Fracture and Friction

School Disorder and complex systems.
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Elsa Bayart (elsa.bayart@ens-lyon.fr)

Single rupture
Linear trajectory
Isotropic, homogeneous, elastic material
Brittle Fracture

I. Scaling laws

- Theoretical strength of a material
- Stress amplification
- Propagation criterion
- Equation of motion

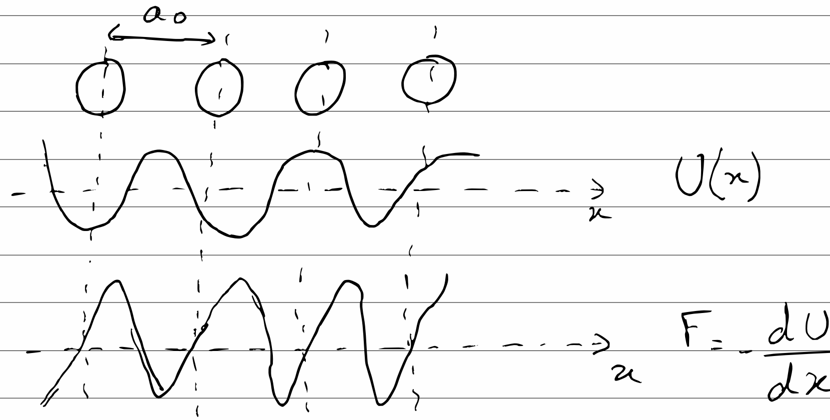
1/ Material strength

Energy needed to create new surfaces:

Lennard-Jones interaction for crystals

$a_0 \sim 10^{-10}$ m

. Imposed displacement to separate atoms



stress $\sigma(x) = \frac{F(x)}{a_0^2}$ periodic.

$$\begin{cases} \sigma(x) = \sigma_{max} \sin\left(2\pi \frac{x - a_0}{a_0}\right) \\ \sigma(x) = E\, \varepsilon(x) \quad \text{Hooke's law.} \end{cases}$$

$$\varepsilon = \frac{\Delta l}{l} = \frac{x - a_0}{a_0} \ll 1$$

Imposing a small deformation ε

$$\sigma(x) = \sigma_{max} \sin(2\pi\varepsilon) \sim 2\pi \sigma_{max} \varepsilon$$

$$\Rightarrow \quad 2\pi \sigma_{max} \varepsilon = E \varepsilon$$

$$\Rightarrow \quad \sigma_{max} = \frac{E}{2\pi} \quad \text{et} \quad \boxed{\varepsilon \simeq \frac{1}{6}} \,!$$

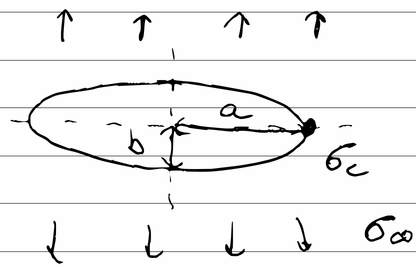
10%

2-3 orders of magnitude > real strength.

Surface creat° ≠ only atomic separation

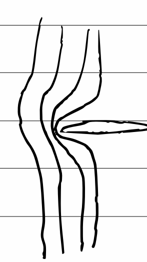
↳ occurs via rupture propagation with stress amplification at the rupture tip.

2) Stress amplification.



Elliptical hole
semi-major axis a
semi-minor axis b

displacement curves



Inglis solution (1913)

$$\sigma_c = \sigma_\infty \left(1 + \frac{2a}{b} \right)$$

crack : $a \gg b$, slender geometry

$$\sigma_c \sim \frac{2a}{b} \sigma_\infty \gg \sigma_\infty$$

Rupture when $\sigma_c = \sigma_{max}^{crystal}$

$$\Rightarrow \quad \sigma_\infty^R = \sigma_{max} \frac{b}{2a} \ll \sigma_{max}^{crystal}$$

In terms of curvature radius $[\rho] = L$

$$\rho_c \approx \frac{b^2}{a}$$

k: amplification factor

$$k = \frac{\sigma_c}{\sigma_\infty} = 1 + \frac{2a}{b} = 1 + 2\sqrt{\frac{a}{\rho}}$$

$$k \approx 2\sqrt{\frac{a}{\rho}} \quad \text{with } \rho \ll a$$

$$\sigma_R \approx \sigma_R^{th}/k \sim \sigma_R^{th}\sqrt{\frac{\rho}{a}}$$

The sharper the crack, the smaller σ_R.

