Evolving GPDs in x space: a new path through Apfel

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Brief introduction on GPDs and their evolution

- Generalised Parton Distributions (GPDs):

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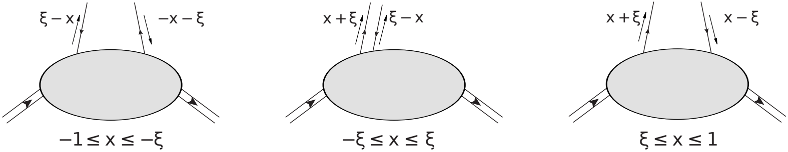
  - "hadron-parton" amplitudes which depend on three variables (x, ξ, t) and a scale μ,



- ★ x: average momentum fraction carried by the active parton
- ★ ξ: skewness parameter $\xi \simeq \frac{x_B}{2-x_B}$
- ★ t: the Mandelstam variable

Generalised Parton Distributions

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  - "hadron-parton" amplitudes which depend on three variables (x, ξ, t) and a scale μ,
  - are defined in terms of a non-local matrix element,

$$\frac{1}{2}\int \frac{e^{ixP^+z^-}}{2\pi}\langle P+\frac{\Delta}{2}|\bar{\psi}^q(-\frac{z}{2})\gamma^+\psi^q(\frac{z}{2})|P-\frac{\Delta}{2}\rangle \mathrm{d}z^-|_{z^+=0,\mathbf{z}=0}$$
$$= \frac{1}{2P^+}\left[H^q(x,\xi,t)\bar{u}\gamma^+u + E^q(x,\xi,t)\bar{u}\frac{i\sigma^{+\alpha}\Delta_\alpha}{2M}u\right].$$

$$\frac{1}{2}\int \frac{e^{ixP^+z^-}}{2\pi}\langle P+\frac{\Delta}{2}|\bar{\psi}^q(-\frac{z}{2})\gamma^+\gamma_5\psi^q(\frac{z}{2})|P-\frac{\Delta}{2}\rangle \mathrm{d}z^-|_{z^+=0,\mathbf{z}=0}$$
$$= \frac{1}{2P^+}\left[\tilde{H}^q(x,\xi,t)\bar{u}\gamma^+\gamma_5 u + \tilde{E}^q(x,\xi,t)\bar{u}\frac{\gamma_5\Delta^+}{2M}u\right].$$

D. Müller *et al.*, Fortsch. Phy. 42 101 (1994)
X. Ji, Phys. Rev. Lett. **78**, 610 (1997)
A. Radyushkin, Phys. Lett. **B380**, 417 (1996)

4 GPDs without helicity transfer + 4 helicity flip GPDs

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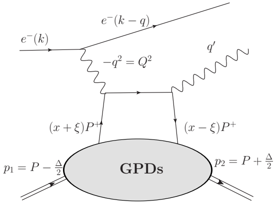
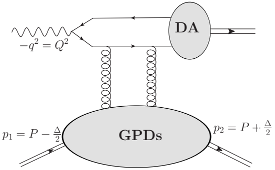
  - are related to PDFs in the forward limit $H(x, \xi = 0, t = 0; \mu) = q(x; \mu)$
  - are universal, *i.e.* are related to the Compton Form Factors (CFFs) of various exclusive processes through convolutions

$$\mathcal{H}(\xi, t) = \int \mathrm{d}x \; C(x, \xi) H(x, \xi, t)$$





Properties

- Polynomiality Property:

$$\int_{-1}^{1} \mathrm{d}x\, x^m H^q(x,\xi,t;\mu) = \sum_{j=0}^{\left[\frac{m}{2}\right]} \xi^{2j} C_{2j}^q(t;\mu) + mod(m,2)\xi^{m+1} C_{m+1}^q(t;\mu)$$

X. Ji, J.Phys.G 24 (1998) 1181-1205
A. Radyushkin, Phys.Lett.B 449 (1999) 81-88

Special case :

$$\int_{-1}^{1} \mathrm{d}x\; H^q(x,\xi,t;\mu) = F_1(t)$$

Lorentz Covariance

Properties

- Polynomiality Property:

Lorentz Covariance

- Positivity property:

$$\left|H^q(x,\xi,t) - \frac{\xi^2}{1-\xi^2}E^q(x,\xi,t)\right| \leq \sqrt{\frac{q\left(\frac{x+\xi}{1+\xi}\right) q\left(\frac{x-\xi}{1-\xi}\right)}{1-\xi^2}}$$

