High-order diagrammatic expansion around BCS

Polarized superfluid phase of the attractive Hubbard model

Gabriele **Spada**
ENS → Trento

Kris **Van Houcke &** Félix **Werner**
ENS

Riccardo **Rossi**
Flatiron Institute → EPFL

Fedor **Simkovic,** Renaud **Garioud &** Michel **Ferrero**
Collège de France & École Polytechnique

[arXiv:2103.12038]

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"Quantum 2021"

- superconductivity: Kamerlingh Onnes, 1911
- BCS theory: 1957
- 2021: BCS theory → exact

[arXiv:2103.12038]

Unbiased approaches for fermionic $\mathcal{N}$-body problems

strongly interacting fermions: *solids & molecules, nuclei & neutron stars, QCD*

$\underline{\textit{Challenge:}}$ $\quad$ large $\mathcal{N}$, $\quad$ thermo. lim. $\mathcal{N} \to \infty$

Unbiased approaches for fermionic $\mathcal{N}$-body problems

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Challenge: large $\mathcal{N}$, thermo. lim. $\mathcal{N} \to \infty$

analytics

1D, small $\mathcal{N}$, weakly correlated limits

Tensor network

1D: 🙂
2D: harder (bond dimension $\to \infty$)
3D: ?

continuous space: ?

“bulk” Monte Carlo approaches

quantum
d dim.
volume L^d

→

“classical”
$d+1$ dim.
“volume” $L^d \times \beta$

- path integral $\mathbf{r}_1(\tau), \ldots, \mathbf{r}_{\mathcal{N}}(\tau)$
- **A**uxiliary **F**ield **QMC** *(lattice QCD)*
- **D**eterminental **D**iagrammatic **MC** *(CT-INT)*

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except in special cases

fermion sign problem: $\quad t_{\mathrm{CPU}} \sim e^{\#\beta\mathcal{N}}$

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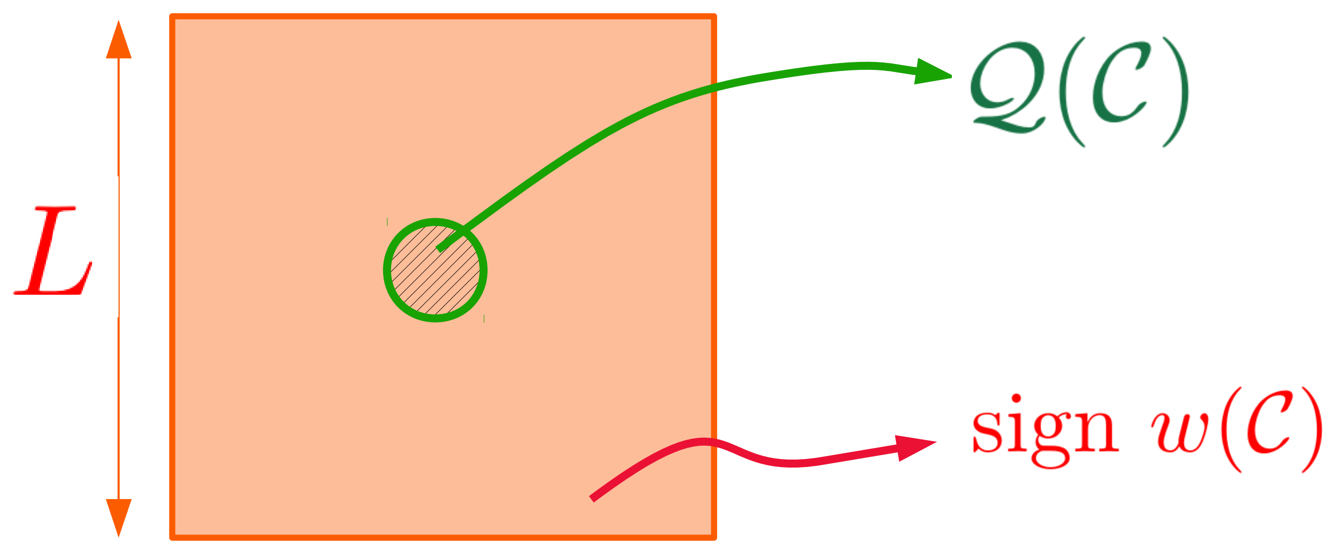

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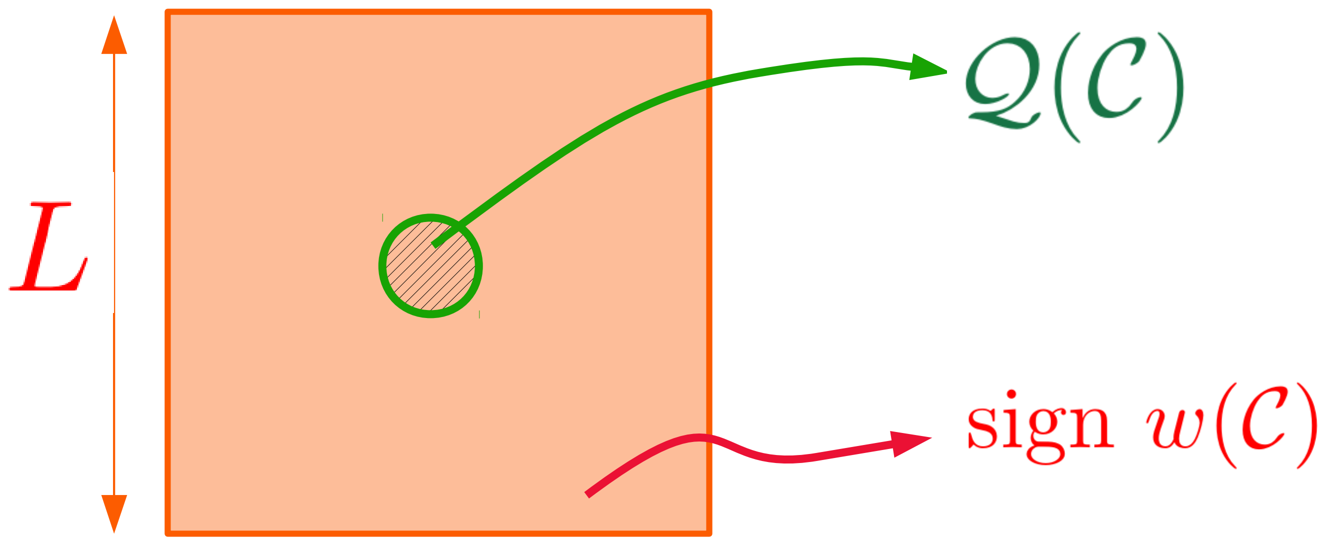

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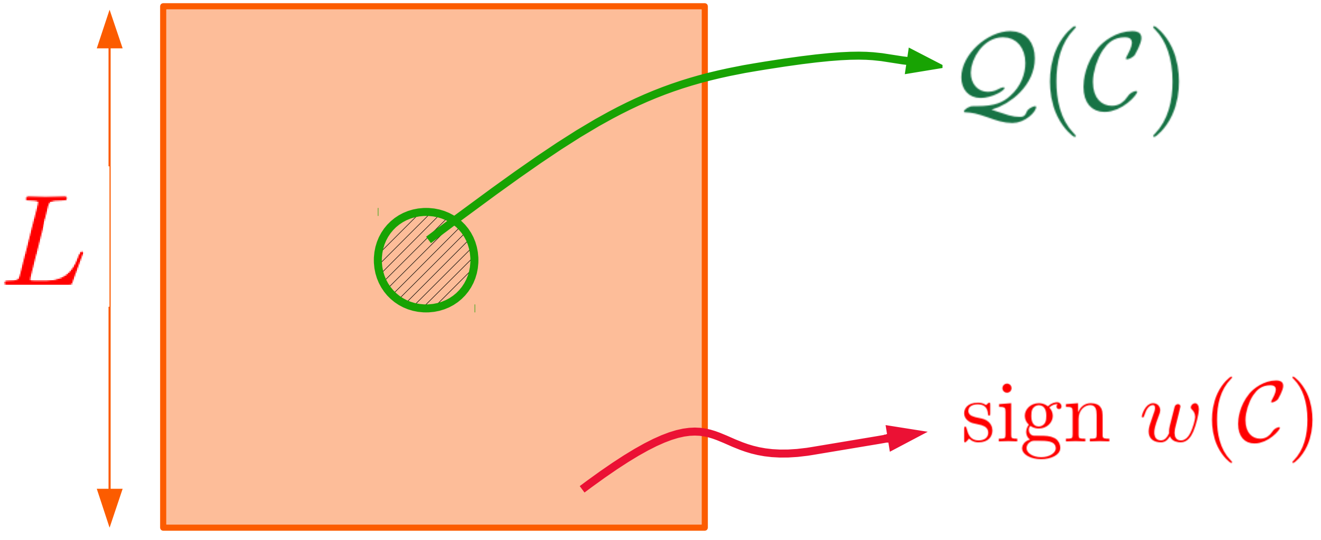

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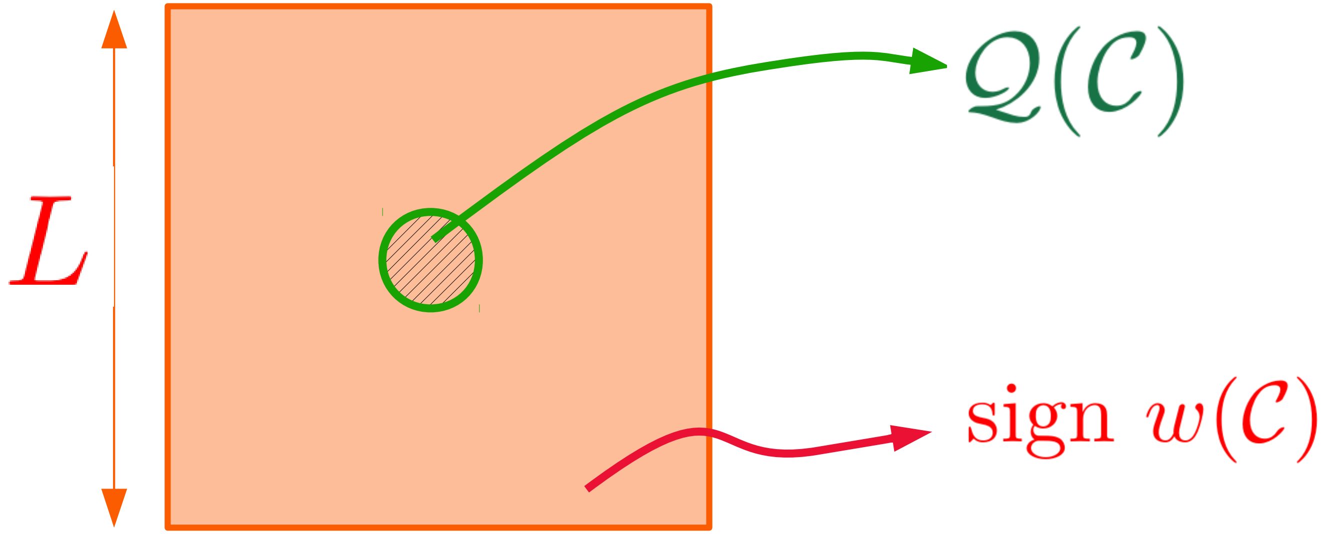

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3D XY-model, $T < T_c$.

$$\langle \vec{S}(\vec{r}) \rangle = \vec{0},$$

$$\lim_{r\to\infty} \left(\lim_{L\to\infty} \right) \langle \vec{S}(\vec{r}) \cdot \vec{S}(\vec{0}) \rangle = M^2$$

$\Rightarrow$ $\vec{S}(\vec{r})$ and $\vec{S}(\vec{0})$ CORRELATED

Broken-symmetry point of view:

external field $\eta \cdot \vec{u}_x$

$$\langle\langle \quad \rangle\rangle := \lim_{\eta\to 0} \lim_{L\to\infty} \langle \quad \rangle$$

$$\langle\langle \vec{S}(\vec{r}) \rangle\rangle = M \cdot \vec{u}_x$$

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$\Rightarrow$ UNCORRELATED

Diagrammatic MC with ***connected diagrams***

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***flexibility* of diagrammatic technique**:

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- reorganize expansion (use *dressed* propagators / vertices)
  $\Rightarrow$ *non-perturbative*

$\Rightarrow$ ***low orders already ~ OK***

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fermionic sign helps:

strong cancellations between diagrams

$\Rightarrow$ ***large orders contributions reduced***

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$$S \rightsquigarrow S_\xi \quad \text{such that} \quad \begin{cases} S_{\xi=0} & \text{quadratic} \\ S_{\xi=1} = S & \\ \xi \mapsto S_\xi & \text{analytic} \end{cases}$$

$$Q = \langle \hat{Q} \rangle_S \rightsquigarrow Q(\xi) = \langle \hat{Q} \rangle_{S_\xi} = \sum_{N=0}^{\infty} Q_N \, \xi^N$$

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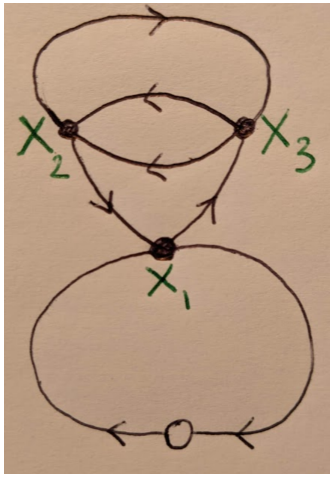

sum of all order-N ***connected*** *Feynman diag.*

well-defined in thermo. limit $\mathcal{N} = \infty$

$$Q_N = \sum_{\substack{\text{connected} \\ \text{topologies } \mathcal{T}}} \int dX_1 \dots dX_N \, \underbrace{\mathcal{D}(\mathcal{T}; X_1 \dots X_N)}_{\xrightarrow[|X_i| \to \infty]{} 0}$$

$$X := (\vec{r}, \tau)$$

$$\int dX := \sum_{\vec{r}} \int_0^\beta d\tau$$

Freedom in choosing S_0

- Mean-field • DMFT • Fully dressed G ("bold")
- add field (bosonic): sum ladders (*pp* or *ph*)

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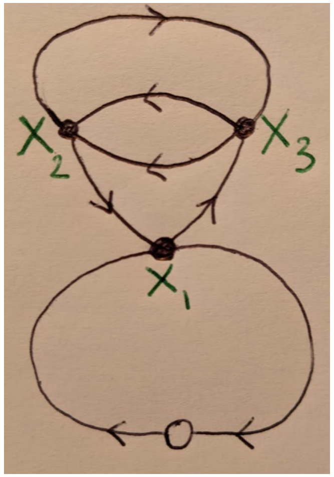

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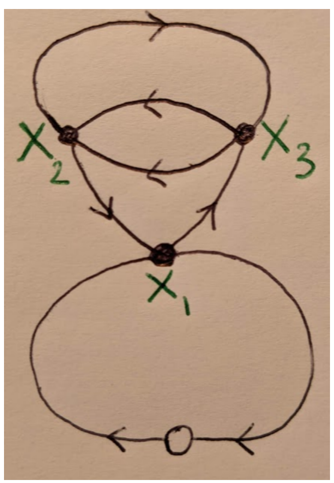
- *symmetry breaking*



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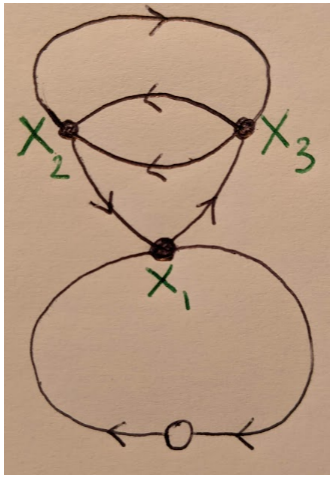

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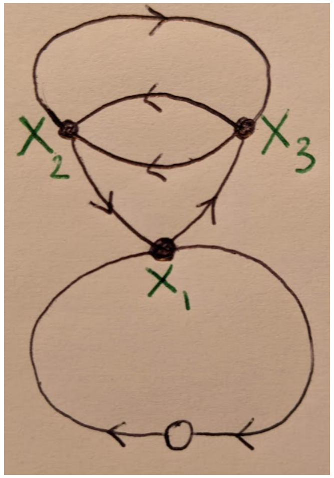

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Monte Carlo algorithms

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  configuration: $\mathcal{C} = (\mathcal{T}, X_1, \dots, X_N)$ probability: $P(\mathcal{C}) \propto |\mathcal{D}(\mathcal{T}; X_1 \dots X_N)|$

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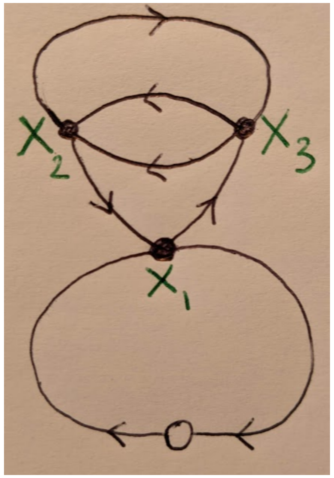

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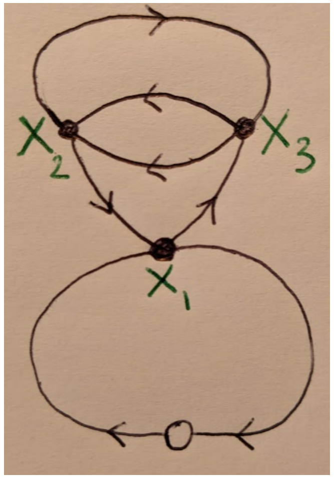

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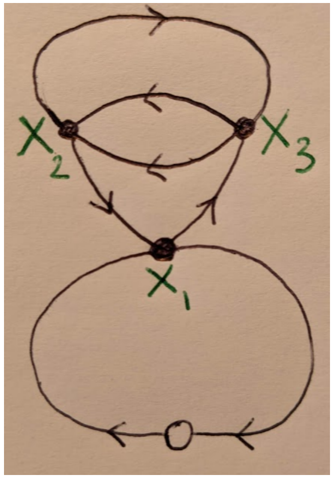
Real-time (Anderson impurity):

- "Keldysh Det" [Profumo *et al.* 2015]

see talk by
X. Waintal



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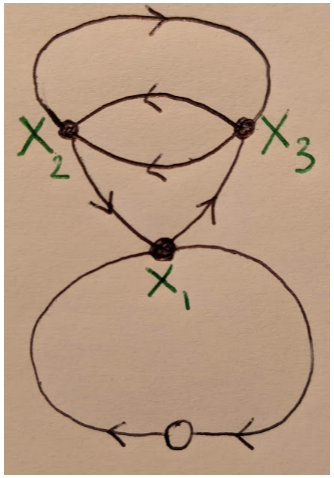

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Computational complexity

- $t_{\text{CPU}} \sim \begin{cases} (N!)^2 & \text{(DiagMC)} \\ e^{\# N} & \text{(CDet)} \end{cases}$

- counteracted by convergence: $|a_N| \sim e^{-\# N}$

$$\Rightarrow \quad t(\epsilon) \sim \epsilon^{-\# \ln(\ln \epsilon^{-1})} \quad \text{(DiagMC)}, \qquad t(\epsilon) \sim \epsilon^{-\alpha} \quad \text{(CDet)}.$$

where ϵ = total error (statistical + truncation)

[Rossi *et al.*, *EPL* 2017]

a hunting board of diagrammatic MC...

repulsive Hubbard model 2D square lattice
supercond. phase diagram (d- & p-wave) $T \to 0$ in Fermi-liq regime [Deng *et al.* 2015]
crossover to AF insulator at half-filling [Simkovic *et al.* 2020 ; Kim *et al.* 2020]
diagonal hopping → *pseudogap physics* [Wu *et al.* 2017 ; Rossi *et al.* 2020]

graphene *Dirac liquid* T=0 [Tupitsyn & Prokof'ev 2017]

phonons+electrons with Coulomb interactions 3D cubic lattice
phonon spectrum (Kohn anomaly) [Tupitsyn *et al.* 2016]

unitary Fermi gas contact interactions, 3D c° space (cold atoms)
eq. of state, contact, non-Fermi-liq n(k) [Van Houcke *et al.* 2012 ; Rossi *et al.* 2018]

electron gas Coulomb interactions, 3D c° space
static response, Fermi-liq params [Chen & Haule 2019 / 2020]

frustrated spins AF Heisenberg model, triangular & pyrochlore lattices
quantum ↔ classical correspondence [Kulagin *et al.* 2013 ; Huang *et al.* 2016]

(...) all in normal phases ($T > T_c$)

Haldane model 2D honeycomb , magnetic field
T=0: *topological phases*, ***magnetically ordered*** [Tupitsyn & Prokof'ev 2019]

here: **superfluid / superconducting** phase

Hubbard model – 3D cubic lattice

$$H = H_{\mathrm{kin}} - \sum_{\sigma} \mu_{\sigma} N_{\sigma} + H_{\mathrm{int}}$$

$$H_{\mathrm{kin}} = -t \sum_{\langle \mathbf{i},\mathbf{j} \rangle \, \sigma} (c^{\dagger}_{\mathbf{i}\sigma} c_{\mathbf{j}\sigma} + h.c.)$$

$$H_{\mathrm{int}} = U \sum_{\mathbf{i}} n_{\mathbf{i}\uparrow} n_{\mathbf{i}\downarrow}$$