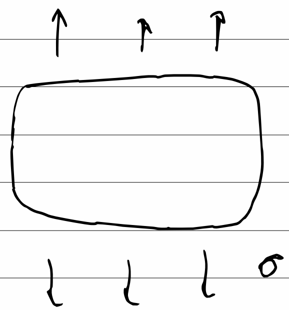
(holes to prevent propagation).

$\Rightarrow$ Rupture strength controlled by existing microcracks

$\hookrightarrow$ control / prediction difficult.

(propagation of existing cracks only).

3) Initiation criterion.



elastic energy density:

$$u_p \sim \frac{1}{2}\sigma\varepsilon \sim \frac{\sigma^2}{2E}$$

Introducing a crack of length l

$$E_{released} = u_p \times S_{crack} \quad / \text{unit thickness}$$

$$S_{crack} = l^2 \rightarrow \text{only lengthscale.}$$

* Energy variation:

$$\Delta\mathcal{E}_p = \mathcal{E}_{with\ crack} - \mathcal{E}_{wo\ crack} < 0$$

$$\Delta\mathcal{E}_p = -\underbrace{\alpha}_{geom.}\frac{\sigma^2 l^2}{\underbrace{2E}_{E_{released}.}}$$

* Surface energy

$$\Delta\mathcal{E}_s \sim \gamma l > 0$$

↳ fracture energy, more general than surface.

- total

$$\Delta \mathcal{E} = \Delta \mathcal{E}_p + \Delta \mathcal{E}_s$$

$$= \underbrace{-\alpha \frac{\sigma^2 l^2}{2E}}_{\text{dominates for large } l} + \underbrace{\gamma l}_{\text{dominates for small } l}$$

- Propagation

$$\left.\frac{\partial \Delta \mathcal{E}}{\partial l}\right|_{l=l_c} = 0 \Rightarrow -\alpha \frac{\sigma^2 l_c}{E} + \gamma = 0$$

$$\boxed{\begin{array}{l} l_c \sim \dfrac{\gamma E}{\sigma^2} \\[2ex] \sigma_c \sim \sqrt{\dfrac{\gamma E}{l}} \end{array}}$$

Griffith length

Griffith criterion (1921)

$\sigma_c \searrow$ when $l \nearrow$

The longest crack controls the material strength.

4) Rupture dynamics (eq. of motion)

Mott theory (1947)

$l = l_c$: nucleation (griffith)

$l > l_c$: rupture accelerates $\Rightarrow$ fracture energy + kinetic energy.

mass / u. thickness $M = \rho l^2$

particular velocity $\propto v$, the crack speed.

$$\Delta \mathcal{E}_k = \beta \frac{1}{2} M v^2 = \frac{1}{2} \beta \rho l^2 v^2$$

Let's write the $\mathcal{E}$ variation of a quasistatic crack :

$$\Delta \mathcal{E}_{qs}(l) = \Delta \mathcal{E}_{qs}(l_c + l - l_c)$$

$$= \Delta \mathcal{E}_{qs}(l_c) - \alpha \frac{\sigma^2}{2E}(l - l_c)^2$$

using the fact that $\Delta \mathcal{E}_{qs}(l_c) = \frac{\gamma l_c}{2}$

$$\Delta \mathcal{E}_{tot}(l) = \Delta \mathcal{E}_{qs}(l) + \Delta \mathcal{E}_k(l)$$
$$= \Delta \mathcal{E}_{qs}(l_c) - \alpha \frac{\sigma^2}{2E}(l - l_c)^2 + \frac{1}{2} \beta \rho l^2 v^2$$

Key point: $l > l_c$, dynamic propagation, no energy input into the system.

$$\Rightarrow \quad \Delta \mathcal{E}_{tot}(l) \sim \Delta \mathcal{E}_{qs}(l_c)$$

$$\Rightarrow \quad \alpha \frac{\sigma^2}{2E}(l - l_c)^2 + \frac{1}{2}\beta\rho\, l^2 v^2 = 0$$

$$\Rightarrow \quad v(t) \sim \underbrace{\sqrt{\frac{\alpha\sigma^2}{\rho\beta E}}}_{v_{max}^{Mott}} \left(1 - \frac{l_c}{l(t)}\right)$$

- Homogeneous loading: cracks accelerate until limiting speed
- no dependence in $\ddot{l}$: no inertia.
- Several tens of years to show this result.

2. Singular elastic fields at the crack tip.

2.1. Linear elasticity

Continuous medium : $l \gg l_{microscopic}$

The material is:

- homogeneous : no internal stress creation by inverting 2 elements
- isotropic : no internal stress creation by rotating 1 element.
- elastic : reversible deformations.

