From Sandpiles to Disordered Elastic Systems

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These are lecture notes for the summer school in June 2022 in Orsay. Notes on my master class and a recent review can can be found here:
Master Class: http://www.phys.ens.fr/∼wiese/masterENS/
Review: http://www.phys.ens.fr/∼wiese/pdf/review.pdf

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1 Discrete Stochastic Processes

1.1 Stochastic noise as a consequence of the discreteness of the state variables

We ask a very general question: Is there a continuous random process $\hat{n}(t)$ which has the same statistics as the discrete process $n(t)$?

Let us consider a more general problem: Be $n(t)$ the number of particles at time t. With rate r_+ the number of particles increases by one, and with rate r_- it decreases by one. This implies that after one time step, as long as $r_\pm \delta t$ are small,

$$\langle n(t+\delta t) - n(t)\rangle = (r_+ - r_-)\delta t, \tag{1}$$

$$\left\langle [n(t+\delta t) - n(t)]^2 \right\rangle = (r_+ + r_-)\delta t. \tag{2}$$

The following *continuous random process* $\hat{n}(t)$ has the same first two moments as $n(t)$,

$$\mathrm{d}\hat{n}(t) = (r_+ - r_-)\mathrm{d}t + \sqrt{r_+ + r_-}\,\mathrm{d}\eta(t) \ , \tag{3}$$

$$\langle \mathrm{d}\eta(t)\mathrm{d}\eta(t')\rangle = \delta(t-t')\mathrm{d}t \ . \tag{4}$$

The notation $\mathrm{d}\hat{n} = ...$ is the stochastic Itô calculus used by Mathematicians. Physicists would just divide both sides by $\mathrm{d}t$, to write

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{n}(t) = (r_+ - r_-) + \xi(t), \tag{5}$$

$$\langle \xi(t)\xi(t')\rangle = (r_+ + r_-)\delta(t-t'). \tag{6}$$

Note that I also moved the amplitude of the noise to its expectation.

This procedure can be modified to include higher cumulants of $n(t+\delta t)-n(t)$, leading to more complicated noise correlations [1].

1.2 Example: Diffusion

If r_+ and r_- are numbers, then

$$\mu = r_+ - r_- \qquad \text{(drift)} \tag{7}$$
$$D = r_+ + r_- \qquad \text{(diffusion constant)} \tag{8}$$

Attention: You will also find the convention $D=(r_+ + r_-)/2$.

1.3 Example: reaction diffusion process

The reaction-annihilation process is defined as

$$A + A \xrightarrow{\nu} A. \tag{9}$$

Then the rate $r_+ = 0$, and r_- is the number of pairs $n(n-1)/2$ times the reaction rate ν upon contact,

$$r_- = \nu \times \frac{\hat{n}(t)(\hat{n}(t)-1)}{2}. \tag{10}$$

The latter, in principle, is only defined on integer $\hat{n}(t)$, but we can use it for all $\hat{n}(t)$. Thus the best we can do to replace the discrete stochastic process with a continuous one is to write

$$\begin{aligned} \frac{\mathrm{d}\hat{n}(t)}{\mathrm{d}t} &= -\frac{\nu}{2}\hat{n}(t)(\hat{n}(t)-1) + \sqrt{\frac{\nu}{2}\hat{n}(t)(\hat{n}(t)-1)}\,\eta(t), \\ \langle \eta(t)\eta(t')\rangle &= \delta(t-t'). \end{aligned} \tag{11}$$

