QED corrections to $\bar{B} \to \bar{K}\ell^+\ell^-$

IJCLab Seminar

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Plan

1. Introduction/Motivation
   Building blocks of the calculation. EFT Lagrangian.
2. Amplitudes and Phase Space
   Gauge Invariance. Choice of differential variables for the rate.
3. IR Divergences
   Phase space slicing. Soft and collinear divergences. Effect of photon energy cuts and choice of differential variables.
4. Results
5. Conclusion and Future Work

Based on 2009.00929, done in collaboration with G.Isidori and R.Zwicky

Introduction/Motivation

Lepton Flavour Universality (LFU) predicted by SM.

We consider the process $\bar{B} \to \bar{K}\ell^+\ell^-$ (Corresponds to FCNCs)

Define the ratio R_K

$$R_K\left[q^2_{\min}, q^2_{\max}\right] = \frac{\int_{q^2_{\min}}^{q^2_{\max}} dq^2 \frac{d\Gamma\left(B \to K\mu^+\mu^-\right)}{dq^2}}{\int_{q^2_{\min}}^{q^2_{\max}} dq^2 \frac{d\Gamma(B \to Ke^+e^-)}{dq^2}},$$

where $q^2 = (\ell^+ + \ell^-)^2$.

R_K is a theoretically *clean observable*.

SM predicts $R_K = 1$ (up to QED corrections, due to kinematic effects).

Introduction/Motivation

However, LHCb reports

$$R_K\left[1.1\mathrm{GeV}^2, 6\mathrm{GeV}^2\right] = 0.846^{+0.042+0.013}_{-0.039-0.012}$$

This represents a 3.1 σ deviation from the SM.

$\implies$ Hints to Physics beyond the SM.

However, need to make sure QED corrections properly accounted for in experiments (PHOTOS).

Despite smallness of $\frac{\alpha}{\pi} \approx 2\cdot 10^{-3}$, QED corrections are important as they can be enhanced by collinear logs of the lepton mass, $\ln\left(m_\ell/m_B\right)$.

Also, precise determination of CKM matrix elements.

Introduction/Motivation

Bordone et al. (*arXiv:1605.07633*) already performed a calculation to estimate QED corrections to R_K.

However, our work represents a more complete treatment since

- We work with the *full amplitudes* (real and virtual). Hence, we can capture effects beyond collinear $\ln m_\ell$ terms, such as $\ln m_K$ which are not necessarily small.
- Results at the *double* differential level are given, and hence they can be used for angular analysis (moments).
- We present a *detailed discussion on IR divergences*, and demonstrate explicitly the conditions under which they cancel.

We use an *EFT*, for $\bar{B}(p_B) \to \bar{K}(p_K)\,\ell^+(\ell_2)\,\ell^-(\ell_1)$.

$$\mathcal{L}^{\rm EFT}_{\rm int} = g_{\rm eff}\, L^\mu V^{\rm EFT}_\mu + {\rm h.c.}\ ,$$

$$V^{\rm EFT}_\mu = \sum_{n\geq 0} \frac{f^{(n)}_\pm(0)}{n!}(-D^2)^n[(D_\mu B^\dagger)K \mp B^\dagger(D_\mu K)]\ ,$$

where D_μ is the covariant derivative and $f^{(n)}_\pm(0)$ denotes the $n^{\rm th}$ derivative of the *$B \to K$ form factor $f_\pm(q^2)$*.

$$\begin{aligned} H^\mu_0(q^2_0) \ &\equiv \langle \bar{K}|V_\mu|\bar{B}\rangle = f_+(q^2_0)(p_B+p_K)^\mu + f_-(q^2_0)(p_B-p_K)^\mu \\ &= \langle \bar{K}|V^{\rm EFT}_\mu|\bar{B}\rangle + \mathcal{O}(e), \end{aligned}$$

$$L_\mu \equiv \bar{\ell}_1\Gamma^\mu\ell_2\,, \quad V_\mu \equiv \bar{s}\gamma_\mu(1-\gamma_5)b\,,$$

$$g_{\rm eff} \equiv -\frac{G_F}{\sqrt{2}}\lambda_{\rm CKM}, \qquad \Gamma^\mu \equiv \gamma^\mu(C_V + C_A\gamma_5) \qquad C_{V(A)} = \alpha\frac{C_{9(10)}}{4\pi}$$

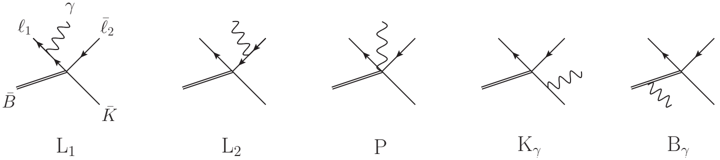
Introduction/Motivation

The radiative amplitude is computed using the ordinary QED Lagrangian for fermions and mesons,

$$\mathcal{L}_{\mathrm{QED}} \equiv \mathcal{L}_{\xi}(A) + \sum_{\psi=\ell_1,\ell_2} \bar{\psi}(i\not{D} - m_\ell)\psi + \sum_{M=B,K} (D_\mu M)^\dagger D^\mu M - m_M^2 M^\dagger M$$

Amplitudes and Phase Space: Real diagrams

We evaluate the *real diagrams*, given by



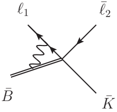
The real amplitude is *gauge invariant*, as expected, thanks to the P diagrams, which are generated by covariant derivatives in $\mathcal{L}_{\mathrm{int}}$.

Keeping the leading terms in the $k \to 0$ limit, i.e. at $\mathcal{O}(1/E_\gamma)$, $\mathcal{A}^{(1)}$ assumes the *Low or eikonal form*,

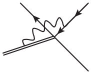
$$\mathcal{A}^{(1)}_{\mathrm{Low}} = e\mathcal{A}^{(0)} \sum_i \hat{Q}_i \frac{\epsilon^* \cdot p_i}{k \cdot p_i} \ .$$

This will be useful when discussing soft divergences.

