Gravitational waves and the eikonal operator

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The framework

The aim is to use a particle-physicist approach to derive classical observables relevant to gravitational binaries

- Model the celestial bodies as "elementary" objects with known couplings to gravity (massless fields in general)
- Use quantum perturbative amplitudes to describe the large-distance scattering and extract the classical PM limit

In the eikonal approach* it is possible to implement this programme by focusing on gauge invariant quantities

* see also: EFT matching, Cheung, Rothstein, Solon; KMOC: Kosower, Maybee, O'Connell; Impulse: Kälin, Porto; many explicit amplitude results: Bern, Cheung, Roiban, Shen, Solon, Herrmann, Parra-Martinez, Ruf, Zeng, Bjerrum-Bohr, Damgaard, Planté, Vanhove,. . .

Classical physics is obtained by resumming an infinite set of contributions which leads to exponentiation

A general approach applicable to various sources / gravitational theories (shockwaves, Kerr / GR, supergravity, string theory . . .)

The plan

We focus first on elastic processes

- The $2 \to 2$ classical amplitudes
- The eikonal exponentiation and the classical observables
- The key message: starting at 3PM order there are non-trivial radiative effects

So we need to consider inelastic processes

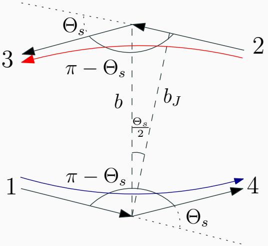
- The $2 \to 3$ classical amplitudes
- Waveforms from scattering amplitudes
- Exponentiation in the inelastic case (the eikonal operator)
- The Zero Frequency Limit (ZFL) revisited

The eikonal exponentiation provides a natural explanation for the different regimes (Post-Newtonian versus Ultra-Relativistic) of the radiation.

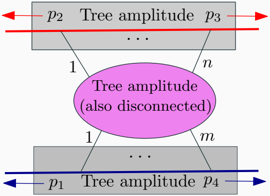
The elastic scattering

The setup

Consider the $2 \to 2$ scattering with $p_1^2 = p_4^2 = -m_1^2$, $p_2^2 = p_3^2 = -m_2^2$



A spacetime picture of the scattering



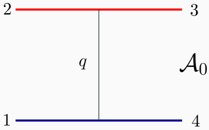
Diagrammatic picture

Key classical quantities:

- The centre-of-mass energy E, $E^2 = s = -(p_1 + p_2)^2 = (m_1^2 + m_2^2 + 2m_1 m_2 \sigma)$
- The angular momentum $J = p\, b_J$, $p = |\vec{p}_i|$, $Ep = m_1 m_2 \sqrt{\sigma^2 - 1}$
- The momentum transferred $Q = p_1 + p_4$, $|Q_s| = 2p \sin\left(\frac{\Theta_s}{2}\right)$

One particle exchange

Let us start from the 1-particle exchange



$$\mathcal{A}_0 = \frac{8\pi G m_1 m_2 \beta}{q^2} + \dots$$

$\beta^{\mathcal{N}=8} = 4m_1 m_2 \sigma^2$

$\beta^{GR} = 4m_1 m_2 \left(\sigma^2 - \frac{1}{D-2}\right)$

q is quantum and the dots contain analytic terms as $q \to 0$

In terms of classical quantity $b \sim \hbar/q$

$$\widetilde{\mathcal{A}}(s,b) = \int \frac{d^{D-2}q}{(2\pi)^{D-2}} \frac{\mathcal{A}(s,q^2)}{4pE} e^{ibq} \,.$$

In $D = 4 - 2\epsilon \to 4$ we have

$$i\widetilde{\mathcal{A}}_0^{\mathcal{N}=8} = \frac{2im_1 m_2 G(\pi b^2)^{\epsilon}\sigma^2\Gamma(-\epsilon)}{\sqrt{\sigma^2-1}} \to -i\frac{Gm_1 m_2}{\hbar}\log(b)\frac{4\sigma^2}{\sqrt{\sigma^2-1}}$$

No well defined classical limit?!

