Hadronic Light-by-Light scattering in the anomalous magnetic moment of the muon

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Virtual breakfast with $(g-2)$,
GDR-Intensity Frontier & IJCLab Flavour GT , 19 May 2021

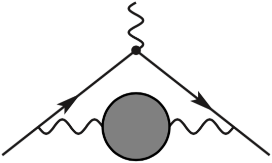
PRISMA Cluster of Excellence

SFB 1044 THE LOW-ENERGY FRONTIER OF THE STANDARD MODEL

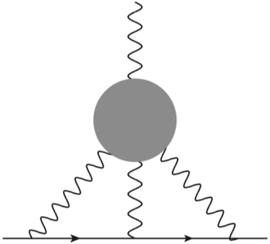
erc European Research Council Established by the European Commission

Source of dominant uncertainties in SM prediction for $(g-2)_\mu$

- After announcement by Fermilab Muon $(g-2)$ experiment and 2020 Theory White Paper: $a_\mu^{\mathrm{exp}} - a_\mu^{\mathrm{SM}} = (251 \pm 59) \cdot 10^{-11}$
- 4.2σ, with practically equal contributions to the error from theory and experiment.
- HVP (O(α^2), about $7000 \cdot 10^{-11}$) target accuracy: $\lesssim 0.5\%$
- HLbL (O(α^3), about $100 \cdot 10^{-11}$) target accuracy: $\lesssim 15\%$.



Hadronic vacuum polarisation (HVP)



Hadronic light-by-light scattering (HLbL)

Approaches to $a_\mu^{\rm HLbL}$

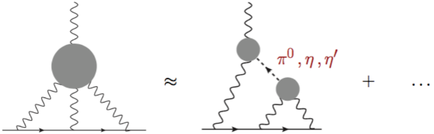
1. **Model calculations:** (the only approach until 2014)
   - based on pole- and loop-contributions of hadron resonances

2. **Dispersive representation:** the Bern approach has been worked out furthest.
   - identify and compute individual contributions
   - determine/constrain the required input (transition form factors, $\gamma^*\gamma^* \to \pi\pi$ amplitudes, ...) dispersively

3. **Experimental program:** provide input for model & dispersive approach, e.g. $(\pi^0, \eta, \eta') \to \gamma\gamma^*$ at virtualities $Q^2 \lesssim 3\,\text{GeV}^2$; active program at BES-III.

4. **Lattice calculations:**
   - RBC-UKQCD T. Blum, N. Christ, M. Hayakawa, T. Izubuchi, L. Jin, Ch. Lehner, ...
   - Mainz N. Asmussen, E.-H. Chao, A. Gérardin, J. Green, J. Hudspith, HM, A. Nyffeler, ...

Models for a_μ^{HLbL}



Contribution	BPP	HKS, HK	KN	MV	BP, MdRR	PdRV	N, JN
π^0, η, η'	85±13	82.7±6.4	83±12	114±10	–	114±13	99 ± 16
axial vectors	2.5±1.0	1.7±1.7	–	22±5	–	15 ± 10	22 ± 5
scalars	−6.8±2.0	–	–	–	–	−7±7	−7±2
π, K loops	−19±13	−4.5±8.1	–	–	–	−19±19	−19±13
π, K loops +subl. N_C	–	–	–	0±10	–	–	–
quark loops	21±3	9.7±11.1	–	–	–	2.3 (c-quark)	21±3
Total	83±32	89.6±15.4	80±40	136±25	110±40	105 ± 26	116 ± 39

BPP = Bijnens, Pallante, Prades '95, '96, '02; HKS = Hayakawa, Kinoshita, Sanda '95, '96; HK = Hayakawa, Kinoshita '98, '02; KN = Knecht, AN '02; MV = Melnikov, Vainshtein '04; BP = Bijnens, Prades '07; MdRR = Miller, de Rafael, Roberts '07; PdRV = Prades, de Rafael, Vainshtein '09; N = AN '09, JN = Jegerlehner, AN '09

Table from A. Nyffeler, PhiPsi 2017 conference

One further estimate: NB. much smaller axial-vector contribution

$$a_\mu^{\text{HLbL}} = (103 \pm 29) \times 10^{-11} \qquad \text{Jegerlehner 1809.07413}$$