$U < 0$

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Diag. expansion in superfluid (superconducting) phase

$$\mathcal{O} := \langle c_{\mathbf{0}\uparrow} c_{\mathbf{0}\downarrow} \rangle$$

unperturbed quadratic Hamiltonian:

$$H_0 = H_{\text{kin}} - \sum_{\sigma} \mu_{0,\sigma} N_\sigma + H^{(\Delta_0)}_{\text{pair}}$$

breaks U(1) symmetry

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$$H_\xi := (1-\xi)\, H_0 + \xi\, H$$

$$Q(\xi) := \langle \hat{Q} \rangle_{H_\xi} \mathrel{\hat{=}} \sum_{N=0}^\infty Q_N\, \xi^N$$

$$\text{pressure: } P = -\Omega/L^3, \quad \Omega = -T \ln \mathrm{Tr} \exp(-\beta H)$$

$$P(\xi) := \frac{T}{L^3} \ln \mathrm{Tr} \exp(-\beta H_\xi) \mathrel{\hat{=}} \sum_{N=0}^\infty P_N\, \xi^N$$

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$$(1-\xi)\,\Delta_0 \Leftrightarrow$$ **symmetry breaking field**

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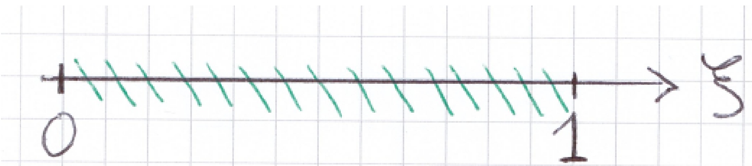
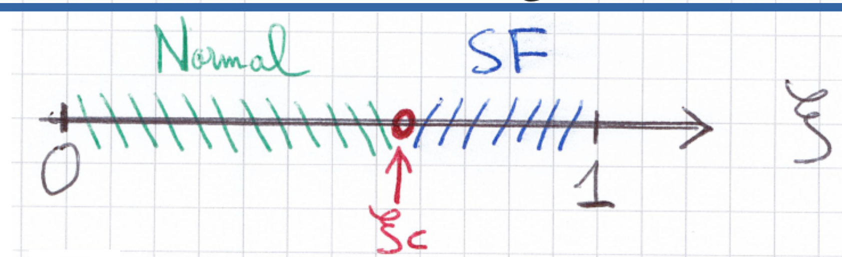
	$T > T_c$	$T < T_c$
$\Delta_0 = 0$	 🙂 $Q \equiv Q(\xi=1) = (\text{Resum}) \sum_{N=0}^{\infty} Q_N$	 🙁 Divergent, not resummable
$\Delta_0 > 0$		

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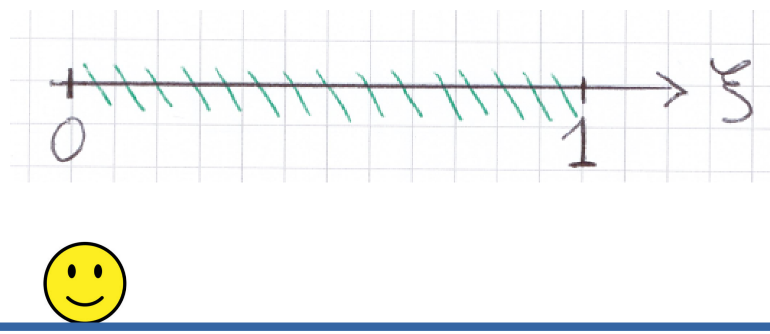
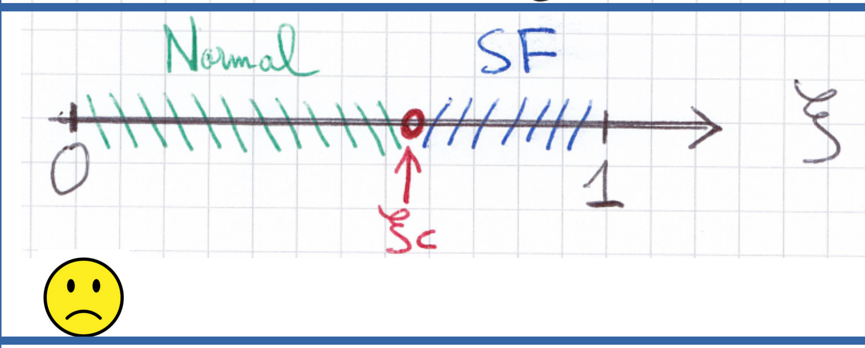
$\Delta_0 = 0$		
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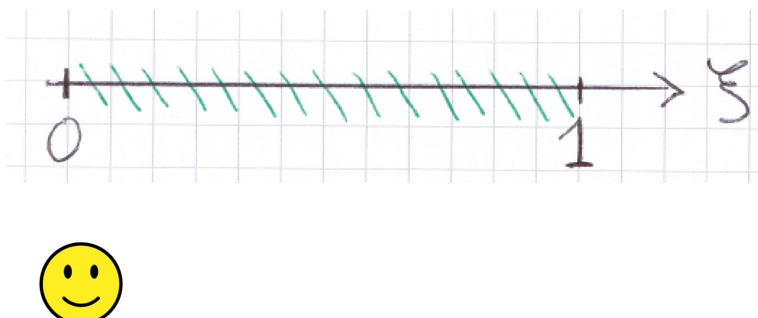
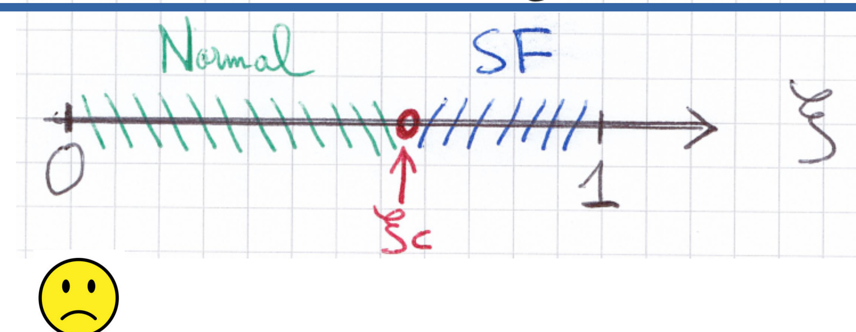
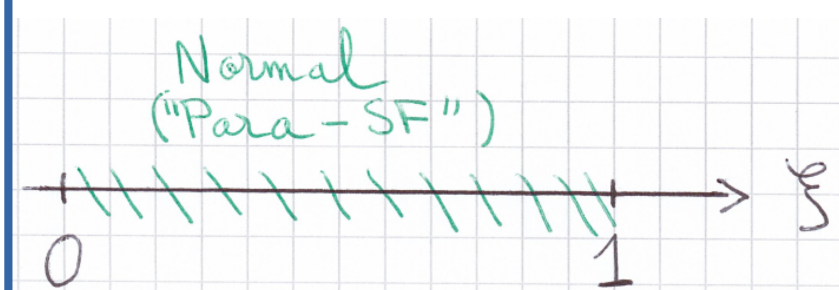
$\Delta_0 = 0$		
$\Delta_0 > 0$		

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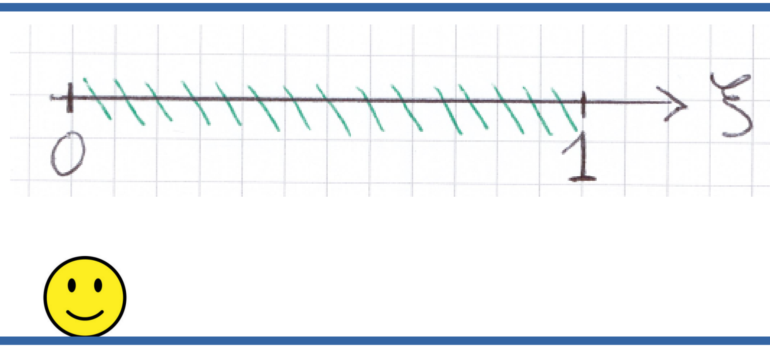
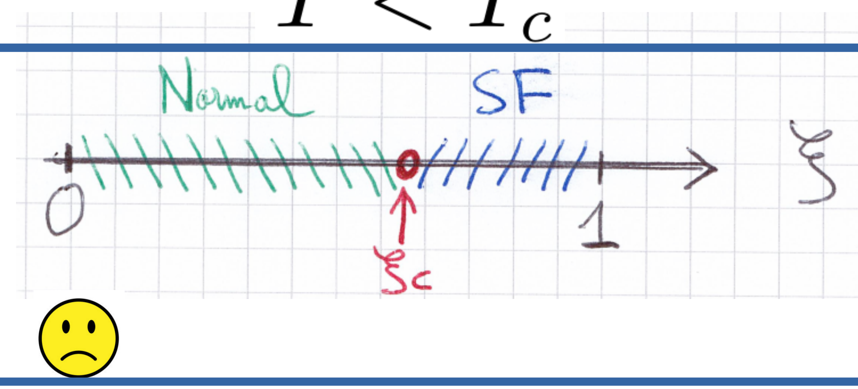
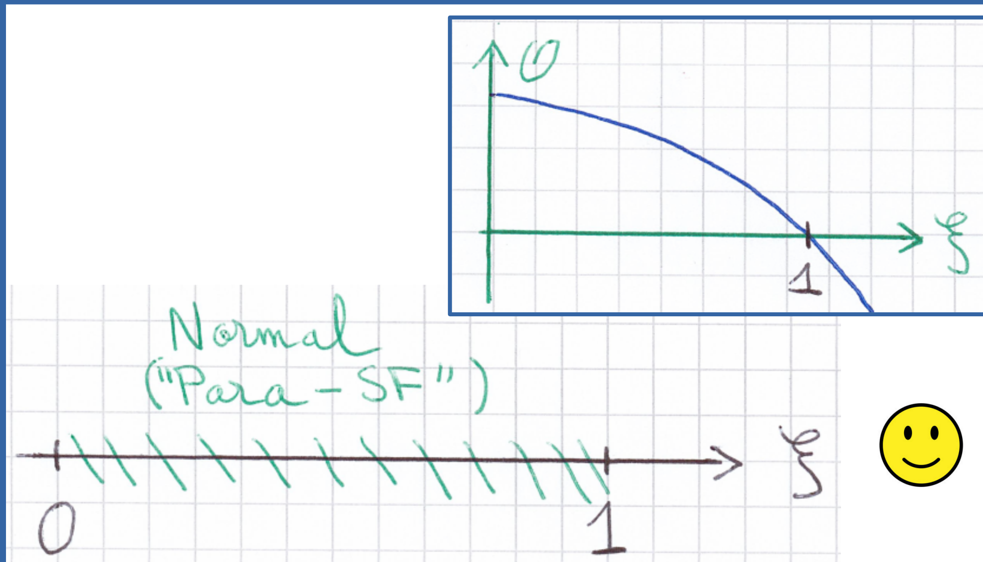
$\Delta_0 = 0$		
$\Delta_0 > 0$		

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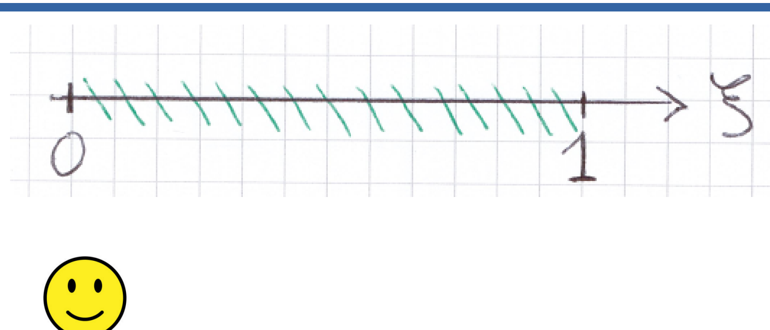
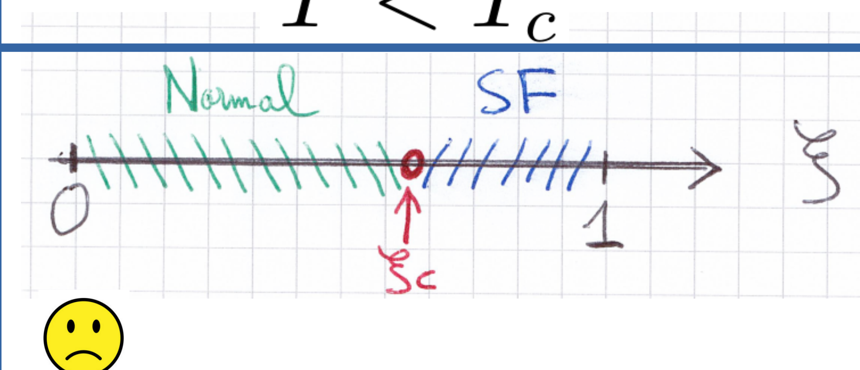
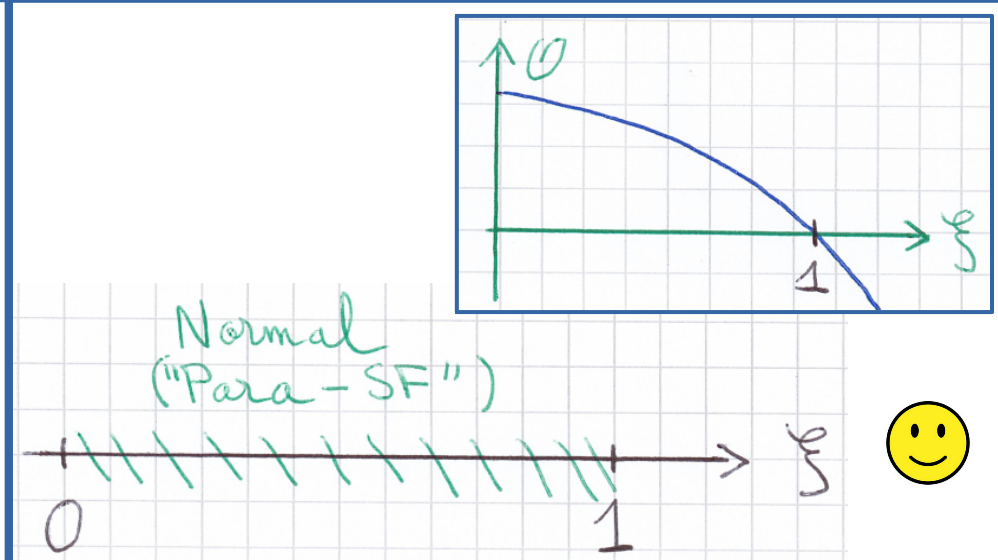
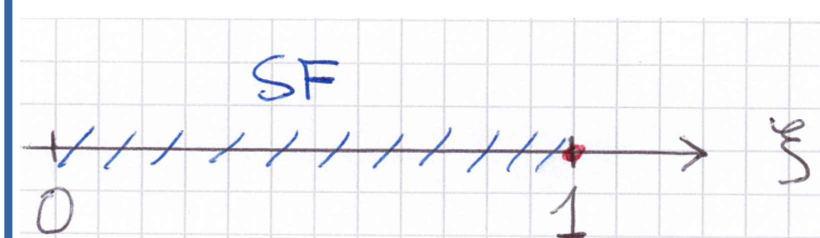
$\Delta_0 = 0$		
$\Delta_0 > 0$		

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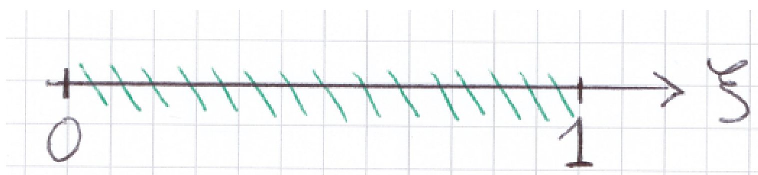
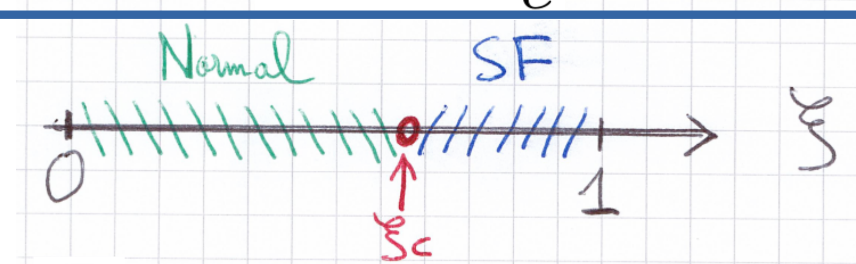
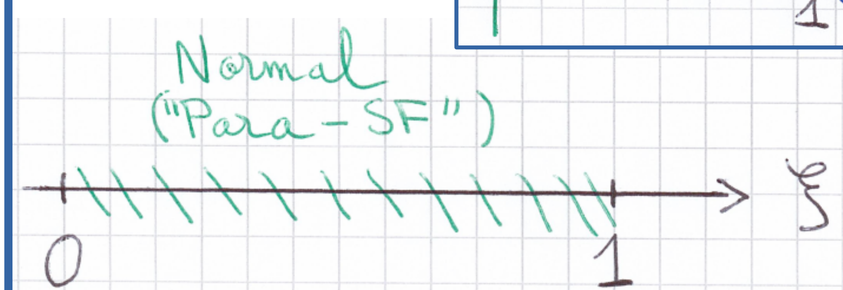
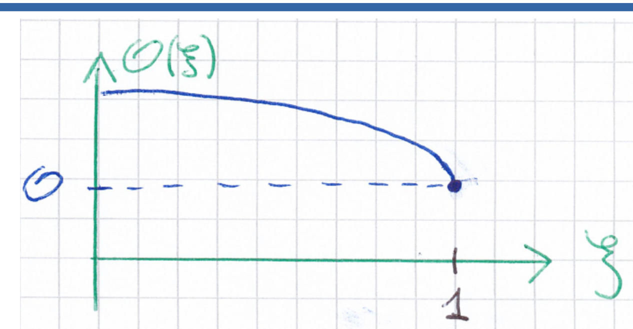
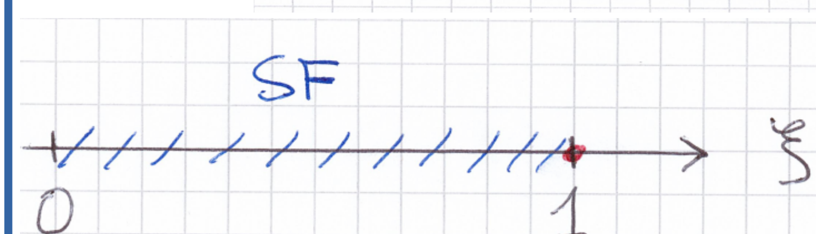
$\Delta_0 = 0$		
$\Delta_0 > 0$	 	 

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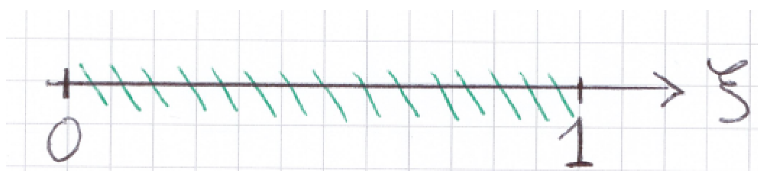
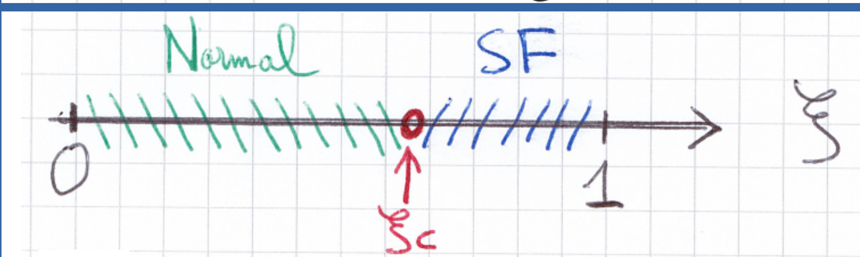
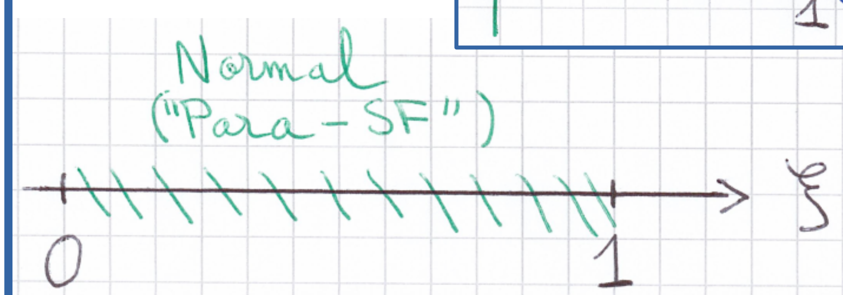
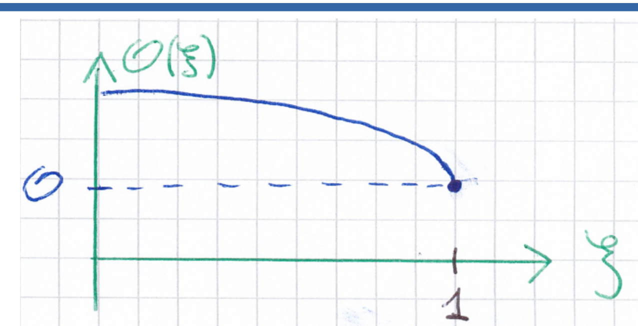
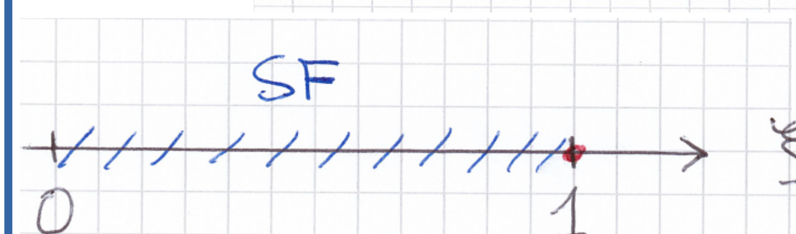
$\Delta_0 = 0$		
$\Delta_0 > 0$		