+ small deformations : $\nabla \vec{u} \ll 1$

__Deformation__

$$\vec{x} \longrightarrow \vec{X} = \vec{x} + \vec{u}(\vec{x})$$

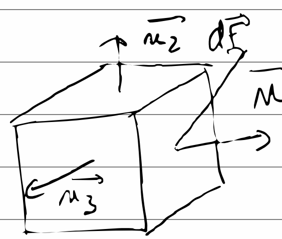
reference — deformed state.

$$\varepsilon = \underbrace{\frac{1}{2}\left(\partial_i u_j + \partial_j u_i\right)}_{\varepsilon_{ij}}$$

tensor elements

symmetric + diagonalizable (frame of the principal def).

__Stress__ : force / unit surface



$$dF_i = \sigma_{ij}\, n_j\, dS$$

vectoriel $\vec{dF} = \overline{\overline{\sigma}}\, \vec{n}\, dS$

σ_{ij} : symmetric tensor (from moment equilibrium)

diagonal terms $\rightarrow$ normal stresses
non-diagonal terms $\rightarrow$ tangential stresses

* Stress / volumic force relationship

$\vec{f}$: volumic force density

$$\int_V \vec{f}\, dV = \int_S \sigma \vec{n} \cdot d\vec{S}$$

Green-Ostrogradsky theorem

$$\int_V \operatorname{div} \vec{E}\, dV = \int \vec{E} \cdot \vec{n}\, dS$$

$$\Rightarrow \quad \vec{f} = \operatorname{div} \sigma$$

i.e. $\quad f_i = \partial_k \sigma_{ki}$

Hooke's law

Constitutive relation of the material:
sollicitation $\Leftrightarrow$ response

$$\sigma_{ij} = C_{ijkl}\,\varepsilon_{kl}$$

$$C_{ijkl} = \frac{\partial^2 W}{\partial \varepsilon_{ij}\,\partial \varepsilon_{kl}}$$ W : elastic energy

$$\boxed{\begin{aligned} \sigma_{ij} &= \frac{E}{1+\nu}\left(\varepsilon_{ij} + \frac{\nu}{1-2\nu}\,\varepsilon_{kk}\,\delta_{ij}\right) \\ \varepsilon_{ij} &= \frac{1+\nu}{E}\,\sigma_{ij} - \frac{\nu}{E}\,\sigma_{kk}\,\delta_{ij} \end{aligned}}$$

The material is characterized by 2 elastic constants

(E, ν) : Young - Poisson

(λ, μ) : Lamé coefficients >0

$$\sigma_{ij} = 2\mu\,\varepsilon_{ij} + \lambda\,\varepsilon_{kk}\,\delta_{ij}$$

and others

$$E = \frac{3\lambda + 2\mu}{\lambda + \mu}\,\mu \quad \text{et} \quad \nu = \frac{1}{2}\,\frac{\lambda}{\lambda + \mu}$$

$$\lambda, \mu > 0 \Rightarrow 0 \leq \nu \leq \frac{1}{2}$$

2D Hooke's law

Need to eliminate the 3rd dimension when
- $L_z \ll L_x, L_y$ (slender structures)
- invariance along one dimension (plate)

- Plane stress – slender structures – faces free of stresses.

$$\sigma_{iz} = 0 \Rightarrow \varepsilon_{iz} = 0 \quad \text{for } i \neq z$$

$$\sigma_{zz} = 0 \Rightarrow \varepsilon_{zz} + \frac{\nu}{1-2\nu}\Big(\underbrace{\varepsilon_{xx} + \varepsilon_{yy}}_{\varepsilon_{kk}^{(2)}} + \varepsilon_{zz}\Big) = 0$$

$$\varepsilon_{zz} = -\frac{\nu}{1-\nu}\,\varepsilon_{kk}^{(2)} \Rightarrow \varepsilon_{kk} = \frac{1-2\nu}{1-\nu}\,\varepsilon_{kk}^{(2)}$$

$$\boxed{\sigma_{ij} = \frac{E}{1+\nu}\left(\varepsilon_{ij} + \frac{\nu}{1-\nu}\,\varepsilon_{kk}^{(2)}\,\delta_{ij}\right)}$$

Invariant inverse relation $\sigma_{kk}^{(2)} = \sigma_{kk}$.

$$\sigma_{ij} = 2\mu\,\varepsilon_{ij} + \frac{2\mu\lambda}{\lambda + 2\mu}\,\varepsilon_{kk}^{(2)}$$

• plane strain - extended object in the 3rd dir, forbidden displacement of faces.

