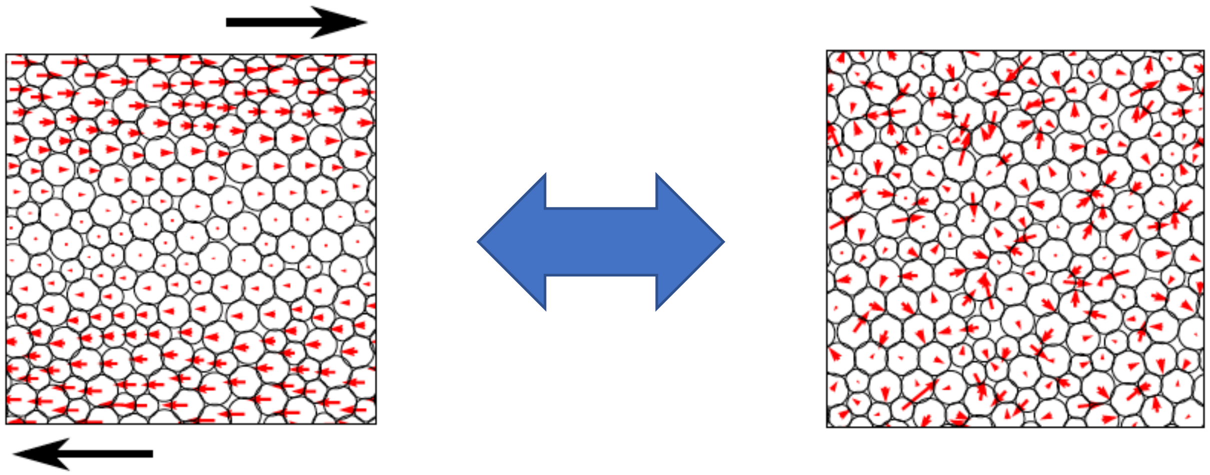


Dense disordered active matter

Lisa Manning

Syracuse University

June 16 and 17

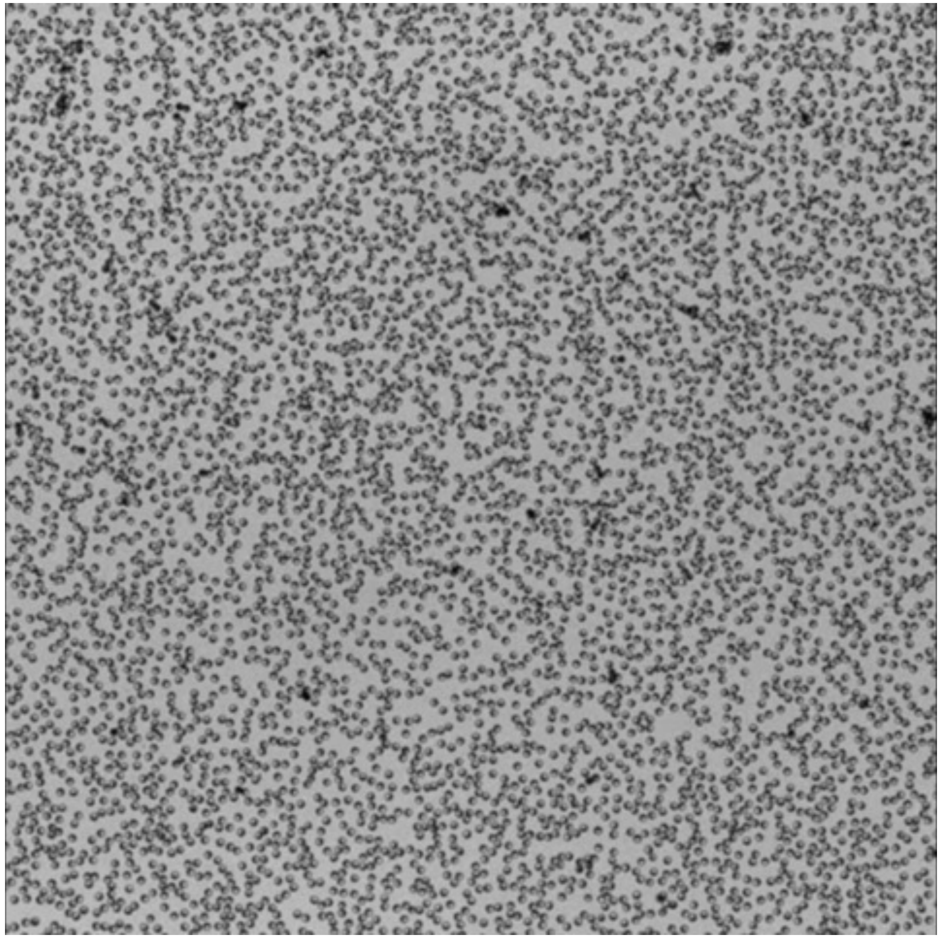
Disorder in complex systems summer school