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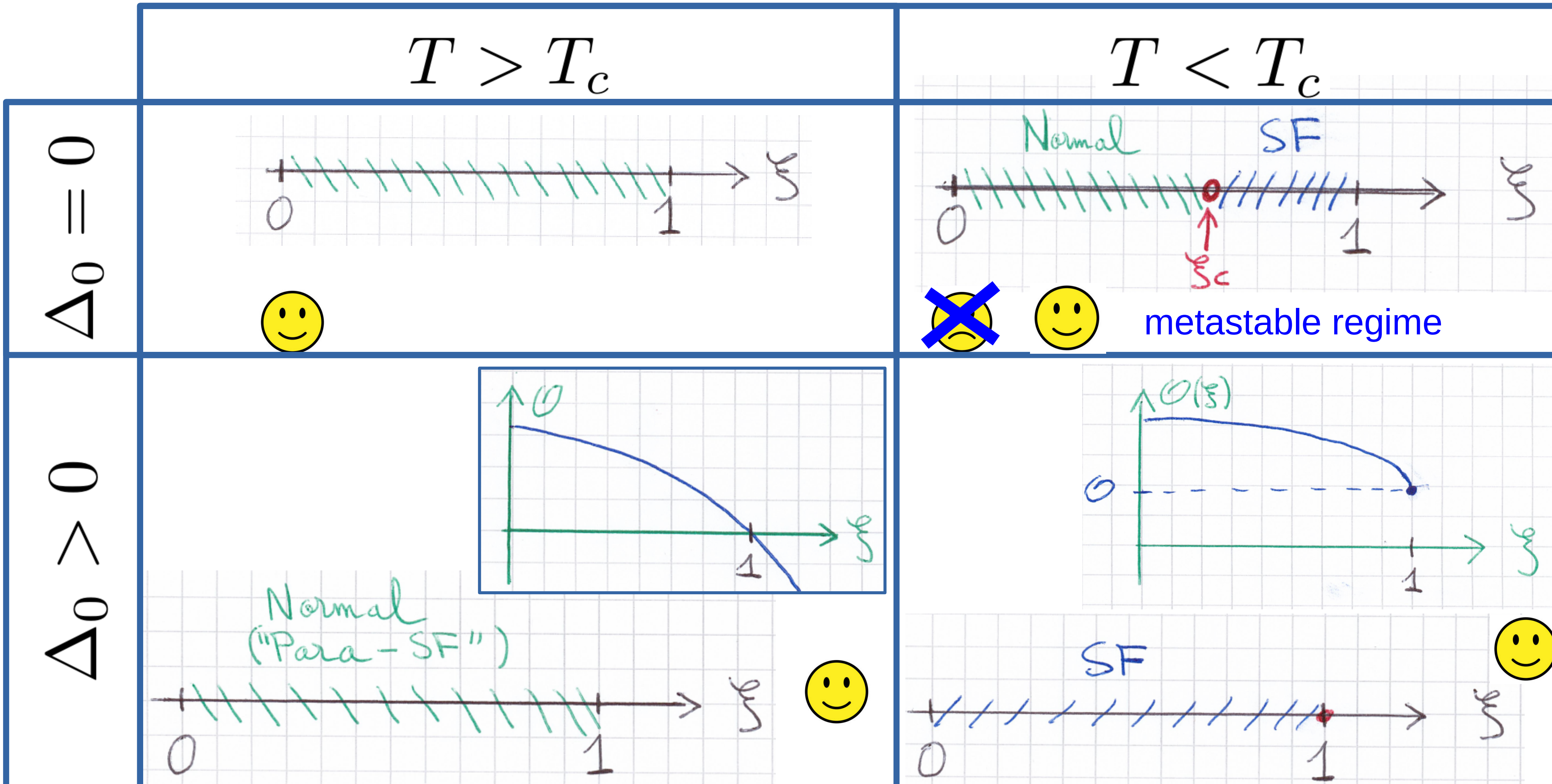

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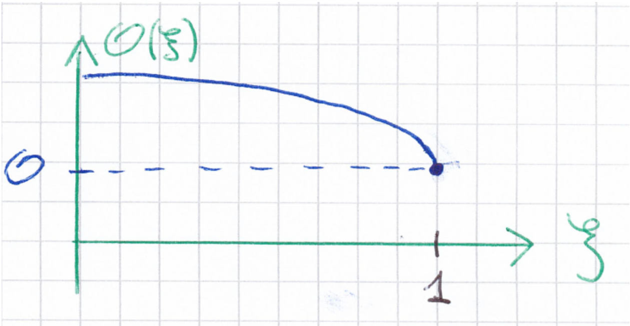

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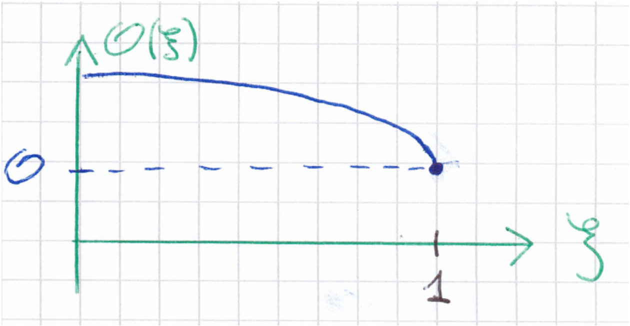

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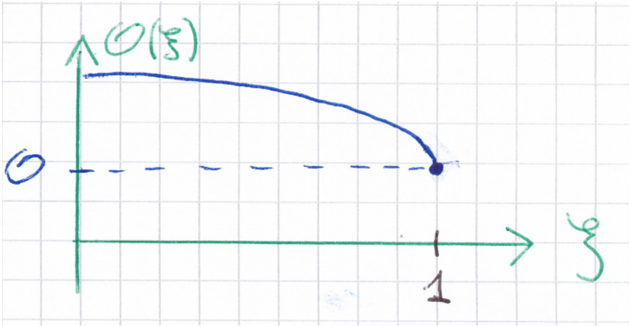

$$\underline{order\ \ parameter}: \ \mathcal{O}(\xi) := \langle c_{\mathbf{0}\uparrow} c_{\mathbf{0}\downarrow} \rangle_{H_\xi} \xrightarrow[\xi \to 1^-]{} \mathcal{O} \neq 0$$

spontaneous symmetry breaking – thermodynamic limit $L \to \infty$ before $\xi \to 1^-$

$$\mathcal{O}(\xi) = \sum_{N=0}^{\infty} \mathcal{O}_N\, \xi^N \qquad \mathcal{O} = \mathcal{O}(\xi \to 1^-) = \sum_{N=0}^{\infty} \mathcal{O}_N$$

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natural choice:
BCS mean-field theory

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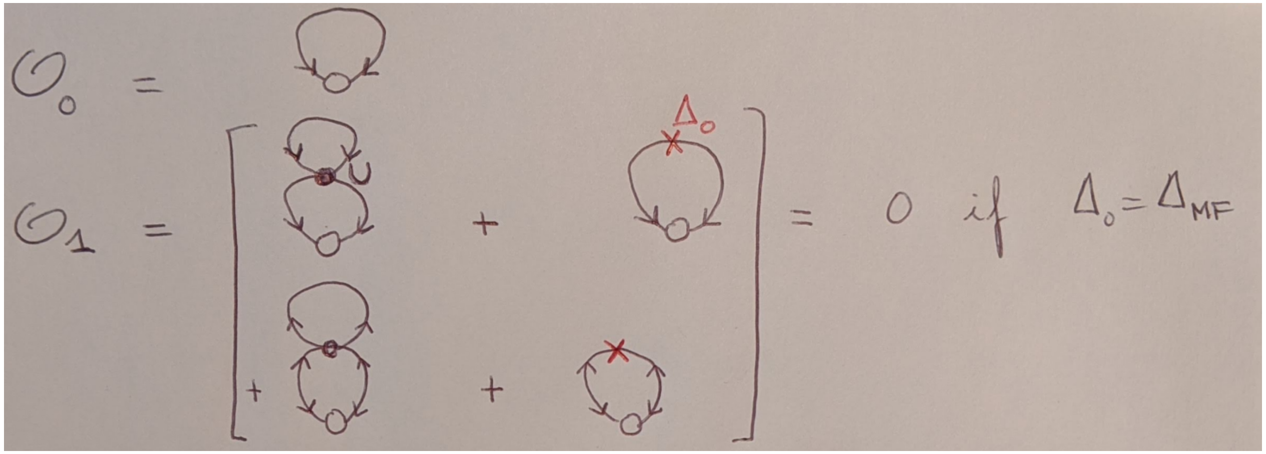

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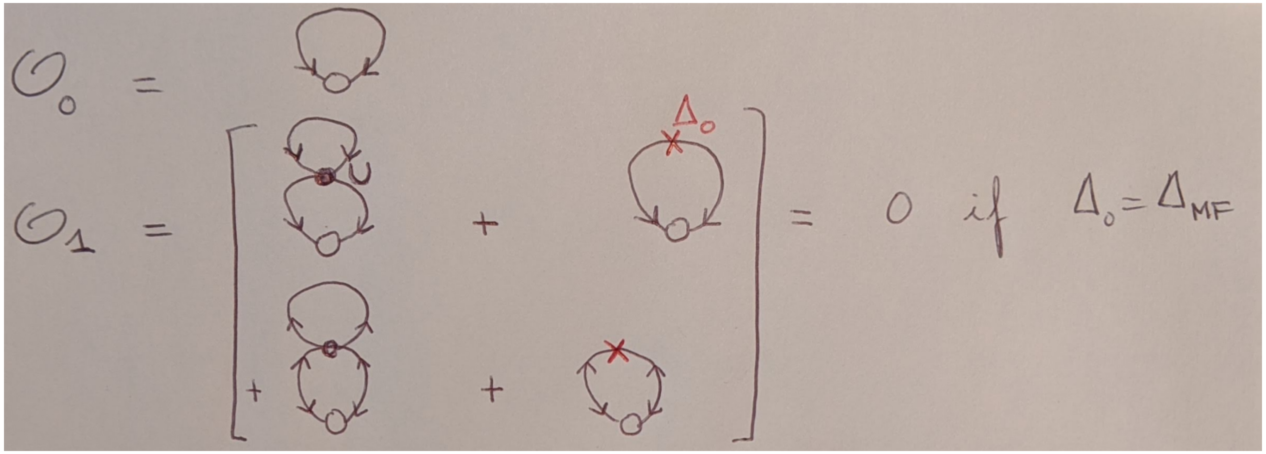

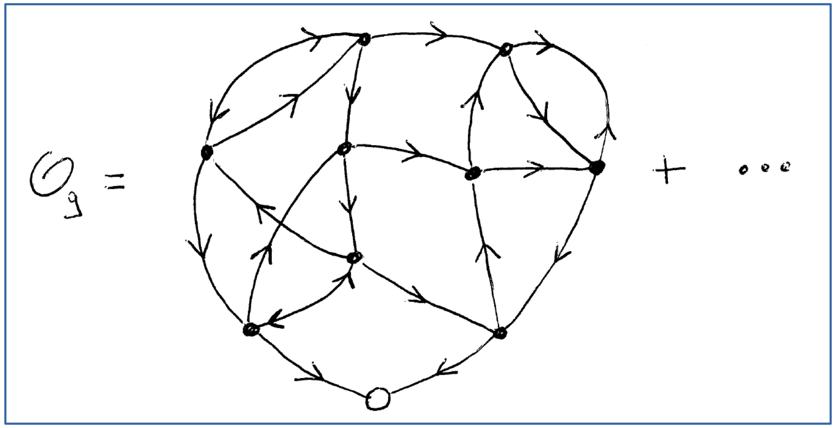

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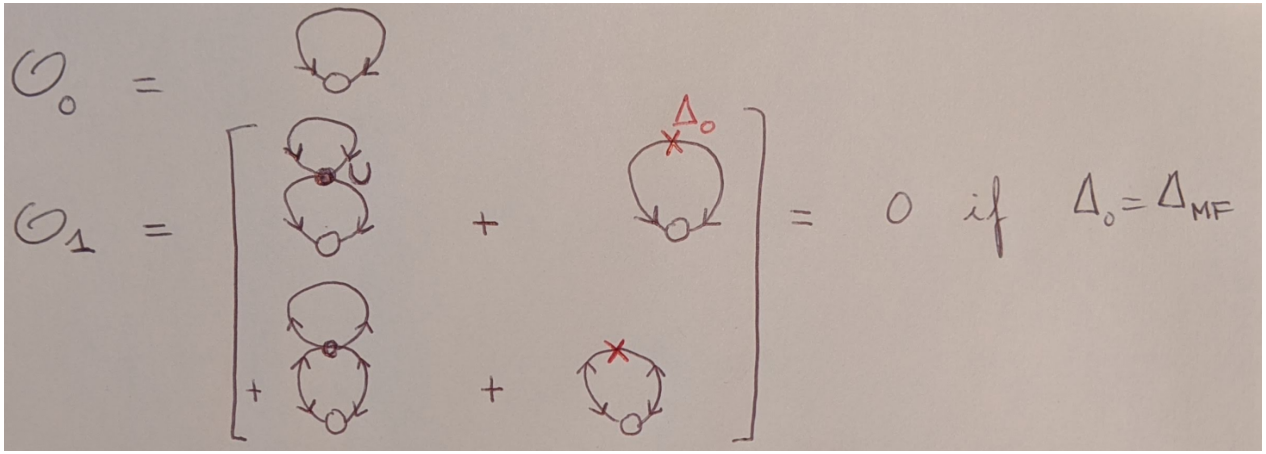

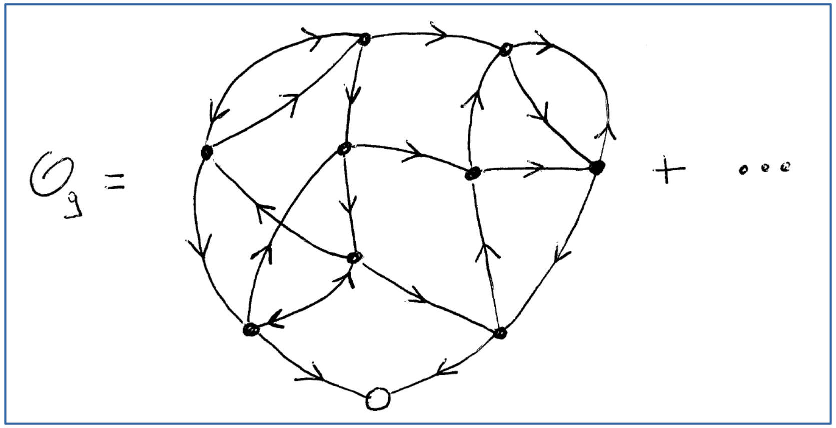

large distances : small contribution

broken symmetry

<u>algorithm:</u> ***CDet*** *[Rossi 2017]* *with* ***Nambu propagators***

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$$\begin{pmatrix} \mathcal{G}_{00}(X-X') & \mathcal{G}_{01}(X-X') \\ \mathcal{G}_{10}(X-X') & \mathcal{G}_{11}(X-X') \end{pmatrix} := - \begin{pmatrix} \left\langle \mathrm{T}\, c^\dagger_\uparrow(X) c_\uparrow(X') \right\rangle_{H_0} & \left\langle \mathrm{T}\, c^\dagger_\uparrow(X) c^\dagger_\downarrow(X') \right\rangle_{H_0} \\ \left\langle \mathrm{T}\, c_\downarrow(X) c_\uparrow(X') \right\rangle_{H_0} & \left\langle \mathrm{T}\, c_\downarrow(X) c^\dagger_\downarrow(X') \right\rangle_{H_0} \end{pmatrix}$$

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$$\mathcal{O}_N = -\frac{(-U)^N}{N!}\int dX_1 \dots dX_N \; \mathrm{cdet}(A)$$

$$\mathrm{cdet}(A) = \det(A) - \sum(\text{disconnected diagrams})$$

$$X = (\mathbf{i}, \tau)$$

$$\begin{pmatrix} \textit{recursively} \\ 3^N \;\; \textit{operations} \end{pmatrix}$$

$$A := \begin{pmatrix} 0 & \delta_{\mathrm{sh}} & \dots & \mathcal{G}_{00}(X_1-X_N) & \mathcal{G}_{01}(X_1-X_N) & \mathcal{G}_{0\alpha}(X_1) \\ \delta_{\mathrm{sh}} & 0 & \dots & \mathcal{G}_{10}(X_1-X_N) & \mathcal{G}_{11}(X_1-X_N) & \mathcal{G}_{1\alpha}(X_1) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \mathcal{G}_{00}(X_N-X_1) & \mathcal{G}_{01}(X_N-X_1) & \dots & 0 & \delta_{\mathrm{sh}} & \mathcal{G}_{0\alpha}(X_N) \\ \mathcal{G}_{10}(X_N-X_1) & \mathcal{G}_{11}(X_N-X_1) & \dots & \delta_{\mathrm{sh}} & 0 & \mathcal{G}_{1\alpha}(X_N) \\ \mathcal{G}_{\alpha'0}(-X_1) & \mathcal{G}_{\alpha'1}(-X_1) & \dots & \mathcal{G}_{\alpha'0}(-X_N) & \mathcal{G}_{\alpha'1}(-X_N) & \mathcal{G}_{\alpha'\alpha}(0) \end{pmatrix}$$

$$\delta_{\mathrm{sh}} = 0 \text{ if } \Delta_0 = \Delta_{\mathrm{MF}}$$

$$(\alpha = 1,\; \alpha' = 0)$$

algorithm: ***CDet** [Rossi 2017]* *with **Nambu propagators***

$$\begin{pmatrix} \mathcal{G}_{00}(X-X') & \mathcal{G}_{01}(X-X') \\ \mathcal{G}_{10}(X-X') & \mathcal{G}_{11}(X-X') \end{pmatrix} := - \begin{pmatrix} \left\langle \mathrm{T}\, c^\dagger_\uparrow(X) c_\uparrow(X') \right\rangle_{H_0} & \left\langle \mathrm{T}\, c^\dagger_\uparrow(X) c^\dagger_\downarrow(X') \right\rangle_{H_0} \\ \left\langle \mathrm{T}\, c_\downarrow(X) c_\uparrow(X') \right\rangle_{H_0} & \left\langle \mathrm{T}\, c_\downarrow(X) c^\dagger_\downarrow(X') \right\rangle_{H_0} \end{pmatrix}$$

$$\mathcal{O}_N = -\frac{(-U)^N}{N!} \int dX_1 \dots dX_N \ \mathrm{cdet}(A)$$

$$\mathrm{cdet}(A) = \det(A) - \sum (\text{disconnected diagrams})$$

$$X = (\mathbf{i}, \tau)$$

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$$A := \begin{pmatrix} 0 & \delta_{\mathrm{sh}} & \dots & \mathcal{G}_{00}(X_1-X_N) & \mathcal{G}_{01}(X_1-X_N) & \mathcal{G}_{0\alpha}(X_1) \\ \delta_{\mathrm{sh}} & 0 & \dots & \mathcal{G}_{10}(X_1-X_N) & \mathcal{G}_{11}(X_1-X_N) & \mathcal{G}_{1\alpha}(X_1) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \mathcal{G}_{00}(X_N-X_1) & \mathcal{G}_{01}(X_N-X_1) & \dots & 0 & \delta_{\mathrm{sh}} & \mathcal{G}_{0\alpha}(X_N) \\ \mathcal{G}_{10}(X_N-X_1) & \mathcal{G}_{11}(X_N-X_1) & \dots & \delta_{\mathrm{sh}} & 0 & \mathcal{G}_{1\alpha}(X_N) \\ \mathcal{G}_{\alpha'0}(-X_1) & \mathcal{G}_{\alpha'1}(-X_1) & \dots & \mathcal{G}_{\alpha'0}(-X_N) & \mathcal{G}_{\alpha'1}(-X_N) & \mathcal{G}_{\alpha'\alpha}(0) \end{pmatrix}$$

$$\delta_{\mathrm{sh}} = 0 \text{ if } \Delta_0 = \Delta_{\mathrm{MF}} \qquad (\alpha = 1,\ \alpha' = 0)$$