The elastic eikonal

The semi-classical limit requires that long range part of $\widetilde{\mathcal{A}}$ takes the form

$$\boxed{1 + i\widetilde{\mathcal{A}}(s,b) = \Big(1 + 2i\Delta(s,b)\Big)\, e^{\frac{i}{\hbar}2\delta(s,b)}}$$

where δ is $\mathcal{O}(\hbar^0)$ and Δ encodes the quantum terms $\mathcal{O}(\hbar^m)$ with $m \geq 0$

The PM expansion reads

$$\delta = \sum_{k=0}^{\infty} \delta_k\ , \quad \Delta = \sum_{k=0}^{\infty} \Delta_k$$

where $\delta_k,\ \Delta_k$ are of order G^{k+1}. $\mathcal{N} = 8$ in $D = 4$, we have

$$2\delta_0 = -\frac{4Gm_1m_2\sigma^2 \log b}{\sqrt{\sigma^2-1}},\ \delta_1 = 0 \text{ Caron-Huot, Zahraee},\ \delta_2 \neq 0 \text{ see G. Veneziano's talk}$$

Analogue results are available for pure GR see C. Heissenberg's talk

Ignoring the quantum terms the inverse FT reads

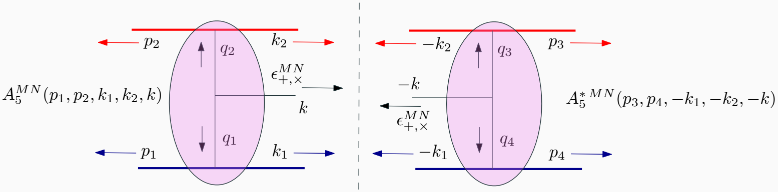
$$i\,\frac{\mathcal{A}(s,Q^2)}{4pE} = \int d^{D-2}b \left(e^{\frac{i}{\hbar}2\delta(s,b)} - 1\right)\, e^{\frac{i}{\hbar}b\cdot Q}$$

and a stationary phase approximation yields $Q_s^\mu = -\frac{\partial\, \mathrm{Re}\, 2\delta(s,b)}{\partial b^\mu}$ and so Θ_s

The inelastic scattering

The 3-particle cut (momentum space)

Generically $\mathrm{Im}(2\delta_2) \neq 0$: why? It is related to the 3-particle cut



Unitarity implies

$$[\mathrm{Im}\, 2A_2]_{3pc} = \int \frac{d^D k}{(2\pi)^D} \frac{d^D k_1}{(2\pi)^D} \frac{d^D k_2}{(2\pi)^D} (2\pi)^D \delta(p_1 + p_2 + k_1 + k_2 + k)$$
$$2\pi\theta(k^0)\,\delta(k^2)\; 2\pi\theta(k_1^0)\,\delta(k_1^2 + m_1^2)\; 2\pi\theta(k_2^0)\,\delta(k_2^2 + m_2^2)\; |A_5|^2$$

In b-space we have $\left[\mathrm{Im}\, \widetilde{A}_2\right]_{3pc} = \mathrm{Im}(2\delta_2)$: so this is a shortcut to the derivation of the imaginary part and Radiation-Reaction effects

see G. Veneziano's talk

The 3-particle cut (impact-parameter space)

In impact parameter space the convolution over k_i becomes a product. We have: $2\,\mathrm{Im}(2\delta_2) = \int \frac{d^{(D-1)}k}{(2\pi)^{(D-1)}2k^0} |\widetilde{A}_{5,i}(b,k)|^2$ with

$$\widetilde{A}_{5,i}(b,k) = \int \frac{d^{D-2}\boldsymbol{q}_-}{(2\pi)^{D-2}} \frac{e^{ib\boldsymbol{q}_-}}{4Ep} A_{5,i}(p_1,p_2,k_1,k_2,k)$$

- $\boldsymbol{q}_- \equiv \frac{1}{2}(\boldsymbol{q}_1 - \boldsymbol{q}_2)$ is a $(D-2)$-vector; the other momenta (except k) are fixed by the onshell/conservation conditions
- $A_{5,i} = A_5^{MN}\epsilon_{MN}^{(i)}$ for the physical polarizations $i = +, \times$ (in the $\mathcal{N} = 8$ case one has also other massless particles: dilaton, . . .)