$\varepsilon_{iz} = 0$ no cumulated displacement in the z direction

$$\boxed{\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu(1+\nu)}{E}\sigma_{kk}^{(2)}\delta_{ij}}$$

Elastostatic

$$f_i = \partial_k \sigma_{ki} \quad \text{volumic force density}$$

Equilibrium: $\vec{f} = \vec{0} \Rightarrow \partial_k \sigma_{ki} = 0$

2D

$$\begin{cases} \partial_x \sigma_{xx} + \partial_y \sigma_{xy} = 0 & (\text{I}) \\ \partial_x \sigma_{xy} + \partial_y \sigma_{yy} = 0 & (\text{II}) \end{cases}$$

Introduction of the Airy function χ such as:

$$\sigma_{xx} = \frac{\partial^2 \chi}{\partial y^2} \qquad \sigma_{yy} = \frac{\partial^2 \chi}{\partial x^2} \qquad \sigma_{xy} = \frac{\partial^2 \chi}{\partial x \partial y}$$

(I) and (II) are satisfied.

Equilibrium : $\mathrm{div}(\sigma) = 0$
+ hooke's law expressed in terms of displacement :

$$\Rightarrow \quad \vec{\nabla}(\vec{\nabla}\cdot\vec{u}) - \frac{1-2\nu}{2(1-\nu)}\vec{\nabla}\wedge(\vec{\nabla}\wedge\vec{u}) = \vec{0} \quad (\ast)$$

(cf Landau)

$\mathrm{div}(\ast)$ and $\vec{\nabla}\cdot(\vec{\nabla}\wedge\vec{A}) = 0$ (vectorial identity).

$$\vec{\nabla}\cdot(\ast) = 0 = \nabla^2(\vec{\nabla}\cdot\vec{u})$$

Remark that: $\mathrm{Tr}(\varepsilon) = \frac{1}{2}\mathrm{Tr}\left(\partial_i u_j + \partial_j u_i\right)$

$$= \frac{1}{2}\, 2 \times \left(\partial_i u_i + \partial_j u_j\right) = \vec{\nabla}\cdot\vec{u}$$

$$\Rightarrow \quad \nabla^2 \left(\mathrm{Tr}(\varepsilon) \right) = 0$$

2D, plane stress

$$\sigma_{ij} = 2\mu \, \varepsilon_{ij} + \frac{2\mu\lambda}{\lambda + 2\mu} \varepsilon_{kk} \, \delta_{ij}$$

$$\Rightarrow \quad \mathrm{Tr}(\sigma) = 2\mu \, \frac{3\lambda + 2\mu}{\lambda + 2\mu} \, \mathrm{Tr}(\varepsilon)$$

$$\Rightarrow \quad \nabla^2 \left(\mathrm{Tr}(\sigma) \right) = 0$$

or $\mathrm{Tr}(\sigma) = \sigma_{xx} + \sigma_{yy} = \nabla^2 \chi$

$$\boxed{\Delta^2 \chi = 0}$$ Airy equation

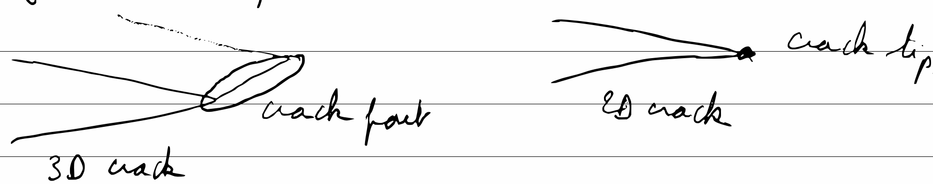
Elastostatic: 2D linear elasticity is all embedded in the ~~A~~ Airy equation (bilaplacian equation).

Fracture mechanics: We only have to solve it with the correct boundary conditions.

Quasi-static crack

crack = creation of free surfaces.

↳ Solve the elastic problem with stress-free faces along the crack path.

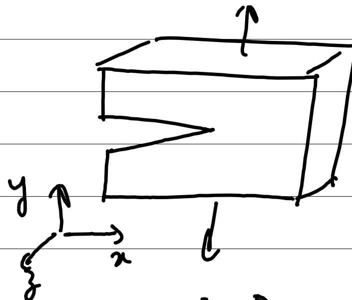


In the following:

- 2D crack
- straight: not limiting as no intrinsic length in linear elasticity → allowed to "zoom on" a straight part.