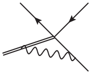
The *virtual diagrams* are given by



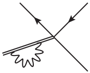
$\mathrm{B}_\gamma\mathrm{L}_1$



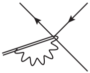
$\mathrm{B}_\gamma\mathrm{L}_2$



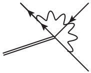
$\mathrm{B}_\gamma\mathrm{K}_\gamma$



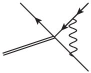
$\mathrm{B}_\gamma\mathrm{B}_\gamma$



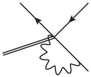
$\mathrm{P}\,\mathrm{B}_\gamma$



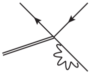
$\mathrm{K}_\gamma\mathrm{L}_1$



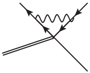
$\mathrm{K}_\gamma\mathrm{L}_2$



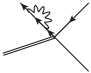
$\mathrm{P}\,\mathrm{K}_\gamma$



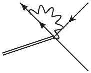
$\mathrm{K}_\gamma\mathrm{K}_\gamma$



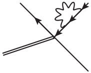
$\mathrm{L}_1\mathrm{L}_2$



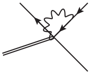
$\mathrm{L}_1\mathrm{L}_1$



$\mathrm{P}\,\mathrm{L}_1$



$\mathrm{L}_2\mathrm{L}_2$



$\mathrm{P}\,\mathrm{L}_2$

Amplitudes and Phase Space

The self-energy diagrams are calculated in the *on-shell scheme*.

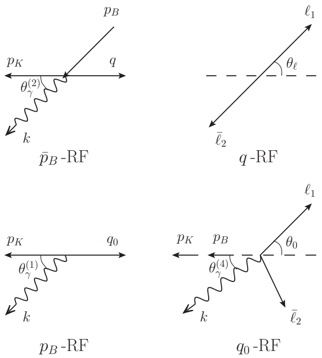
Like the real amplitude, the virtual amplitude is also *gauge invariant*, as expected.

We use *dimensional regularisation* to regulate soft divergences, as well as the UV divergences.

The UV divergences are treated using a "minimal subtraction" type scheme, and therefore the final result contains ambiguous finite terms.

$\Longrightarrow$ motivates further work to compute counterterms (and structure-dependent corrections) [ongoing].

Amplitudes and Phase Space: Differential Variables



$$\{q_a^2, c_a\} = \begin{cases} q_\ell^2 = (\ell_1+\ell_2)^2, & c_\ell = -\left(\frac{\vec{\ell_1}\cdot\vec{p}_K}{|\vec{\ell_1}||\vec{p}_K|}\right)_{q-\mathrm{RF}} \quad [\text{“Hadron collider”}]\ , \\ q_0^2 = (p_B - p_K)^2\ , & c_0 = -\left(\frac{\vec{\ell_1}\cdot\vec{p}_K}{|\vec{\ell_1}||\vec{p}_K|}\right)_{q_0-\mathrm{RF}} \quad [\text{“B-factory”}]\ , \end{cases}$$

where $q - \mathrm{RF}$ and $q_0 - \mathrm{RF}$ denotes the rest frames of $q \equiv \ell_1 + \ell_2$ and $q_0 \equiv p_B - p_K = q + k$ respectively.

Amplitudes and Phase Space: Radiative Rate

The *radiative rate* $\bar{B} \to \bar{K}\ell_1\bar{\ell}_2\gamma$ is given by

$$d^2\Gamma_{\bar{B}\to\bar{K}\ell_1\bar{\ell}_2\gamma} = \frac{1}{m_B}\left(\int \rho_a \left[|\mathcal{A}^{(1)}|^2 + \mathcal{O}(e^4)\right] d\Phi_\gamma\right) dq_a^2 dc_a \ ,$$

where $a = \{\ell, 0\}$.

Implement a *cut-off on the photon energy*,

$$\bar{p}_B^2 > m_B^2(1-\delta_{\rm ex})$$

where

$$\bar{p}_B^2 = (p_B - k)^2 = (\ell_1 + \ell_2 + p_K)^2.$$

The larger $\delta_{\rm ex}$ is, the more photon inclusive we are.

Amplitudes and Phase Space: Non-Radiative Rate

The *non-radiative* $\bar{B} \to \bar{K}\ell_1\bar{\ell}_2$ rate is given by

$$d^2\Gamma_{\bar{B}\to\bar{K}\ell_1\bar{\ell}_2} = \frac{\rho_\ell|_{\bar{p}_B^2\to m_B^2}}{m_B}\left\{|\mathcal{A}^{(0)}|^2 + 2\mathrm{Re}[\mathcal{A}^{(0)}(\mathcal{A}^{(2)})^*]\right\} dq^2 dc_\ell$$

Since there is *no photon-emission*, in this case there is no difference between the $\{q^2, c_\ell\}$- and $\{q_0^2, c_0\}$-variables.