Institut Pascal

Motivation: Would like a theory of active matter

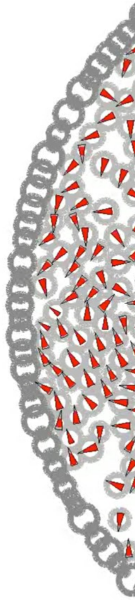
From talk by Julien Tailleur

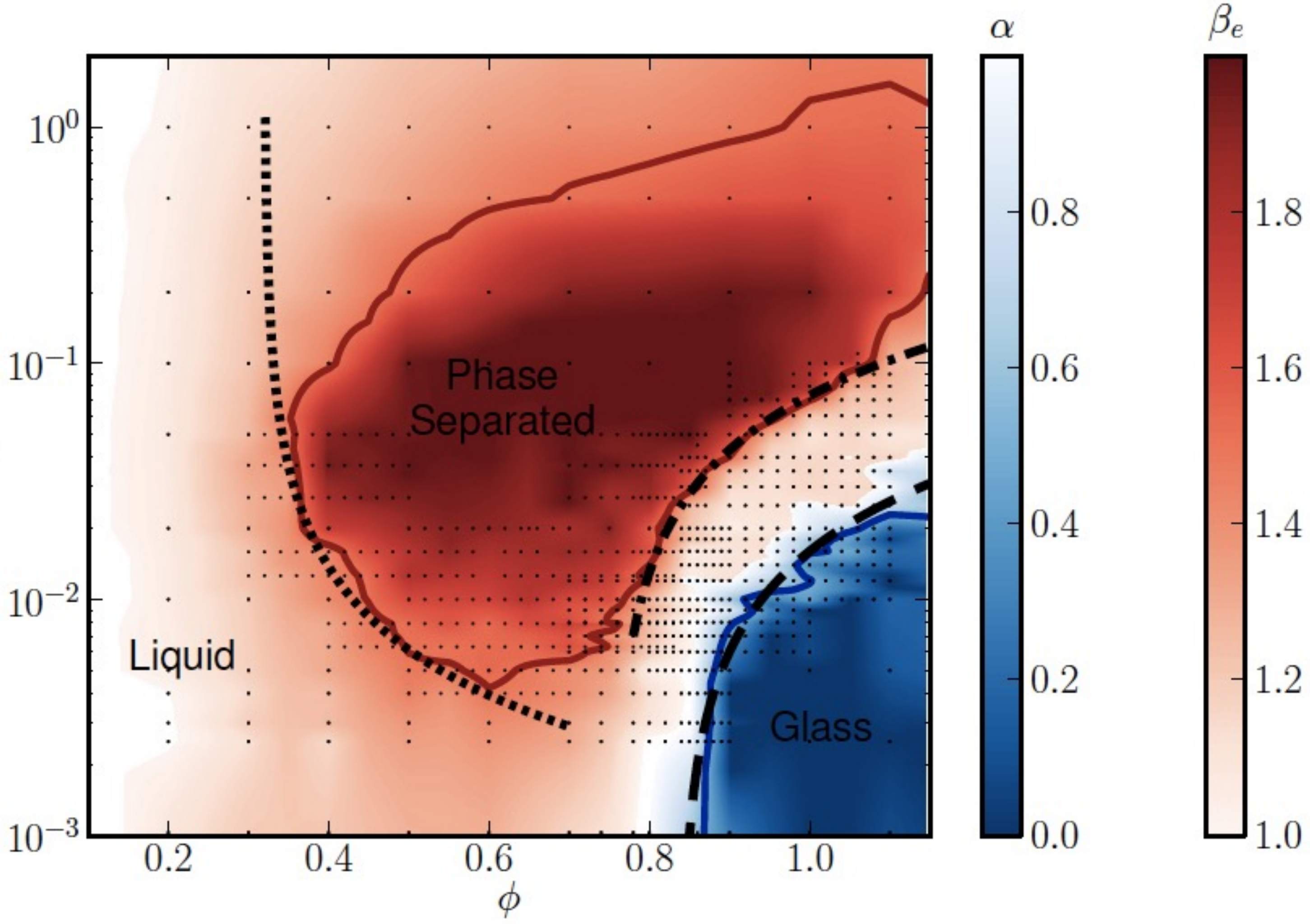
- breaking time-reversal invariance generates beautiful new types of behavior:
  - Giant number fluctuations
  - Motility induced phase separation

