Leading hadronic contribution to the muon magnetic moment from lattice QCD

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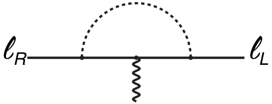
Budapest-Marseille-Wuppertal collaboration [BMWc]

Borsanyi, Fodor, Guenther, Hoelbling, Katz, LL, Lippert, Miura, Szabo, Parato, Stokes, Toth, Torok, Varnhorst

Nature 593 (2021) 51, online 7 April 2021 → BMWc '20
PRL 121 (2018) 022002 (Editors' Selection) → BMWc '17
& Aoyama et al., Phys. Rept. 887 (2020) 1-166 → WP '20

Context and motivation

Why so excited about the muon magnetic moment?



$$\longrightarrow \quad \frac{a_\ell}{2m_\ell} eF^{\mu\nu}[\bar{\ell}_L \sigma_{\mu\nu} \ell_R]$$

- Actually, interested in $a_\ell = (g_\ell - 2)/2$
- Loop induced $\Rightarrow$ sensitive to dofs that may be too heavy or too weakly coupled to be produced directly
- CP and flavor conserving, chirality flipping $\Rightarrow$ complementary to: EDMs, s and b decays, LHC direct searches, ...
- As early as 1956, Berestetskii noted a_μ typically $(m_\mu/m_e)^2 \sim 40,000$ times more sensitive to heavy dofs than a_e

  $\Rightarrow$ a_μ sensitive to possibly unknown, heavy dofs
- Despite $\tau_\mu \sim 2\,\mu$s, a_μ measured in 1960 [Garwin et al '60]

  $\rightarrow$ measurements progressed in // with the development of the SM, each new experiment probing theory to a new level
- Early 2000s, BNL measured a_μ to 0.54 ppm: EW contribution seen at 3σ level

  $\rightarrow$ But also excess over SM prediction $\sim 2\times$ EW contribution

Why so excited about the muon magnetic moment?

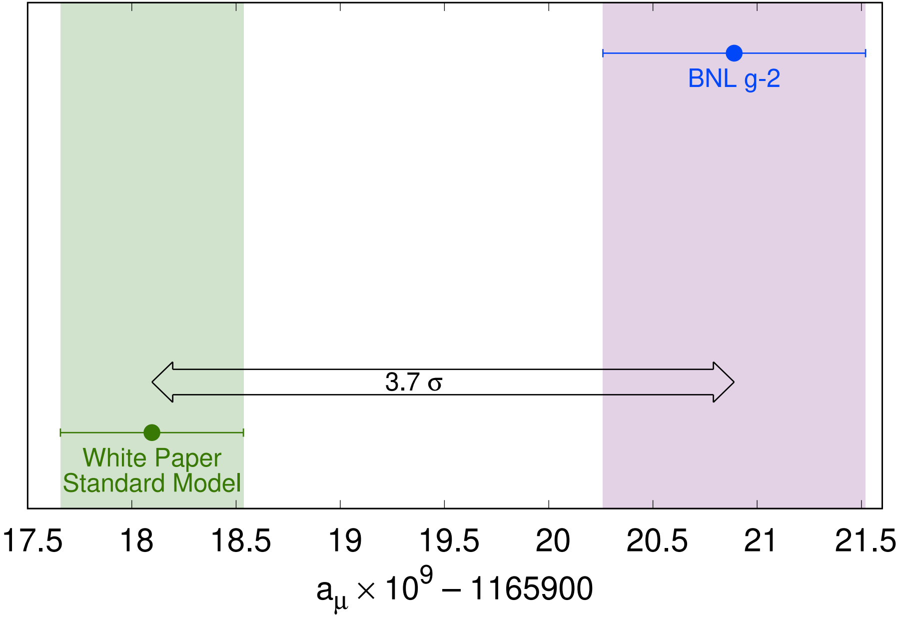
- Since then, persistent tension between measurement & SM $> 3.5\sigma$
- To decide on possible presence of BSM physics:
  - significant upgrade of BNL experiment @ FNAL w/ goal to reduce measurement error by $\times 4$
  - important theoretical effort to improve SM prediction to same level

$\Rightarrow$ White Paper from the muon $g-2$ Theory Initiative posted on arXiv in June 2020 w/ reference SM prediction [Aoyama et al '20 = WP '20]

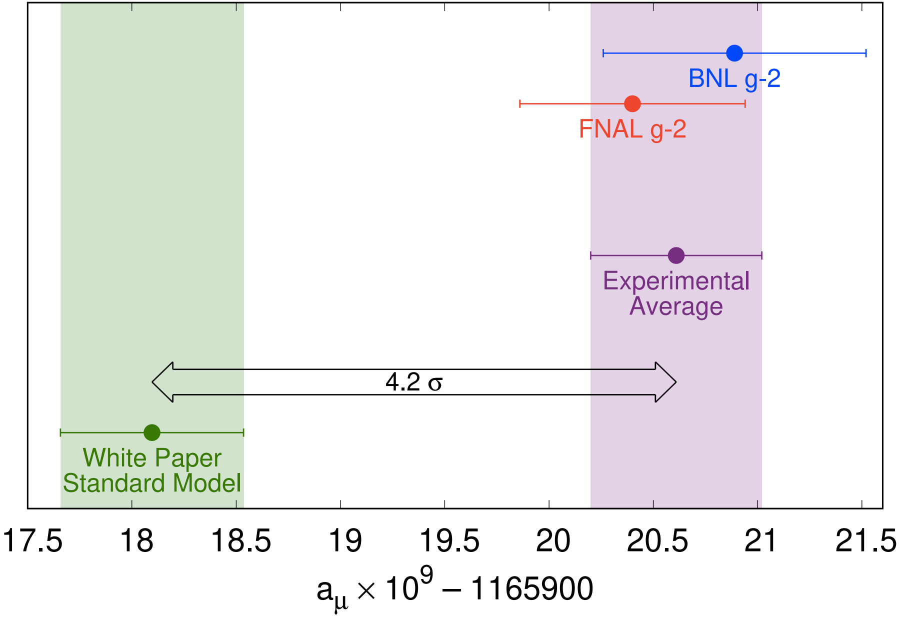
$\Rightarrow$ Presentation and publication on April 7 of FNAL's first results
$\to$ tour de force measurement confirms BNL result w/ already improved precision
$\to$ reduces WA error to 0.35 ppm and increases tension w/ SM to 4.2σ