Wisdom gained from model calculations Prades, de Rafael, Vainshtein 0901.0306

- heavy (charm) quark loop makes a small contribution

$$a_\mu^{\rm HLbL} = (\frac{\alpha}{\pi})^3 \, N_c \mathcal{Q}_c^4 c_4 \frac{m_\mu^2}{m_c^2} + \ldots, \qquad c_4 \approx 0.62.$$

- Light-quarks: (A) charged pion loop is negative, proportional to m_π^{-2}:

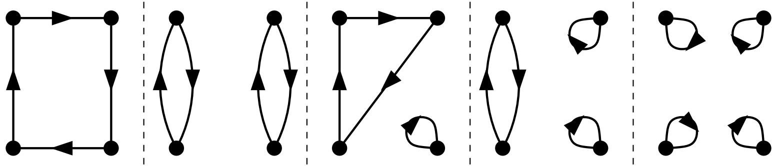
$$a_\mu^{\rm HLbL} = (\frac{\alpha}{\pi})^3 \, c_2 \frac{m_\mu^2}{m_\pi^2} + \ldots, \qquad c_2 \approx -0.065.$$

  (B) The neutral-pion exchange is positive, $\log^2(m_\pi^{-1})$ divergent:
  Knecht, Nyffeler, Perrottet, de Rafael PRL88 (2002) 071802

$$a_\mu^{\rm HLbL} = (\frac{\alpha}{\pi})^3 \, N_c \frac{m_\mu^2}{48\pi^2(F_\pi^2/N_c)} \left[\log^2 \frac{m_\rho}{m_\pi} + {\rm O}\Big(\log \frac{m_\rho}{m_\pi}\Big) + {\rm O}(1)\right].$$

- For real-world quark masses: using form factors for the mesons is essential, and resonances up to 1.5 GeV can still be relevant ⇒ medium-energy QCD.

Quark-line contraction topologies



First two classes of diagrams thought to be dominant, with a cancellation between them:

	Weight factor of:	fully connected	(2,2) topology
SU(2)$_\mathrm{f}$:	isovector-meson exchange	$34/9 \approx 3.78$	$-25/9 \approx -2.78$
$m_s = \infty$	isoscalar-meson exchange	0	1
SU(3)$_\mathrm{f}$:	octet-meson exchange	3	-2
$m_s = m_{ud}$	singlet-meson exchange	0	1

Large-N_c argument by J. Bijnens, 1608.01454; see also 1712.00421; Fig. by J. Green.

Dispersive methods: the Bern approach

Full HLbL tensor:

$$\Pi^{\mu\nu\lambda\sigma}(q_1,q_2,q_3) = i^3 \int_{x,y,z} e^{-i(q_1 x+q_2 y+q_3 z)} \langle 0|T\{j_x^\mu j_y^\nu j_z^\lambda j_0^\sigma\}|0\rangle = \sum_{i=1}^{54} T_i^{\mu\nu\lambda\sigma}\Pi_i,$$

e.g. $T_1^{\mu\nu\lambda\sigma} = \epsilon^{\mu\nu\alpha\beta}\,\epsilon^{\lambda\sigma\gamma\delta}\,q_{1\alpha}\,q_{2\beta}\,q_{3\gamma}\,(q_1+q_2+q_3)_\delta$,
where the 54 structures are really **seven** combined with **crossing symmetry**.

Computing $(g-2)_\mu$ using the projection technique (directly at $q=0$):

$$a_\mu^{\rm HLbL} = -e^6 \int \frac{d^4q_1}{(2\pi)^4}\frac{d^4q_2}{(2\pi)^4} \frac{\sum_{i=1}^{12}\hat{T}_i(q_1,q_2;p)\hat{\Pi}_i(q_1,q_2,-q_1-q_2)}{q_1^2 q_2^2 (q_1+q_2)^2[(p+q_1)^2-m_\mu^2][(p-q_2)^2-m_\mu^2]}$$

with $\hat{\Pi}_i$ linear combinations of the Π_i.