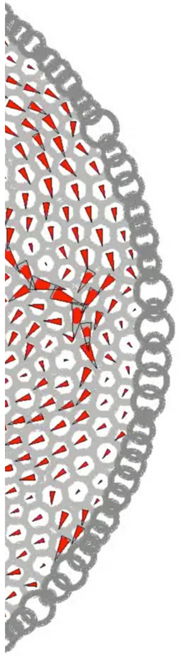


van der Linden et al PRL, 123, 098001 (2019)

At high densities, dynamics change

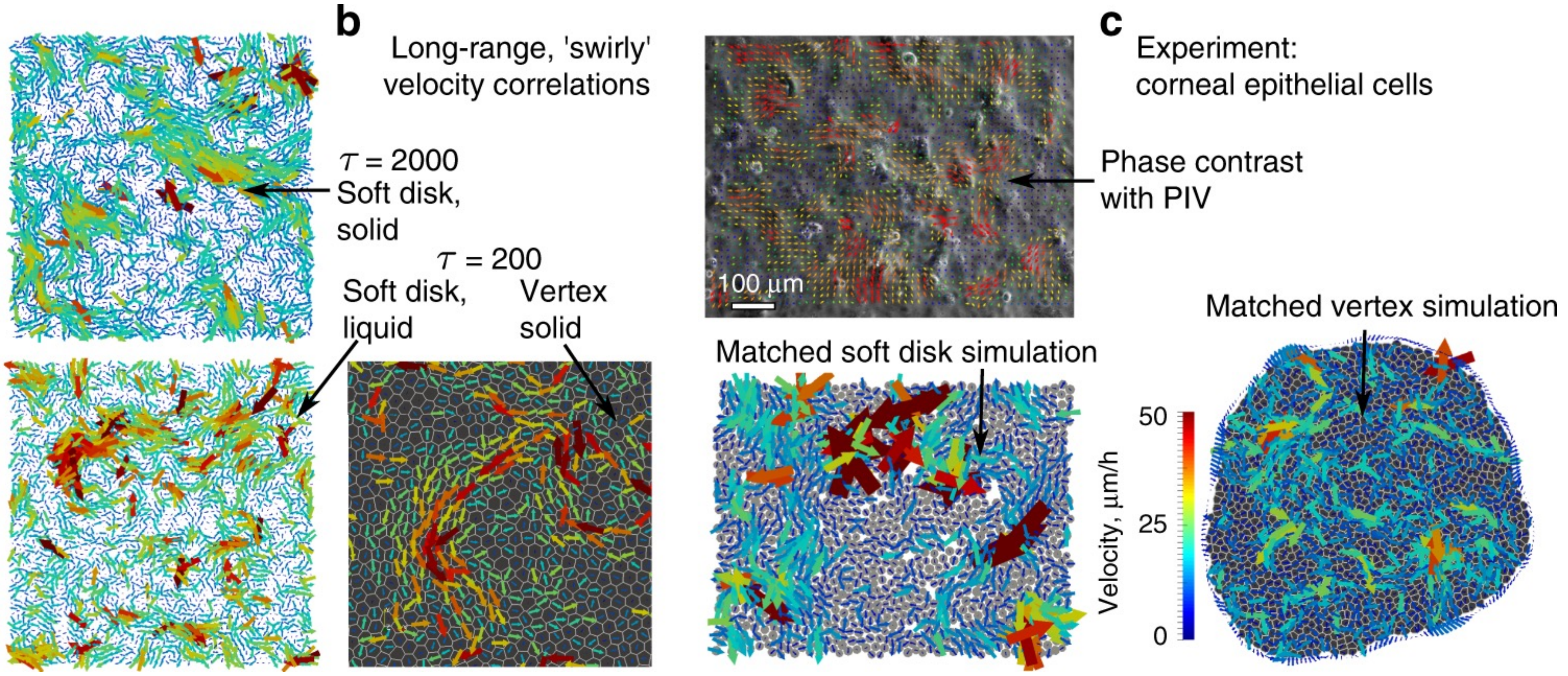