Fracture modes

mode I

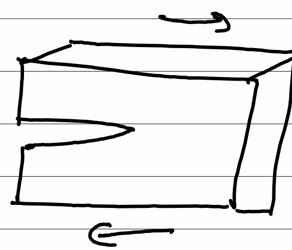


tensile loading.

$$u_x(x,-y) = u_x(x,y)$$

$$u_y(x,-y) = -u_y(x,y)$$

mode II

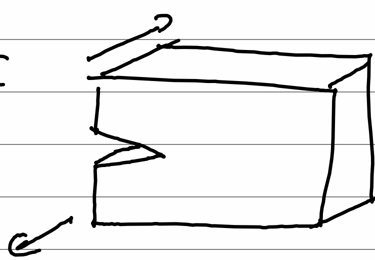


shear loading

$$u_x(x,-y) = -u_x(x,y)$$

$$u_y(x,-y) = u_y(x,y)$$

mode III



antiplane shear loading

$$u_z(x,-y) = -u_z(x,y)$$

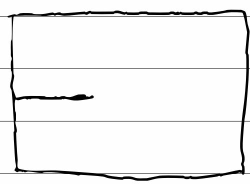
Remark: mode II is unstable in an intact material.

Elastic fields surrounding the crack tip

William's expansion (1980)

→ rigorous result corresponding to stress amplification (Ingles 1913)

2D solid with a straight 1D crack



2D elasticity ⟹ Airy equation.

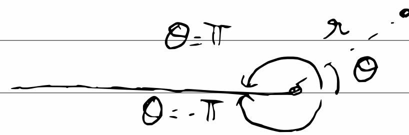
$$\Delta^2 \chi = 0$$

with $\sigma_{xx} = \dfrac{\partial^2 \chi}{\partial y^2}$ $\sigma_{yy} = \dfrac{\partial^2 \chi}{\partial x^2}$ $\sigma_{xy} = \dfrac{\partial^2 \chi}{\partial x \partial y}$

Boundary conditions:

crack with free surfaces ≡ stress-free along the crack.

Polar coordinates



$$\sigma_{\theta\theta}(r, \theta = \pm\pi) = \sigma_{r\theta}(r, \theta = \pm\pi) = 0$$

(σ_{rr}, compression/dilat. allowed).

Relations $\sigma - \chi$ in polar coordinates

$$\begin{cases} \sigma_{rr} = \frac{1}{r^2} \partial^2_{\theta\theta} \chi + \frac{1}{r} \partial_r \chi \\ \sigma_{\theta\theta} = \partial^2_{rr} \chi \\ \sigma_{r\theta} = - \partial_r \left(\frac{1}{r} \partial_\theta \chi \right) \end{cases}$$

Airy equation:

$$(*) \quad \Delta^2 \chi(r,\theta) = \left(\partial_{rr} + r^{-1} \partial_r + r^{-2} \partial_{\theta\theta} \right)^2 \chi(r,\theta) = 0$$

$$\begin{cases} \text{Bound. cond. on } \chi: \\ \sigma_{\theta\theta}(r, \theta = \pm\pi) = \partial^2_{rr} \chi(r, \theta = \pm\pi) = 0 \\ \sigma_{\theta r}(r, \theta = \pm\pi) = - \partial_r \left(\frac{1}{r} \partial_\theta \chi(r, \theta = \pm\pi) \right) = 0 \end{cases}$$

Asymptotic solution for χ

$$\chi(r,\theta) = r^{\lambda+2} f(\theta) \qquad \lambda \in \mathbb{R}$$

Introduction in $\Delta^2 \chi = 0$

$$f^{(4)}(\theta) + \left(\lambda^2 + (\lambda+2)^2\right) f''(\theta) + \lambda^2 (\lambda+2)^2 f(\theta) = 0$$

Solution of the form : $f(\theta) = e^{\alpha\theta}$

$$\Rightarrow \left(\lambda^2 + \alpha^2\right)\left(\left(\lambda+2\right)^2 + \alpha^2\right) = 0$$

solutions $\alpha = \pm i\lambda$ and $\alpha = \pm i(\lambda+2)$

$$\Rightarrow f(\theta) = \left.\begin{array}{l} a \sin\lambda\theta + b\cos\lambda\theta + c\sin(\lambda+2)\theta \\ \qquad + d\cos(\lambda+2)\theta \end{array}\right\}$$