- Same day, *Nature* published our *ab-initio* calculation of hadronic vacuum polarization contribution to the SM prediction that brings it much closer to measurement of a_μ

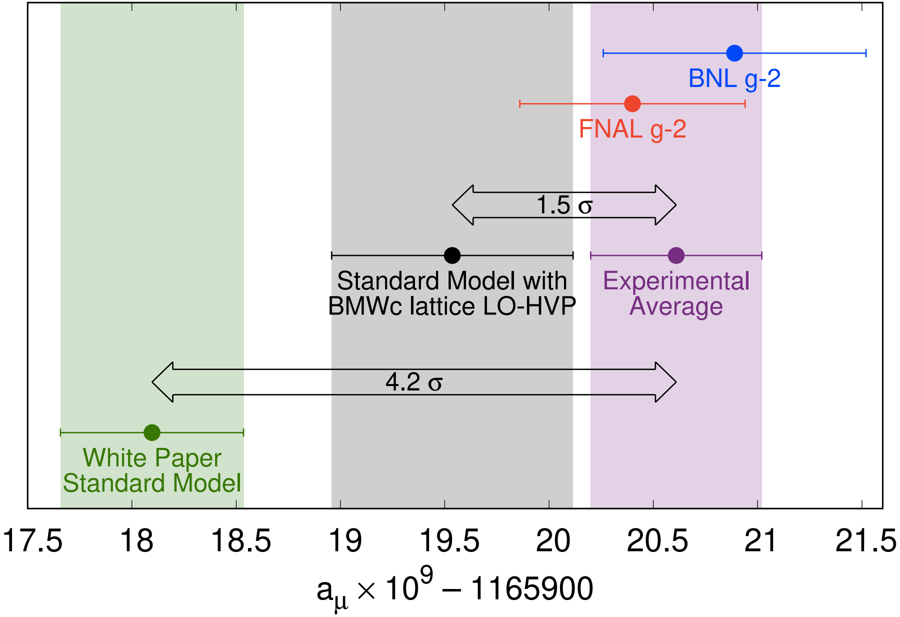
Situation ca. June 2020



Fermilab plot, April 7 2021



Fermilab plot, April 7 2021, BMWc version



Standard model calculation of a_μ

At needed precision: all three interactions and most SM particles

$$
\begin{aligned}
a_\mu^{\rm SM} &= a_\mu^{\rm QED} + a_\mu^{\rm had} + a_\mu^{\rm EW} \\
&= O\left(\frac{\alpha}{2\pi}\right) + O\left(\left(\frac{\alpha}{\pi}\right)^2\left(\frac{m_\mu}{M_\rho}\right)^2\right) + O\left(\left(\frac{e}{4\pi\sin\theta_W}\right)^2\left(\frac{m_\mu}{M_W}\right)^2\right) \\
&= O\left(10^{-3}\right) + O\left(10^{-7}\right) + O\left(10^{-9}\right)
\end{aligned}
$$

SM prediction vs experiment on April 7, 2021 (v1)

SM contribution	$a_\mu^{\text{contrib.}} \times 10^{10}$	Ref.
HVP LO (R-ratio)	692.8 ± 2.4	[KNT '19]
	694.0 ± 4.0	[DHMZ '19]
	692.3 ± 3.3	[CHHKS '19]
	693.1 ± 4.0	[WP '20]
HVP LO (lattice<2021)	711.6 ± 18.4	[WP '20]
HVP NLO	-9.83 ± 0.07	
		[Kurz et al '14, Jegerlehner '16, WP '20]
HVP NNLO	1.24 ± 0.01	[Kurz '14, Jeger. '16]
HLbyL LO (pheno)	9.2 ± 1.9	[WP '20]
HLbyL LO (lattice<2021)	$7.8 \pm 3.1 \pm 1.8$	[RBC '19]
HLbyL LO (lattice 2021)	$10.7 \pm 1.1 \pm 0.9$	[Mainz '21]
HLbyL LO (avg)	9.0 ± 1.7	[WP '20]
HLbyL NLO (pheno)	0.2 ± 0.1	[WP '20]
QED [5 loops]	11658471.8931 ± 0.0104	[Aoyama '19, WP '20]
EW [2 loops]	15.36 ± 0.10	[Gnendiger '15, WP '20]
HVP Tot. (R-ratio)	684.5 ± 4.0	[WP '20]
HLbL Tot.	9.2 ± 1.8	[WP '20]
SM [0.37 ppm]	11659181.0 ± 4.3	[WP '20]
Exp [0.35 ppm]	11659206.1 ± 4.1	[BNL '06 + FNAL '21]
Exp − SM	25.1 ± 5.9 [4.2σ]	

SM prediction vs experiment on April 7, 2021 (v2)