IR Divergences

Split the differential rate as follows

$$d^2\Gamma_{\bar{B}\to\bar{K}\ell_1\bar{\ell}_2}(\delta_{\rm ex}) = d^2\Gamma^{\,\rm LO} + \frac{\alpha}{\pi}\sum_{i,j}\hat{Q}_i\hat{Q}_j\left(\mathcal{H}_{ij} + \mathcal{F}_{ij}^{(a)}(\delta_{\rm ex})\right)\, dq_a^2 dc_a,$$

where $d^2\Gamma^{\,\rm LO}$ corresponds to the zeroth order differential rate and $\mathcal{H}$ and $\mathcal{F}$ stand for the virtual and real contributions respectively.

$$\begin{aligned} \frac{\alpha}{\pi}\sum_{i,j}\hat{Q}_i\hat{Q}_j\mathcal{H}_{ij} &= \frac{1}{m_B}\rho_\ell|_{\bar{p}_B^2\to m_B^2} 2{\rm Re}[\mathcal{A}^{(2)*}\mathcal{A}^{(0)}]\ , \\ \frac{\alpha}{\pi}\sum_{i,j}\hat{Q}_i\hat{Q}_j\mathcal{F}_{ij}^{(a)} &= \frac{1}{m_B}\int d\Phi_\gamma\,\rho_a\,|\mathcal{A}^{(1)}|^2\ , \end{aligned}$$

IR Divergences

The integrals are split into *divergent parts* which can be done *analytically* and a necessarily regular part which is dealt with numerically.

$$\mathcal{H}_{ij} = \frac{d^2\Gamma^{\rm LO}}{dq^2 dc_\ell}\left(\tilde{\mathcal{H}}_{ij}^{(s)} + \tilde{\mathcal{H}}_{ij}^{(hc)}\right) + \Delta\mathcal{H}_{ij} \; ,$$

$$\mathcal{F}_{ij}^{(a)}(\delta_{\rm ex}) = \frac{d^2\Gamma^{\rm LO}}{dq^2 dc_\ell}\tilde{\mathcal{F}}_{ij}^{(s)}(\omega_s) + \tilde{\mathcal{F}}_{ij}^{(hc)(a)}(\underline{\delta}) + \Delta\mathcal{F}_{ij}^{(a)}(\underline{\delta}) \; ,$$

with $\tilde{\mathcal{H}}_{ij}^{(s)}$ ($\tilde{\mathcal{H}}_{ij}^{(hc)}$) and $\tilde{\mathcal{F}}_{ij}^{(s)}$ ($\tilde{\mathcal{F}}_{ij}^{(hc)(a)}$), containing all *soft* (*hard-collinear*) singularities, whereas $\Delta\mathcal{H}$ and $\Delta\mathcal{F}$ are regular.

We adopt the *phase space slicing method*, which requires the introduction of two auxiliary (unphysical) cut-offs $\omega_{s,c}$,

$$\underline{\delta} \equiv \{\delta_{\rm ex}, \omega_s, \omega_c\} \; , \quad \omega_s \ll 1 \; , \quad \frac{\omega_c}{\omega_s} \ll 1 \; .$$

IR Divergences

Phase Space slicing conditions

$$\bar{p}_B^2 \geq m_B^2(1-\omega_s) \iff E_\gamma^{p_B-\mathrm{RF}} \leq \frac{\omega_s m_B}{2}$$

$$k\cdot\ell_{1,2} \leq \omega_c m_B^2$$

In these regions of the phase space, the integrals become simple enough so that they can be done analytically.

In what follows, hard-collinear divergences should be understood as logs of the lepton mass, $\ln m_\ell$

IR Divergences: Soft

The *soft part* of the *real amplitude* ($E_\gamma^{p_B-\mathrm{RF}} \leq \frac{\omega_s m_B}{2}$), namely the *Low part* of the amplitude, is given by

$$\tilde{\mathcal{F}}_{ij}^{(s)}(\omega_s) \;=\; (2\pi)^2 \int_{\omega_s} \frac{-p_i \cdot p_j}{(k\cdot p_i)(k\cdot p_j)} d\Phi_\gamma$$

The sum $\tilde{\mathcal{H}}_{ij}^{(s)} + \tilde{\mathcal{F}}_{ij}^{(s)}(\omega_s)$ is free from *soft divergences* ($\frac{1}{\epsilon_{\mathrm{IR}}}$), as well as *soft collinear divergences* ($\frac{1}{\epsilon_{\mathrm{IR}}} \ln m_\ell, \ln^2 m_\ell$).

$\Longrightarrow$ Ensures their *cancellation at the differential level*.

This result is independent of the choice of differential variables, and on the value of the cut on the photon energy.

Of course, terms proportional to $\ln \omega_s$ survive, and only cancel in the end when all contributions to the rate are added.

IR Divergences: Hard Collinear from Virtual

We now turn to *hard collinear divergences,* $\ln m_{\ell_1}$.

We follow largely the method in the review paper *Harris and Owens '02* performed in *dim reg*, which we then adapted to *mass reg*.

For the sake of illustration, we focus on the contribution to $\ln m_{\ell_1}$ ($\ln m_{\ell_2}$ can be obtained in a completely analogous fashion).

FIRST, the contribution to $\ln m_{\ell_1}$ from the virtual diagrams can be easily collected, and reads

$$\tilde{\mathcal{H}}^{(hc)} = \left(\frac{3}{2} - 2\right) \hat{Q}_{\ell_1}^2 \ln\left(\frac{m_{\ell_1}}{\mu}\right)$$

IR Divergences: Hard Collinear from Soft

SECOND, we consider the contribution coming from the *soft* region of the real integral ($E_\gamma^{p_B-\mathrm{RF}} \leq \frac{\omega_s m_B}{2}$).

$$\mathcal{F}_{ij}^{(s)}(\omega_s) = \frac{(\pi\mu^2)^\epsilon}{2\pi}\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}\int_0^{(E_\gamma^{(n)})^{\max}} \frac{dE_\gamma^{(n)}}{\left(E_\gamma^{(n)}\right)^{1+2\epsilon}}$$

$$\times \int_0^\pi \frac{d\theta_\gamma}{\sin^{2\epsilon-1}\theta_\gamma}\int_0^\pi \frac{d\phi_\gamma}{\sin^{2\epsilon}\phi_\gamma}\left[\frac{-(E_\gamma^{(n)})^2\, p_i\cdot p_j}{(k\cdot p_i)(k\cdot p_j)}\right] .$$

In particular, the angular integrals are needed up to $\mathcal{O}(\epsilon)$, and for some specific values of i and j, the results are not known in the literature!