Introduction of the crack via the BC :

$$\sigma_{\theta\theta}(r, \theta = \pm\pi) = 0 \Rightarrow \partial^2_{rr}\chi(\theta = \pm\pi) = 0$$

$$\Rightarrow \ (\lambda+2)(\lambda+1)\, r^{\lambda} f(\pm\pi) = 0 \qquad (1)$$

$$\sigma_{r\theta}(r, \theta = \pm\pi) = 0 \ \Rightarrow \ -\partial_r\left(\frac{1}{r}\partial_\theta \chi\right) = 0$$

$$\Rightarrow \ (\lambda+1)\, r^{\lambda} f'(\pm\pi) = 0 \qquad (2)$$

$\lambda = -1, -2 \ \Rightarrow$ trivial solutions

$$\Rightarrow \ \lambda \neq -1, -2.$$

(1) and (2) $\Rightarrow \ f(\pm\pi) = f'(\pm\pi) = 0$

$$\begin{cases}
f(\pi) = a\sin(\lambda\pi) + b\cos(\lambda\pi) + c\sin(\lambda\pi) + d\cos(\lambda\pi) \\
\quad = (a+c)\sin\lambda\pi + (b+d)\cos\lambda\pi \\[1ex]
f(-\pi) = -(a+c)\sin(\lambda\pi) + (b+d)\cos(\lambda\pi) \\[1ex]
f'(\pi) = \lambda a\cos(\lambda\pi) - \lambda b\sin(\lambda\pi) + (\lambda+2)c\cos(\lambda\pi) \\
\qquad\qquad - (\lambda+2)d\sin(\lambda\pi) \\
\quad = \left(a\lambda + c(\lambda+2)\right)\cos(\lambda\pi) - \left(b\lambda + d(\lambda+2)\right)\sin(\lambda\pi) \\[1ex]
f'(-\pi) = \left(a\lambda + c(\lambda+2)\right)\cos(\lambda\pi) + \left(b\lambda + d(\lambda+2)\right)\sin(\lambda\pi)
\end{cases}$$

$f(\pi) + f(-\pi) = 0 \ \Rightarrow \ (b+d)\cos\lambda\pi = 0$ 1

$f(\pi) - f(-\pi) = 0 \ \Rightarrow \ (a+c)\sin\lambda\pi = 0$ 2

$f'(\pi) - f'(-\pi) = 0 \ \Rightarrow \ \left(a\lambda + c(\lambda+2)\right)\cos\lambda\pi = 0$ 3

$f'(\pi) - f'(-\pi) = 0 \ \Rightarrow \ \left(b\lambda + d(\lambda+2)\right)\sin\lambda\pi = 0$ 4

Or $a, b, c, d = 0 \not\Rightarrow$ trivial solutions

Or $\begin{cases} \sin \lambda\pi = 0 \\ \cos \lambda\pi = 0 \end{cases} \Rightarrow \begin{array}{ll} \lambda = n & n \in \mathbb{Z} \\ \lambda = n + \frac{1}{2} & n \in \mathbb{Z} \end{array}$

$\lambda = n \Rightarrow$ (1) $(b+d)\underbrace{\cos(n\pi)}_{=\pm 1} = 0 \Rightarrow \boxed{b = -d}$

(3) $a n + c(n+2) = 0 \Rightarrow \boxed{c = -\frac{n}{n+2} a}$

$\lambda = n + \frac{1}{2} \Rightarrow$ (2) $(a+c)\underbrace{\sin\left(n\pi + \frac{\pi}{2}\right)}_{\pm 1} \Rightarrow \boxed{c = -a}$

(4) $b\left(n + \frac{1}{2}\right) + d\left(n + \frac{5}{2}\right) = 0$

$\Rightarrow \boxed{d = -b \frac{n + 1/2}{n + 5/2}}$

$\chi(r, \theta) = r^{\lambda+2} f(\theta)$ with $\lambda \neq -1, \lambda \neq -2$
and $\{\lambda = n \; ; \; \lambda = n + \frac{1}{2}\}$

χ is a superposition of solutions
for $n = [-\infty, +\infty] \in \mathbb{Z}$ then σ too.

William's expansion

$$\sigma_{ij}(r, \theta) = \sum_{n=-\infty}^{+\infty} a_n r^{n/2} f_{ij}^{(n)}(\theta)$$

← n order

a_m are the coefficients a, b, c, d.

$f_{ij}^{(m)}(\theta)$ are the trigonometric functions for $\lambda = m$.

<u>Remarks</u>

1) dev. in $\square^{1/2}$ due to the stress discontinuity induced by the crack.