One extra FT yields the result in terms of the retarded time $u = t - r$ rather than the frequency $\omega = k^0$ (and $\hat{k} = k/\omega$)

$$\widetilde{A}_{5,i}(b,u,\hat{k}) = \int \frac{d\omega}{2\pi} e^{-i\omega u} \widetilde{A}_{5,i}(b,\omega,\hat{k})$$

The classical $2 \to 3$ amplitude

A unified GR and $\mathcal{N}=8$ expression for the $2 \to 3$ classical amplitude

Goldberger, Ridgway; Luna, Nicholson, O'Connell, White; Mogull, Plefka, Steinhoff

$$A_5^{MN} = (8\pi G)^{\frac{3}{2}} \Bigg\{ \frac{8\big(P_1 k P_2^M - P_2 k P_1^M\big)\big(P_1 k P_2^N - P_2 k P_1^N\big)}{q_1^2 q_2^2}$$
$$+8P_1P_2\left[\frac{P_1^M P_1^N \frac{kP_2}{kP_1} - P_1^{(M}P_2^{N)}}{q_2^2} + \frac{P_2^M P_2^N \frac{kP_1}{kP_2} - P_1^{(M}P_2^{N)}}{q_1^2} - 2\frac{P_1 k P_2^{(M}q_1^{N)} - P_2 k P_1^{(M}q_1^{N)}}{q_1^2 q_2^2}\right]$$
$$+m_1 m_2 \beta\left[-\frac{P_1^M P_1^N (kq_1)}{(P_1 k)^2 q_2^2} + \frac{P_2^M P_2^N (kq_1)}{(P_2 k)^2 q_1^2} + 2\left(\frac{P_1^{(M}q_1^{N)}}{(P_1 k) q_2^2} - \frac{P_2^{(M}q_1^{N)}}{(P_2 k) q_1^2} + \frac{q_1^M q_1^N}{q_1^2 q_2^2}\right)\right]\Bigg\}$$

← Leading term in the Weinberg limit $k \ll q_i$

$\mathcal{N}=8$ setup: P_i, K_i are 10D momenta, q and k 4D

$$P_1 = (p_1; 0,0,0,0,0,m_1)\,, \quad P_1^2 = 0$$
$$P_2 = (p_2; 0,0,0,0,m_2,0)\,, \quad P_2^2 = 0$$

GR setup: all vectors are 4D

$$P_1 = (p_1; 0,0,0,0,0,0)\,, \quad P_1^2 = p_1^2 = -m_1^2$$
$$P_2 = (p_2; 0,0,0,0,0,0)\,, \quad P_2^2 = p_2^2 = -m_2^2$$

We parametrise the 4D kinematics in terms of

$$p_1^\mu = -\overline{m}_1 u_1^\mu + \frac{1}{2}\boldsymbol{q}_-^\mu\,, \quad p_2^\mu = -\overline{m}_2 u_2^\mu - \frac{1}{2}\boldsymbol{q}_-^\mu\,, \quad u_i^\mu = (\cosh y_i, 0, \sinh y_i)$$

$$y_\sigma = -(u_1 u_2)\,, \quad \omega_1 = -(u_1 k)\,, \quad \omega_2 = -(u_2 k)\,, \quad \boldsymbol{q}_-^2\,, \quad (\boldsymbol{q}_- \boldsymbol{k})\,,$$

$$(u_i \boldsymbol{q}_-) = 0\,, \quad \overline{m}_i^2 = m_i^2 + \frac{\boldsymbol{q}_-^2}{4}\,, \quad y_\sigma = \frac{\sigma - \frac{\boldsymbol{q}_-^2}{m_1 m_2}}{\sqrt{1+\frac{\boldsymbol{q}_-^2}{m_1^2}}\sqrt{1+\frac{\boldsymbol{q}_-^2}{m_2^2}}}$$

Waveforms

The $\widetilde{A}_{5,i}(b,k)$'s are directly related to waveforms in the frequency domain. The standard recipe in $D=4$ for the graviton is

$$W_{\mu\nu} \equiv g_{\mu\nu} - \eta_{\mu\nu} = \frac{4G}{\kappa r}\widetilde{A}_{5\mu\nu}(b,k) \qquad \text{[rec.]}$$