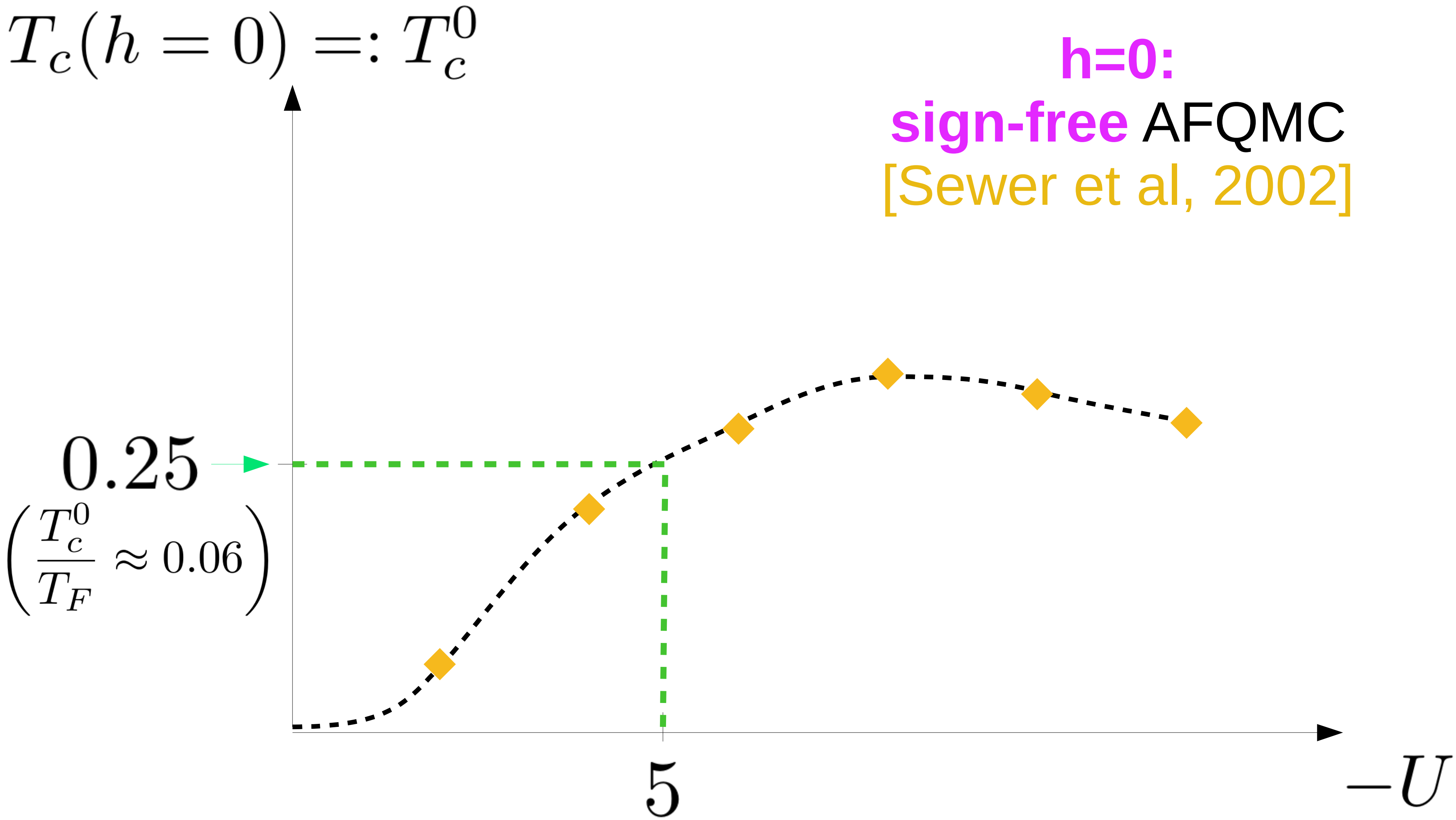
implementation: ***F**ast **F**eynman **D**iagrammatics* *library* *[Rossi & Simkovic]*
with ***Many Configuration MC*** *[Simkovic & Rossi 2021]*

RESULTS

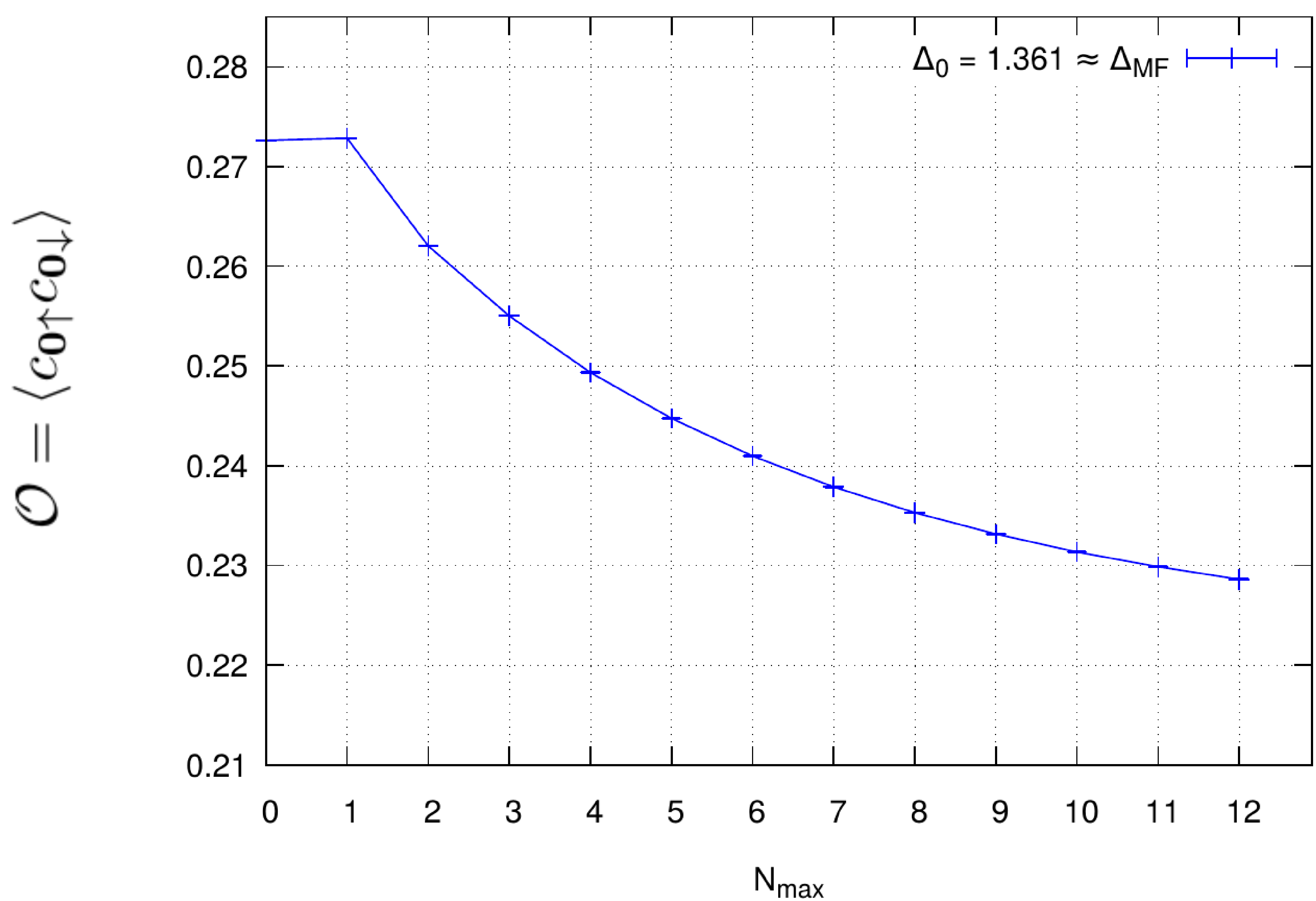
$$\mu_\uparrow = \mu + h, \quad \mu_\downarrow = \mu - h$$

$$t \equiv 1, \boxed{U = -5}$$

$$\mu = -3.38 \Rightarrow \langle n_\uparrow + n_\downarrow \rangle \simeq 0.5 \,(\text{quarter filling})$$



$$T = 1/8 \approx T_c^0/2, \quad h = 0$$



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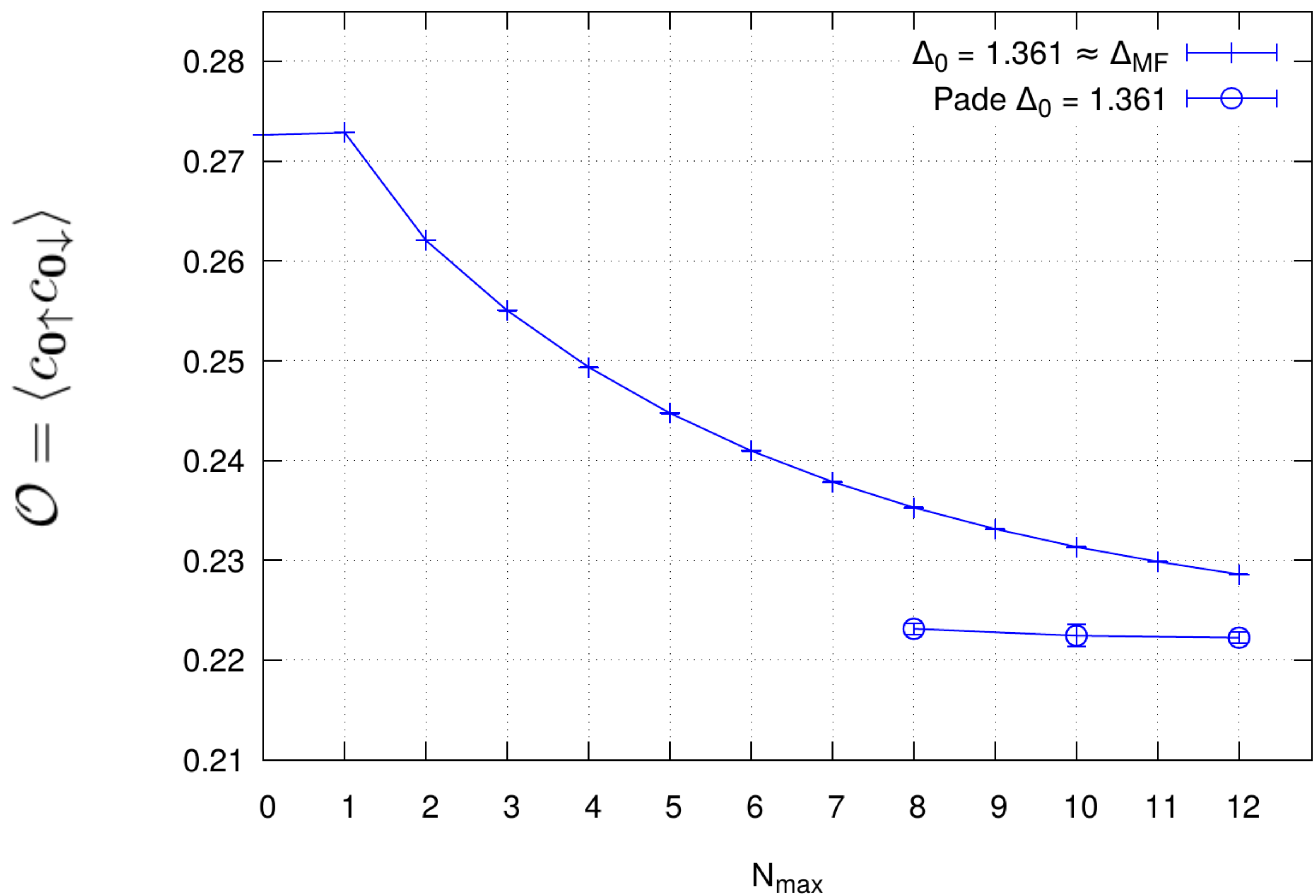

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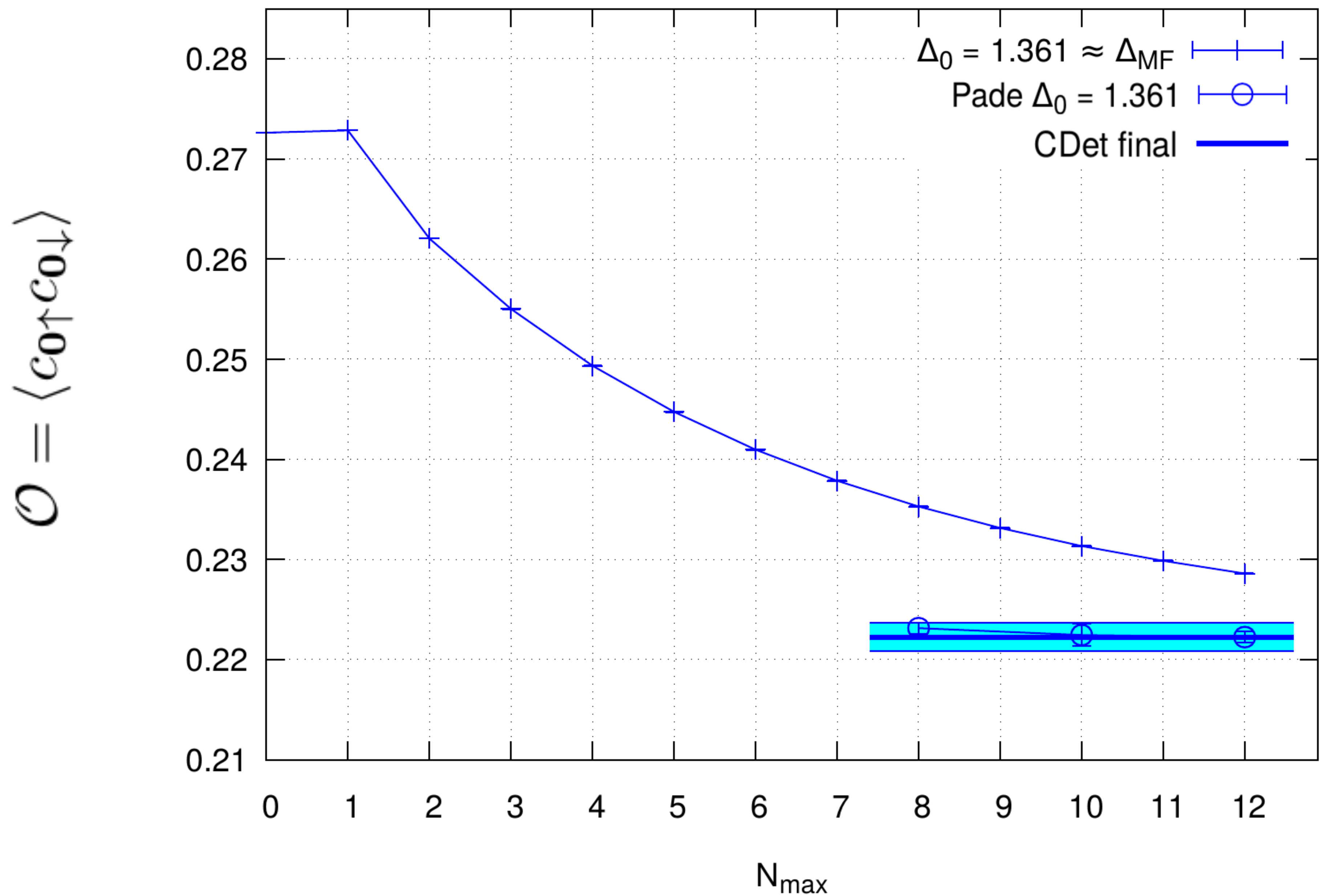

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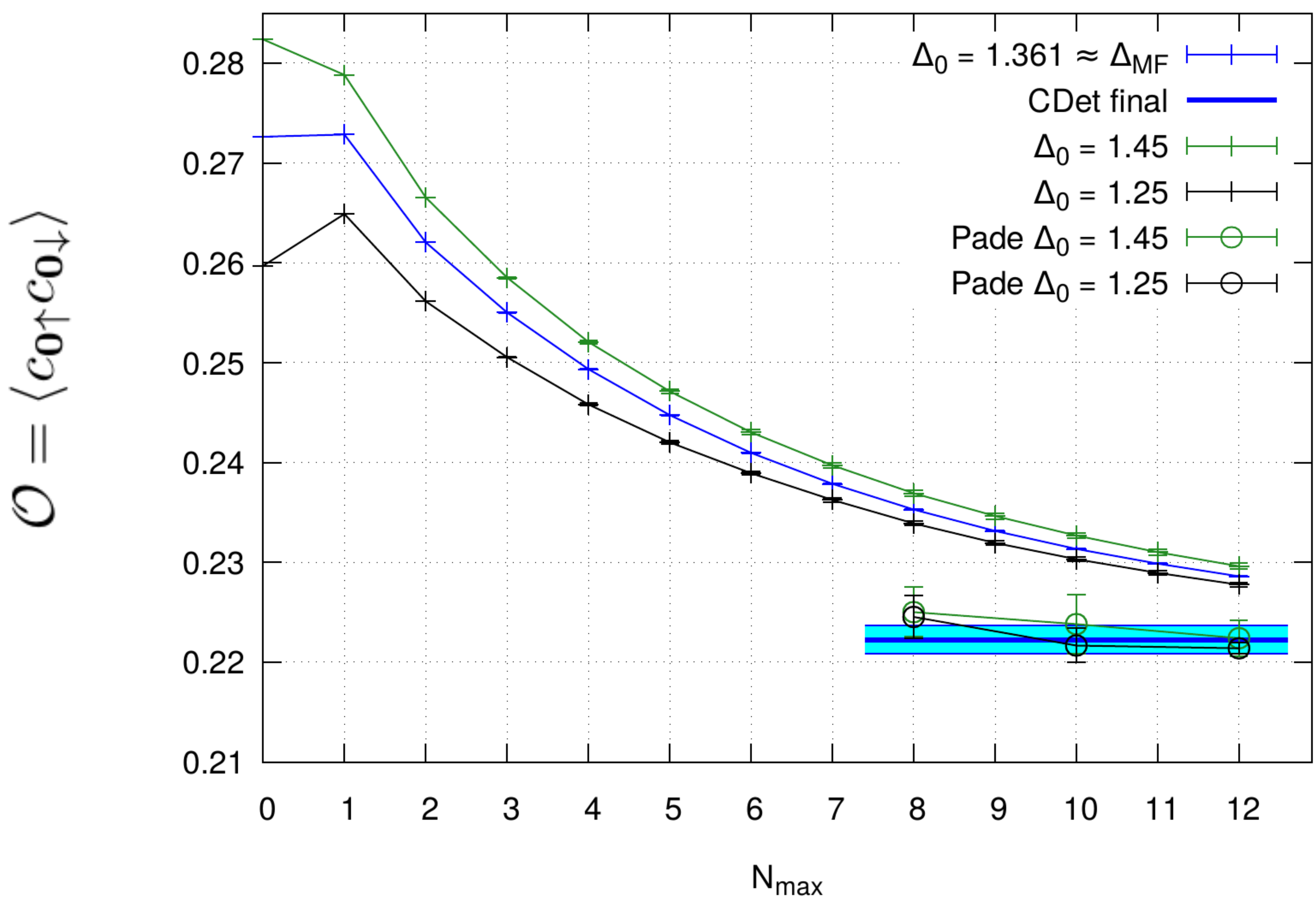

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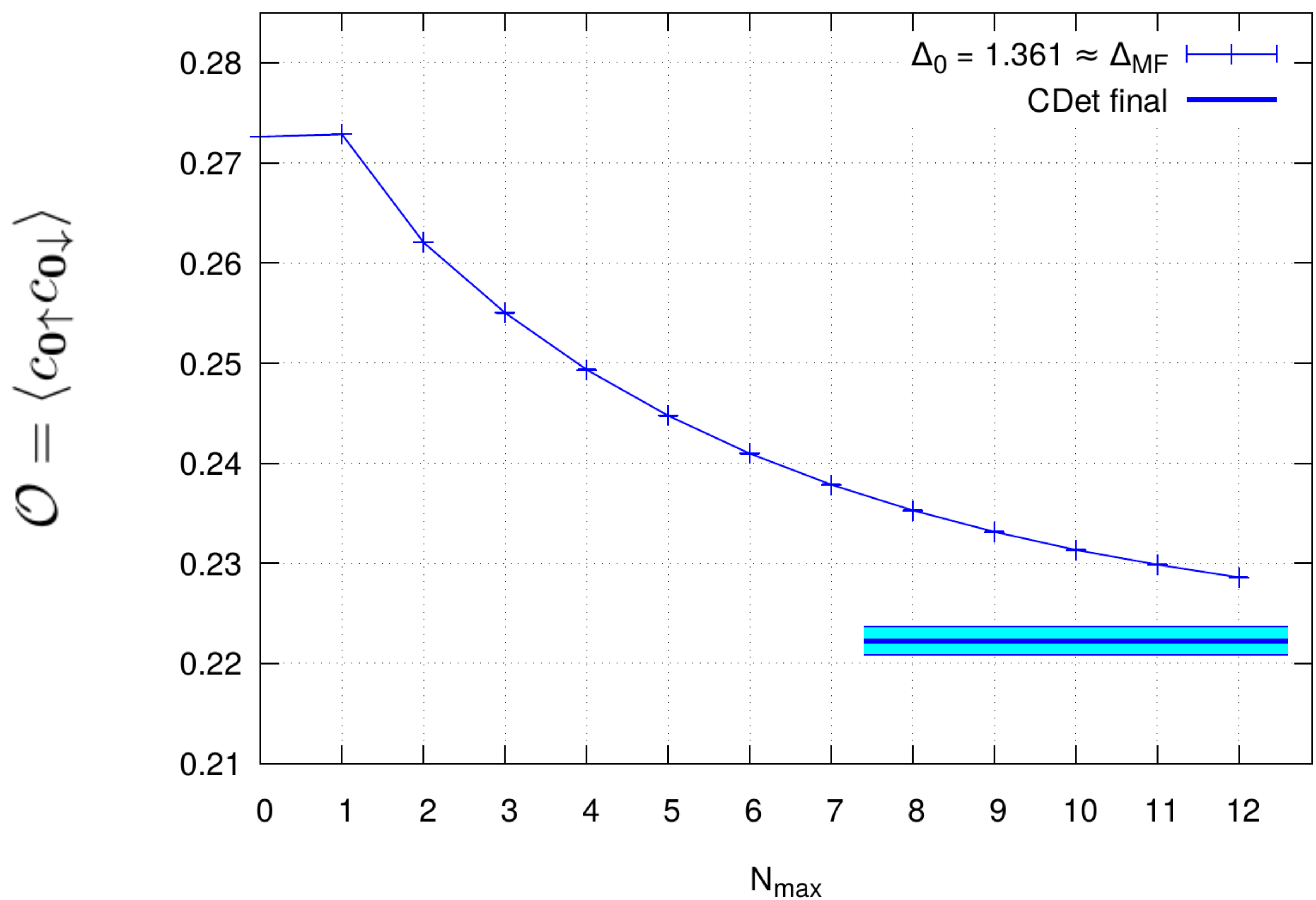
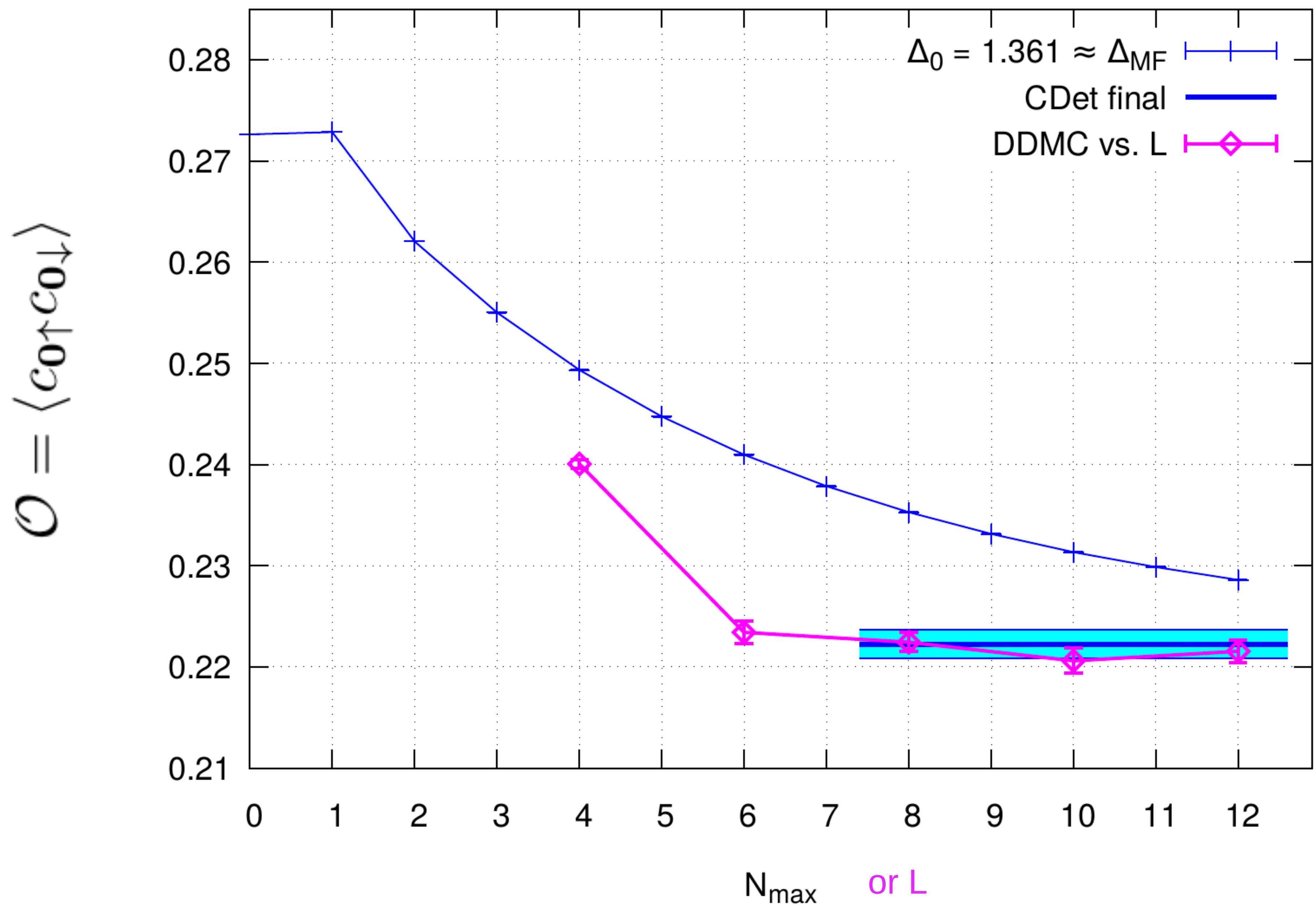