SM contribution	$a_\mu^{\rm contrib.} \times 10^{10}$	Ref.
HVP LO (R-ratio)	692.8 ± 2.4	[KNT '19]
	694.0 ± 4.0	[DHMZ '19]
	692.3 ± 3.3	[CHHKS '19]
	693.1 ± 4.0	[WP '20]
HVP LO (lattice)	707.5 ± 5.5	[BMWc '20]
HVP NLO	-9.83 ± 0.07	
		[Kurz et al '14, Jegerlehner '16, WP '20]
HVP NNLO	1.24 ± 0.01	[Kurz '14, Jeger. '16]
HLbyL LO (pheno)	9.2 ± 1.9	[WP '20]
HLbyL LO (lattice<2021)	$7.8 \pm 3.1 \pm 1.8$	[RBC '19]
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HLbyL LO (avg)	9.0 ± 1.7	[WP '20]
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QED [5 loops]	11658471.8931 ± 0.0104	[Aoyama '19, WP '20]
EW [2 loops]	15.36 ± 0.10	[Gnendiger '15, WP '20]
HVP Tot. (lat. + R-ratio)	698.9 ± 5.5	[WP '20, BMWc '20]
HLbL Tot.	9.2 ± 1.8	[WP '20]
SM [0.49 ppm]	11659195.4 ± 5.7	[WP '20 + BMWc '20]
Exp [0.35 ppm]	11659206.1 ± 4.1	[BNL '06 + FNAL '21]
Exp − SM	10.7 ± 7.0 [1.5σ]	

Hadronic contributions to a_μ: quark and gluon loops

$$a_\mu^{\text{exp}} - a_\mu^{\text{QED}} - a_\mu^{\text{EW}} = 718.9(4.1) \times 10^{-10} \stackrel{?}{=} a_\mu^{\text{had}}$$

Clearly right order of magnitude:

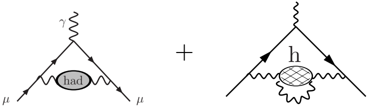
$$a_\mu^{\text{had}} = O\left(\left(\frac{\alpha}{\pi}\right)^2 \left(\frac{m_\mu}{M_\rho}\right)^2\right) = O\left(10^{-7}\right)$$

(already Gourdin & de Rafael '69 found $a_\mu^{\text{had}} = 650(50) \times 10^{-10}$)

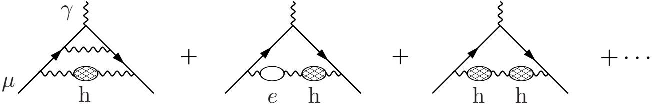
Write

$$a_\mu^{\text{had}} = a_\mu^{\text{LO-HVP}} + a_\mu^{\text{HO-HVP}} + a_\mu^{\text{HLbyL}} + O\left(\left(\frac{\alpha}{\pi}\right)^4\right)$$

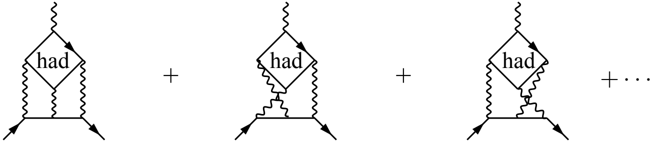
Hadronic contributions to a_μ: diagrams



$\rightarrow a_\mu^{\text{LO-HVP}} = O\left(\left(\frac{\alpha}{\pi}\right)^2\right)$



$\rightarrow a_\mu^{\text{NLO-HVP}} = O\left(\left(\frac{\alpha}{\pi}\right)^3\right)$



$\rightarrow a_\mu^{\text{HLbL}} = O\left(\left(\frac{\alpha}{\pi}\right)^3\right)$

A very brief introduction to lattice QCD

What is lattice QCD (LQCD)?

To describe matter w/ sub-% precision, QCD requires $\geq$ 104 numbers at every spacetime point

$\rightarrow \infty$ number of numbers in our continuous spacetime

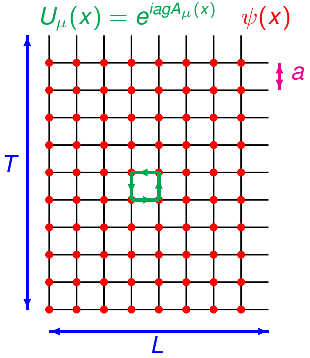
$\rightarrow$ must temporarily "simplify" the theory to be able to calculate *(regularization)*

$\Rightarrow$ Lattice gauge theory $\longrightarrow$ mathematically sound definition of NP QCD:

- UV (& IR) cutoff $\rightarrow$ well defined path integral in Euclidean spacetime:

$$\langle O \rangle = \int \mathcal{D}U\mathcal{D}\bar{\psi}\mathcal{D}\psi\, e^{-S_G-\int \bar{\psi}D[M]\psi}\, O[U,\psi,\bar{\psi}]$$
$$= \int \mathcal{D}U\, e^{-S_G}\, \det(D[M])\, O[U]_{\mathrm{Wick}}$$

- $\mathcal{D}Ue^{-S_G}\det(D[M]) \geq 0$ & finite # of dofs $\rightarrow$ evaluate numerically using stochastic methods



LQCD is QCD when $m_q \rightarrow m_q^{\mathrm{ph}}$, $a \rightarrow 0$ (after renormalization), $L \rightarrow \infty$ (and stats $\rightarrow \infty$)

HUGE conceptual and numerical ($O(10^9)$ dofs) challenge

Our “accelerators”

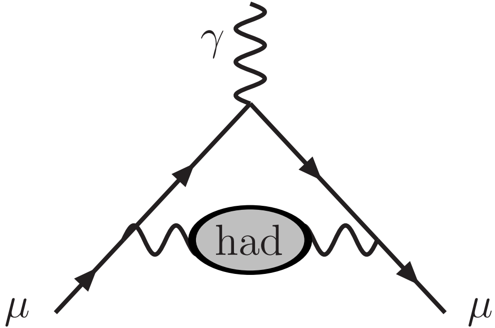
Such computations require some of the world’s most powerful supercomputers

- copyright Photothèque CNRS/Cyril Frésillon

- 1 year on supercomputer
  $\sim$ 100 000 years on laptop

- In Germany, those of the Forschungszentrum Jülich, the Leibniz Supercomputing Centre (Munich), and the High Performance Computing Center (Stuttgart); in France, Turing and Jean Zay at the Institute for Development and Resources in Intensive Scientific Computing (IDRIS) of the CNRS, and Joliot-Curie at the Very Large Computing Centre (TGCC) of the CEA, by way of the French Large-scale Computing Infrastructure (GENCI).