IR Divergences: Hard Collinear from Soft

However, through private communication with Gabor Somogyi (unpublished work), we were able to obtain the necessary results, in terms of *generalised polylogs of weight 2*.

When expanded in small m_{ℓ_1}, the collinear $\ln m_{\ell_1}$ can be collected.

After a very long and painful calculation, assembling all bits and pieces and using charge conservation, we have

$$\mathcal{F}^{(s)}(\omega_s)|_{\ln m_{\ell_1}} = \hat{Q}_{\ell_1}^2 \ln m_{\ell_1} \left[-1 - 2\ln\left(\bar{z}(\omega_s)\right)\right]$$

where

$$\bar{z}(\omega_s) = \frac{\omega_s m_B^2}{m_B^2 - (p_K + \ell_2)^2} \,.$$

IR Divergences: Hard Collinear

FINALLY, we compute the $\ln m_{\ell_1}$ contribution from the collinear region ($k\cdot\ell_{1,2} \leq \omega_c m_B^2$) of the phase space of the real radiation.

$$
\begin{aligned}
\frac{\alpha}{\pi}\tilde{\mathcal{F}}^{(hc,a)}(\underline{\delta}) &= \frac{\alpha}{\pi}\sum_{i,j}\hat{Q}_i\hat{Q}_j\tilde{\mathcal{F}}_{ij}^{(hc,a)}(\underline{\delta}) \\
&= \frac{1}{m_B}\int_{\omega_s}^{\delta_{\rm ex}} \rho_a^{\ell_1||\gamma}(\omega_c)\,|\mathcal{A}^{(1)}_{\ell_1||\gamma}|^2 d\Phi_\gamma
\end{aligned}
$$

where $|\mathcal{A}^{(1)}_{\ell_1||\gamma}|^2$ is the part of $|\mathcal{A}^{(1)}|^2$ proportional to $1/(k\cdot\ell_1)$ when $m_{\ell_1}\to 0$ which includes contributions beyond the Low term.

IR Divergences: Hard Collinear

In the collinear region, one decomposes the phase space as follows

$$d\Phi_{\bar{B}\to\bar{K}\ell_1\bar{\ell}_2\gamma} = d\Phi_{\bar{B}\to\bar{K}\ell_{1\gamma}\bar{\ell}_2}\frac{1}{16\pi^2}dz\, d\ell_{1\gamma}^2 \ .$$

where the parametrisation

$$\begin{aligned} \ell_1 &= z\ell_{1\gamma} \\ k &= (1-z)\ell_{1\gamma} \equiv \bar{z}\ell_{1\gamma} \\ \Longrightarrow\ \ell_{1\gamma} &\equiv \ell_1 + k \end{aligned}$$

has been used.

$d\Phi_{\bar{B}\to\bar{K}\ell_{1\gamma}\bar{\ell}_2}$ represents the non-radiative phase space factor, with $\ell_{1\gamma}$ considered to be one final state particle.

IR Divergences: Hard Collinear

Furthermore, the real amplitude squared simplifies to

$$|\mathcal{A}^{(1)}_{\ell_1||\gamma}|^2 = \frac{e^2}{(k\cdot\ell_1)}\hat{Q}^2_{\ell_1}\left(\tilde{P}_{f\to f\gamma}(z) - \frac{m^2_{\ell_1}}{k\cdot\ell_1}\right)|\mathcal{A}^{(0)}_{\bar{B}\to\bar{K}\ell_{1\gamma}\bar{\ell}_2}|^2,$$

where $\tilde{P}_{f\to f\gamma}(z)$ is the collinear emission part of the splitting function for a fermion to a photon

$$\tilde{P}_{f\to f\gamma}(z) \equiv \left(\frac{1+z^2}{1-z}\right) \ ,$$

Note that while the $m^2_{\ell_1}/(k\cdot\ell_1)$ term is immaterial for the $\ln m_{\ell_1}$ contribution per se, it is essential for the numerics as it contributes to $\ln\omega_s$ terms, which have to cancel in the sum of all contributions to the decay rate.

IR Divergences: Hard Collinear

The $d\ell_{1\gamma}^2 \equiv 2d(k\cdot\ell_1)$ integral gives the $\ln m_{\ell_1}$ term:

$$\int_{\frac{1-z}{2z}m_{\ell_1}^2}^{\omega_c m_B^2} \frac{d(k\cdot\ell_1)}{k\cdot\ell_1} = \ln\frac{2\omega_c z}{\hat{m}_{\ell_1}^2(1-z)}$$

Note that in dim reg, this integral would instead produce a pole in ϵ.

The integration boundaries on $d\ell_{1\gamma}^2$ correspond to the phase space slicing condition.

Hatted quantities are normalised w.r.t. the m_B mass, i.e. $\hat{m}_K = m_K/m_B$.

IR Divergences: Hard Collinear in $\{q_0^2, c_0\}$-vars

In the case of the $\{q_0^2, c_0\}$-variables, the $z-$integration (*from $z(\delta_{ex})$ to $z(\omega_s)$*) factorises completely, and can be easily performed analytically.

$$z(\delta) = 1 - \frac{\delta m_B^2}{m_B^2 - (p_K + \ell_2)^2} \ ,$$

$$\tilde{\mathcal{F}}^{(hc,0)}(\underline{\delta}) = \frac{\lambda^{1/2}(m_B^2, q_0^2, m_K^2)}{2^9 \pi^3 m_B^3} |\mathcal{A}^{(0)}(q_0^2, c_0)|^2 \, A(\delta_{\rm ex}, \omega_s) \hat{Q}_{\ell_1}^2 \ln m_{\ell_1} \ ,$$

where

$$\begin{aligned} A(\delta_{\rm ex}, \omega_s) \quad &= \quad \frac{1}{2} \bar{z}(\delta_{\rm ex})(3 + z(\delta_{\rm ex})) + 2 \ln \frac{\bar{z}(\omega_s)}{\bar{z}(\delta_{\rm ex})} \\ &\stackrel{z(\delta_{\rm ex}) \to 0}{\longrightarrow} \ \frac{3}{2} + 2 \ln \bar{z}(\omega_s) \ , \end{aligned}$$

The second line is the result in the fully photon inclusive case.