A. Radysuhkin, Phys. Rev. **D59**, 014030 (1999)
B. Pire *et al.*, Eur. Phys. J. **C8**, 103 (1999)
M. Diehl *et al.*, Nucl. Phys. **B596**, 33 (2001)
P.V. Pobilitsa, Phys. Rev. **D65**, 114015 (2002)

Positivity of Hilbert space norm

Properties

- Polynomiality Property:

Lorentz Covariance

- Positivity property:

Positivity of Hilbert space norm

- Support property:

$$x \in [-1; 1]$$

M. Diehl and T. Gousset, Phys. Lett. **B428**, 359 (1998)

Relativistic quantum mechanics

Properties

- Polynomiality Property:

  Lorentz Covariance

- Positivity property:

  Positivity of Hilbert space norm

- Support property:

  Relativistic quantum mechanics

- Scale evolution property
  $\rightarrow$ generalisation of DGLAP and ERBL evolution equations

  D. Müller *et al.*, Fortschr. Phys. 42, 101 (1994)

  Renormalization

Requirement

An evolution code must conserve all these properties

GPD evolution: Status

- Important effort in the late 1990s and the 2000s
  - Evolution kernels derived for all GPD types at LO by different groups independently and at NLO by Belitsky and Müller

    see *e.g.* A. Belitsky and D. Müller, Phys.Lett., 1999, B464, 249-256
  - Efforts on numerical implementation (one public code by Vinnikov) and assessment of impact

    A. Freund and M. McDermott, Phys.Rev.D 65 (2002) 056012
    A. Vinnikov, hep-ph/0604248
    M. Diehl and W. Kugler, Phys.Lett.B 660 (2008) 202-211
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- GPD evolution left the front stage in the following years
  - JLab data have been obtained in a very narrow Q^2 region
- Renewing interest with the forthcoming EIC facility
  - new derivation of higher order kernels

V. Braun *et al.*, JHEP, 2019, 02, 191
V. Braun *et al.*, JHEP, 2017, 06, 037

  - fast evolution code required for EIC impact studies

Technological choices

Technological choices I

x-space vs Conformal space

- Momentum space evolution

$$\mu^2 \frac{\partial H}{\partial \mu^2}(x,\xi,\mu) = \int \mathrm{d}y \underbrace{V(x,y,\xi)}_{pQCD} H(y,\xi,\mu)$$

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- Conformal space evolution (non-singlet case)

$$\mathcal{C}_n(\xi,\mu^2) = \xi^n \int_{-1}^{1} \mathrm{d}x \; C_n^{(3/2)}\left(\frac{x}{\xi}\right) H(x,\xi,\mu^2)$$

$$\mu^2 \frac{\partial \mathcal{C}_n}{\partial \mu^2}(\xi,\mu) = \sum_i M_{n,i}\mathcal{C}_i(\xi,\mu)$$

  - M diagonal at LO but not at NLO in general
  - Reconstruction from the conformal moments (Mellin-Barnes transform)
    D. Müller and A. Schäfer, Nucl.Phys.B 739 (2006)

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Choice: momentum space

- Model available in PARTONS mostly in x-space
- Reusing technology available from the PDF community

Technological choices II

Modular architecture

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Rely on existing PDF codes

- Less redesigning efforts
- Reuse existing features
- Facilitate parts of the code validation

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We turned ourselves to Apfel++ and extended it for GPDs

Implementation and Validation

Splitting functions

$$\frac{\partial H^{(-)}}{\partial \ln \mu^2}(x,\xi,\mu) = \frac{\alpha_s(\mu)}{4\pi}\int_x^\infty \frac{\mathrm{d}y}{y}\mathcal{P}^{-,(0)}\left(\frac{x}{y},\kappa\right)H^{(-)}(y,\xi,\mu), \quad \kappa = \frac{\xi}{x}$$

$$\mathcal{P}^{-,(0)}(y,\kappa) = \underbrace{\Theta(1-y)\mathcal{P}_1^{-,(0)}(y;\kappa)}_{\text{Generalised DGLAP kernel}} + \underbrace{\Theta(\kappa-1)\mathcal{P}_2^{-,(0)}(y;\kappa)}_{\text{Genuine ERBL contribution}}$$