2) Other BC not accounted for by the asymptotic solution. They will determined the $\{a_m\}$ coefficients.

Physical terms of the expansion

Need for the elastic energy convergence.

$$E_{el} = \frac{1}{2}\int_S \sigma_{ij}\,\varepsilon_{ij}\,ds = \frac{1}{2}\iint \sigma_{ij}\,\varepsilon_{ij}\,r\,dr\,d\theta$$

$$\sigma_{ij} \propto \varepsilon_{ij} \sim r^{n/2}$$

$$\Rightarrow E_{el} \sim \int r^{n+1}\,dr$$

E_{el} finite when $r \to 0 \Rightarrow n+1 \geq 0$

$$\Rightarrow \boxed{n \geq -1}$$

$$\boxed{\sigma_{ij}(r,\theta) = \sum_{n=-1}^{+\infty} a_n\, r^{n/2}\, f_{ij}^{(n)}(\theta)}$$

The 1st term is singular in $r \to 0$ as $\dfrac{1}{\sqrt{r}}$.

$\to$ stress amplification at the crack tip.

Introduction of the Stress Intensity Factor (SIF)
such that $a_m \hookrightarrow \frac{K}{\sqrt{2\pi}}$

$$\sigma_{ij} = \sum_{\ell = I, II, III} \frac{K_\ell}{\sqrt{2\pi r}} f_\ell(\theta)$$

$[K] = [\sigma] L^{1/2}$ $\quad L \hookrightarrow$ system (no intrinsic length)

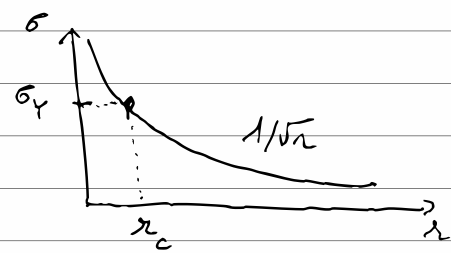
K contains the informat about loading, geometries..

dynamic crack:

$$\sigma_{ij} = \sum_{I\,II\,III} \frac{K_\ell(t)}{\sqrt{2\pi r}} f(\theta, v)$$

(different resolution as Airy doesn't hold, dynamic boundary value problem).

<u>Singularity regularization</u>



$$r_c \sim \left(\frac{K}{\sigma_Y}\right)^2$$

for plastic: $K = \sqrt{\Gamma E} = 10^6\ Pa \cdot m^{1/2}$

$\sigma_Y = 100\ MPa$

$r_c = 10^{-4}\ m$.

- SIF dimensions $\quad [K] = [\sigma\sqrt{r}]$ $\quad Pa.m^{1/2}$

$$= [\sigma]\, L^{1/2}$$

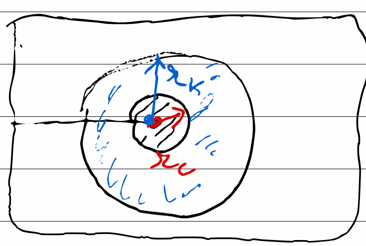
linear elasticity = no intrinsic length $\Rightarrow$ L is macroscopic (rupture length, system size)

- $K_I, K_{II}, K_{III} \Longleftrightarrow a, b$
  carry the information about loading and system geometry.

The singular term is a superposition of the contribution of each mode:

$$\sigma_{ij} = \sum_{l = I, II, III} \frac{K_l}{\sqrt{2\pi r}} f_l(\theta)$$

Small scale yielding hypothesis



r_c : process zone

r_K : singular zone

r_L : system size

$$r_c \ll r_K \ll r_L$$

3. Propagation condition

Energy release rate

During propagation: released elastic energy ⊖
dissipated fracture energy ⊕

Dissipated rate of energy

$$\phi = -\frac{\partial \mathcal{E}}{\partial l}\,\dot{l} > 0$$

$\mathcal{E}$: elastic stored energy per unit length
l : rupture length.

$-\frac{\partial \mathcal{E}}{\partial l}$ is a generalized force.

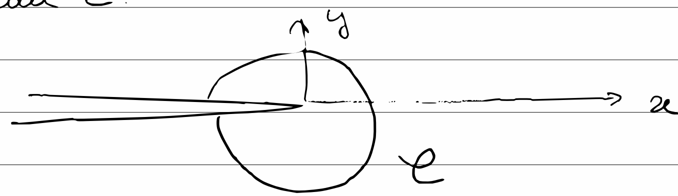
Definition of the energy release rate:

$$G \equiv -\frac{\partial \mathcal{E}}{\partial l} \qquad [G] = [E]\cdot L^{-2}$$

It measures the energy flux from the loading scale to the crack tip.