IR Divergences: Cancellation of hc logs in $\{q_0^2, c_0\}$

Putting the above results together, one has

$$d^2\Gamma^{(0)}\Big|^{(hc)}_{\ell_1||\gamma,\,\ln m_{\ell_1}} = d^2\Gamma^{\mathrm{LO}}_{\bar{B}\to\bar{K}\ell_{1\gamma}\bar{\ell}_2}\left(\frac{\alpha}{\pi}\right)\hat{Q}^2_{\ell_1}\left[\frac{3}{2} + 2\ln\bar{z}(\omega_s)\right]\ln m_{\ell_1}\ ,$$

Summing all $\ln m_{\ell_1}$ contributions, one has

$$\frac{d^2\Gamma}{dq_0^2\,dc_0}\bigg|_{\ln m_{\ell_1}} = \frac{d^2\Gamma^{\mathrm{LO}}}{dq_0^2\,dc_0}\left(\frac{\alpha}{\pi}\right)\hat{Q}^2_{\ell_1}\ln m_{\ell_1}\times C^{(0)}_{\ell_1}\ ,$$

where

$$C^{(0)}_{\ell_1} = \left[\frac{3}{2} + 2\ln\bar{z}(\omega_s)\right]_{\tilde{\mathcal{F}}^{(hc)}} + \left[-1 - 2\ln\bar{z}(\omega_s)\right]_{\tilde{\mathcal{F}}^{(s)}} + \left[\frac{3}{2} - 2\right]_{\tilde{\mathcal{H}}} = 0$$

$\Longrightarrow$ Vanishes in fully photon inclusive limit!

IR Divergences: Hard Collinear Real in $\{q^2, c_\ell\}$-vars

We now consider the same calculation in the $\{q^2, c_\ell\}$*-variables*.

The virtual contribution and the contribution from the soft region of the phase space both remain unchanged.

$\Longrightarrow$ *Only the contribution from the collinear region needs to be computed, ie.* $\tilde{\mathcal{F}}^{(hc,\ell)}$

Setting $m_K \to 0$, for simplicity in the illustration, we have

$$q_0^2 = \frac{q^2}{z} \; , \quad c_0 = \frac{c_\ell(1+z)+1-z}{c_\ell(1-z)+1+z} \; ,$$

and

$$dq_0^2 dc_0 \; = \; 4(c_\ell(1-z)+1+z)^{-2} dq^2 dc_\ell$$

IR Divergences: Hard Collinear Real in $\{q^2, c_\ell\}$-vars

The $z-$integral does *not* factorise in this case, and we have

$$\tilde{\mathcal{F}}^{(hc,\ell)}(\underline{\delta}) = -\frac{\hat{Q}_{\ell_1}^2}{2^7\pi^3 m_B^3}\int_{\max(z_{\rm inc}(c_\ell), z_{\delta_{\rm ex}}(c_\ell))}^{\max(z_{\rm inc}(c_\ell), z_{\omega_s}(c_\ell))} dz$$

$$\times \left[\frac{|\mathcal{A}^{(0)}(q_0^2, c_0)|^2 \lambda^{1/2}(q_0^2, m_B^2, 0)}{(c_\ell(1-z)+1+z)^2}\tilde{P}_{f\to f\gamma}(z)\,\ln m_{\ell_1}\right] ,$$

where $c_0 = c_0(c_\ell)$, and $z_\delta(c_\ell)$ implements the photon energy cut. The boundaries for the z-integral are given by

$$z_{\rm inc}(c_\ell)|_{m_K\to 0} = \hat{q}^2 \ , \quad z_\delta(c_\ell)|_{m_K\to 0} = \frac{1+\hat{q}^2-\delta+c_\ell(1-\hat{q}^2-\delta)}{1+\hat{q}^2+\delta+c_\ell(1-\hat{q}^2-\delta)}$$

IR Divergences: Cancellation of hc logs in $\{q^2, c_\ell\}$

This time, adding all the contributions, one finds

$$\left.\frac{d^2\Gamma}{dq^2 dc_\ell}\right|_{\ln m_{\ell_1}} = \frac{\alpha}{\pi} \hat{Q}_{\ell_1}^2 K_{\mathrm{hc}}(q^2, c_\ell) \ln m_{\ell_1},$$

where $K_{\mathrm{hc}}(q^2, c_\ell)$ is a non-vanishing function, *even in the fully photon inclusive limit*.

However, upon integration over q^2 and c_ℓ, it vanishes, ie.

$$\int_0^{m_B^2} dq^2 \int_{-1}^{1} dc_\ell \; K_{\mathrm{hc}}(q^2, c_\ell) = 0,$$

as expected.

IR Divergences: Structure-dependent terms (Real)

We also showed that all collinear logs $\ln m_\ell$ are captured by the EFT used.

The argument relies on the *gauge invariance* of the real amplitude $\mathcal{A}^{(1)}$ ($k \cdot \mathcal{A}^{(1)} = 0$), and on the fact that in the collinear region, $k - \ell_1 = \mathcal{O}(m_{\ell_1}^2)$.

The above two conditions then implies $\ell_1 \cdot \mathcal{A}^{(1)} = \mathcal{O}(m_{\ell_1}^2)$ in the collinear region.