Performing all "kinematic" integrals using Gegenbauer-polynomial technique after Wick rotation:

$$a_\mu^{\rm HLbL} = \frac{2\alpha^3}{48\pi^2}\int_0^\infty dQ_1^4 \int_0^\infty dQ_2^4 \int_{-1}^1 d\tau \sqrt{1-\tau^2}\sum_{i=1}^{12} T_i(Q_1,Q_2,\tau)\bar{\Pi}_i(Q_1,Q_2,\tau)$$

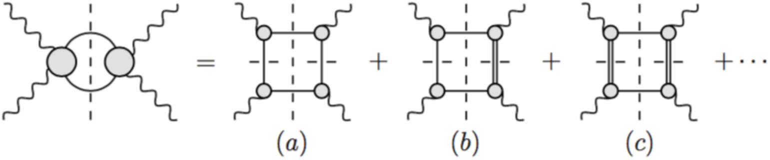
Colangelo, Hoferichter, Procura, Stoffer (2015)

Dispersive methods (II): sample results

- Dispersive analysis of the $\pi^0 \to \gamma^*\gamma^*$ transition form factor leads to $a_\mu^{\pi^0} = 62.6^{+3.0}_{-2.5} \cdot 10^{-11}$ Kubis et al. PRL121, 112002 (2018)

- Charged-pion contributions: Colangelo et al. PRL118, 232001 (2017)

$$a_\mu^{\pi\,\mathrm{box}} + a_{\mu,J=0}^{\pi\pi,\pi-\mathrm{poleLHC}} = -24(1) \cdot 10^{-11}$$



Compilation of White Paper (2006.04822):

$$a_\mu^{\mathrm{HLbL}} = (89 \pm 19) \times 10^{-11}$$

Omitting here the charm-quark contribution. Uncertainty dominated by axial-vector meson exchanges and short-distance contributions.

Direct lattice calculation of HLbL in $(g-2)_\mu$

At first, this was thought of as a QED+QCD calculation
[pioneered in Hayakawa et al., hep-lat/0509016].

Today's viewpoint: the calculation is considered a QCD four-point Green's function, to be integrated over with a weighting kernel which contains all the QED parts.

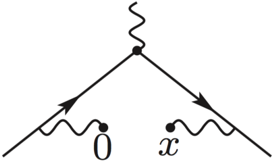
RBC-UKQCD: calculation of $a_\mu^{\rm HLbL}$ using coordinate-space method in muon rest-frame; photon+muon propagators:

- either on the $L \times L \times L$ torus (QED$_L$) (1510.07100–present)
- or in infinite volume (QED$_\infty$) (1705.01067–present).

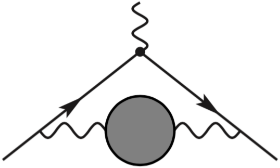
Mainz:

- manifestly covariant QED$_\infty$ coordinate-space approach, averaging over muon momentum using the Gegenbauer polynomial technique (1510.08384–present).

Analogy: hadronic vacuum polarization in x-space HM 1706.01139



$\Rightarrow$



QED kernel $H_{\mu\nu}(x)$

$a_\mu^{\rm hvp}$

$$a_\mu^{\rm hvp} = \int d^4x \; H_{\mu\nu}(x) \; \Big\langle j_\mu(x) j_\nu(0) \Big\rangle_{\rm QCD},$$

$$j_\mu = \frac{2}{3}\bar u\gamma_\mu u - \frac{1}{3}\bar d\gamma_\mu d - \frac{1}{3}\bar s\gamma_\mu s + \dots; \qquad H_{\mu\nu}(x) = -\delta_{\mu\nu}\mathcal{H}_1(|x|) + \frac{x_\mu x_\nu}{x^2}\,\mathcal{H}_2(|x|)$$

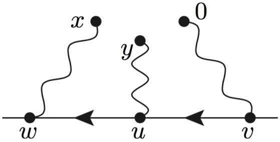
Kernel known in terms of Meijer's functions: $\mathcal{H}_i(|x|) = \frac{8\alpha^2}{3m_\mu^2} f_i(m_\mu|x|)$ with

$$f_2(z) = \frac{G^{2,2}_{2,4}\left(z^2 \Big| \begin{smallmatrix} \frac{7}{2},\, 4 \\ 4,\, 5,\, 1,\, 1 \end{smallmatrix}\right) - G^{2,2}_{2,4}\left(z^2 \Big| \begin{smallmatrix} \frac{7}{2},\, 4 \\ 4,\, 5,\, 0,\, 2 \end{smallmatrix}\right)}{8\sqrt{\pi} z^4},$$