Lattice QCD calculation of a_μ^{HVP}



All quantities related to a_μ will be given in units of 10^{-10}

HVP from LQCD: introduction

Consider in Euclidean spacetime, i.e. spacelike $q^2 = -Q^2 \leq 0$ [Blum '02]

$$\Pi_{\mu\nu}(Q) =$$

$$= \int d^4x\, e^{iQ\cdot x} \langle J_\mu(x) J_\nu(0) \rangle$$

$$= \left(Q_\mu Q_\nu - \delta_{\mu\nu} Q^2\right) \Pi(Q^2)$$

w/ $J_\mu = \frac{2}{3}\bar{u}\gamma_\mu u - \frac{1}{3}\bar{d}\gamma_\mu d - \frac{1}{3}\bar{s}\gamma_\mu s + \frac{2}{3}\bar{c}\gamma_\mu c + \cdots$

Then [Lautrup et al '69, Blum '02]

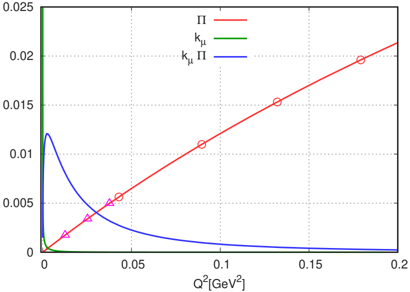
$$a_\ell^{\text{LO-HVP}} = \alpha^2 \int_0^\infty \frac{dQ^2}{m_\ell^2} k(Q^2/m_\ell^2) \hat{\Pi}(Q^2)$$

w/ $\hat{\Pi}(Q^2) \equiv \left[\Pi(Q^2) - \Pi(0)\right]$ and

$$k(r) = \left[r + 2 - \sqrt{r(r+4)}\right]^2 / \sqrt{r(r+4)}$$

Integrand peaked for $Q \sim (m_\ell/2) \sim 50\,\mathrm{MeV}$ for μ

However, $Q_{\text{min}} \equiv \frac{2\pi}{T} \sim 110\,\mathrm{MeV}$ for lattice w/ $T \sim 11\,\mathrm{fm}$



$(k_\mu(Q^2) = (\pi/m_\mu)^2 k(Q^2))$

Low-Q^2 challenges in finite volume (FV)

A. Must subtract $\Pi_{\mu\nu}(Q=0) \neq 0$ in FV that contaminates $\Pi(Q^2) \sim \Pi_{\mu\nu}(Q)/Q^2$ for $Q^2 \to 0$ w/ very large FV effects

B. On-shell renormalization requires $\Pi(0)$ which is problematic (see above)

C. Need $\hat{\Pi}(Q^2)$ interpolation due to $Q_{\text{min}} = 2\pi/T \sim 135\,\text{MeV} > \frac{m_\mu}{2} \sim 50\,\text{MeV}$ for $T \sim 9\,\text{fm}$

$$\Downarrow$$

- Compute on $T \times L^3$ lattice in $N_f = 2+1+1$ QCD

$$C_{TL}^{\text{iso}}(t) = \frac{a^3}{3} \sum_{i=1}^{3} \sum_{\vec{x}} \langle J_i(x) J_i(0) \rangle$$

- Decompose ($C_{TL}^{I=1} = \frac{9}{10} C_{TL}^{ud}$)

$$C_{TL}^{\text{iso}}(t) = C_{TL}^{ud}(t) + C_{TL}^{s}(t) + C_{TL}^{c}(t) + C_{TL}^{\text{disc}}(t) = C_{TL}^{I=1}(t) + C_{TL}^{I=0}(t)$$

- Define $\forall Q_0 \in \mathbb{R}$ [Bernecker et al '11, BMWc '13, Feng et al '13, Lehner '14, ...] (ad A, B, C) [see also Charles et al '17]

$$\hat{\Pi}_{TL}^{f}(Q^2) \equiv \Pi_{TL}^{f}(Q^2) - \Pi_{TL}^{f}(0) = \frac{1}{3} \sum_{i=1}^{3} \frac{\Pi_{ii,TL}^{f}(0) - \Pi_{ii,TL}^{f}(Q)}{Q^2} - \Pi_{TL}^{f}(0) = a \sum_{t=0}^{T-a} \text{Re}\left[\frac{e^{iQt} - 1}{Q^2} + \frac{t^2}{2}\right] \text{Re}\, C_{TL}^{f}(t)$$

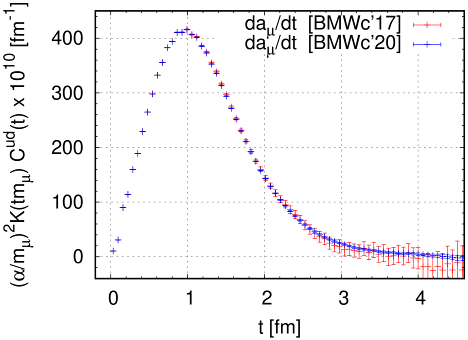
Our lattice definition of $a_{\ell,f}^{\text{LO-HVP}}$