Results

We consider *relative* QED corrections. For a single differential in $\frac{d}{dq_a^2}$,

$$\Delta^{(a)}(q_a^2;\delta_{\rm ex}) = \left(\frac{d\Gamma^{\rm LO}}{dq_a^2}\right)^{-1}\frac{d\Gamma(\delta_{\rm ex})}{dq_a^2}\Big|_{\alpha}\,,$$

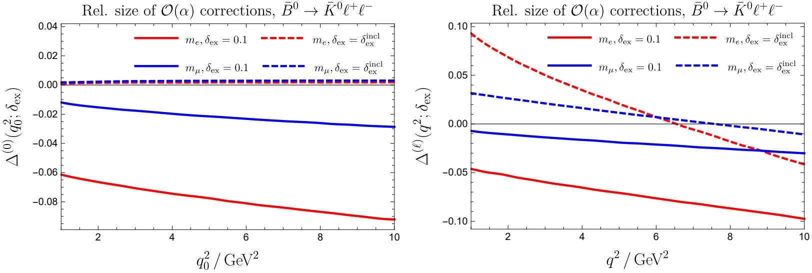
where the numerator and denominator are integrated separately over $\int_{-1}^{1} dc_a$ respectively. In addition, we define the single differential in $\frac{d}{dc_a}$

$$\Delta^{(a)}(c_a,[q_1^2,q_2^2];\delta_{\rm ex}) = \left(\int_{q_1^2}^{q_2^2}\frac{d^2\Gamma^{\rm LO}}{dq_a^2 dc_a}dq_a^2\right)^{-1}\int_{q_1^2}^{q_2^2}\frac{d^2\Gamma(\delta_{\rm ex})}{dq_a^2 dc_a}dq_a^2\Big|_{\alpha}\,,$$

where the non-angular variable is binned.

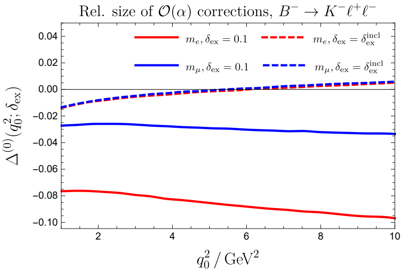
It is important to integrate the QED correction and the LO separately as this corresponds to the experimental situation.

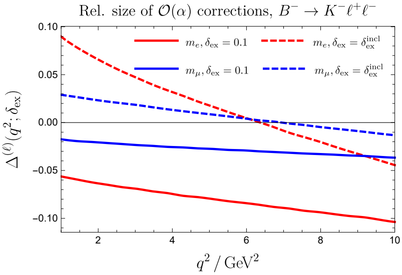
Results: $\bar{B}^0 \to \bar{K}^0 \ell^+ \ell^-$ in q_a^2



- In photon-inclusive case ($\delta_{\rm ex} = \delta_{\rm ex}^{\rm inc}$, dashed lines), all IR sensitive terms cancel in the q_0^2 variable locally.
- (Approximate) lepton universality on the plots on the left.
- Effects due to the photon energy cuts are sizeable since hard-collinear logs do not cancel in that case. More pronounced for electrons.

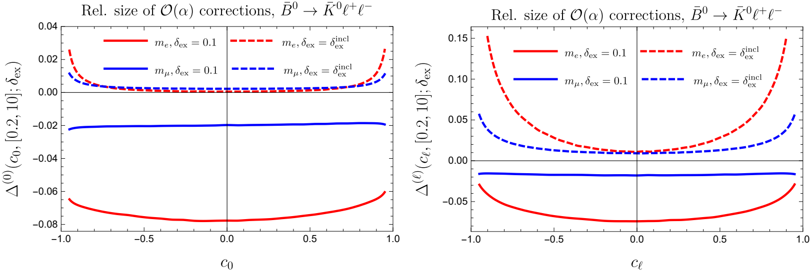
Results: $B^- \to K^- \ell^+ \ell^-$ in q_a^2





- Same comments as above apply.
- In the charged case, however, we see finite effects of the $\mathcal{O}(2\%)$ due to $\ln \hat{m}_K$ "collinear logs" which do not cancel.

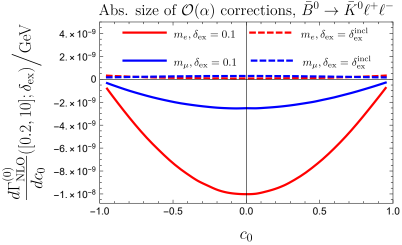
Results: $\bar{B}^0 \to \bar{K}^0 \ell^+ \ell^-$ in c_a

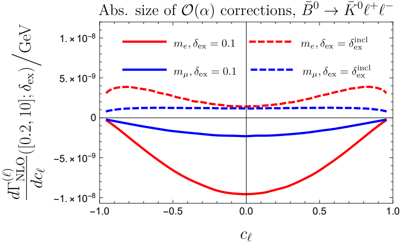


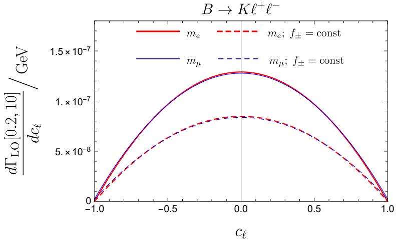
Enhanced effect towards the endpoints $\{-1, 1\}$ is partly due to the special behaviour of the LO differential rate which behaves like $\propto (1 - c_\ell^2) + \mathcal{O}(m_\ell^2)$ and explains why the effect is less pronounced for muons.

Even in c_ℓ. Almost even in c_0 (up to non-collinear effects).