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- Numerical implementation needs to be carefully done due to spurious divergencies:

$$\lim_{y\to 1/\kappa}(1-\kappa^2y^2)\mathcal{P}_1^{-,(0)}(y;\kappa) = -2C_F\frac{1+\kappa}{\kappa}$$

$$\lim_{y\to 1/\kappa}(1-\kappa^2y^2)\mathcal{P}_2^{-,(0)}(y;\kappa) = 2C_F\frac{1+\kappa}{\kappa}$$

Forward limit and DGLAP equations

- When $\kappa \to 0$, one recovers the forward DGLAP equations
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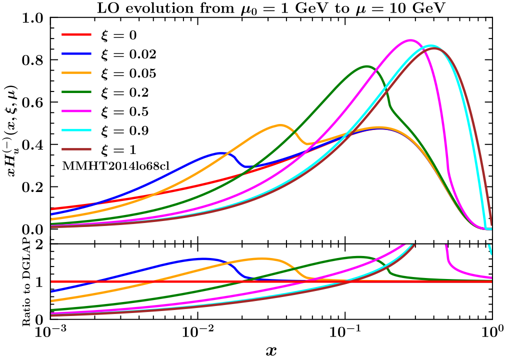

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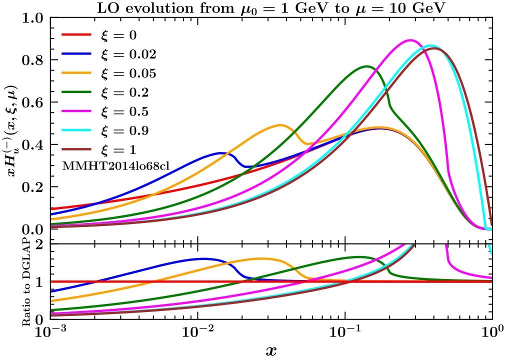

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First validations

$x = \xi$ continuity, DLGAP limit and spurious divergences handling

ERBL Limit and Gegenbauer polynomials

- When $\kappa \to 1/x$ (*i.e.* $\xi \to 1$) one recovers the ERBL kernel
  - Eigen basis known $\to$ 3/2-Gegenbauer Polynomials
  - Direct (albeit restricted to $\xi = 1$) comparison between x-space and conformal space evolution

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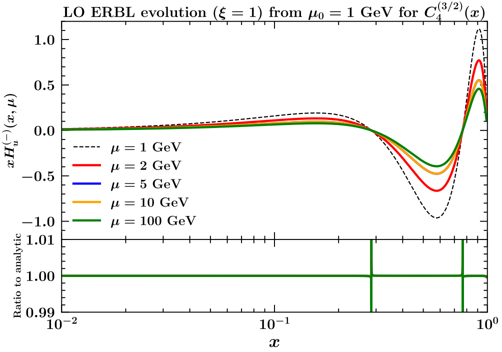

- $H(x,1) \propto (1-x^2)C_4^{(3/2)}(x)$
- $\frac{H(x,1,\mu)}{H(x,1,\mu_0)} = \left(\frac{\alpha_S(\mu)}{\alpha_s(\mu_0)}\right)^{\gamma_4/\beta_0}$
- Ratio is independent of x
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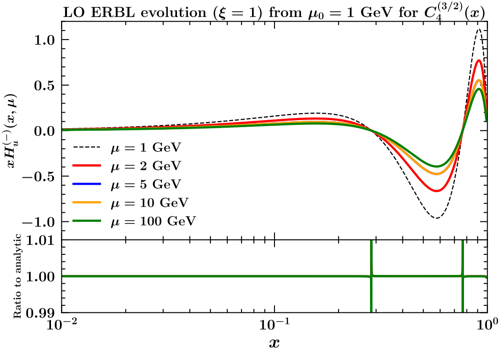

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Additional validations

conformal evolution when $\xi \to 1$ guaranteeing the ERBL limit

Polynomiality

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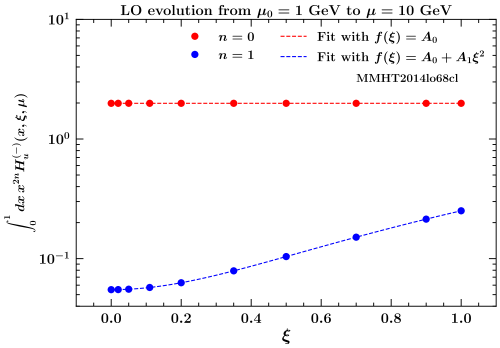