Combining everything, get $a_{\ell,f}^{\text{LO-HVP}}$ from $C_{TL}^f(t)$:

$$a_{\ell,f}^{\text{LO-HVP}}(Q^2 \leq Q^2_{\max}) = \lim_{a\to 0,\, L\to\infty,\, T\to\infty} \alpha^2 \left(\frac{a}{m_\ell^2}\right) \sum_{t=0}^{T/2}{}' K(tm_\ell, Q^2_{\max}/m_\ell^2)\ \mathrm{Re}C_{TL}^f(t)$$

where

$$K(\tau, r_{\max}) = \int_0^{r_{\max}} dr\, k(r)\left(\tau^2 - \frac{4}{r}\sin^2\frac{\tau\sqrt{r}}{2}\right)$$

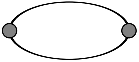


(144×96^3, $a \sim 0.064$ fm, $M_\pi \sim 135$ MeV)

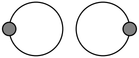
Simulation challenges

D. $\pi\pi$ contribution very important $\rightarrow$ have physically light π

E. Two types of contributions



quark-connected (qc)



quark-disconnected (qd)

where qd contributions are $SU(3)_f$ and Zweig suppressed but very challenging

F. $\langle J_\mu^{ud}(x) J_\nu^{ud}(0)\rangle_{\rm qc}$ & disc. have very poor signal at large $\sqrt{x^2}$ + need high-precision results

$\rightarrow$ many algorithmic improvements + very high statistics + rigorous bounds

G. Must control $\langle J_\mu(x) J_\nu(0)\rangle$ at $\sqrt{x^2} \gtrsim 2/m_\mu$ $\rightarrow L = 6.1 \div 6.6\,\text{fm},\ T = 8.6 \div 11.3\,\text{fm}$

H. Need controlled continuum limit $\rightarrow$ have 6 a's: $0.134 \rightarrow 0.064\,\text{fm}$

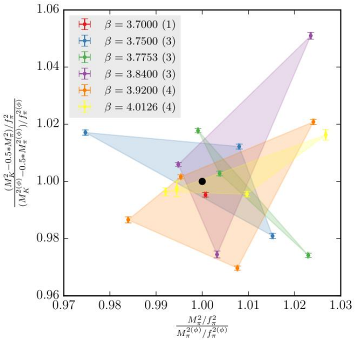
$\rightarrow$ improve approach to continuum limit w/ phenomenological models (SRHO, SMLLGS) w/ 2-loop SU(2) SχPT for systematic error

Simulation details: ad D - H

31 high-statistics simulations w/ $N_f = 2+1+1$ flavors of 4-stout staggered quarks:

- Bracketing physical m_{ud}, m_s, m_c
- 6 a's: $0.134 \rightarrow 0.064\,\mathrm{fm}$
- $L = 6.1 \div 6.6\,\mathrm{fm}$, $T = 8.6 \div 11.3\,\mathrm{fm}$
- Conserved EM current

β	a [fm]	$T \times L$	#conf
3.7000	0.1315	64×48	904
3.7500	0.1191	96×56	2072
3.7753	0.1116	84×56	1907
3.8400	0.0952	96×64	3139
3.9200	0.0787	128×80	4296
4.0126	0.0640	144×96	6980



For sea-quark QED corrections

β	a [fm]	$T \times L$	#conf
3.7000	0.1315	24×48	716
		48×64	300
3.7753	0.1116	28×56	887
3.8400	0.0952	32×64	4253

- State-of-the-art techniques:
  - EigCG
  - Low mode averaging [Neff et al '01, Giusti et al '04, ...]
  - All mode averaging [Blum et al '13]
  - Solver truncation [Bali et al '09]

$\Rightarrow$ Over 25,000 gauge configurations

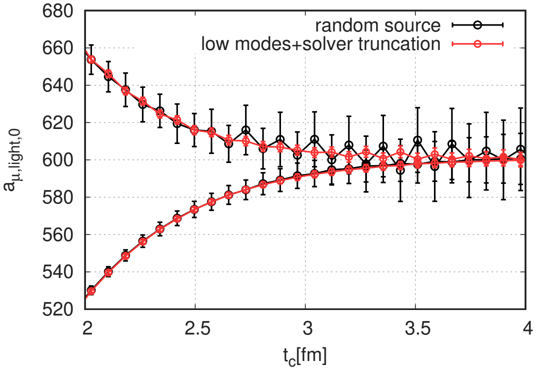
$\Rightarrow$ 10's of millions of measurements

Noise reduction: ad F-G

N/S in $C_L^{ud}(t)$ grows like $e^{(M_\rho - M_\pi)t}$

- LMA: use exact (all-to-all) quark propagators in IR and stochastic in UV [Neff et al '01, Giusti et al '04]
- Decrease noise by replacing $C_L^{ud}(t)$ by average of rigorous upper/lower bounds above $t_c = 4\,\mathrm{fm}$ [Lehner '16, BMWc '17]

  $$0 \leq C_L^{ud}(t) \leq C_L^{ud}(t_c)\, e^{-E_{2\pi}(t-t_c)}$$



$\Rightarrow \times 5$ in precision: few pemil accuracy on each ensemble

More challenges

I. Need $\hat{\Pi}(Q^2)$ for $Q^2 \in [0, +\infty[$, but $\frac{\pi}{a} \sim 9.7\,\mathrm{GeV}$ for $a \sim 0.064\,\mathrm{fm}$