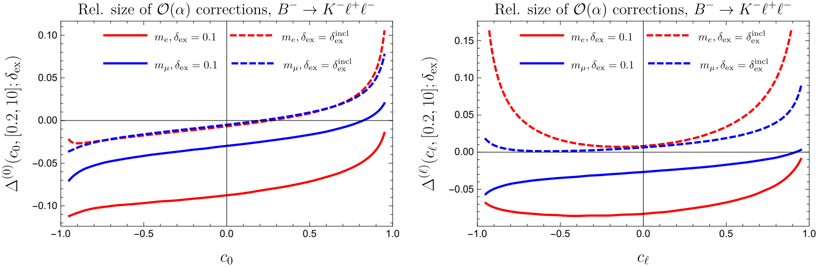
Results: $\bar{B}^0 \to \bar{K}^0 \ell^+ \ell^-$ in c_a







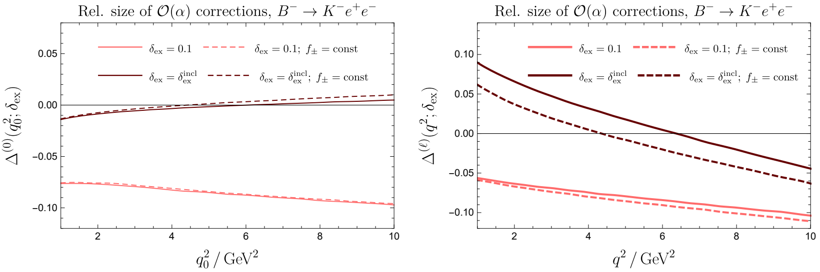
Results: $B^- \to K^-\ell^+\ell^-$ in c_a



- Same comments as before apply.
- More enhanced than the neutral meson case.
- 'Collinear' $\ln m_K$ odd in c_0/c_ℓ.

Results: Distortion of the $\bar{B} \to \bar{K}\ell^+\ell^-$ spectrum

Distortion of the $\bar{B} \to \bar{K}\ell^+\ell^-$ spectrum due to γ-radiation



Effects are more prominent in the photon-inclusive case ($\delta_{\rm ex} = \delta_{\rm ex}^{\rm inc}$, in brown) since there is more phase space for the q^2- and q_0^2-variables to differ.

$\Longrightarrow$ *Best to report results in q_0^2*

Results: Distortion of the $\bar{B} \to \bar{K}\ell^+\ell^-$ spectrum

To understand the distortion better, consider the following analysis in the collinear region:

$$|\mathcal{A}^{(0)}(q_0^2, c_0)|^2 \propto f_+(q_0^2)^2 = f_+(q^2/z)^2.$$

Since $z < 1$ in general, it is clear that momentum transfers of a higher range are probed.

Results: Distortion of the $\bar{B} \to \bar{K}\ell^+\ell^-$ spectrum

For example, when $c_\ell = -1$, maximising the effect, one gets

$$z_{\delta_{\rm ex}}(q^2)\Big|_{c_\ell=-1} = \frac{q^2}{q^2 + \delta_{\rm ex} m_B^2} \ , \quad (q_0^2)_{\rm max} = q^2 + \delta_{\rm ex} m_B^2 \ ,$$

For $\delta_{\rm ex} = 0.15$, $q^2 = 6\,{\rm GeV}^2$ one has $(q_0^2)_{\rm max} = 10.18\,{\rm GeV}^2$

$\Longrightarrow$ *Problematic for probing R_K in $q^2 \in [1, 6]$ GeV^2 range, due to charmonium resonances!*

Furthermore, in photon-inclusive case, the lower boundary for z becomes $z_{\rm inc}(c_\ell)|_{m_K \to 0} = \hat{q}^2$ such that $(q_0^2)_{\rm max} = m_B^2$.

$\Longrightarrow$ *Entire spectrum is probed for any fixed value of q^2*

Results: LFU and R_K

Recall

$$R_K\big|_{q_0^2\in[q_1^2,q_2^2]\,\mathrm{GeV}^2} = \frac{\Gamma[\bar{B}\to\bar{K}\mu^+\mu^-]}{\Gamma[\bar{B}\to\bar{K}e^+e^-]}\Big|_{q_0^2\in[q_1^2,q_2^2]\,\mathrm{GeV}^2} \approx 1+\Delta_{\mathrm{QED}}R_K .$$

The net QED correction that should be applied to R_K according to our analysis amounts to

$$\Delta_{\mathrm{QED}}R_K \approx \frac{\Delta\Gamma_{K\mu\mu}}{\Gamma_{K\mu\mu}}\Bigg|^{m_B^{\mathrm{rec}}=5.175\,\mathrm{GeV}}_{q_0^2\in[1,6]\,\mathrm{GeV}^2} - \frac{\Delta\Gamma_{Kee}}{\Gamma_{Kee}}\Bigg|^{m_B^{\mathrm{rec}}=4.88\,\mathrm{GeV}}_{q_0^2\in[1,6]\,\mathrm{GeV}^2} \approx +1.7\%$$

Results: LFU and R_K

Well below experimental errors:

$$R_K\left[1.1\mathrm{GeV}^2, 6\mathrm{GeV}^2\right] = 0.846^{+0.042+0.013}_{-0.039-0.012}$$

However, effect of cuts can be significant. In Bordone et al. (*arXiv:1605.07633*), in addition to the above energy cuts, a tight angle cut was also used, and a correction to R_K of

$$\Delta_{\mathrm{QED}} R_K \approx +3.0\% \quad ,$$

was reported.

$\Longrightarrow$ *Highlights the importance of building a MC to cross-check the experimental analysis (ongoing work)*

Summary and Conclusion

- *Soft* and *soft collinear* divergences always cancel at the differential level, independent of differential variables, and experimental cut-off on the photon energy.
- Without a cut-off on the photon energy, *hard collinear logs* cancel at the differential level if $\{q_0^2, c_0\}$-variables used. Get approximate LFU.
- If $\{q^2, c_\ell\}$-variables are used, hard collinear logs survive at the differential level, and only cancel in the total rate.
- By gauge invariance, structure-dependent terms do not give rise to further collinear logs.

Summary and Conclusion

- With a cut-off, hard-collinear logs *always* survive. To prevent distortion of the spectrum, it is best to report results in the $\{q_0^2, c_0\}$-variables.
- LFU ratios such as R_K are under control w.r.t. $\ln \frac{m_\ell}{m_B}$ from the theory side.