- $n = 0$ case $\to$ moment is ξ and μ independent (proportional to $\mathcal{C}_0$)
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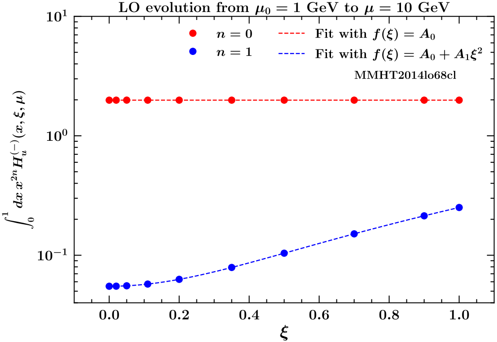

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Polynomiality and evolution

Apfel++ seems to conserve polynomiality as expected

Ongoing work

- Pursue the validation with a ξ-dependent model
  $\rightarrow$ Radyushkin Double Distribution Ansatz (RDDA)

A. Mukherjee *et al.*, PRD 67, 073014 (2003)

$$H_{\mathbf{DD}}(x,\xi) = \int_{-1}^{1} \mathrm{d}\beta \int_{-1+|\beta|}^{1-|\beta|} \mathrm{d}\alpha\delta(x-\beta-\alpha\xi)\left(f(\beta,\alpha)+\xi\delta(\beta)D(\alpha)\right)$$
$$f(\beta,\alpha) = q(\beta)\pi(\beta,\alpha)$$

- Systematic comparisons with the Vinnikov code

A. Vinnikov, hep-ph/0604248

- Guidance from previous papers using also the RDDA.

M. Diehl and W. Kugler, Phys.Lett.B 660 (2008) 202-211

Comparison with conformal evolution

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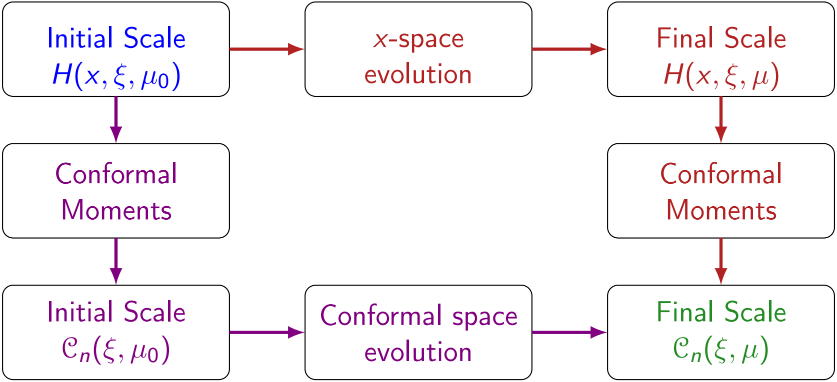

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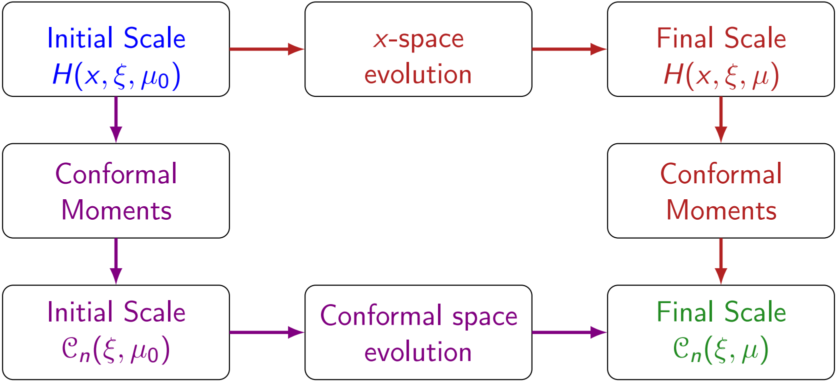

Conformal and x-space agreement would be a strong validation !

Conclusion

Summary

- New open-source code implementing GPD LO evolution
- Thought to be PARTONS-compatible (modularity)
- Validation in progress and very encouraging for now

Perspective

- Complete validation and proceed with joined PARTONS-Apfel++ release
- Splitting functions for polarised GPDs need to be implemented
- NLO evolution desirable in the forthcoming years (EIC studies)
  $\rightarrow$ code architecture should be flexible enough

Thank you for your attention

Back up slides