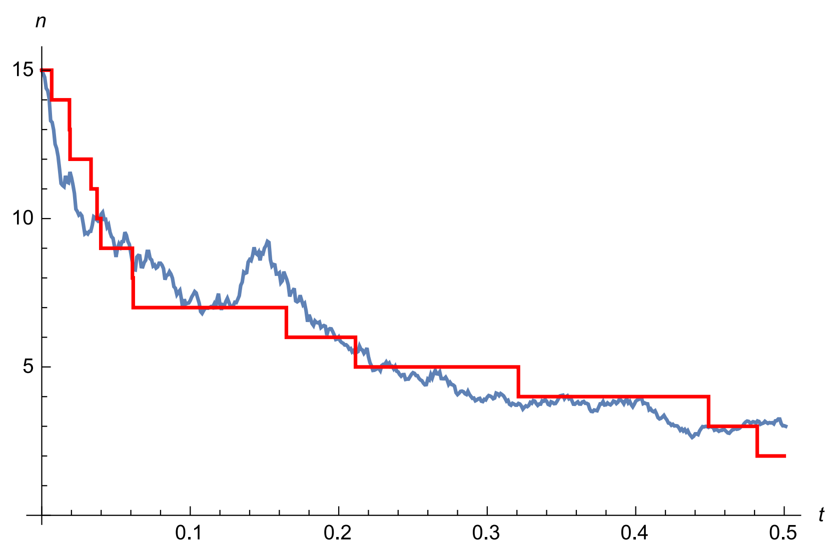

Figure 1: One trajectory for the process n_t, i.e. a direct numerical simulation of $A + A \to A$ with rate one (red, with jumps). One simulation for the continuous process $\hat{n}_t$, Eq. (11) (blue-gray, continuous, rough). We have chosen two trajectories which look "similar". Note that $\hat{n}_t$ is not monotonically decreasing.

Using $n_{\rm i} = 15$, and $\nu = 1$, we have shown two typical trajectories on figure 1 (right), one for the process $n(t)$ (red, with jumps), and one for the process $\hat n_t$ (blue-grey, rough). While by construction both processes have (almost) the same first two moments, clearly $\hat n(t)$ looks different: It is continuous, which $n(t)$ is not, and it can increase in time, which $n(t)$ can not.

1.4 Example: Depinning in force fields which are random walks

You have seen the equation of motion (here written for one monomer)

$$\partial_t \dot u(t) = m^2[v - \dot u(t)] \; + \partial_t \mathcal{F}(t), \tag{12}$$
$$\partial_t \mathcal{F}(t) = \sqrt{\dot u(t)}\,\eta(t), \tag{13}$$
$$\langle \eta(t)\eta(t')\rangle = \delta(t-t'). \tag{14}$$

What are the correlations of the force?

$$\begin{aligned}\left\langle [\mathcal{F}(t)-\mathcal{F}(0)]^2\right\rangle &= \left\langle \int_0^t \partial_{t_1}\mathcal{F}(t_1)\mathrm{d}t_1 \int_0^t \partial_{t_2}\mathcal{F}(t_2)\mathrm{d}t_2 \right\rangle \\ &= \left\langle \int_0^t \sqrt{\dot u(t_1)}\eta(t_1)\mathrm{d}t_1 \int_0^t \sqrt{\dot u(t_2)}\eta(t_2)\mathrm{d}t_2 \right\rangle \\ &= \int_0^t \dot u(t_1)\mathrm{d}t_1 = u(t)-u(0) \equiv |u(t)-u(0)|. \end{aligned}\tag{15}$$

The forces perform a random walk as a function of u,

$$\left\langle [F(u)-F(u')]^2\right\rangle = |u-u'|. \tag{16}$$

2 A phenomenological derivation of the stochastic field theory for the Manna model

In this section, we apply our considerations to a non-trivial example, the stochastic Manna model. We will see that our formalism permits a systematic derivation of the effective stochastic equations of motion. While the result is known in the literature [2, 3, 4, 5], it was there derived by symmetry principles, which are convincing only up to a certain degree. Furthermore, they leave undetermined all coefficients. While many of them can be eliminated by rescaling, our derivation will "land" on a particular line of parameter space, characterised by the absence of additional memory terms, see section 2.4.

2.1 Basic Definitions

The Manna sandpile was introduced in 1991 by S.S. Manna [6], as a stochastic version of the Bak-Tang-Wiesenfeld (BTW) sandpile [7].