$\rightarrow$ match onto perturbation theory

$$a_{\ell,f}^{\text{LO-HVP}} = a_{\ell,f}^{\text{LO-HVP}}(Q \leq Q_{\text{max}}) + \gamma_\ell(Q_{\text{max}})\,\hat{\Pi}^f(Q^2_{\text{max}}) + \Delta^{\text{pert}} a_{\ell,f}^{\text{LO-HVP}}(Q > Q_{\text{max}})$$

using $O(\alpha_s^4)$ results from `rhad` package [Harlander et al '03]

J. Include c quark for higher precision and good matching onto perturbation theory $\rightarrow$ done

K. Even in large volumes w/ $L \gtrsim 6.1\,\mathrm{fm}$ & $T \geq 8.7\,\mathrm{fm}$, finite-volume (FV) effects significant

$\rightarrow$ 1-loop SU(2) χPT [Aubin et al '16] suggests 2% even in our large volumes

$\rightarrow$ leading source of systematic in all previous $a_\mu^{\text{LO-HVP}}$ lattice calculations

$\rightarrow$ perform dedicated FV study w/ even larger volumes $(\sim 11\,\mathrm{fm})^4$

$\rightarrow$ check and supplement w/ 2-loop χPT [Bijnens et al '99, BMWc '20], ρ-π-γ EFT (RHO) [Sakurai '60, Jegerlehner et al '11, Chakraborty et al '17], Gounaris-Sakurai inspired model (MLLGS) [GS '68, Lellouch & Lüscher '01, Meyer '11, Francis et al '13], Hansen-Patella (HP) [Hansen et al '19, '29]

L. Our $N_f = 2 + 1 + 1$ calculation has $m_u = m_d$ and $\alpha = 0$

$\Rightarrow$ missing effects compared to HVP from dispersion relations that are relevant at permil-level precision

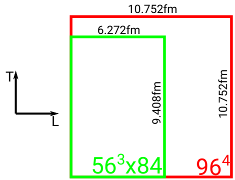
$\rightarrow$ perform lattice calculation of ALL $O(\alpha)$ and $O(\delta m = m_d - m_u)$ effects on ALL quantities computed

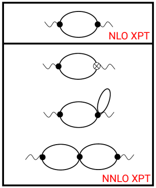
Finite-volume corrections: ad K

Early estimate of these e^{-LM_π} effects [Aubin et al '16]: 2% on $a_\mu^{\text{LO-HVP}}$ in our $L = 6$ simulations

$\rightarrow$ Perform dedicated lattice study

- 4 very-high statistics $N_f = 2 + 1$, super-smeared (4HEX) simulations
- Tuned so that staggered M_π^{HMS} brackets physical M_π
- L up to 11 fm ($a \simeq 0.112$ fm)!





$\rightarrow$ Check w/ EFTs and models: dominated by long-distance $\pi\pi$ effects

- NNLO (2-loop) χPT [Bijnens '99, Aubin et al '19, BMWc '20]
- Meyer-Lellouch-Lüscher formalism w/ Gounaris-Sakurai model (MLLGS) [Lellouch & Lüscher '01, Meyer '11, Francis '13, Giusti et al '18, BMWc '20]
- QFT relation to Compton scattering (HP) [Hansen et al '19-'20]
- ρ-π-γ EFT (RHO) [Sakurai '60, Jegerlehner & Szafron '11, HPQCD '17]

$[\times 10^{-10}]$	lattice	NLO	NNLO	MLLGS	HP	RHO
$a_\mu^{\text{LO-HVP}}(\text{big}) - a_\mu^{\text{LO-HVP}}(\text{ref})$	18.1(2.0)(1.4)	11.6	15.7	17.8	16.7	15.2

Model validation $\Rightarrow$ $a_\mu^{\text{LO-HVP}}(\infty) - a_\mu^{\text{LO-HVP}}(\text{big}) = 0.6(3) \times 10^{-10}$ from NLO & NNLO χPT

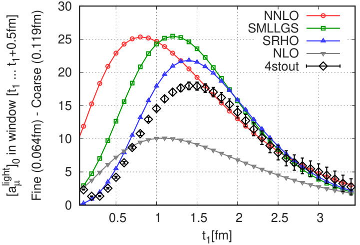
$$a_\mu^{\text{LO-HVP}}(\infty) - a_\mu^{\text{LO-HVP}}(\text{ref}) = 18.7(2.0)_{\text{stat}}(1.4)_{\text{cont}}(0.3)_{\text{big}}(0.6)_{I=0}(0.1)_{\text{qed}}[2.5]$$

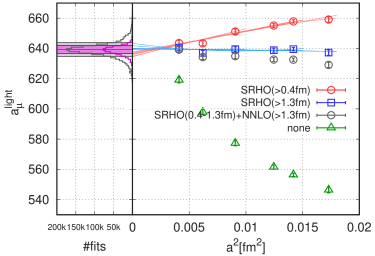
Continuum extrapolation: ad H

Long-distance discretization effects in $a_{\mu,ud}^{\text{LO-HVP}}$ due to taste violations in $\pi\pi$ states [HPQCD '17]

Correct w/ SRHO [HPQCD '17] (consistent w/ SMLLGS [BMWc '20] and NNLO SχPT at larger t)

- Parameters fixed w/ experiment
- Reproduces observed discretization effects well
- Corrections vanish in continuum limit
- 6 a's $\rightarrow$ full control over continuum limit





