, and $C = \sqrt{1 - 8\tau^2\delta\Omega^2/3}$
- $\rightarrow$ nonphysical oscillations around 0 for $t > 3/(2\delta\Omega^2\tau)$

Partial summation

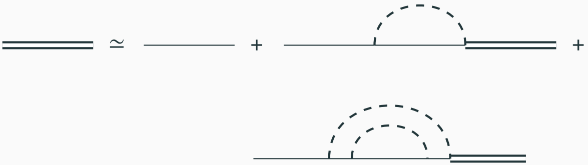
- Next scheme: Kraichnan propagator
- $\rightarrow$ substitute the free propagator inside the dotted loop for the Bourret propagator:

- $\rightarrow$ sum of unconnected/nested diagrams
- Decoupling in Laplace-transform space:

$$\begin{aligned}
\left[U^{(2)}(p)\right]^{-1} \quad &= \quad p - \sum_{n\geq 1} \frac{\left(-2\,\delta\Omega^2/3\right)^n}{(p+\tau^{-1})^n (p+2\tau^{-1})^{n-1}} \\
&- \sum_{n\geq 3} \frac{\left(-2\,\delta\Omega^2/3\right)^n}{(p+\tau^{-1})^2 (p+2\tau^{-1})^{n-1} (p+3\tau^{-1})^{n-2}} \\
&- \sum_{n\geq 4} \frac{\left(-2\,\delta\Omega^2/3\right)^n}{(p+\tau^{-1})^2 (p+2\tau^{-1})^2 (p+3\tau^{-1})^{n-2} (p+4\tau^{-1})^{n-3}} \\
&- \cdots
\end{aligned}$$

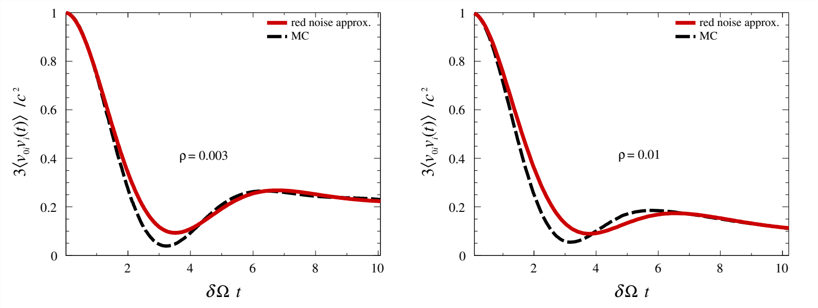
Partial summation

- n formally sent to infinity, but physical solution with a truncation to $n \leq 2$:

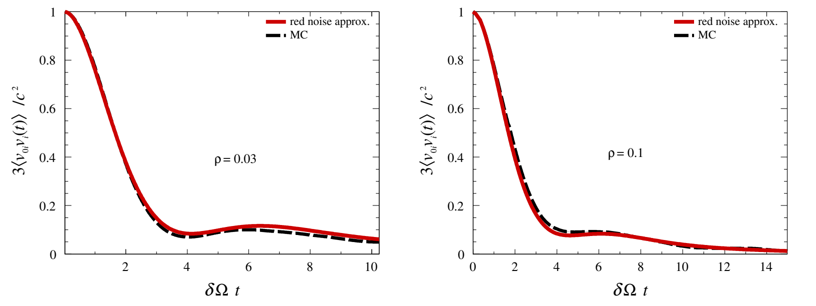


- Applicable in both the “high-rigidity” and “gyro-resonant” regimes (white-noise limit applicable to the high-rigidity regime only)

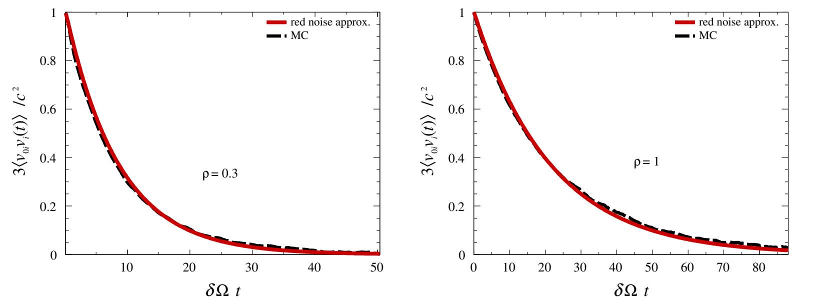
Velocity decorrelations



Velocity decorrelations

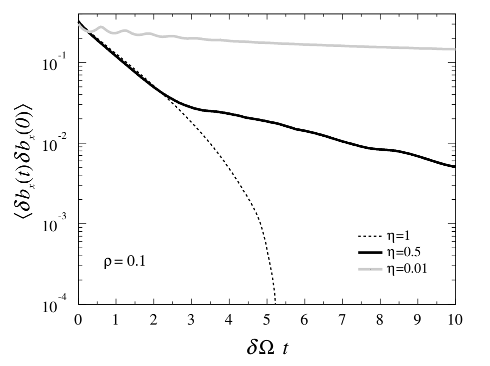


Velocity decorrelations



iv. Hints in presence of a mean field

2-pt correlation function of the experienced magnetic field



- Mean field is oriented in the *z* direction, perpendicular to the initial directions of the test particles
- Longer time-scale memory on top of that stemming from the turbulence, the intensity of which relative to that of the turbulence increasing with decreasing values of η
- Red-noise approximation is no longer valid

V. Outlook

Outlook

- Red-noise approximation valid for both the gyro-resonant and the high-rigidity regimes
- Regime of Larmor radius smaller than the smallest scale of the turbulence: modeling of the $\langle \delta b_i(t) \delta b_j(0) \rangle$ functions beyond an exponential fall-off required and/or improved partial summation scheme (including contribution from crossed diagrams)
- Better modelings of the $\langle \delta b_i(t) \delta b_j(0) \rangle$ functions necessary in the case of the presence of a mean field