The Abelian Sandpile Model (ASM) or Bak-Tang-Wiesenfeld (BTW) model. Randomly throw grains on a square lattice. If the number of grains (=height) at one point is greater or equal to $2d$ (i.e. 4 in 2 dimensions), then move a grain to each of the $2d$ nearest neighbors.

An alternative is the Manna model.

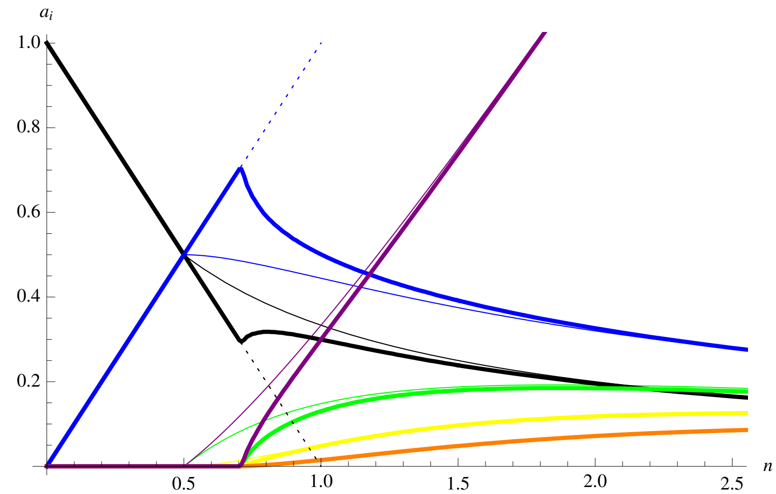

Figure 2: Thick lines: The order parameters of the Manna model, as a function of n, the average number of grains per site, obtained from a numerical simulation of the stochastic Manna model on a grid of size 150×150 with periodic boundary conditions. We randomly update a site for 10^7 iterations, and then update the histogram 500 times every 10^5 iterations. Plotted are the fraction of sites that are: unoccupied (black), singly occupied (blue), double occupied (green), triple occupied (yellow), quadruple occupied (orange). The activity $\rho = \sum_{i>1} a_i(i-1)$ is plotted in purple. No data were calculated for $n < 0.5$, where $a_0 = e = 1 - n$, $a_1 = n$, and $a_{i>2} = 0$ (inactive phase). Note that before the transition, $a_0 = 1 - n$ and $a_1 = n$. The transition is at $n = n_c = 0.702$.
Thin lines: The MF phase diagram, as given by Eqs. (26) ff. for $n \leq \frac{1}{2}$, and by Eqs. (27) ff. for $n \geq \frac{1}{2}$. We checked the latter with a direct numerical simulation.

Manna Model (MM). Randomly throw grains on a square lattice. If the height at one point is greater or equal to two, then with rate 1 move two grains from this site to randomly chosen neighbouring sites. (Both grains may end up on the same site.)

We start by analysing the phase diagram. We denote by a_i the fraction of sites with i grains. It satisfies the sum rule

$$\sum_i a_i = 1 \ . \tag{17}$$

In these variables, the number of grains n per site can be written as

$$n := \sum_i a_i \, i \ . \tag{18}$$

The empty sites are

$$e := a_0 \ . \tag{19}$$

The fraction of active sites is

$$a := \sum_{i \geq 2} a_i \ . \tag{20}$$

We also define the (weighted) activity as

$$\rho := \sum_{i \geq 2} a_i (i-1) \ . \tag{21}$$

Note that ρ satisfies the sum rule

$$n - \rho + e = 1 \ . \tag{22}$$

In order to take full advantage of this definition, one may change the toppling rules of the Manna model to those of the

Weighted Manna Model (wMM) : If a site contains $i \geq 2$ grains, randomly move these grains to neighbouring sites with rate $(i-1)$.

2.2 MF solution

In order to make analytical progress, we now study the *topple-away* or Mean Field solution of the stochastic Manna sandpile, which we can solve analytically. We define:

Mean-Field Manna Model (MF-MM). If a site contains two or more grains, move these grains to any randomly chosen other site of the system.