$$T = 1/8 \approx T_c^0/2, \quad h = 0$$

benchmark vs.
Determinant **D**iagrammatic **MC**
[Burovski's code]



Polarized regime

$h \neq 0 \quad \left(h \equiv \frac{\mu_\uparrow - \mu_\downarrow}{2} \right)$

no unbiased results available (sign problem)

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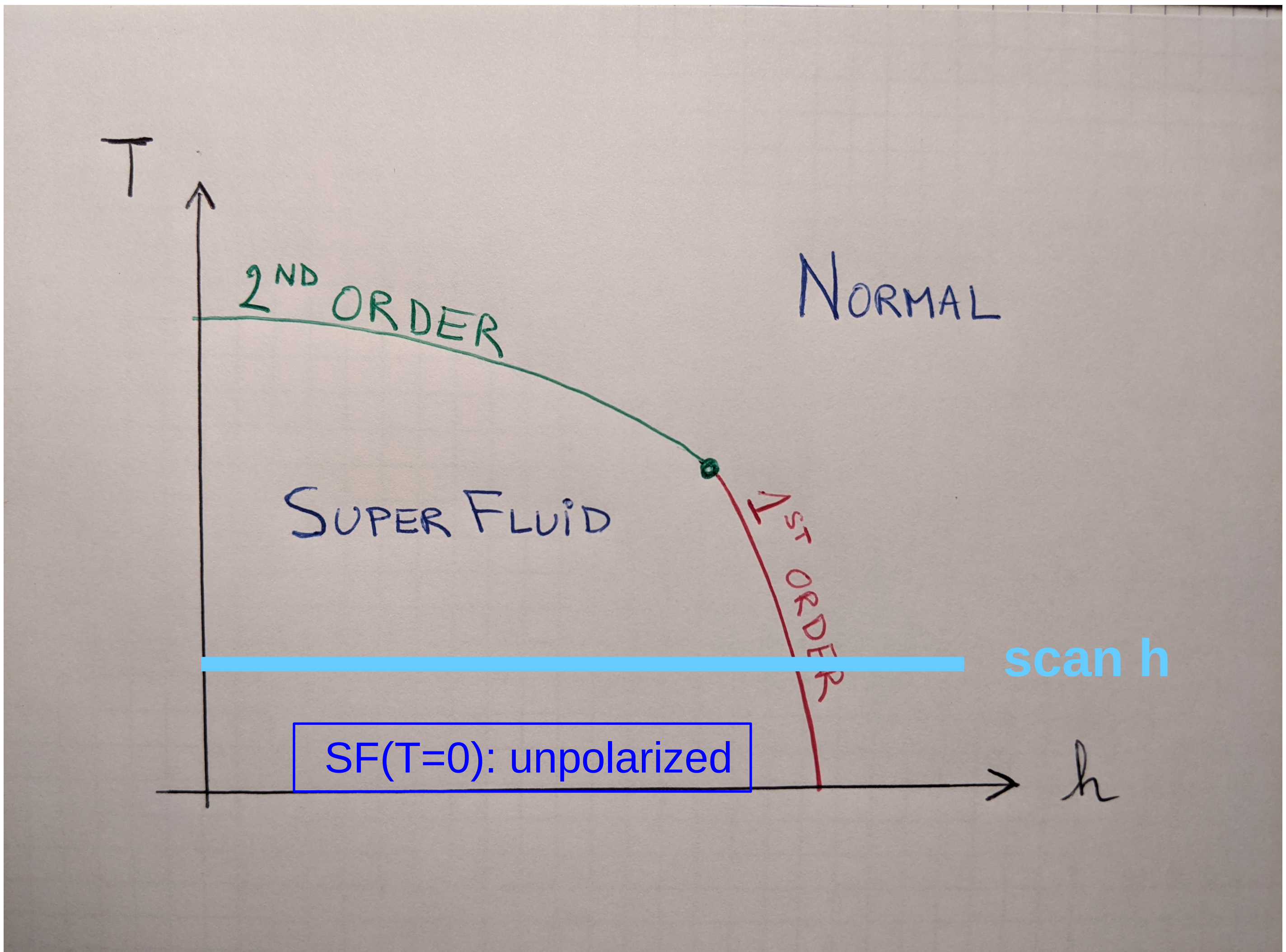
expected phase-diagram topology (U not very large & discarding FFLO)
from BCS-MF; DMFT [Dao *et al.* 2008; Koga & Werner 2010]
& cold-atom experiments in c° space [ENS & MIT, 2008-2010]



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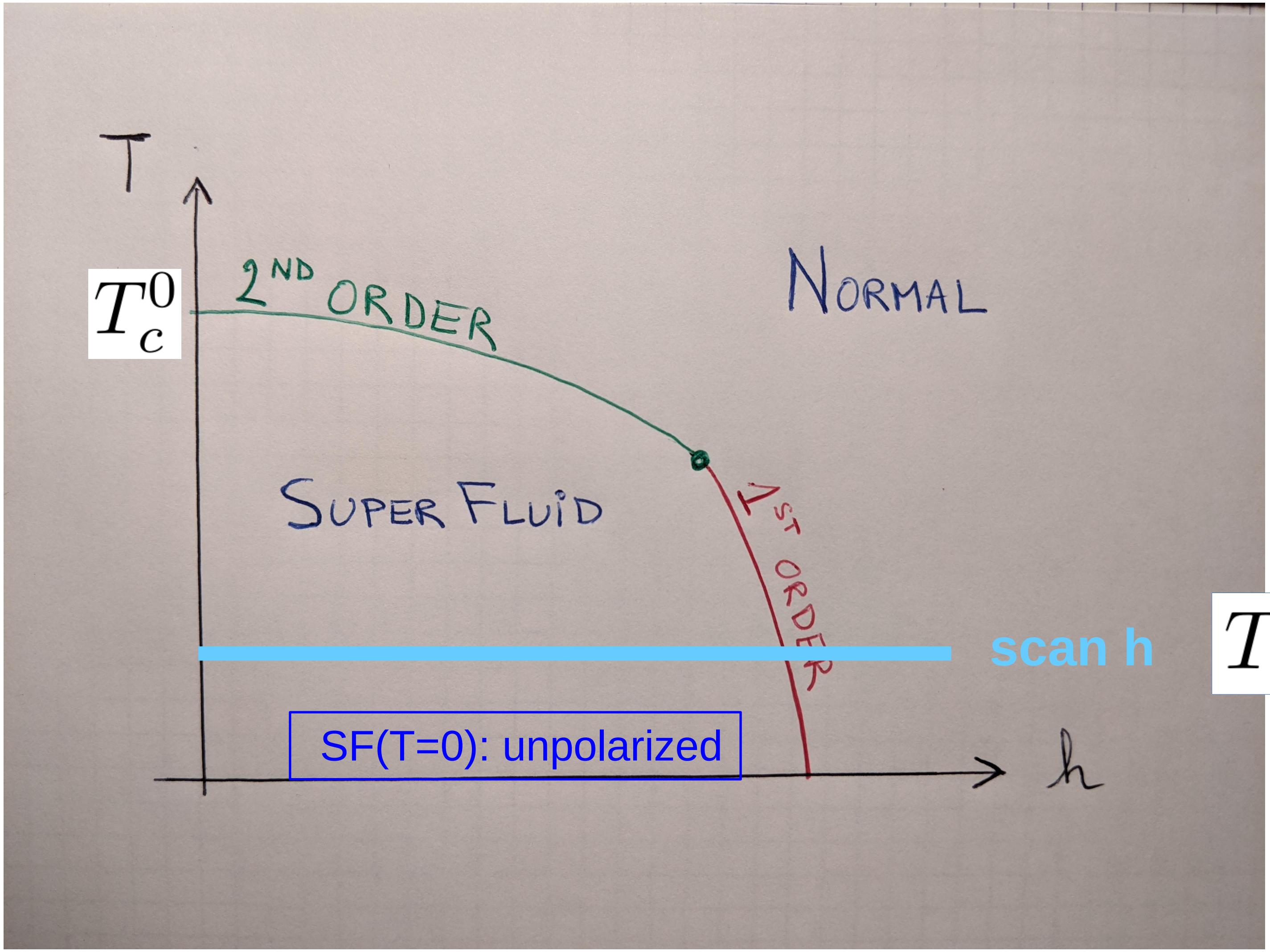
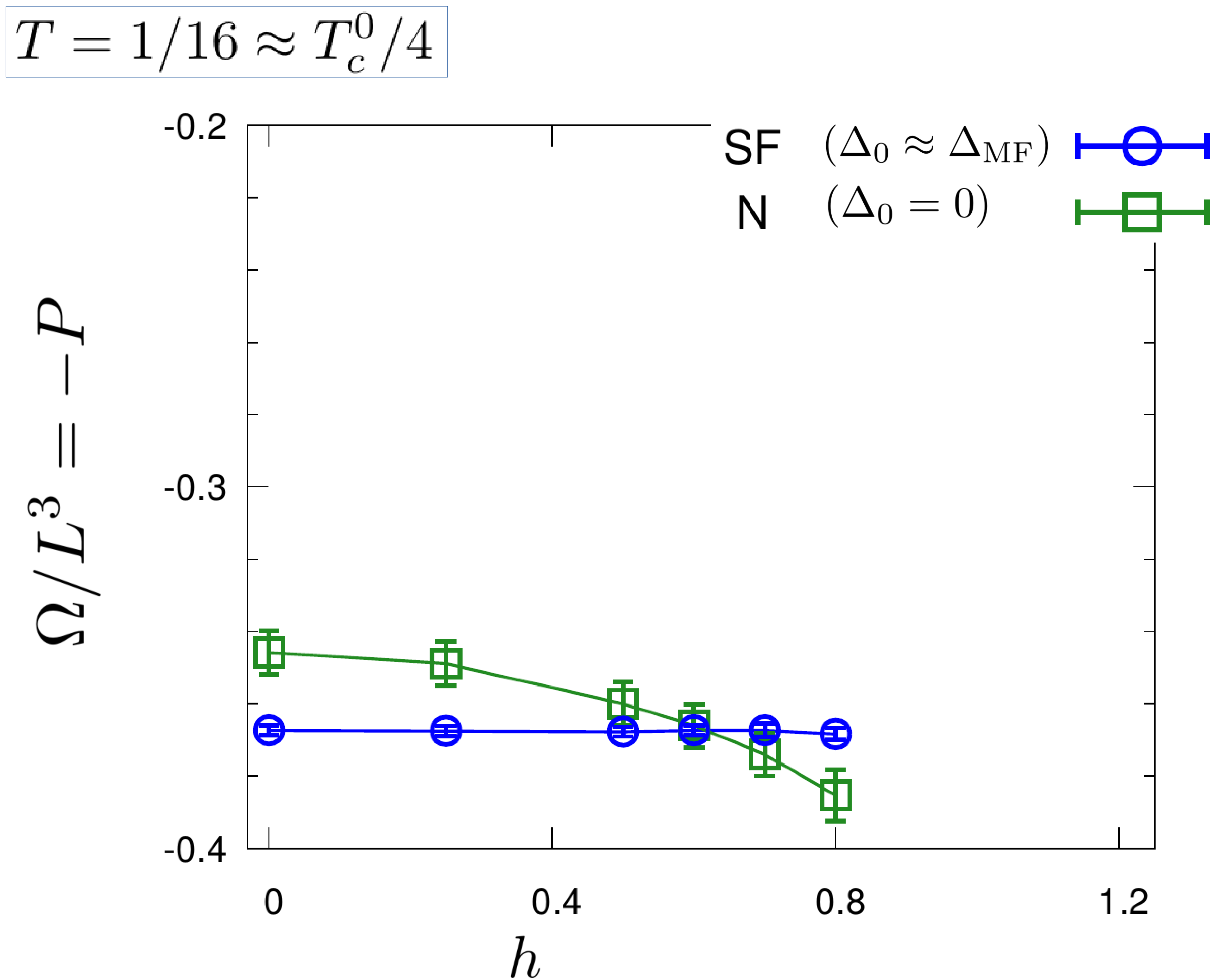
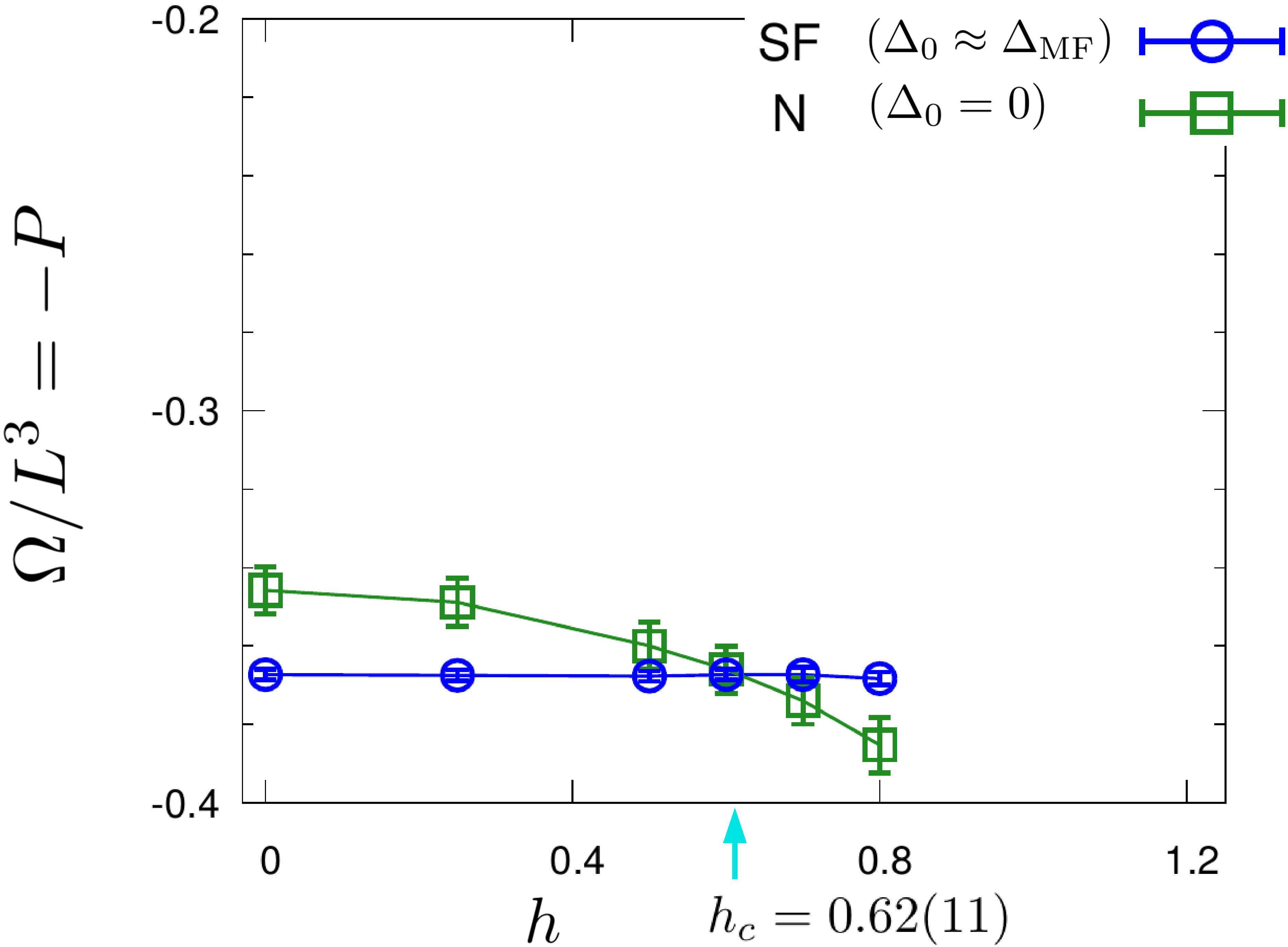