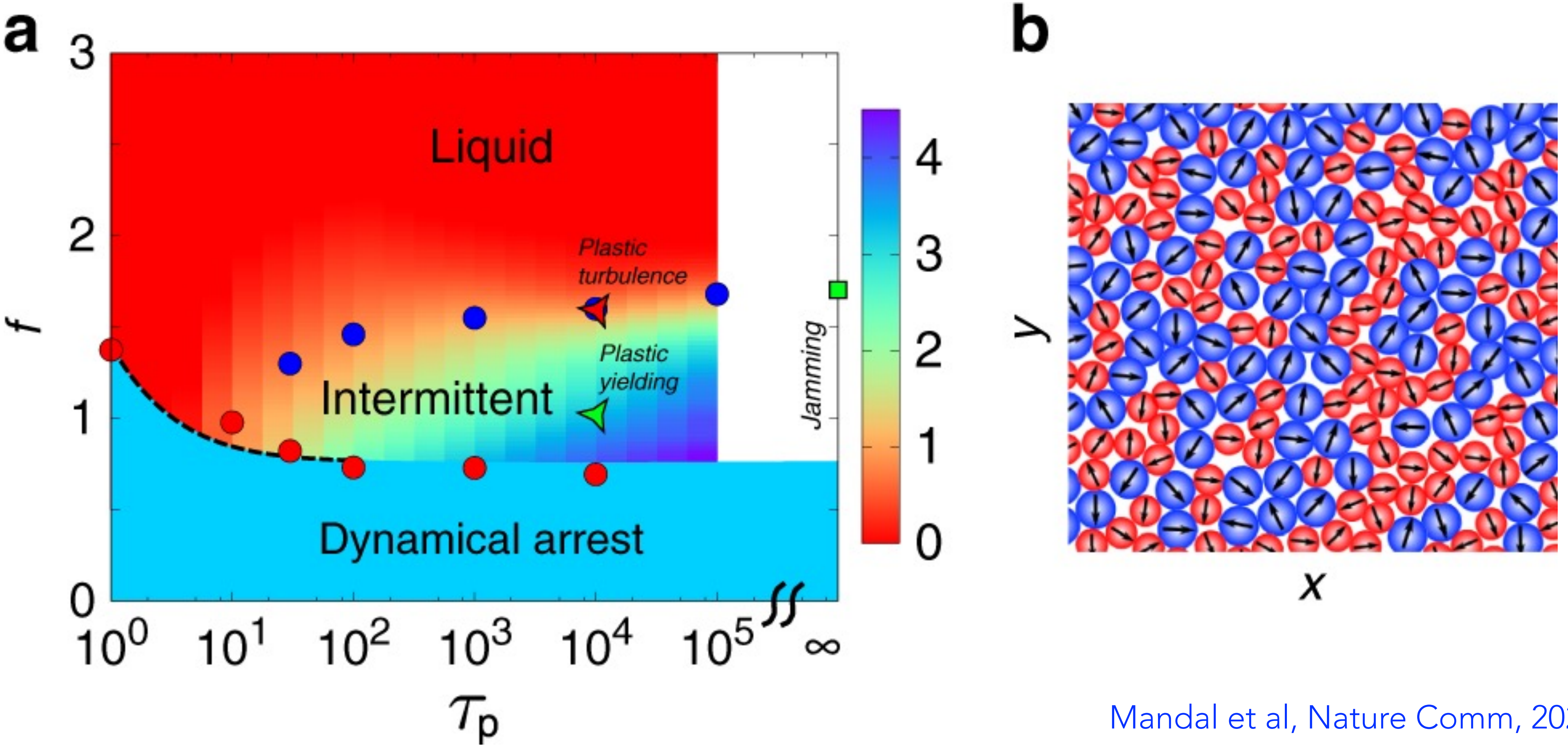
Henkes, Fily, Marchetti, PRE 2011

The dynamics become “swirly”: long-range velocity correlations