- Improves approach to continuum limit $\Rightarrow$ reduced uncertainties
- Does NOT modify this limit $\Rightarrow$ NO model dependence of result
- Vary time range of SRHO correction : $t \geq t_{\text{sep}} = 0.4, 0.7, 1.0, 1.3\,\text{fm}$
- Systematics: • cuts on a; • SRHO ($t \geq t_{\text{sep}}$) & none ($t < t_{\text{sep}}$); SRHO or NNLO SχPT ($t \geq 1.3\,\text{fm}$); • different window boundaries

Including isospin breaking on the lattice: ad I

$$S_{\rm QCD+QED} = S_{\rm QCD}^{\rm iso} + \frac{1}{2}\delta m \int (\bar{d}d - \bar{u}u) + ie \int A_\mu J_\mu, \qquad J_\mu = \bar{q}Q\gamma_\mu q, \qquad \delta m = m_d - m_u$$

- Separation into isospin limit results and corrections requires an unambiguous definition of this limit (scheme and scale)
- Must be included not only in calculation of $\langle J_\mu J_\nu \rangle$ correlator **BUT ALSO** of all quantities used to fix quark masses and QCD scale

(1) operator insertion method [RM123 '12, '13, ...]

$$\langle \mathcal{O} \rangle_{\rm QCD+QED} = \langle \mathcal{O}_{\rm Wick} \rangle_{G_\mu}^{\rm iso} - \frac{\delta m}{2} \langle [\mathcal{O} \int (\bar{d}d - \bar{u}u)]_{\rm Wick} \rangle_{G_\mu}^{\rm iso} - \frac{e^2}{2} \langle [\mathcal{O} \int_{xy} J_\mu(x) D_{\mu\nu}(x-y) J_\nu(y)]_{\rm Wick} \rangle_{G_\mu}^{\rm iso}$$
$$+ e^2 \langle \langle \left[\mathcal{O} \partial_e \frac{\det D[G_\mu, eA_\mu]}{\det D[G_\mu, 0]} |_{e=0} \int_x J_\mu(x) A_\mu(x) - \frac{1}{2} \mathcal{O} \partial_e^2 \frac{\det D[G_\mu, eA_\mu]}{\det D[G_\mu, 0]} |_{e=0} \right]_{\rm Wick} \rangle_{A_\mu} \rangle_{G_\mu}^{\rm iso}$$

(2) direct method [Eichten et al '97, BMWc '14, ...]

Include $m_u \neq m_d$ and QED directly in calculation of observables and generation of gauge configurations

(3) combinations of (1) & (2) [BMWc '20]

We include ALL $O(e^2)$ and $O(\delta m)$ effects

For valence e^2 effects use easier (2), and for δm and e^2 sea effects, (1)

Yet more challenges

M. Need permil determination of QCD scale in our simulations

$$\to a_\mu^{\text{LO-HVP}} \sim m_\mu^2 \left(\frac{\Pi'(0)}{a^2}\right)_{\text{lat}} \times a^2 \Rightarrow \frac{\delta a_\mu^{\text{LO-HVP}}}{a_\mu^{\text{LO-HVP}}} \sim 2 \times \frac{\delta a}{a}$$

$\Rightarrow$ 2‰ calculation of Ω^- baryon mass

$\Rightarrow$ Calculate and use Wilson-flow scale [Lüscher '10, BMWc '12] $w_0 = 0.17236(29)(66)$ for defining isospin limit

N. Need thorough and robust determination of statistical and systematic errors

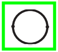
- Stat. err.: resampling methods
- Syst. err.: extended frequentist approach [BMWc '08, '14]
  - Hundreds of thousands of different analyses of correlation functions
  - Weighted by AIC weight

$$\text{AIC} \sim \exp\left[-\frac{1}{2}(\chi^2 + 2n_{\text{par}} - n_{\text{data}})\right]$$

  - Simplify w/ importance sampling
  - Use median of distribution for central values
  - Use $16 \div 84\%$ confidence interval to get total error

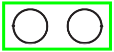
Summary of contributions to $a_\mu^{\text{LO-HVP}}$

Isospin symmetric

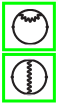

connected light
633.7(2.1)(4.2)

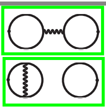

connected strange
53.393(89)(68)


connected charm
14.6(0)(1)


disconnected
-13.36(1.18)(1.36)

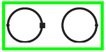
QED isospin-breaking: valence


connected -1.23(40)(31)

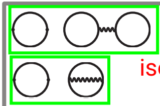

disconnected -0.55(15)(10)

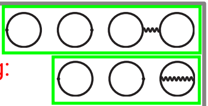
Strong isospin-breaking


connected
6.60(63)(53)


disconnected
-4.67(54)(69)

QED isospin-breaking: sea


connected 0.37(21)(24)

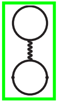

disconnected -0.040(33)(21)

Etc.

bottom; higher order; perturbative

0.11(4)

QED isospin-breaking: mixed


connected -0.0093(86)(95)


disconnected 0.011(24)(14)

Finite-size effects

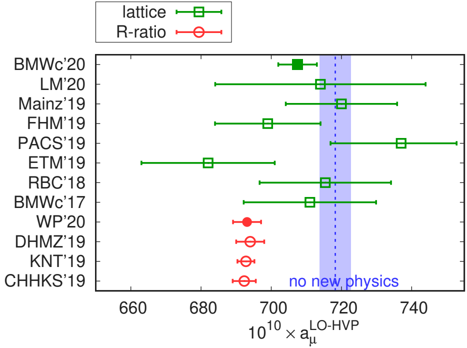
isospin-symmetric
18.7(2.5)

isospin-breaking
0.0(0.1)

$$10^{10}\times a_\mu^{\text{LO-HVP}} = 707.5(2.3)_{\text{stat}}(5.0)_{\text{sys}}[5.5]_{\text{tot}}$$