Future Work

- $\bar{B} \to \bar{K}\ell^+\ell^-$ differential distribution through Monte Carlo (ongoing).
- Structure-dependent corrections (ongoing).
- Fixing ambiguities in the UV counterterms (ongoing).
- Analysis of moments of the angular distribution. Higher moments sensitive to QED corrections (ongoing).
- Calculation can be extended to other spin final states, such as K^*.
- Charged-current semileptonic decays ($\bar{B} \to D\ell\nu$). Unidentified neutrino in final state makes it hard to reconstruct B meson and to apply a cut-off on photon energy.

The END

BACKUP SLIDES

IR Divergences: Structure-dependent terms

The real amplitude can be decomposed,

$$\mathcal{A}^{(1)} = \hat{Q}_{\ell_1} a_{\ell_1}^{(1)} + \delta\mathcal{A}^{(1)} \ ,$$

into a term $\hat{Q}_{\ell_1} a_{\ell_1}^{(1)}$ with all terms proportional to $\hat{Q}_{\ell_1}$, and the remainder $\delta\mathcal{A}^{(1)}$.

$$a_{\ell_1}^{(1)} = -e g_{\text{eff}} \bar{u}(\ell_1) \left[\frac{2\epsilon^* \cdot \ell_1 + \not{\epsilon}^* \not{k}}{2k \cdot \ell_1} \Gamma \cdot H_0(q_0^2) \right] v(\ell_2) \ ,$$

which contains all $1/(k \cdot \ell_1)$-terms.

The structure-dependence of this term is encoded in the form factor H_0.

IR Divergences: Structure-dependent terms

The amplitude square is given by

$$\sum_{\mathrm{pol}} |\mathcal{A}^{(1)}|^2 = \sum_{\mathrm{pol}} |\delta\mathcal{A}^{(1)}|^2 - \hat{Q}_{\ell_1}^2 \sum_{\mathrm{pol}} |a_{\ell_1}^{(1)}|^2 + 2\hat{Q}_{\ell_1} \mathrm{Re}[\sum_{\mathrm{pol}} \mathcal{A}^{(1)} a_{\ell_1}^{(1)*}] \,,$$

where it will be important that $\mathcal{A}^{(1)}$ is gauge invariant.

The *first term* is manifestly free from hard-collinear logs $\ln m_{\ell_1}$.

We use *gauge invariance* and set $\xi = 1$ under which the polarisation sum

$$\sum_{\mathrm{pol}} \epsilon_\mu^* \epsilon_\nu = (-g_{\mu\nu} + (1-\xi) k_\mu k_\nu / k^2) \to -g_{\mu\nu}$$

collapses to the metric term only.

IR Divergences: Structure-dependent terms

The *second term* evaluates to

$$\int d\Phi_\gamma \, \hat{Q}_{\ell_1}^2 \sum_{\text{pol}} |a_{\ell_1}^{(1)}|^2 = \int d\Phi_\gamma \, \hat{Q}_{\ell_1}^2 \, \frac{\mathcal{O}(m_{\ell_1}^2) + \mathcal{O}(k\cdot\ell_1)}{(k\cdot\ell_1)^2} = \mathcal{O}(1) \, \hat{Q}_{\ell_1}^2 \ln m_{\ell_1}$$

where we used $k - \ell_1 = \mathcal{O}(m_{\ell_1}^2)$, valid in the collinear region.

IR Divergences: Structure-dependent terms

We now turn to the *third term*.

Using anticommutation relations, $k - \ell_1 = \mathcal{O}(m_{\ell_1}^2)$ in the collinear limit, and the EoMs, we rewrite $a_{\ell_1}^{(1)}$ as

$$a_{\ell_1}^{(1)} = -eg_{\text{eff}}\bar{u}(\ell_1)\left[\frac{4\epsilon^*\cdot\ell_1 + m_{\ell_1}\not{\epsilon}^*}{2k\cdot\ell_1}\Gamma\cdot H_0(q_0^2)\right]v(\ell_2)\ ,$$

Gauge invariance $k\cdot\mathcal{A}^{(1)} = 0$ implies $\ell_1\cdot\mathcal{A}^{(1)} = \mathcal{O}(m_{\ell_1}^2)$ in the collinear region

IR Divergences: Structure-dependent terms

Therefore, the first part of $a_{\ell_1}^{(1)}$ contributes to

$$\hat{Q}_{\ell_1} \mathrm{Re}[\sum_{\mathrm{pol}} \mathcal{A}^{(1)} a_{\ell_1}^{(1)*}] \to c_1 \hat{Q}_{\ell_1}^2 \frac{\mathcal{O}(m_{\ell_1}^2)}{(k \cdot \ell_1)^2} + c_2 \hat{Q}_{\ell_1} \hat{Q}_X \frac{\mathcal{O}(m_{\ell_1}^2)}{(k \cdot \ell_1)}$$

where $X \in \{\bar{B}, \bar{K}, \bar{\ell}_2\}$.

The second part of $a_{\ell_1}^{(1)}$ contributes to

$$\hat{Q}_{\ell_1} \mathrm{Re}[\sum_{\mathrm{pol}} \mathcal{A}^{(1)} a_{\ell_1}^{(1)*}] \to c_1' \hat{Q}_{\ell_1}^2 \frac{\mathcal{O}(m_{\ell_1}^2)}{(k \cdot \ell_1)^2} + c_2' \hat{Q}_{\ell_1} \hat{Q}_X \frac{\mathcal{O}(m_{\ell_1})}{(k \cdot \ell_1)}$$

Thus, using gauge invariance, one concludes that $\delta\mathcal{A}^{(1)}$ (indicated by terms $\propto \hat{Q}_X$ in the above) does not lead to collinear logs.