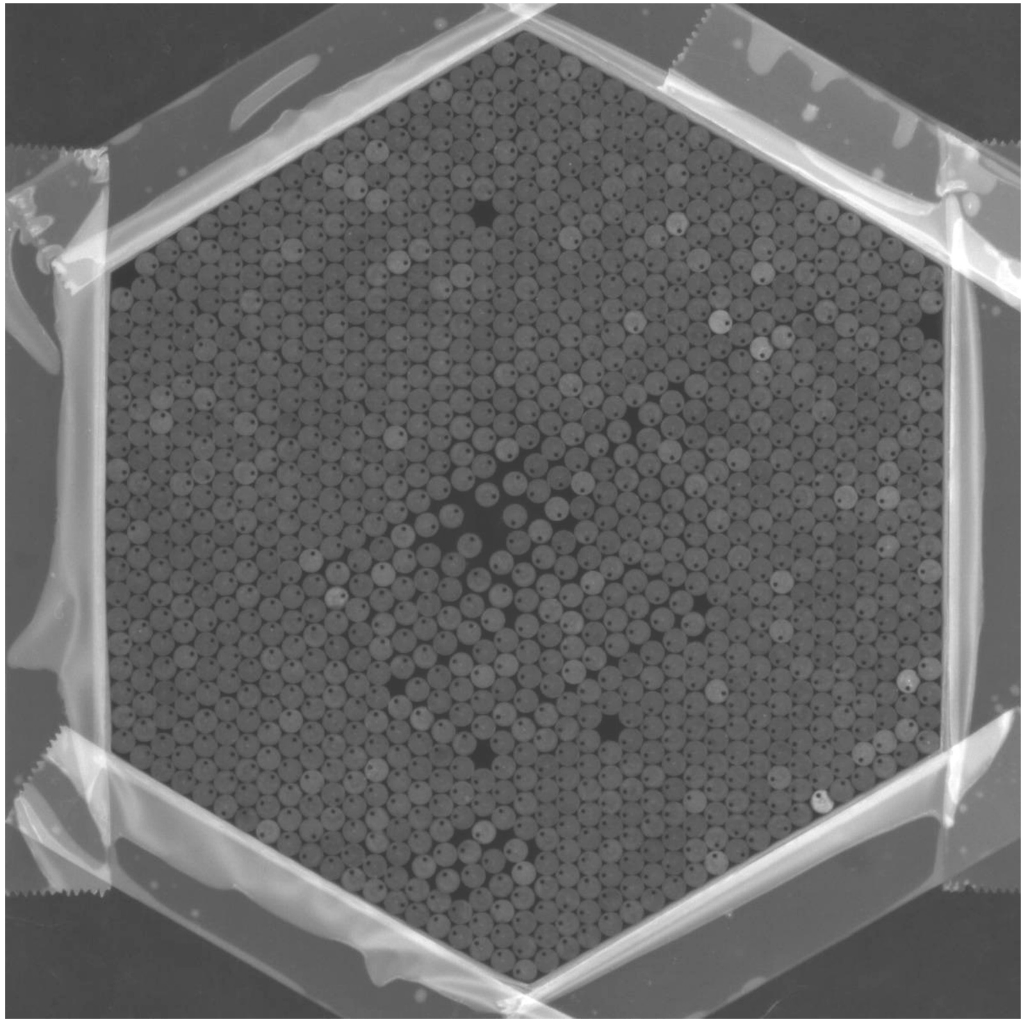
Henkes et al Nature communications (2020)

dynamics share some features with fluids, some with solids (c.f. fluid turbulence talks last week)