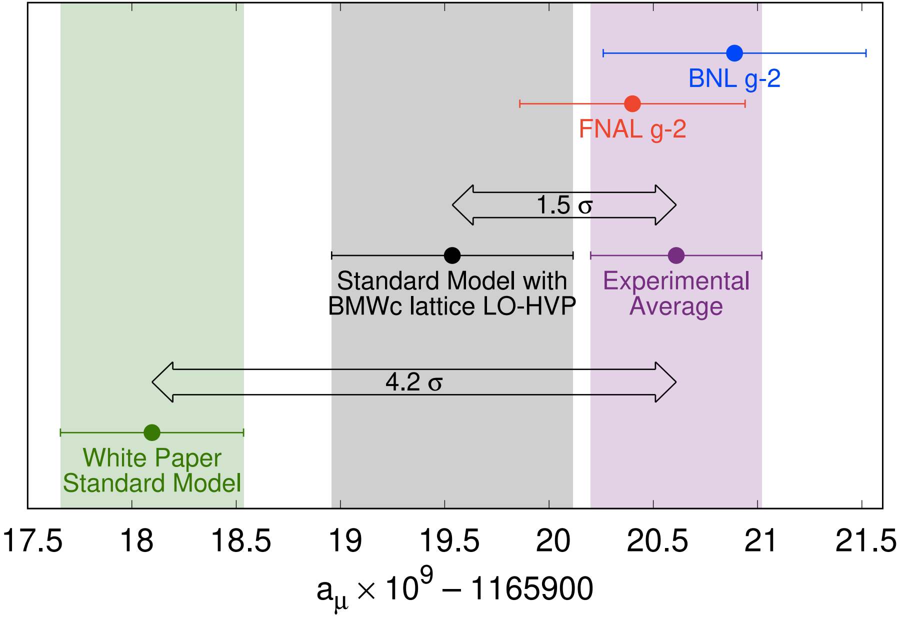
Comparison and outlook

Comparison



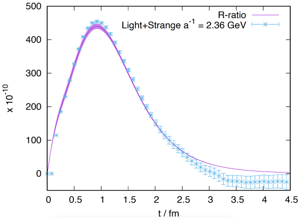
- Consistent with other lattice results
- Total uncertainty is $\sim \div 3$ . . .
- . . . and comparable to R-ratio and experiment
- Consistent w/ experiment @ 1.5σ ("no new physics" scenario) !
- 2.1σ larger than R-ratio average value [WP '20]

Fermilab plot, April 7 2021, BMWc version

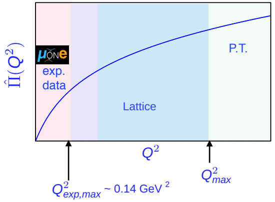


What next?

- FNAL to reduce WA error by factor of 2.5 in coming years
- HLbL error must be reduced by factor of $1.5 \div 2$
- Must reduce ours by factor of 4 !
- Will experiment still agree with our prediction ?
- Must be confirmed by other lattice groups
- If confirmed, must understand why lattice doesn't agree with R-ratio
- If disagreement can be fixed, combine LQCD and phenomenology to improve overall uncertainty [RBC/UKQCD '18]
- Important to pursue $e^+e^- \to$ hadrons measurements [BaBar, CMD-3, Belle III, . . .]
- $\mu e \to \mu e$ experiment MUonE very important for experimental crosscheck and complementarity w/ LQCD
- Important to build J-PARC $g_\mu - 2$ and pursue a_e experiments



[RBC/UKQCD '18]

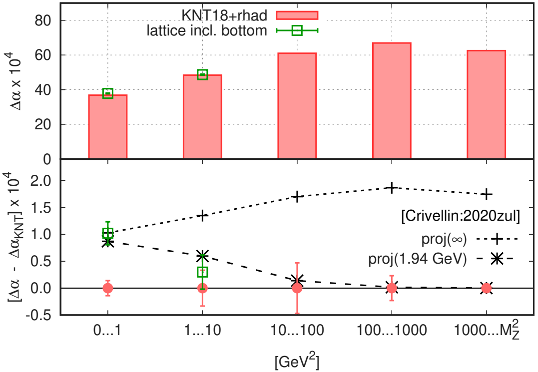


[Marinkovic et al '19]

BACKUP

Do our results imply NP @ EW scale?

- Passera et al '08: first exploration of connection $a_\mu^{\text{LO-HVP}} \leftrightarrow \Delta_{\text{had}}^{(5)}\alpha(M_Z^2)$
- Crivellin et al '20, most aggressive scenario (see also Keshavarzi et al '20, Malaescu et al '20): our results suggest a 4.2σ overshoot in $\Delta_{\text{had}}^{(5)}\alpha(M_Z^2)$ compared to result of fit to EWPO
- Assume 2.8% relative deviation in R-ratio for all s ($\sim$ excess we found in $a_\mu^{\text{LO-HVP}}$)
- Hypothesis is not consistent w/ BMWc '17 nor new result



$a_\mu^{\text{LO-HVP}}$ and $\Delta\alpha_{\text{had}}^{(5)}(M_Z^2)$ vs $\sqrt{s}$

$$a_\mu^{\text{LO-HVP}} = \left(\frac{\alpha m_\mu}{3\pi}\right)^2 \int_{s_{\text{th}}}^{\infty} ds\, \frac{K(s)}{s^2} R(s) \qquad \Delta\alpha_{\text{had}}^{(5)}(M_Z^2) = \frac{\alpha M_Z^2}{3\pi} \int_{s_{\text{th}}}^{\infty} ds\, \frac{R(s)}{s(M_Z^2 - s)}$$