J- integral and G calculation

Propagation of a crack at speed v in dir. x.
Contour $\mathcal{C}$.



Energy flux through $\mathcal{C}$:

- Work of traction forces applied to the material inside $\mathcal{C}$
- kinetic and potential energies due to material transport by translation of $\mathcal{C}$.

Equation of motion:

$$\frac{\partial \sigma_{ij}}{\partial x_j} - \rho \frac{\partial^2 u_i}{\partial t^2} = 0$$

$\times \frac{\partial u_i}{\partial t} \rightarrow$ energy density

$$\frac{\partial}{\partial x_j}\left(\sigma_{ij} \frac{\partial u_i}{\partial t}\right) - \sigma_{ij} \frac{\partial^2 u_i}{\partial x_j \partial t} - \frac{1}{2} \rho \frac{\partial}{\partial t}\left(\frac{\partial u_i}{\partial t} \frac{\partial u_i}{\partial t}\right) = 0$$

$$\underbrace{\frac{\partial}{\partial x_j}\left(\sigma_{ij} \frac{\partial u_i}{\partial t}\right)}_{\text{traction}} - \frac{\partial}{\partial t}\left[\underbrace{\int_{-\infty}^{t} \sigma_{ij} \frac{\partial^2 u_i}{\partial x_j \partial t'} dt'}_{= U \text{ potential}} + \underbrace{\frac{1}{2} \rho \frac{\partial u_i}{\partial t} \frac{\partial u_i}{\partial t}}_{= T_{\text{kinetic}}} \right] = 0$$

$$\frac{\partial}{\partial t} = - v \frac{\partial}{\partial x} \quad \text{for a constant } v.$$

$$\Rightarrow \quad \frac{\partial}{\partial x_j}\left(\sigma_{ij} \frac{\partial u_i}{\partial t}\right) + v \frac{\partial}{\partial x_j}\left(U + T \right) = 0$$

$$\Rightarrow \quad \operatorname{div}\left(\sigma_{ij} \frac{\partial u_i}{\partial t} + v \left(U + T \right)\right) = 0$$

Green-Ostrogradski theorem

$$\int_V \operatorname{div}(\vec{A})\, dV = \int_S \vec{A} \cdot \vec{n}\, dS$$

Let's defined the J-integral / unit length :

$$J(\mathcal{C}) = \int_{\mathcal{C}} \left[\sigma_{ij} \frac{\partial \dot{u}_i}{\partial t} n_j + (U+T) v n_x \right] ds$$

arclength

quantifying the instantaneous $\mathcal{E}$ rate through $\mathcal{C}$.

$$G(v) = \lim_{\mathcal{C} \to 0} \frac{J(\mathcal{C})}{v}$$

G should be independant of $\mathcal{C}$. Not true in general. True when :

- quasi-static crack
- $\mathcal{C}$ is taken in the singular zone (K-field zone).

Thoughness and fracture energy

2 types of propagation criterion:

- in stress amplitude $\rightarrow$ toughness K_c
- in energy $\rightarrow$ fracture energy Γ

Equivalent.

- Toughness K_c

$$\sigma_{ij} = \sum_{l = I, II, III} \frac{K_l}{\sqrt{2\pi r}} f_l(\theta, \nu)$$

propagation when $K > K_c$

Of practical use for industry: test a sample under loading and measure the rupture stress.

$K \sim \sigma \sqrt{L} \longrightarrow$ determination of K_c.

$K = K_c$: quasi-static

$K > K_c$: dynamic

- Fracture energy

Energy flux $\geqslant$ dissipated energy.

$$G \geqslant \Gamma$$

Γ = surface energy + any type of dissipation.

$\Rightarrow \Gamma$ is hardly predictable.

Propagation criterion is $G = \Gamma$
sometimes written as $G = G_c$.

- $K_c - \Gamma$ relationship

$$[J(\mathcal{C})] = \left[v \int E_{el}\, ds \right] \sim v \left[\frac{\sigma^2}{E} \right] [r\, d\theta]$$

$$\sim \left[v \frac{K^2}{E} \right]$$

$$G = \lim_{\mathcal{C} \to 0} \frac{J(\mathcal{C})}{v} \Rightarrow G \propto \frac{K^2}{E}$$

Propagation for $\Gamma \sim \dfrac{K_c^2}{E} \Rightarrow K_c \sim \sqrt{\Gamma E}$