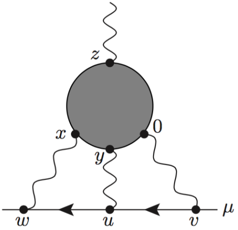
$$f_1(z) = f_2(z) - \frac{3}{16\sqrt{\pi}} \cdot \left[G^{2,3}_{3,5}\left(z^2 \Big| \begin{smallmatrix} 1,\, \frac{3}{2},\, 2 \\ 2,\, 3,\, -2,\, 0,\, 0 \end{smallmatrix}\right) - G^{2,3}_{3,5}\left(z^2 \Big| \begin{smallmatrix} 1,\, \frac{3}{2},\, 2 \\ 2,\, 3,\, -1,\, -1,\, 0 \end{smallmatrix}\right) \right].$$

Coordinate-space approach to $a_\mu^{\rm HLbL}$, Mainz version

QED kernel $\bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y)$



$\Rightarrow$



$$a_\mu^{\rm HLbL} = \frac{me^6}{3} \underbrace{\int d^4y}_{=2\pi^2|y|^3 d|y|} \Big[\int d^4x\, \underbrace{\bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y)}_{\rm QED}\; \underbrace{i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y)}_{\rm =QCD\ blob} \Big].$$

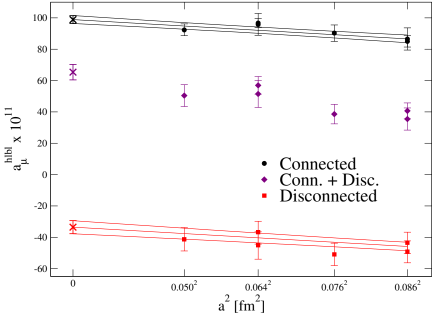
$$i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y) = -\int d^4z\, z_\rho \Big\langle\, j_\mu(x)\, j_\nu(y)\, j_\sigma(z)\, j_\lambda(0) \Big\rangle.$$

- $\bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y)$ computed in the continuum & infinite-volume
- no power-law finite-volume effects & only a 1d integral to sample the integrand in $|y|$.

[Asmussen, Gérardin, Green, HM, Nyffeler 1510.08384, 1609.08454]

$a_\mu^{\rm HLbL}$ at $m_\pi = m_K \simeq 415$ MeV

[Chao, Gérardin, Green, Hudspith, HM 2006.16224 (EPJC)]



$$a_\mu^{\rm hlbl,SU(3)_f} = (65.4 \pm 4.9 \pm 6.6) \times 10^{-11}.$$

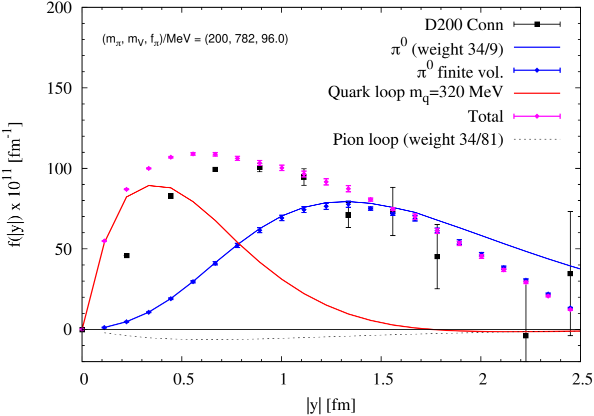
Guesstimating the result at physical quark masses: correct for π^0 exchange

$$a_\mu^{\rm hlbl,SU(3)_f} - a_\mu^{{\rm hlbl},\pi^0,{\rm SU(3)_f}} + a_\mu^{{\rm hlbl},\pi^0,{\rm phys}} = (104.1 \pm 9.1) \times 10^{-11}.$$

Estimate based on lattice QCD calculation of $\pi^0 \to \gamma^*\gamma^*$ transition form factor [Gérardin, HM, Nyffeler 1903.09471 (PRD)].