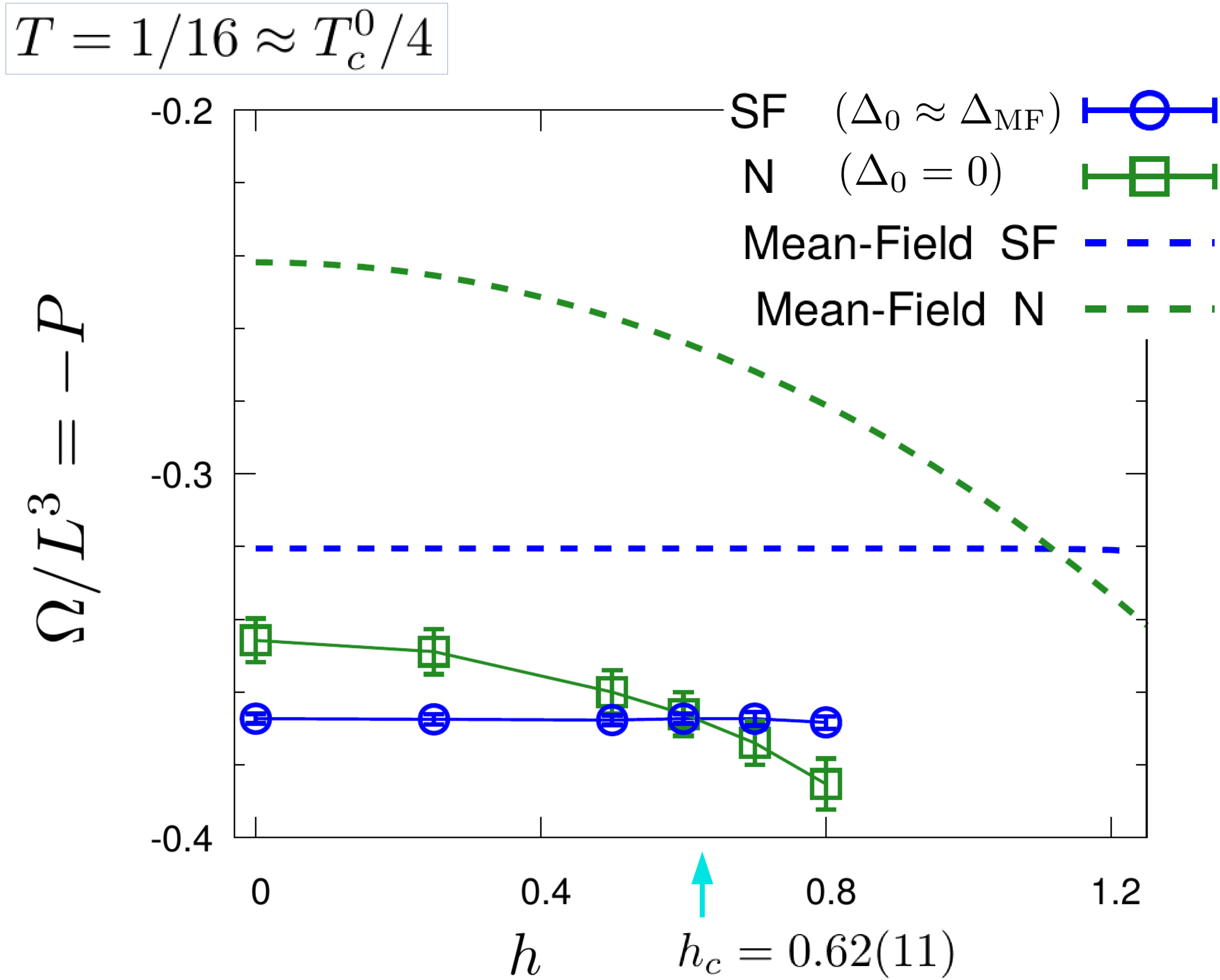
$T \approx T_c^0/4$



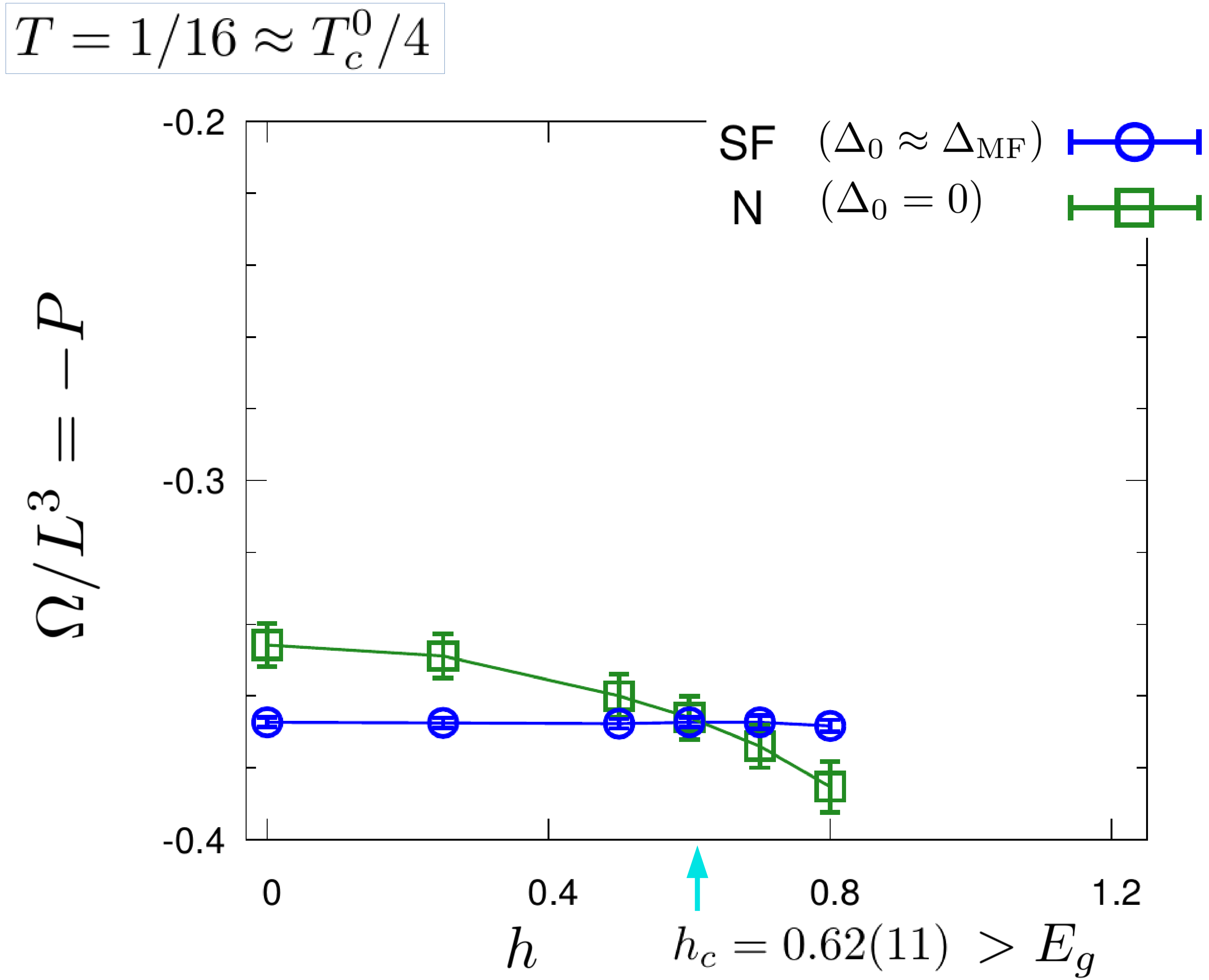
$T = 1/16 \approx T_c^0/4$
$\Omega/L^3 = -P$
-0.2
-0.3
-0.4
0
0.4
0.8
1.2
h
SF $(\Delta_0 \approx \Delta_{\mathrm{MF}})$
N $(\Delta_0 = 0)$

$T = 1/16 \approx T_c^0/4$

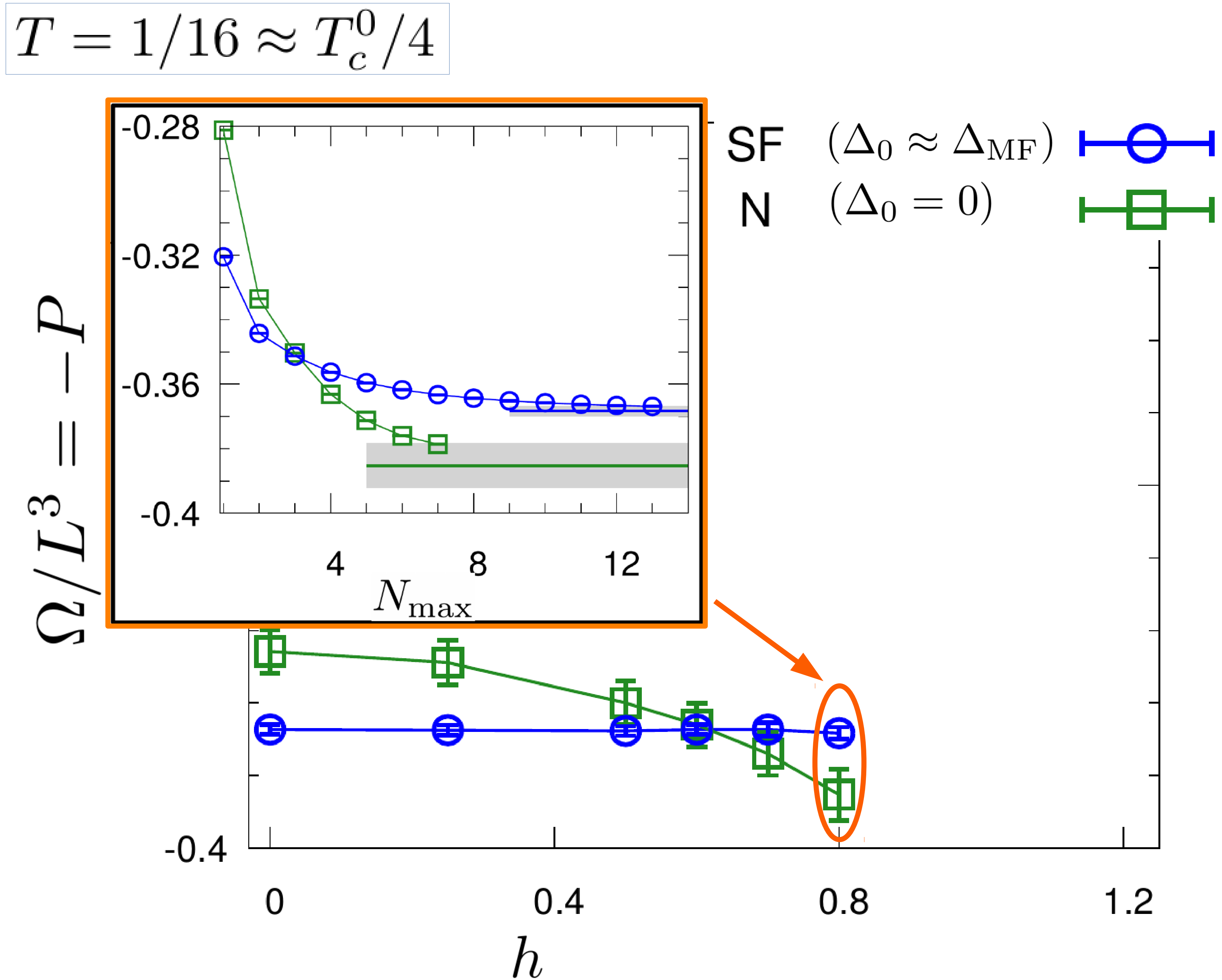




$T = 1/16 \approx T_c^0/4$
SF $(\Delta_0 \approx \Delta_{\rm MF})$
N $(\Delta_0 = 0)$
Mean-Field SF
Mean-Field N
-0.2
-0.3
-0.4
$\Omega/L^3 = -P$
0
0.4
0.8
1.2
h
$h_c = 0.62(11)$

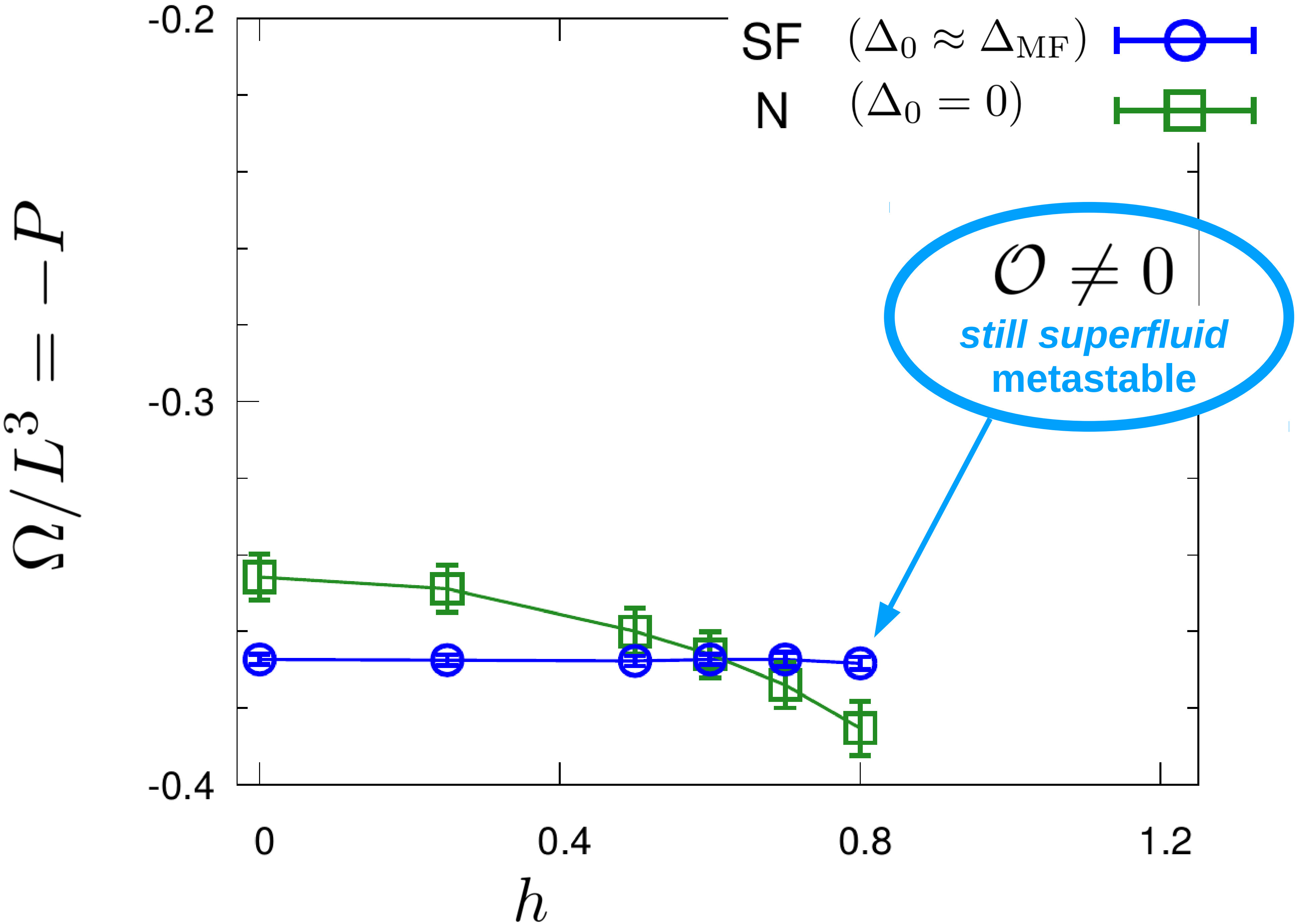


$T = 1/16 \approx T_c^0/4$
SF $(\Delta_0 \approx \Delta_{\rm MF})$
N $(\Delta_0 = 0)$
$\Omega/L^3 = -P$
-0.2
-0.3
-0.4
0
0.4
0.8
1.2
h
$h_c = 0.62(11) > E_g$



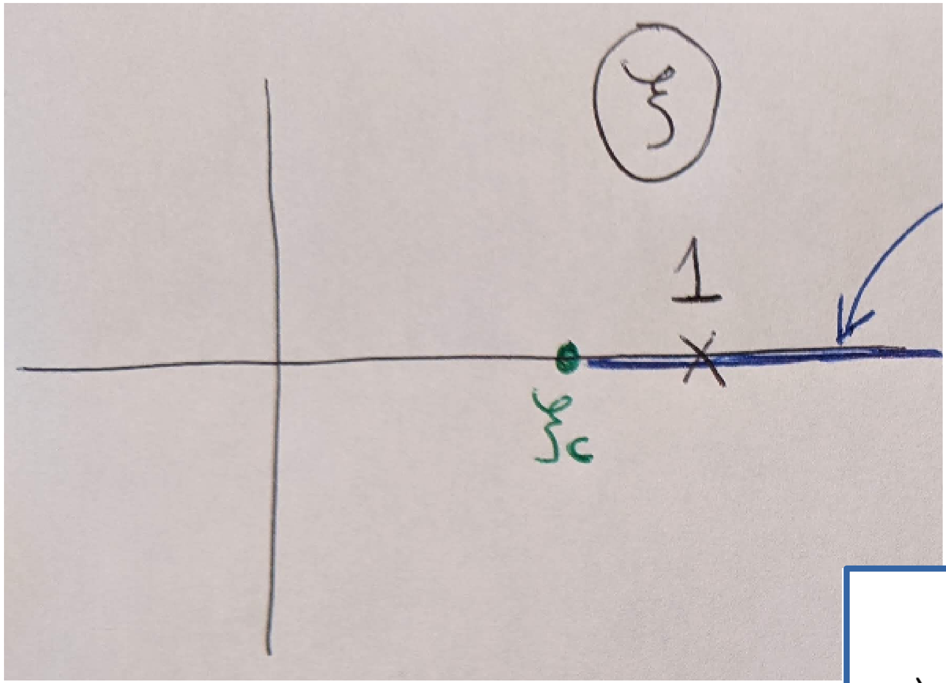
$T = 1/16 \approx T_c^0/4$
SF $(\Delta_0 \approx \Delta_{\rm MF})$
N $(\Delta_0 = 0)$
-0.28
-0.32
-0.36
-0.4
4
8
12
$N_{\rm max}$
$\Omega/L^3 = -P$
-0.4
0
0.4
0.8
1.2
h

$T = 1/16 \approx T_c^0/4$



Metastable regime: *slowly* divergent series

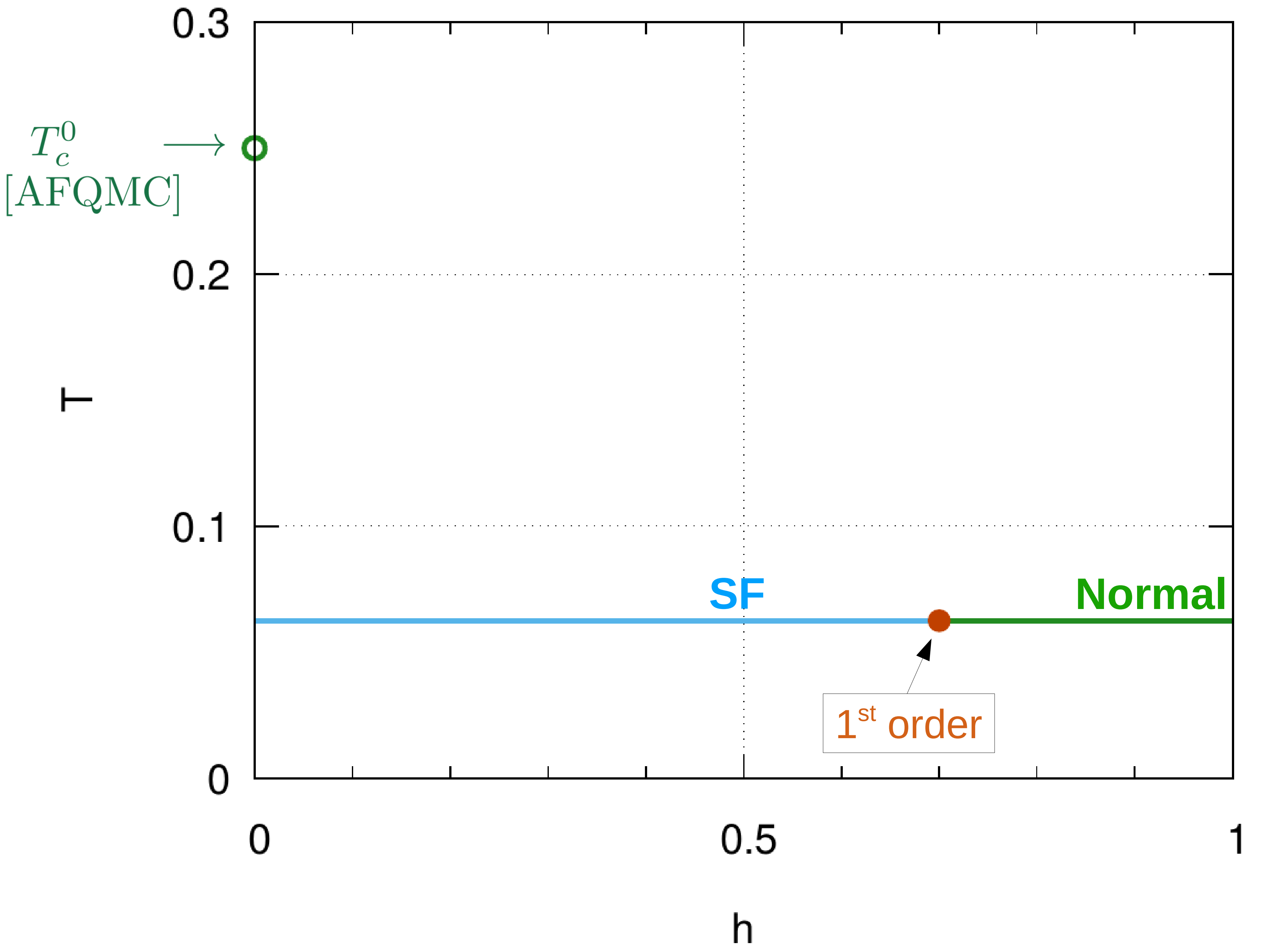
near (1st order) transition point
large energy barrier
→ *weak* essential singularity



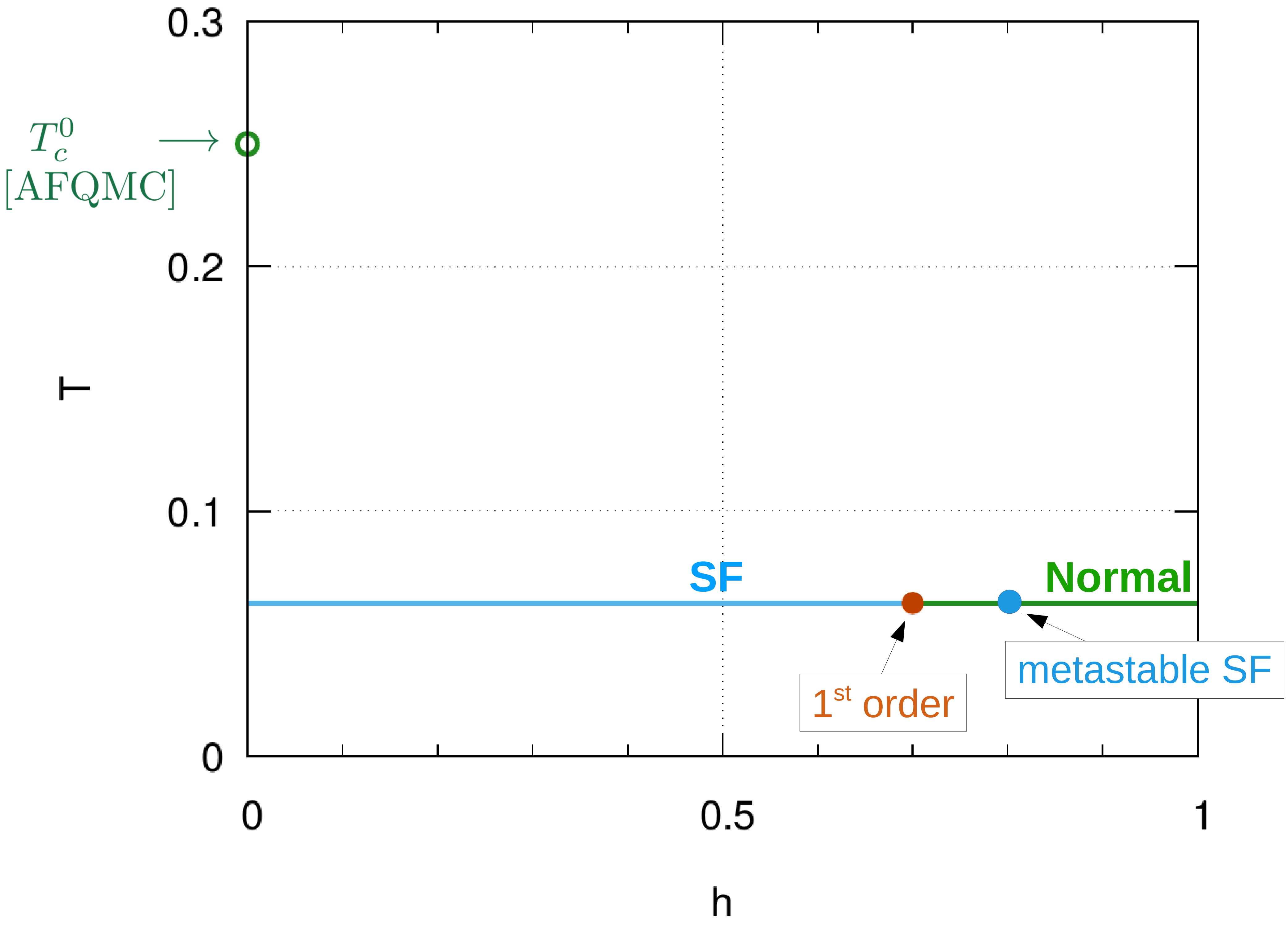
$$\text{Disc}\; Q(\xi) \underset{\xi\to\xi_c}{\sim} \exp\left(-\frac{\#}{|\xi-\xi_c|^{\#}}\right)$$

$$\Rightarrow \quad Q_N \underset{N\to\infty}{\sim} \left(\frac{1}{\xi_c}\right)^N \exp(-\#\, N^a)$$