The rate equations are, setting for convenience $a_{-1} := 0$:

$$\partial_t a_i = -a_i \Theta(i \geq 2) + a_{i+2} + 2\Big[\sum_{j\geq 2} a_j\Big](a_{i-1} - a_i) \,. \tag{23}$$

Using the sum rule (17), they can be rewritten as

$$\partial_t a_i = -a_i \Theta(i \geq 2) + a_{i+2} + 2(1 - a_0 - a_1)(a_{i-1} - a_i) \,. \tag{24}$$

We are interested in the steady state $\partial_t a_i = 0$. One can solve these equations by introducing a generating function. An alternative solution consists in realising that for $i \geq 2$, Eq. (24) admits a steady-state solution of the form

$$a_i = a_2 \kappa^{i-2} \,, \qquad i > 2 \,. \tag{25}$$

This reduces the number of independent equations $\partial_t a_i = 0$ in Eq. (24) from infinity to three. Furthermore, there are the equations $\sum_{i=0}^{\infty} a_i = 1$, and $\sum_{i=0}^{\infty} i\, a_i = n$. Thus there are 5 equations for the 4 variables a_0, a_1, a_2, and κ. The reason we apparently have one redundant equation is due to the fact that we already used the normalisation condition (17) to go from Eq. (23) to Eq. (24).

These equations have two solutions: For $0 < n < 1$, there is always the solution for the *inactive* or *absorbing state*,

$$a_0 = 1 - n \,, \qquad a_1 = n \,, \qquad a_{i\geq 2} = 0 \,. \tag{26}$$

For $n > 1/2$, there is a second non-trivial solution,

$$a_0 = \frac{1}{1+2n} \,, \qquad a_{i>0} = \frac{4n \left(\frac{2n-1}{2n+1}\right)^i}{4n^2 - 1} \,. \tag{27}$$

(Note that a_2/a_1 has the same geometric progression as a_{i+1}/a_i for $i > 2$, which we did note suppose in our ansatz.) Thus the probability to find $i > 0$ grains on a site is given by the exponential distribution

$$p(i) = \frac{4n}{4n^2 - 1} \exp\left(-i\alpha_n\right) \,, \quad \alpha_n = \log\left(\frac{2n+1}{2n-1}\right) \,. \tag{28}$$

Using these two solutions, we get the MF phase diagram plotted on figure 2 (thin lines). This has to be compared with the simulation of the Manna model on the same figure (thick lines). One sees that for $n \geq 2$, MF solution and simulation are getting almost indistinguishable.

2.3 The complete effective equations of motion for the Manna model

In this section, we give the effective equations of motion for the Manna model. Let us start from the mean-field equations for $\rho(t)$ and $n(t)$. For simplicity of expressions, we use the weighted Manna model. The physics close to the transition should not depend on it, as the probability for higher occupancy is low. Let us start from the hierarchy of MF equations for the weighted Manna model. These are similar to Eq. (24), and can be rewritten as

$$\partial_t a_i = (1-i)a_i\Theta(i \geq 2) + (i+1)a_{i+2} + 2\rho(a_{i-1} - a_i) \ . \tag{29}$$

For convenience, let us write explicitly the rate equation for the fraction of empty sites $e \equiv a_0$,

$$\partial_t e = a_2 - 2\rho e \approx \rho(1-2e). \tag{30}$$

The first term, the gain $r_+ = a_2$ comes from the sites with two grains, toppling away, and leaving an empty site. The second term, the loss term, is the rate at which one of the toppling grains lands on an empty site, $r_- = 2\rho e$.

We now follow the formalism developed in section 1.1, Eqs. (1)–(6), to find the necessary noise. This yields

$$\partial_t e = a_2 - 2\rho e + \sqrt{a_2 + 2\rho e}\,\bar{\eta}_t \approx \rho(1-2e) + \sqrt{\rho}\sqrt{1+2e}\,\bar{\eta}_t \ , \tag{31}$$

where $\langle \bar{\eta}_t \bar{\eta}_{t'} \rangle = \delta(t-t')/\ell^d$, and ℓ is the size of the box which we consider. For the second equality we used again that close to the transition, $a_2 \approx \rho$.