Connected integrand for $a_\mu^{\rm HLbL} = \int_0^\infty d|y|\, f(|y|)$ vs. hadronic models

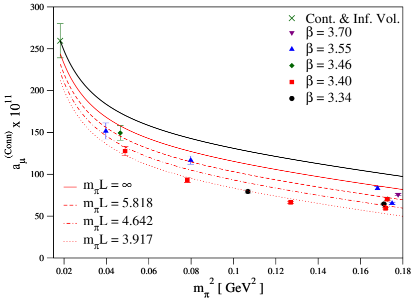
$m_\pi = 200$ MeV, $m_K = 480$ MeV: $(64^3 \times 128$ lattice, $a = 0.064$ fm$)$



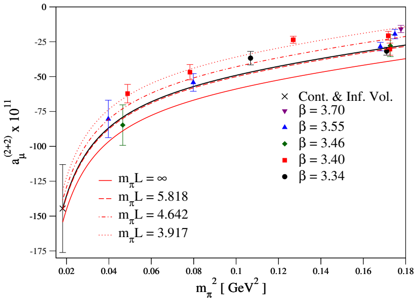
En-Hung Chao, Renwick Hudspith, Antoine Gérardin, Jeremy Green, HM, Konstantin Ottnad
2104.02632

Chiral, continuum, volume extrapolation

Connected contribution

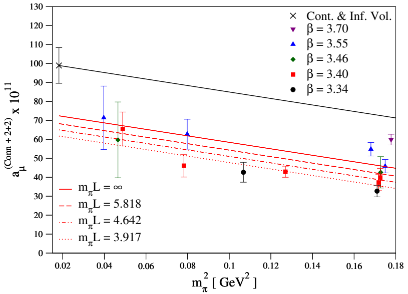


disconnected contribution



Total light-quark contribution:

- vol. dependence: $\propto \exp(-m_\pi L/2)$
- pion-mass dependence fairly mild (!)



Overview table

Contribution	Value$\times 10^{11}$
Light-quark fully-connected and $(2+2)$	107.4(11.3)(9.2)
Strange-quark fully-connected and $(2+2)$	$-0.6(2.0)$
$(3+1)$	0.0(0.6)
$(2+1+1)$	0.0(0.3)
$(1+1+1+1)$	0.0(0.1)
Total	106.8(14.7)

- error dominated by the statistical error and the continuum limit.
- all subleading contributions have been tightly constrained and shown to be negligible.

[Chao et al, 2104.02632]

Method based on QED$_L$ pursued by RBC/UKQCD collaboration

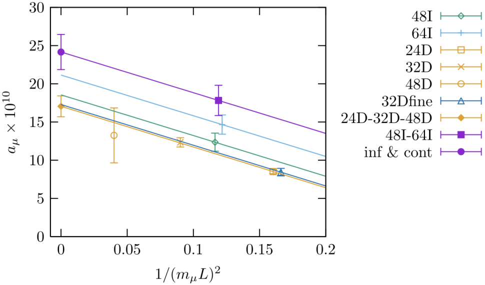
- Photon and muon propagators computed with lattice action.
- Photon $\boldsymbol{q} = 0$ spatial zero-mode removed 'by hand'.
- magnetic moment computed with formula of the type $\boldsymbol{\mu} = \frac{1}{2}\int d^3r\, \boldsymbol{r} \times \boldsymbol{j}$.
- Use domain-wall fermions practically at physical quark masses.
- Gluons: use gauge ensembles with two different actions
- Largest spatial lattice used: 64^3
- Extrapolation of the type

$$a_\mu^{\rm HLbL}(L,a) = a_\mu\Big(1 - \frac{b_2}{(m_\mu L)^2} + \frac{b_3}{(m_\mu L)^3}\Big)\Big(1 - c_1(m_\mu a)^2 + c_2(m_\mu a)^4\Big).$$

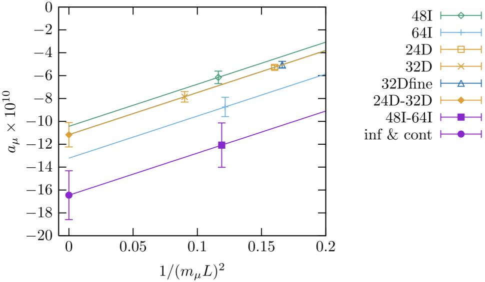
[Blum et al. 1911.08123 (PRL)]

RBC/UKQCD (QED$_L$): final extrapolation [Blum et al. 1911.08123 (PRL)]