r is the distance to the far-away observer

Then for the $\times$ polarisation in GR we have — Alert: preliminary

$$\tilde{A}_{5,\times} = \frac{\kappa^3}{4pE(2\pi)} i(e_\phi \hat{b}) \Bigg\{ m_1 m_2 \beta \Bigg[-\frac{\omega_2 - \omega_1 y_\sigma}{|\mathbf{k}|(y_\sigma^2-1)} e^{ibk/2} K_1\Big(\frac{b\omega_1}{\sqrt{y_\sigma^2-1}}\Big) - \frac{\omega_1 - \omega_2 y_\sigma}{|\mathbf{k}|(y_\sigma^2-1)} e^{-ibk/2} K_1\Big(\frac{b\omega_2}{\sqrt{y_\sigma^2-1}}\Big) + \frac{\omega_1\omega_2}{|\mathbf{k}|\sqrt{y_\sigma^2-1}} \int_0^1 dx e^{i\frac{kb}{2}(1-2x)} b K_0(b|\mathbf{k}|\sqrt{f}) \Bigg] - 4p_1 p_2 \overline{m}_1 \overline{m}_2 |\mathbf{k}| \sqrt{y_\sigma^2-1} \int_0^1 dx e^{i\frac{kb}{2}(1-2x)} b K_0(b|\mathbf{k}|\sqrt{f}) \Bigg\}$$

Bold vectors and b are (D-2)-dimensional

$$|\mathbf{k}|\sqrt{f} = \sqrt{\mathbf{k}^2 x(1-x) + \frac{\omega_2^2}{y_\sigma^2-1}x + \frac{\omega_1^2}{y_\sigma^2-1}(1-x)}\,, \quad (e_\phi u_i) = 0\,, \quad e_\theta^\mu = \frac{\omega_2 u_1^\mu - \omega_1 u_2^\mu}{|\mathbf{k}|\sqrt{y_\sigma^2-1}}$$

This result is written in a covariant form

The FT to u-space is straightforward and then the x-integral is simple

We are in the process of comparing with Jakobsen, Mogull, Plefka, Steinhoff; Mougiakakos, Riva, Vernizzi

The eikonal operator

Why does [rec.] work? An eikonal-based explanation:

Introduce an eikonal operator in the Fock space of the emitted gravitons

$$\frac{2\hat{\delta}(s,b)}{\hbar} = \frac{2\delta_r(s,b)}{\hbar} + \int \frac{d^{(D-1)}k}{(2\pi)^{D-1}2\omega} \frac{a^\dagger(k)\widetilde{A}_5^{(0)}(b,k) + a(k)(\widetilde{A}_5^{(0)}(b,k))^*}{\hbar} + \dots$$

with $[a(k), a^\dagger(k')] = 2\hbar\omega\delta^{D-1}(\frac{p-p'}{2\pi\hbar})$ and δ real (Lorentz indices are understood)

The classical radiation is the expectaction value of the graviton field in the final state $e^{2i\hat{\delta}}|0\rangle$

$$W \sim \langle 0|e^{-2i\hat{\delta}}\left(a^\dagger(k) + a(k)\right)e^{2i\hat{\delta}}|0\rangle$$

Similarly the spectrum of the radiated energy in $D = 4$ is

$$\int \frac{dE^{\text{rad}}}{d\omega d\Omega}\, d\omega d\Omega = \int \frac{d^3k}{(2\pi)^3 2\omega} \langle 0|e^{-2i\hat{\delta}}\left(\omega a^\dagger(k)a(k)\right)e^{2i\hat{\delta}}|0\rangle$$

Using Baker–Campbell–Hausdorff we get $e^{-\,\text{Im}(2\delta_2)}$ from a hermitian $\hat{\delta}$

see C. Heissenberg's talk

The Zero-Frequency-Limit

The ZFL $\omega \to 0$ is determined by the Soft Theorems (ST).

A dynamics-agnostic approach is to fix both the initial and the final state and use the ST to extract the radiation at order $\frac{1}{\omega}$, $\ln \omega$, $\omega \ln^2 \omega$

Laddha, Saha, Sahoo, Sen; Addazi, Bianchi, Veneziano

Write the final momenta in terms of the initial ones and Q_s (which is determined by the eikonal phase) to obtain the waveform

We get perfect agreement with the $(1/\omega)$ and $(\ln \omega)$ terms of our W at leading order in G and finite σ

The eikonal expression should hold for any ω and σ. For instance

$$\lim_{\omega \to 0} \left[\frac{dE^{\rm rad}}{d\omega} \right] = \frac{2G\beta^2(\sigma)}{\pi b^2(\sigma^2-1)^2} \left[\frac{8-5\sigma^2}{3} - \frac{\sigma(3-2\sigma^2)}{(\sigma^2-1)^{\frac{1}{2}}} \cosh^{-1}(\sigma) \right]$$

P Di Vecchia, C Heissenberg, R. R., G Veneziano: 2101.05772

In the $\sigma \to 1$ it reproduces Smarr's result. However ...