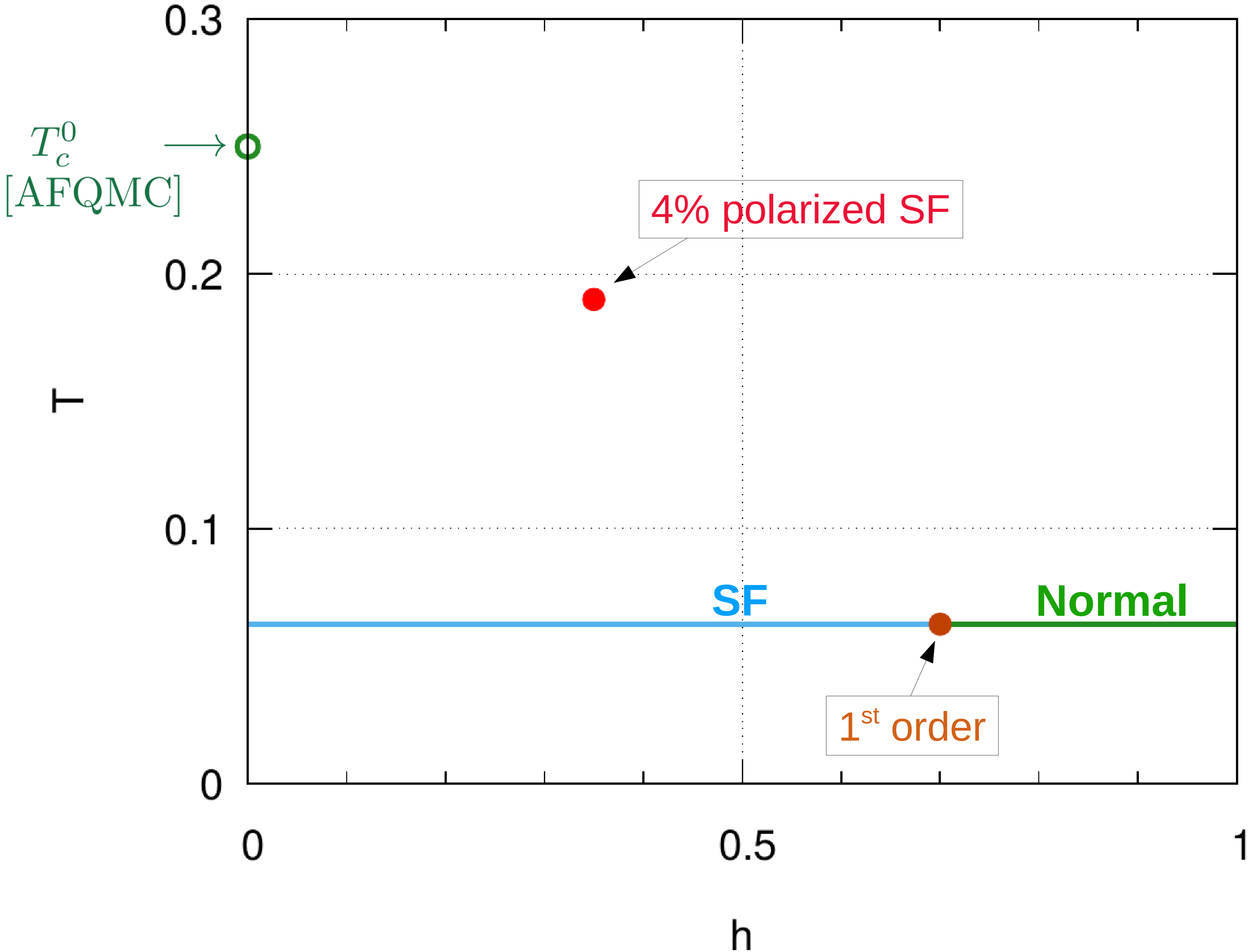
$$a < 1$$



0.3
0.2
0.1
0
T_c^0
[AFQMC]
T
SF
Normal
1st order
0
0.5
1
h



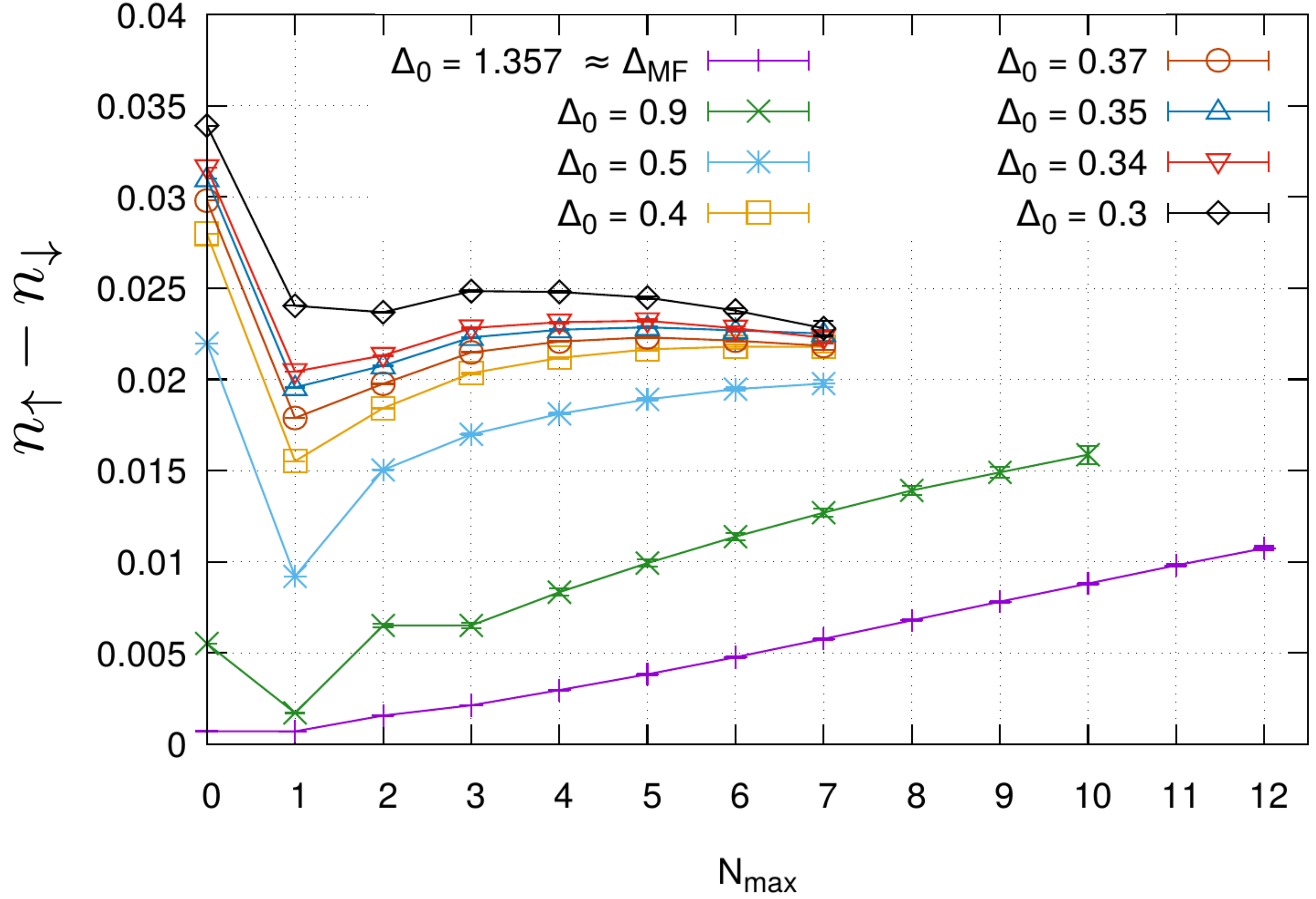
T_c^0 [AFQMC]
0.3
0.2
0.1
0
T
SF
Normal
1st order
metastable SF
0
0.5
1
h



T_c^0
[AFQMC]
4% polarized SF
SF
Normal
1st order
T
h
0.3
0.2
0.1
0
0
0.5
1

Polarized superfluid

$$T = 0.19 \approx \frac{3}{4}\, T_c^0, \quad h = 0.35$$



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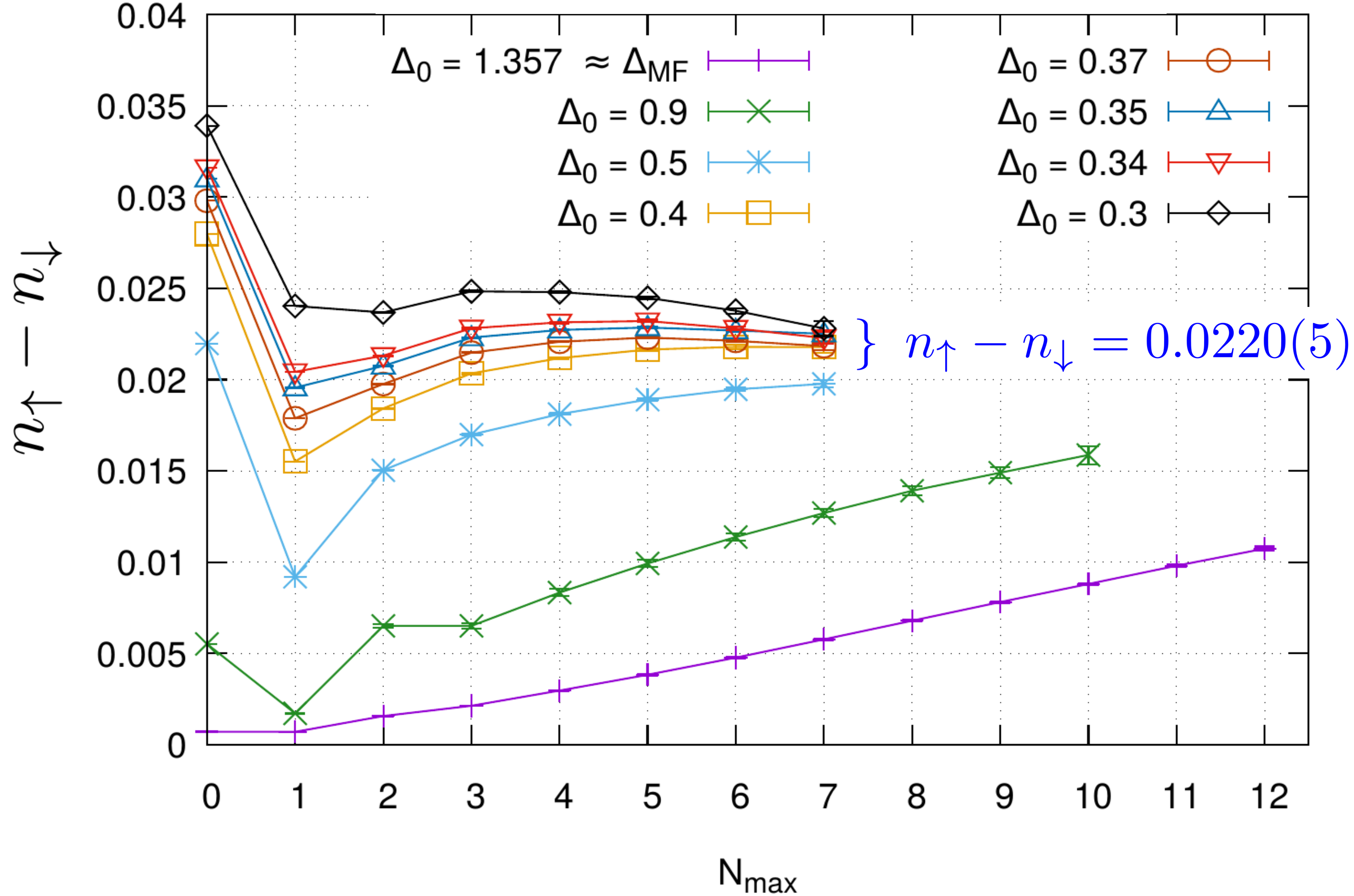

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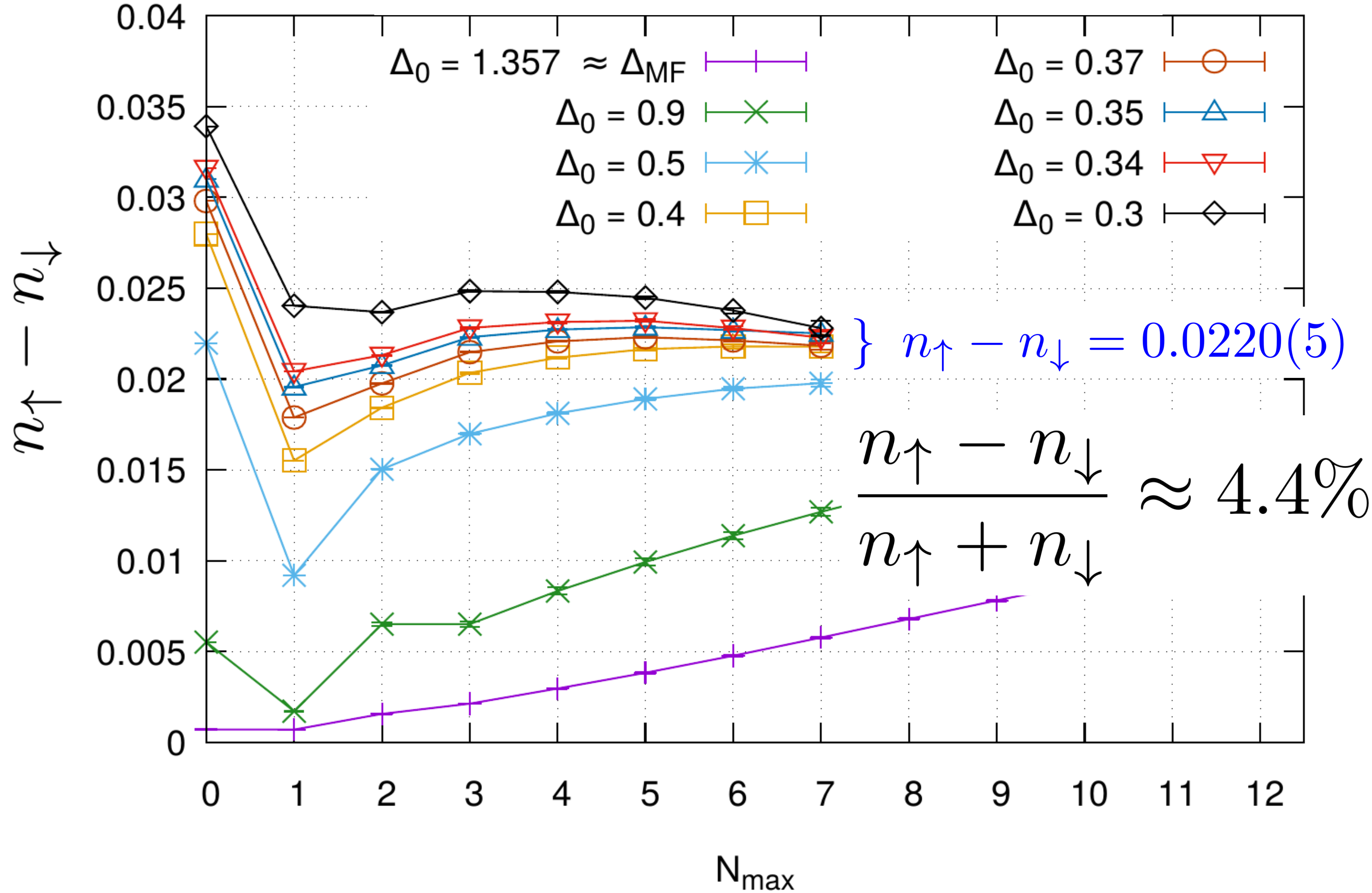

$$\frac{n_\uparrow - n_\downarrow}{n_\uparrow + n_\downarrow} \approx 4.4\%$$

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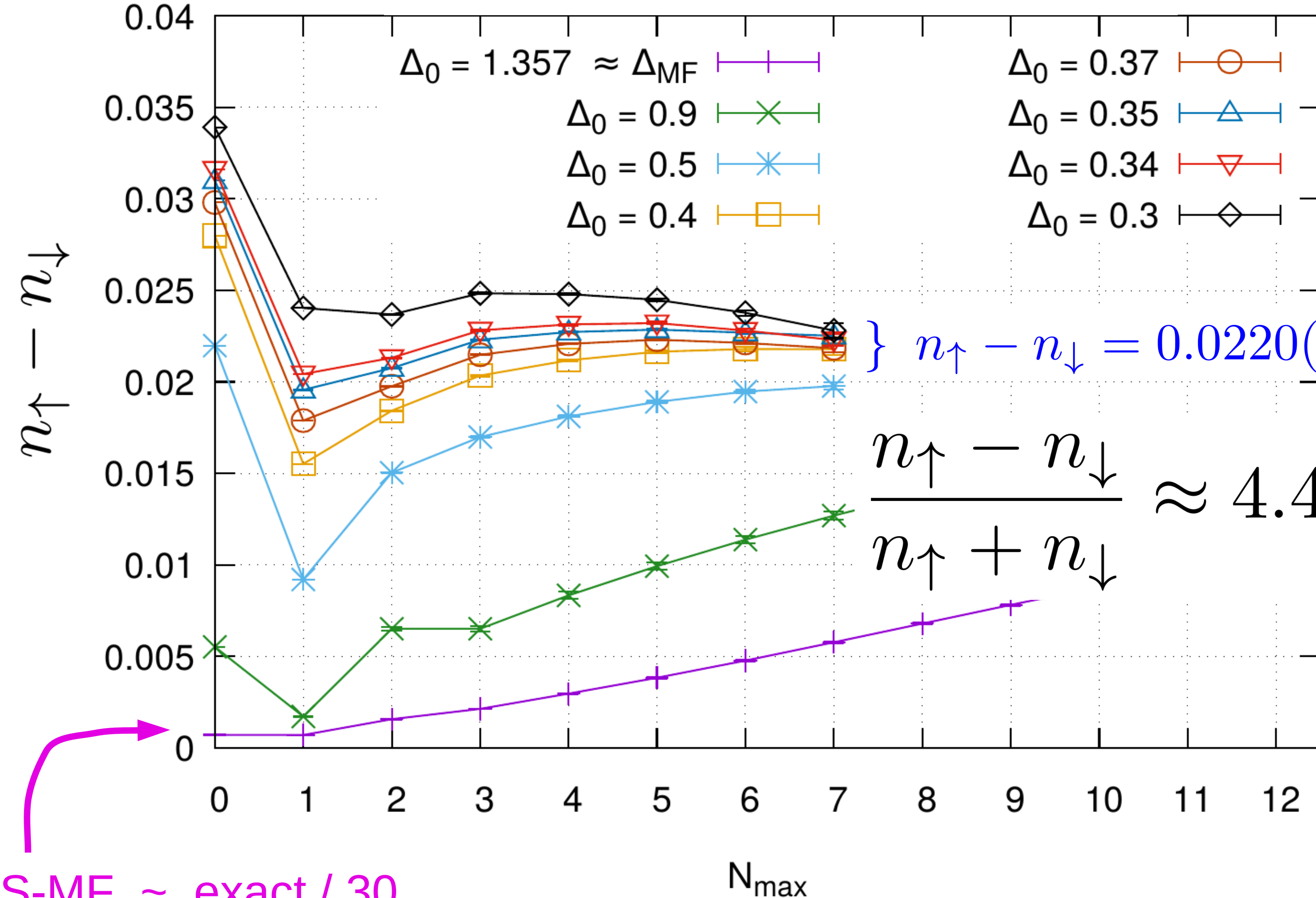

$$\frac{n_\uparrow - n_\downarrow}{n_\uparrow + n_\downarrow} \approx 4.4\%$$

BCS-MF ~ exact / 30

Polarized superfluid

Quasi-particle description

$$H_{eff} = \sum_{\substack{\vec{k} \\ \sigma=\pm 1}} \left(E_{\vec{k}} - h \cdot \sigma \right) b^{+}_{\vec{k}\sigma} \; b_{\vec{k}\sigma}$$