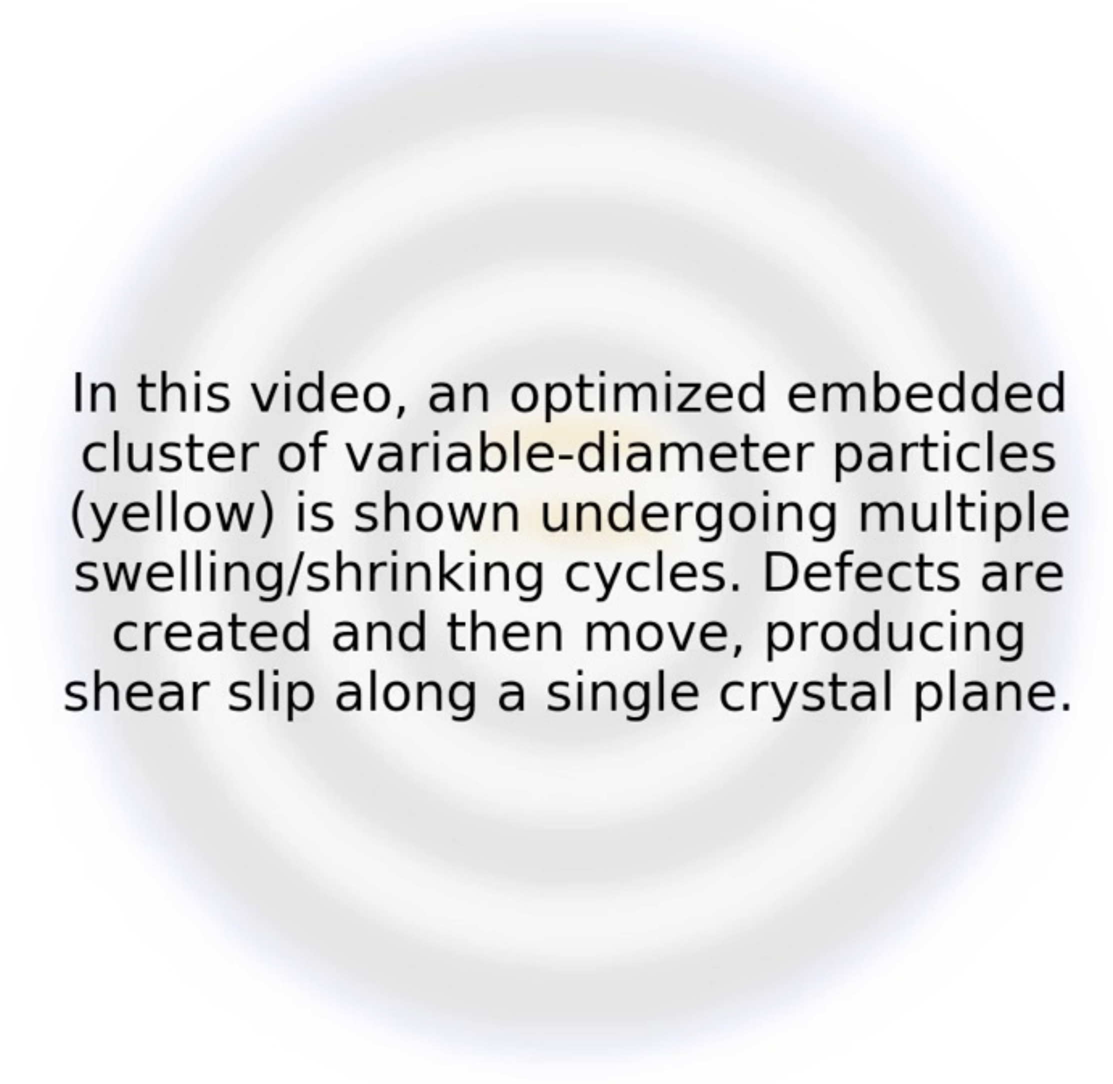
Mandal et al, Nature Comm, 2020

Displacements clearly related to underlying solid-like structure, looks "self-shearing"



G. Briand and O. Dauchot, PRL, (2016)

In ordered systems, one can design function using defects:

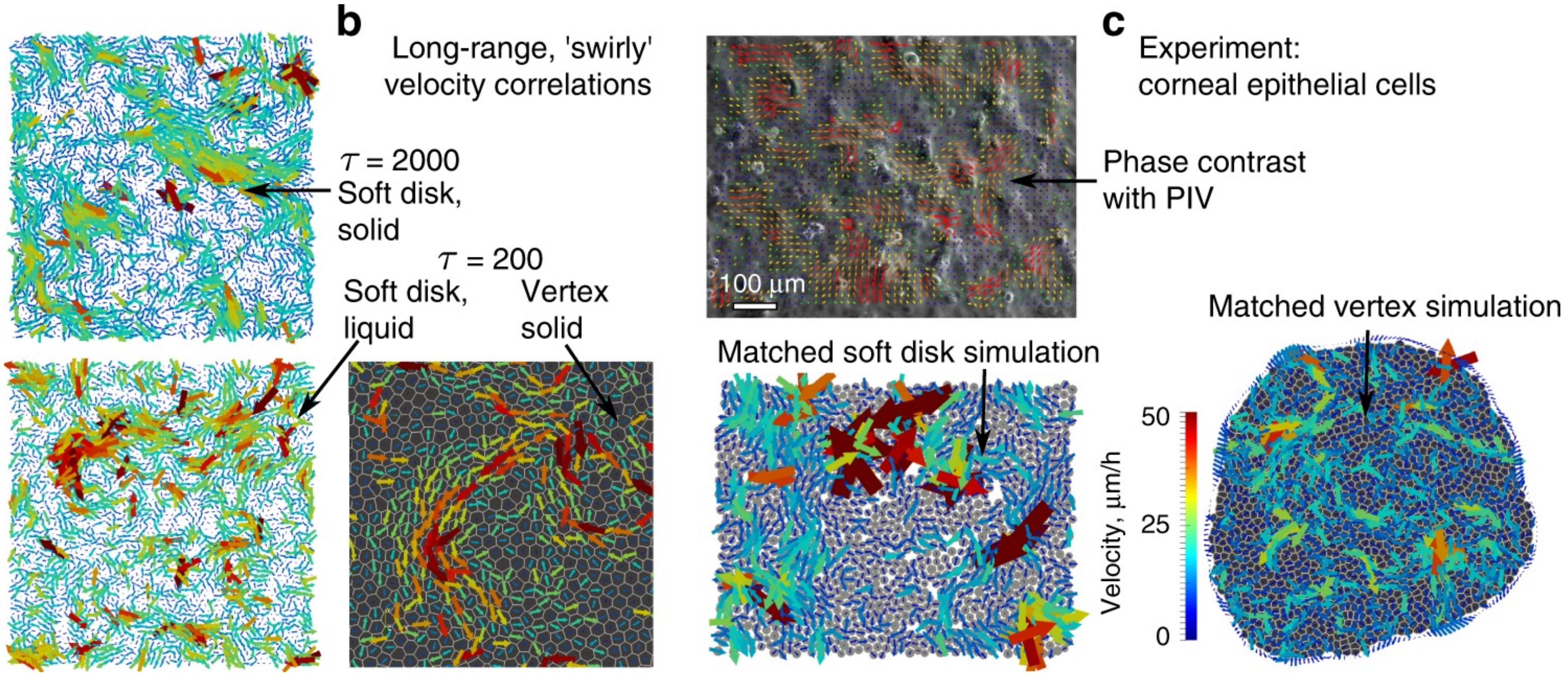


VanSaders and Glotzer, PNAS 2021

Today: long-range velocity correlations
Tomorrow: intermittency and avalanches

These are images from simulations that are "solid-like" in the passive state

soft discs with packing fraction above jamming or vertex model with shape index below rigidity



Henkes et al Nature communications (2020)