The Ultra-Relativistic (UR) regime

The result above for $E^{\rm rad}$ breaks down in the $\sigma \to \infty$ limit, since

$$\cosh^{-1}(\sigma) \to \ln\left(\frac{s}{m_1 m_2}\right)$$

The spectrum has a divergence in the UR limit?!

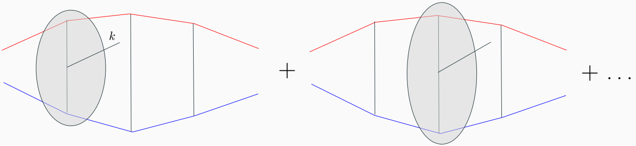
No, the result from the ST is finite

$$\lim_{\omega\to 0}\left[\frac{1}{\sqrt{s}}\frac{dE^{\rm rad}}{d\omega}\right] = \frac{\Theta_s^2(G\sqrt{s})}{\pi}\ln(\Theta_s^{-2})$$

Grudzinov, Veneziano; Ciafaloni, Colferai, Veneziano
Laddha, Saha, Sahoo, Sen

What is the origin of the problem? When does the new UR regime set in?

There is a crucial interplay between elastic and inelastic exponentiation



The eikonal operator revisited

We should write the eikonal operator by separating the inelastic part ($\hat{\delta}_{in}$) and the real elastic eikonal (δ_r)

$$e^{2i\hat{\delta}_{in}(s,b,\partial_b,a^\dagger,a)}\, e^{2i\delta_r(s,b)}$$

where the ∂_b's act only on the eikonal phase. Then there are two regimes

$$1+\frac{\mathbf{q}_-{}^2}{4m_i^2} \to 1+\frac{p^2\Theta_s^2}{4m_i^2}$$

Use $\mathbf{q}_- \to -i\partial_b(2\delta_0) \sim -p\,\Theta_s$

- 1 if $\Theta_s \ll \frac{m_i}{p}$ (Relativistic) $\quad \Theta_s \ll \sqrt{\frac{m_i}{m_j}}\,\sigma^{-\frac{1}{2}}$
- Transition at $\sigma^{\frac{1}{2}}\Theta_s^{\text{rest}} = \Theta_s \sim \sigma^{-\frac{1}{2}}$ D'Eath; Kovacs and Thorne
- $\frac{p^2\Theta_s^2}{4m_i^2}$ if $\Theta_s \gg \frac{2m_i}{p}$ (UR) $\quad \Theta_s \gg \sqrt{\frac{m_i}{m_j}}\,\sigma^{-\frac{1}{2}}$

The effect on y_σ

$$y_\sigma \to \frac{\sigma - \frac{(-i\partial_b)^2}{4m_1m_2}}{\sqrt{1+\frac{(-i\partial_b)^2}{4m_1^2}}\sqrt{1+\frac{(-i\partial_b)^2}{4m_2^2}}}$$

- $\to \sigma$ (Relativistic)
- $\to \sim \frac{1}{\Theta_s^2}$ (Ultra-Relativistic)

The UR ZFL is $\lim_{\omega\to 0}\left[\frac{dE^{\text{rad}}}{d\omega}\right] \simeq \frac{4\sigma^2 G\beta^2}{\pi b^2\sigma^4}\cosh^{-1}(y_\sigma) \to \frac{\Theta_s^2(Gs)}{\pi}\ln(\Theta_s^{-2})$

in agreement with the previous result

Conclusions

The eikonal provides a conceptually simple approach to gravitational binaries: it leads directly to classical, observable quantities starting from a full quantum framework

We included real radiation effects: eikonal exponentiation seems essential in the UR regime

What about other approaches such as KMOC?

Work in progress:

- Complete the derivation of the waveforms interpolating between the relativistic and the UR regimes
- Revisit the total E^{rad} in the UR regimes

  Herrmann, Parra-Martinez, Ruf, Zeng

  Solve the energy crisis

  Restore UR universality

More general questions

Go beyond the leading PM waveforms, study the nature of the classical radiation (coherent? squeezed?) ...

Does the eikonal operator play any role in the elastic 4PM problem?