$E_{\vec{k}}$ → h - independent

$b^{+}_{\vec{k}\sigma} \; b_{\vec{k}\sigma}$: Fermionic

$$\left[b_{\vec{k},\sigma} \, ; N_{\uparrow} - N_{\downarrow} \right] = \sigma \cdot b_{\vec{k},\sigma}$$

Polarized superfluid

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$E_{\vec{k}}$ → h - independent

$$\left[b_{\vec{k},\sigma} \, , N_{\uparrow} - N_{\downarrow}\right] = \sigma \cdot b_{\vec{k},\sigma}$$

$$n_{\uparrow} - n_{\downarrow} = \frac{1}{L^3} \sum_{\vec{k}} \left[\frac{1}{e^{\beta(E_{\vec{k}} - h)} + 1} - \frac{1}{e^{\beta(E_{\vec{k}} + h)} + 1} \right]$$

Polarized superfluid

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$$= \underbrace{\left(\frac{1}{L^3} \sum_{\vec{k}} e^{-\beta E_{\vec{k}}} \right)}_{n_{qp}(T)} \left(e^{\beta h} - e^{-\beta h} \right)$$

Polarized superfluid

Quasi-particle description

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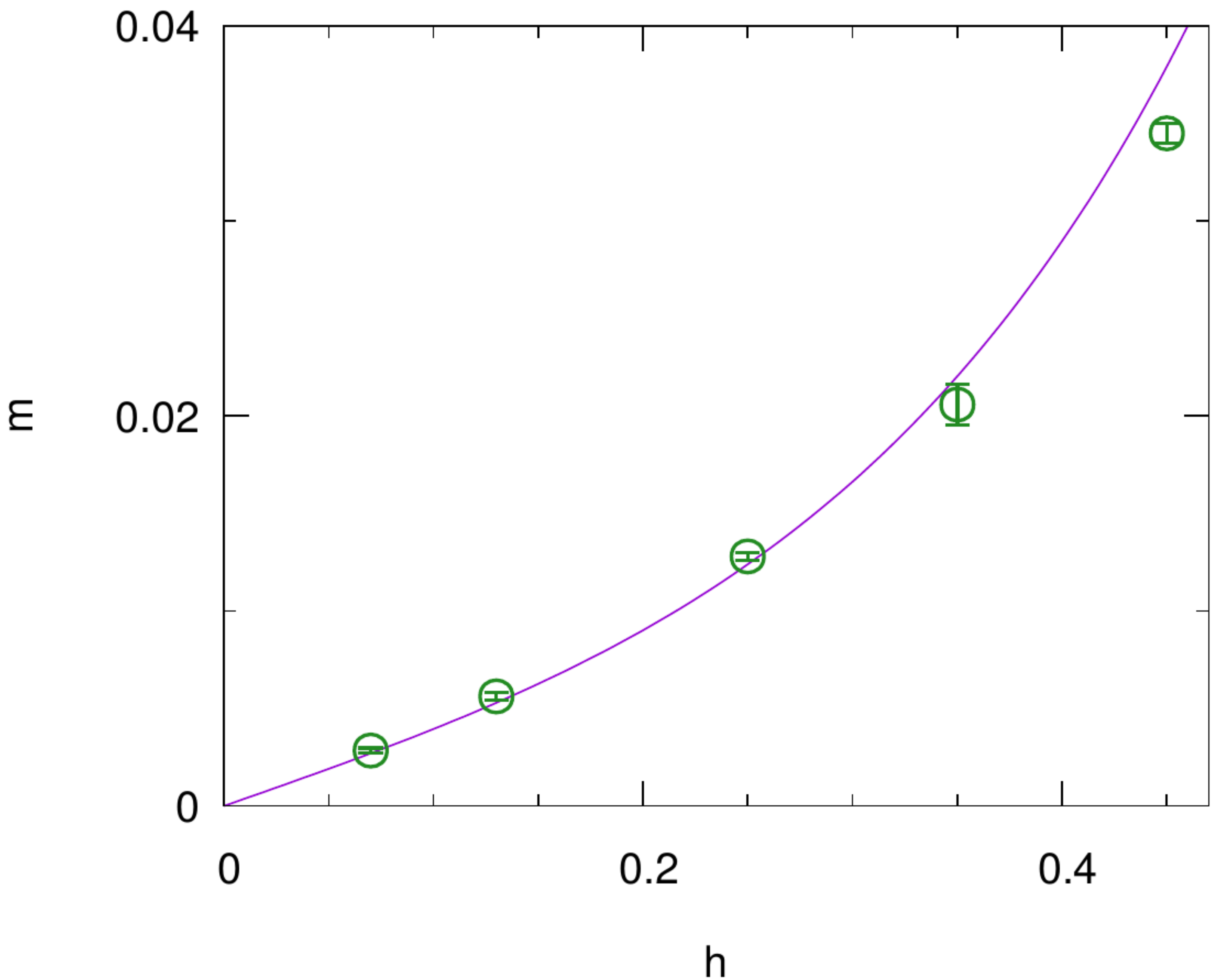
$$= \underbrace{\left(\frac{1}{L^3} \sum_{\vec{k}} e^{-\beta E_{\vec{k}}} \right)}_{n_{qp}(T)} \left(e^{\beta h} - e^{-\beta h} \right)$$

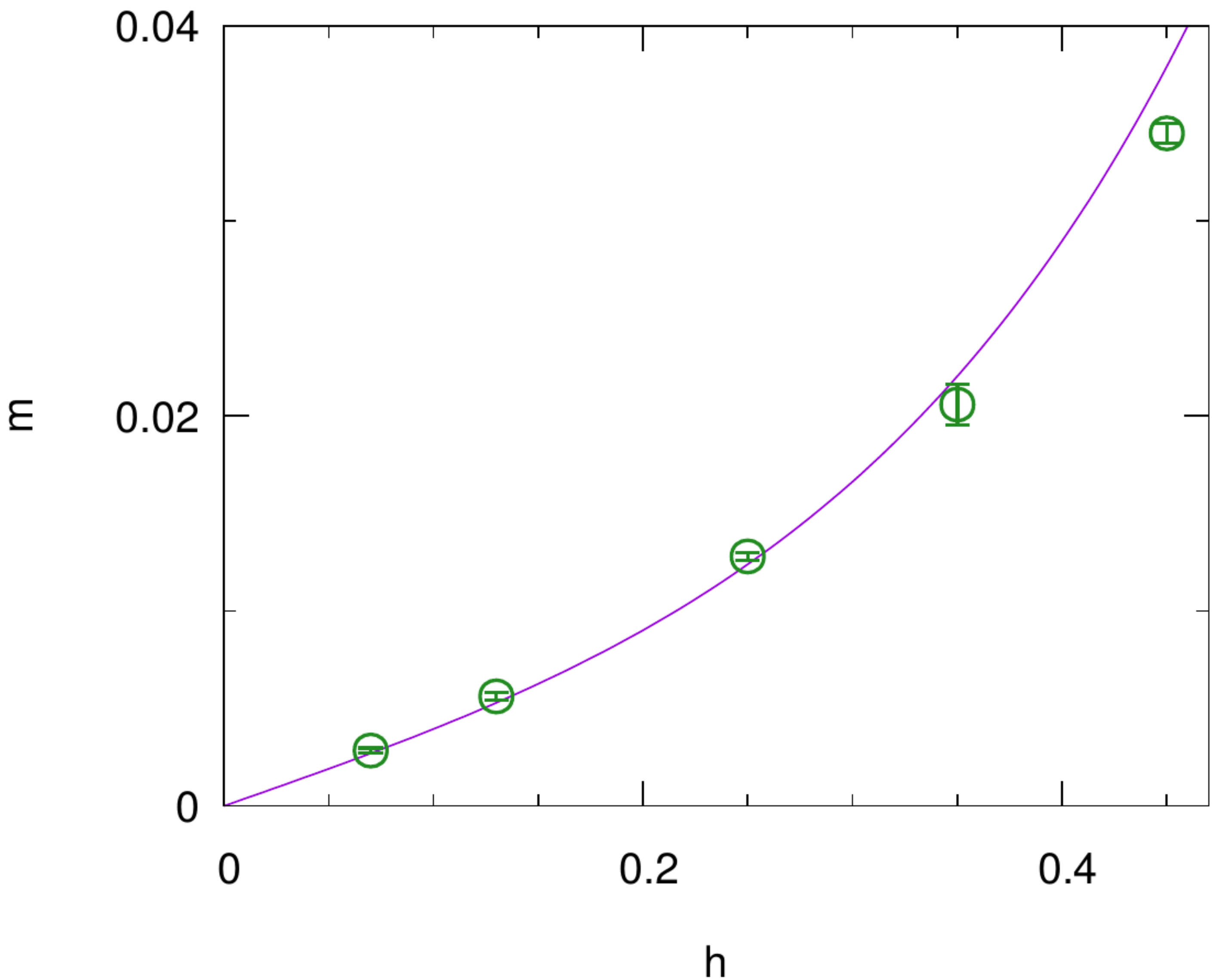
$$E_{\vec{k}} \geqslant E_g \quad \text{(gap)}$$

$$\Rightarrow \quad n_{qp}(T) < e^{-\beta E_g}$$



Fit: $m = n_{qp} \cdot (e^{\beta h} - e^{-\beta h})$

$n_{qp} \simeq 0,0036$



Fit: $m = n_{qp} \cdot (e^{\beta h} - e^{-\beta h})$

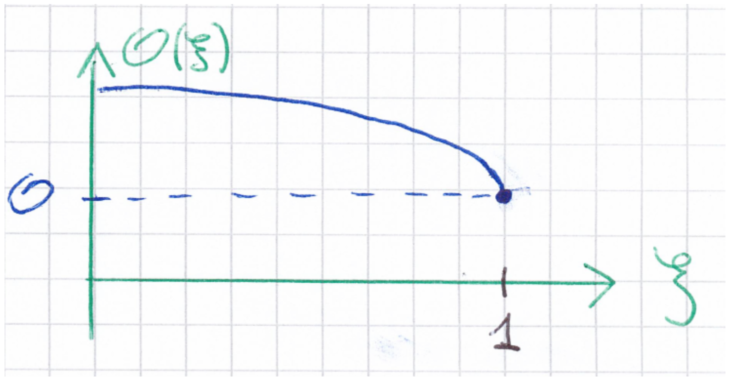
$n_{qp} \simeq 0{,}0036 \Rightarrow E_g < 1{,}07$

Large-order behavior of SF expansion

$(1-\xi) \Leftrightarrow$ **symmetry breaking field**

Goldstone singularity [Patashinskii-Pokrovskii / Brézin-Wallace, 1973]

$$\mathcal{O}(\xi) \underset{\xi\to 1^-}{=} \mathcal{O} + \text{cst}\,\sqrt{1-\xi} + \dots \qquad (T < T_c)$$



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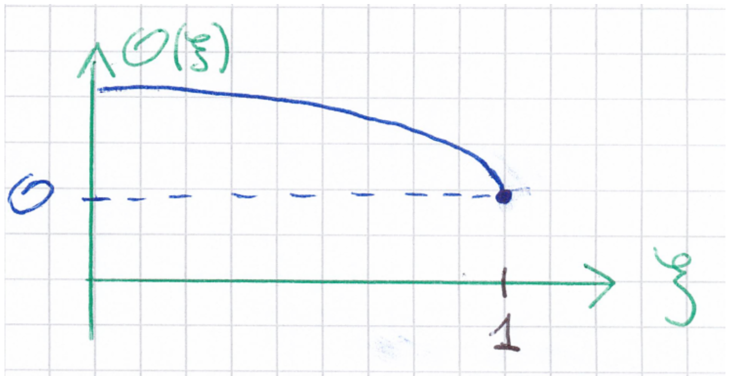
$$\frac{T}{\pi}\left(\frac{\mathcal{O}}{2\,D_s}\right)^{3/2}\sqrt{\Delta_0}$$

SF stiffness



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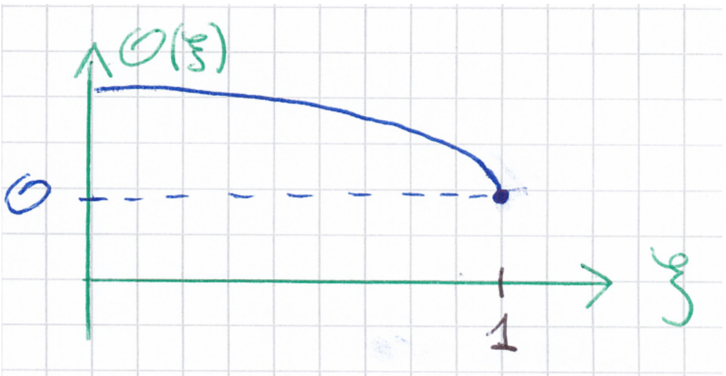
$$\text{cst} = \frac{T}{\pi}\left(\frac{\mathcal{O}}{2\,D_s}\right)^{3/2}\sqrt{\Delta_0}$$

D_s: SF stiffness



$$\Rightarrow \quad \mathcal{O}_N \underset{N\to\infty}{\sim} \frac{\text{cst}}{N^{3/2}}$$

$$P_N \underset{N\to\infty}{\sim} \frac{\text{cst}}{N^{5/2}}$$

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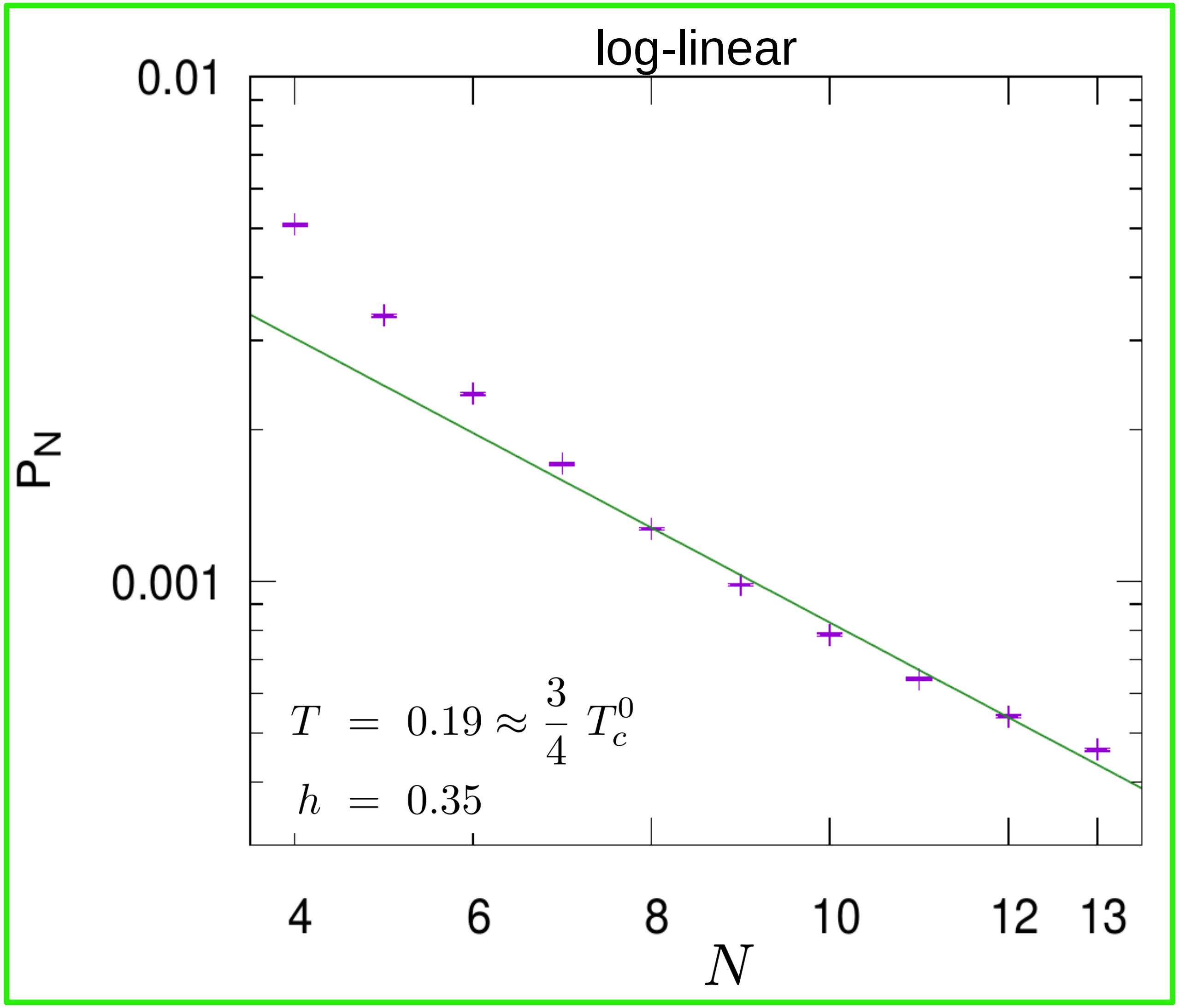

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D_s: SF stiffness

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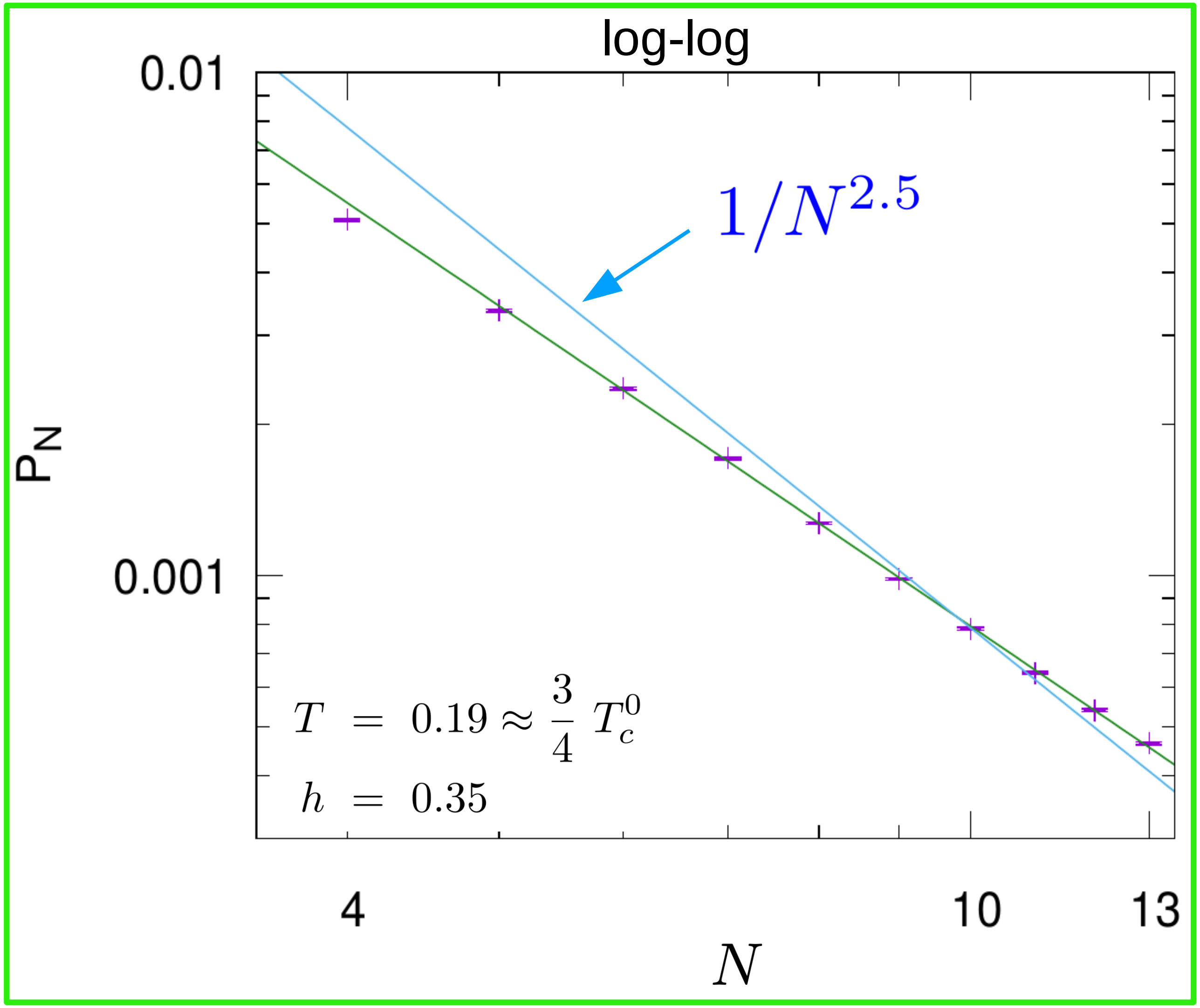
$$P_N \underset{N\to\infty}{\sim} \frac{\text{cst}}{N^{5/2}}$$



OUTLOOK

- bare vertex U → **ladders** [CDet, Normal: Simkovic *et al.* 2020]
  → strong coupling
  polarized SF at T=0 ? (gapless SF, “breached-pair phase”)
- **FFLO?** MF: yes $H_0 \ni \sum_{\vec{r}} \Delta_0(\vec{r})\, c^+_{\vec{r}\uparrow}\, c^+_{\vec{r}\downarrow} + h.c.$
- **2D**: BKT, algebraic order
- d-wave supercond. phase: $H_0 \ni \sum_{\vec{k}} \Delta_0(\vec{k})\, a^+_{\vec{k},\uparrow}\, a^+_{-\vec{k},\downarrow} + h.c.$
- **c° space**: unitary gas
  - filling → 0
  - directly in c° :
    zero convergence radius, but resummable
    as in Normal phase [Rossi *et al.* 2018] ?
- **nuclear phys:** expansion around Hartree-Fock-Bogolioubov (~ BCS-MF)
  yields promising results already at $N_{max} = 3$ [Tichai *et al.* 2018]

our suggestion: MC in *position*-representation
[≠ Li-Wallenberger-Gull 2020]