Connected $\longrightarrow$

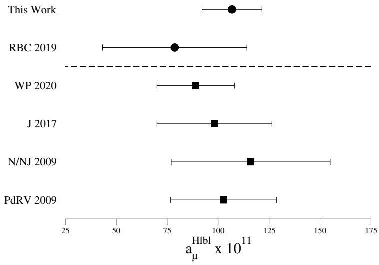


Disconnected $\longrightarrow$



$$\text{Total}: \quad a_\mu^{\text{HLbL}} = (78.7 \pm (30.6)_{\text{stat}} \pm (17.7)_{\text{syst}}) \cdot 10^{-11}.$$

Conclusion on $a_\mu^{\rm HLbL}$



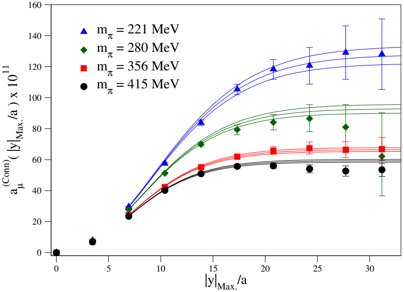
[Fig. from 2104.02632]

- Results from the Bern dispersive framework and from two independent lattice QCD calculations are in good agreement and have comparable uncertainties.
- It is now practically excluded that $a_\mu^{\rm HLbL}$ can by itself explain the tension between the SM prediction and the experimental value of a_μ.
- Epilogue: $a_\mu^{\rm HLbL}$ is a tale of many cancellations, both between the exchange of different mesons and also between Wick-contraction topologies in lattice QCD.

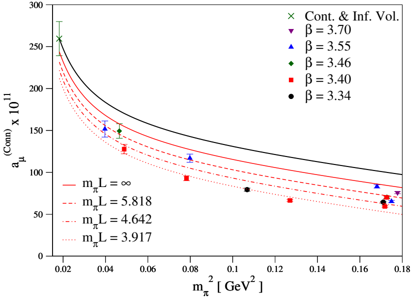
Backup Slides

The connected contribution

Cumulated $a_\mu^{\rm HLbL} = \int_0^{|y|_{\max}} d|y|\, f(|y|)$

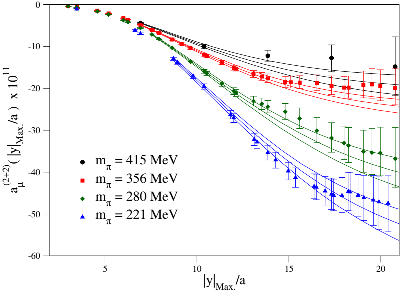


Chiral, continuum, vol. extrapolation

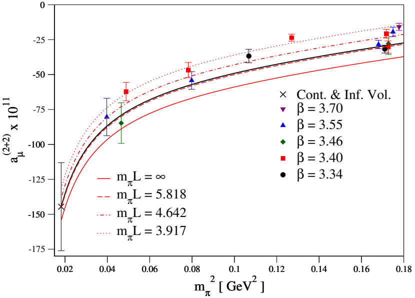


The disconnected contribution

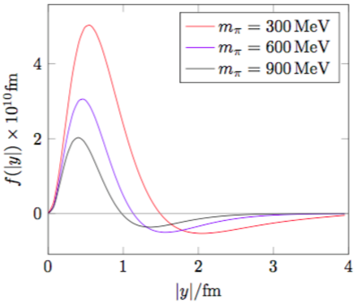
Cumulated $a_\mu^{\rm HLbL} = \int_0^{|y|_{\max}} d|y|\, f(|y|)$



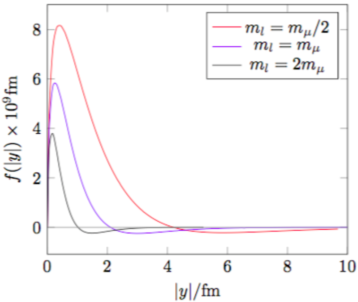
Chiral, continuum, vol. extrapolation



Continuum tests: contribution of the π^0 and lepton loop to $a_\mu^{\rm HLbL}$



Integrand of the pion-pole contribution with VMD transition form factor.

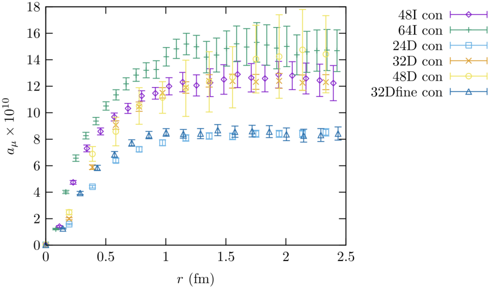


Integrand of the lepton-loop contribution.