Due to Eq. (22), the combination $n - \rho + e = 1$, and since n is conserved this implies $\partial_t e \equiv \partial_t \rho$. It is customary to write equation (31) for $\partial_t \rho$, instead of $\partial_t e$. Next we approximate $\sqrt{1+2e}$ by the value of e at the transition, i.e. $e \to e_{\rm c}^{\rm MF} = \frac{1}{2}$, see the mean-field phase diagram in Fig. 2.

$$\partial_t e \approx \rho(1-2e) + \sqrt{2\rho}\,\bar{\eta}_t \ . \tag{32}$$

Note that this equation gives back $e_{\rm c}^{\rm MF} = \frac{1}{2}$, used above in the simplification of the noise term.

Finally, let us suppose we have not a single box of size ℓ, but a lattice of boxes, labeled by a d-dimensional label x. Each toppling event moves two grains from a site to the neighbouring sites, equivalent to a current

$$J(x,t) = -D\nabla\rho(x,t) + \sqrt{2D\rho(x,t)}\eta(x,t), \tag{33}$$

with diffusion constant $D = 2 \times \frac{1}{2d} = \frac{1}{d}$. The first factor of 2 is due to the fact that two grains topple. The factor of $\frac{1}{2d}$ is due to the fact that each grain can topple in any of the $2d$ directions, thus the rate D per direction is $\frac{1}{2d}$, resulting into $D = 1/d$. As discussed above, we will drop the noise term as subdominant.

This current changes both the activity $\rho(x,t)$, as the number of grains $n(x,t)$, resulting into a contribution for both $\partial_t \rho(x,t)$, and $\partial_t n(x,t)$. It does not couple to the density of empty sites. Using the sum-rule (22) $n - \rho + e = 1$, implies the consistency relation $\partial_t \rho(x,t) \equiv \partial_t n(x,t) + \partial_t e(x,t)$ for the current; this confirms that both $\rho(x,t)$ and $n(x,t)$ must couple to the same current.

Thus, we finally arrive at the following set of equations

$$\partial_t e(x,t) = \big[1 - 2e(x,t)\big]\rho(x,t) + \sqrt{2\rho(x,t)}\,\eta(x,t) \tag{34}$$

$$\partial_t \rho(x,t) = \frac{1}{d}\nabla^2\rho(x,t) + \partial_t e(x,t) \tag{35}$$

$$\langle \eta(x,t)\eta(x',t')\rangle = \delta^d(x-x')\delta(t-t') \tag{36}$$

We could also write equations for ρ and n instead of ρ and e,

$$\partial_t\rho(x,t) = \frac{1}{d}\nabla^2\rho(x,t) + \big[2n(x,t)-1\big]\rho(x,t) - 2\rho(x,t)^2 + \sqrt{2\rho(x,t)}\,\eta(x,t)\ , \tag{37}$$

$$\partial_t n(x,t) = \frac{1}{d}\nabla^2\rho(x,t)\ , \qquad \langle\eta(x,t)\eta(x',t')\rangle = \delta^d(x-x')\delta(t-t')\ . \tag{38}$$

This is known as the conserved directed percolation (C-DP) class[1].

2.4 Mapping to disordered elastic manifolds

In [8] it had been proposed to use these equations as a basis for mapping the effective field theory of the Manna model derived above onto driven disordered elastic systems. The identifications are

$$\rho(x,t) = \partial_t u(x,t) \qquad \text{the velocity of the interface} \tag{39}$$

$$e(x,t) = \mathcal{F}(x,t) \qquad \text{the force acting on the interface} \quad . \tag{40}$$

The second equation (35) is the time derivative of the equation of motion of an interface, subject to a random force $\mathcal{F}(x,t)$,

$$\partial_t u(x,t) = \frac{1}{d}\nabla^2 u(x,t) + \mathcal{F}(x,t)\ . \tag{41}$$