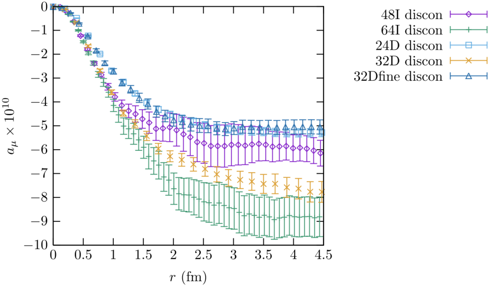
- Even more freedom in choosing best lattice implementation than in HVP.
- The form of the $|y|$-integrand depends on the precise QED kernel used: can perform subtractions (Blum et al. 1705.01067; $\mathcal{L} \to \mathcal{L}^{(2)}$), impose Bose symmetries on $\bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y)$ or add a longitudinal piece $\partial_\mu^{(x)} f_{\rho;\nu\lambda\sigma}(x,y)$.

RBC/UKQCD (QED$_L$): cumulative contributions to a_μ^{HLbL}

Connected $\longrightarrow$



Disconnected $\longrightarrow$



[Blum et al. 1911.08123 (PRL)]

Explicit form of the QED kernel

$$\bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y) = \sum_{A=\mathrm{I,II,III}} \mathcal{G}^{A}_{\delta[\rho\sigma]\mu\alpha\nu\beta\lambda} T^{(A)}_{\alpha\beta\delta}(x,y),$$

with e.g.

$$\mathcal{G}^{\mathrm{I}}_{\delta[\rho\sigma]\mu\alpha\nu\beta\lambda} \equiv \frac{1}{8}\mathrm{Tr}\Big\{\Big(\gamma_\delta[\gamma_\rho,\gamma_\sigma] + 2(\delta_{\delta\sigma}\gamma_\rho - \delta_{\delta\rho}\gamma_\sigma)\Big)\gamma_\mu\gamma_\alpha\gamma_\nu\gamma_\beta\gamma_\lambda\Big\},$$

$$\begin{aligned}
T^{(\mathrm{I})}_{\alpha\beta\delta}(x,y) &= \partial^{(x)}_\alpha(\partial^{(x)}_\beta + \partial^{(y)}_\beta)V_\delta(x,y),\\
T^{(\mathrm{II})}_{\alpha\beta\delta}(x,y) &= m\partial^{(x)}_\alpha\Big(T_{\beta\delta}(x,y) + \frac{1}{4}\delta_{\beta\delta}S(x,y)\Big)\\
T^{(\mathrm{III})}_{\alpha\beta\delta}(x,y) &= m(\partial^{(x)}_\beta + \partial^{(y)}_\beta)\Big(T_{\alpha\delta}(x,y) + \frac{1}{4}\delta_{\alpha\delta}S(x,y)\Big),
\end{aligned}$$

$$\begin{aligned}
S(x,y) &= \int_u G_{m_\gamma}(u-y)\Big\langle J(\hat\epsilon,u)J(\hat\epsilon,x-u)\Big\rangle_{\hat\epsilon}, \quad J(\hat\epsilon,y) \equiv \int_u G_0(y-u)\, e^{m\hat\epsilon\cdot u} G_m(u)\\
V_\delta(x,y) &= x_\delta \bar{\mathfrak{g}}^{(1)}(|x|,\hat x\cdot\hat y,|y|) + y_\delta \bar{\mathfrak{g}}^{(2)}(|x|,\hat x\cdot\hat y,|y|),\\
T_{\alpha\beta}(x,y) &= (x_\alpha x_\beta - \frac{x^2}{4}\delta_{\alpha\beta})\,\bar{\mathfrak{l}}^{(1)} + (y_\alpha y_\beta - \frac{y^2}{4}\delta_{\alpha\beta})\,\bar{\mathfrak{l}}^{(2)} + (x_\alpha y_\beta + y_\alpha x_\beta - \frac{x\cdot y}{2}\delta_{\alpha\beta})\,\bar{\mathfrak{l}}^{(3)}.
\end{aligned}$$

The QED kernel $\bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y)$ is parametrized by six weight functions.