Since $\rho(x,t)$ is positive, $u(x,t)$ is for each x monotonously increasing. Instead of parameterizing $\mathcal{F}(x,t)$ by space x and time t, it can be written as a function of space x and *interface position* $u(x,t)$. Setting $\mathcal{F}(x,t) \to F\big(x,u(x,t)\big)$, the first equation (34) becomes

$$\begin{aligned}\partial_t\mathcal{F}(x,t) \to \partial_t F\big(x,u(x,t)\big) &= \partial_u F\big(x,u(u,t)\big)\partial_t u(x,t)\\ &= \Big[1-2F\big(x,u(x,t)\big)\Big]\partial_t u(x,t) + \sqrt{2\partial_t u(x,t)}\eta(x,t)\ .\end{aligned} \tag{42}$$

For each x, this equation is equivalent to the Ornstein-Uhlenbeck [9] process $F(x,u)$, defined by

$$\partial_u F(x,u) = 1 - 2F(x,u) + \sqrt{2}\,\eta(x,u)\ , \tag{43}$$

$$\langle\eta(x,u)\eta(x',u')\rangle = \delta^d(x-x')\delta(u-u')\ . \tag{44}$$

It is a Gaussian Markovian process with mean $\langle F(x,u)\rangle = 1/2$, and variance in the steady state (see appendix A) of

$$\Big\langle\big[F(x,u)-\tfrac12\big]\,\big[F(x',u')-\tfrac12\big]\Big\rangle = \frac12\delta^d(x-x')\mathrm{e}^{-2|u-u'|}\ . \tag{45}$$

Writing the equation of motion (41) as

$$\partial_t u(x,t) = \frac{1}{d}\nabla^2 u(x,t) + F\big(x,u(x,t)\big)\ , \tag{46}$$

it can be interpreted as the motion of an interface with *position* $u(x,t)$, subject to a disorder force $F\big(x,u(x,t)\big)$. The latter is δ-correlated in the x-direction, and short-ranged correlated in the u-direction. In other words,

[1] The above equations for ρ and n were obtained in the literature [2, 3, 4, 5] by means of symmetry principles, but never properly derived. Evoking symmetry principles also leaves all coefficients undefined, and does not ensure that Eq. (34) is valid on a single site, i.e. is free of spatial derivatives. This locality will prove essential in the next section.

this is a disordered elastic manifold subject to Random-Field disorder. It can be treated via field theory [10].

Also note that Eq. (37) has a quite peculiar symmetry, namely the factor of 2 in front of both $n(x,t)\rho(x,t)$ and $-\rho(x,t)^2$. As a consequence, Eq. (34) does not contain a term $\sim \rho^2(x,t)$, which would spoil the simple mapping presented above. The absence of this term *can not* be induced on symmetry arguments only. How this additional term, if present, can be treated is discussed in Ref. [8].

A Correlations of an Ornstein-Uhlenbeck process

The equation of motion of the Ornstein-Uhlenbeck process

$$\partial_u F(u) = 1 - 2F(u) + \sqrt{2}\,\eta(u), \qquad \langle \eta(u)\eta(u')\rangle = \delta(u-u'). \tag{47}$$

is solved by

$$F(u) = \frac{1}{2} + \sqrt{2}\int_{-\infty}^{u} \mathrm{d}w\, \mathrm{e}^{-2(u-w)}\eta(w). \tag{48}$$

It leads to force correlations

$$\begin{aligned}
&\Delta(u-u') := \left\langle \left[F(u) - \tfrac{1}{2}\right] \left[F(u') - \tfrac{1}{2}\right] \right\rangle \\
&= 2\int_{-\infty}^{u} \mathrm{d}w \int_{-\infty}^{u'} \mathrm{d}w'\, \mathrm{e}^{-2(u+u'-w-w')}\, \langle \eta(w)\eta(w')\rangle \\
&= 2\int_{-\infty}^{\min(u,u')} \mathrm{d}w\, \mathrm{e}^{-2(u+u'-2w)} \\
&= \frac{1}{2}\,\mathrm{e}^{-2|u-u'|}.
\end{aligned} \tag{49}$$

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