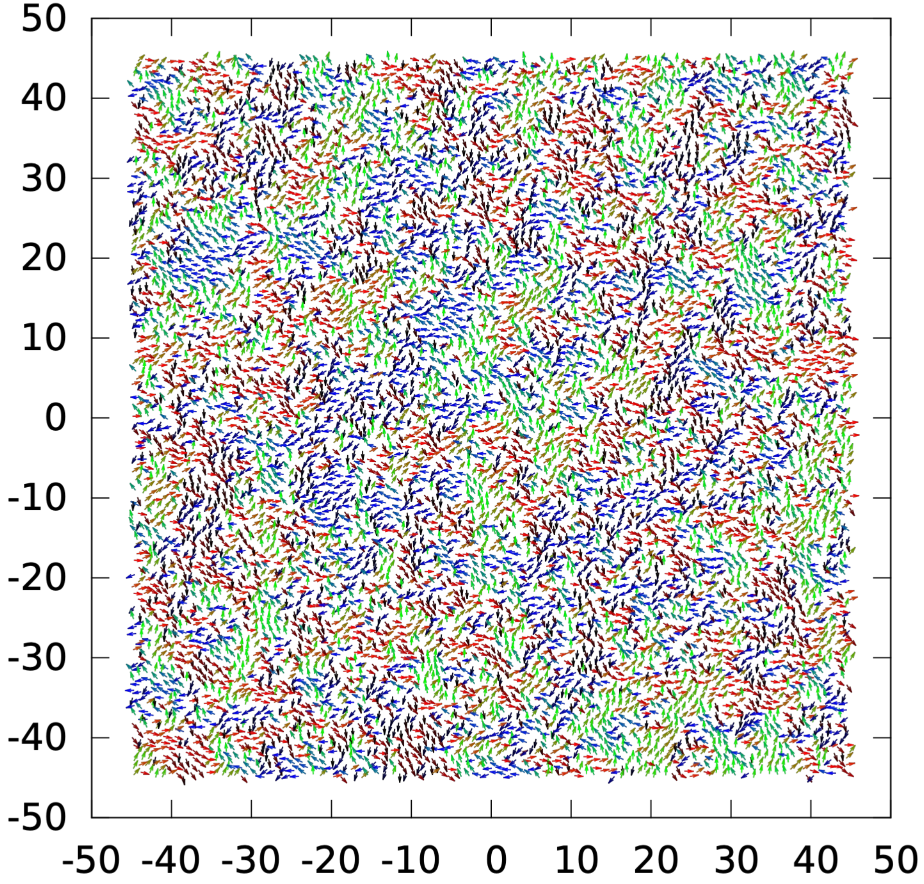
Dense Fluid Systems

Increasing velocity correlation with increasing persistence time.

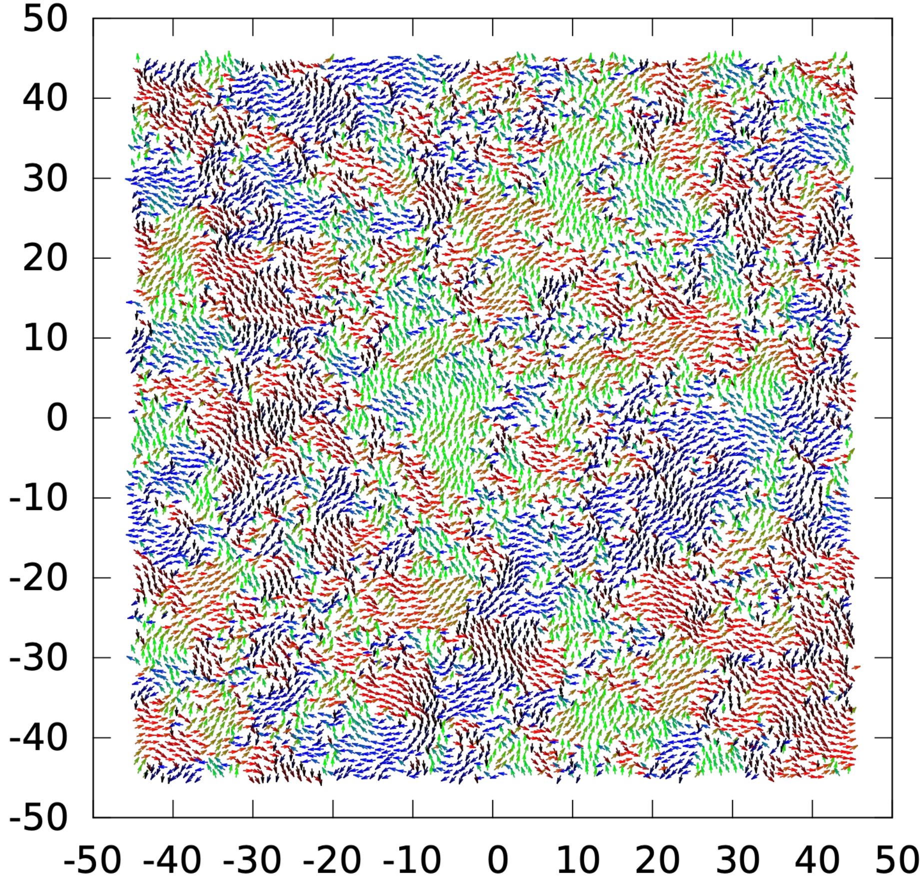
Longitudinal $\rightarrow$ $\omega_{\parallel}(q) = \frac{1}{N}\langle\left|\hat{\mathbf{q}}\cdot\mathbf{v}(\mathbf{q})\right|^2\rangle$

$\omega_{\perp}(q) = \frac{1}{N}\langle\left|\mathbf{v}(\mathbf{q}) - \hat{\mathbf{q}}(\hat{\mathbf{q}}\cdot\mathbf{v}(\mathbf{q}))\right|^2\rangle$ $\rightarrow$ Transverse

τ = 0.14



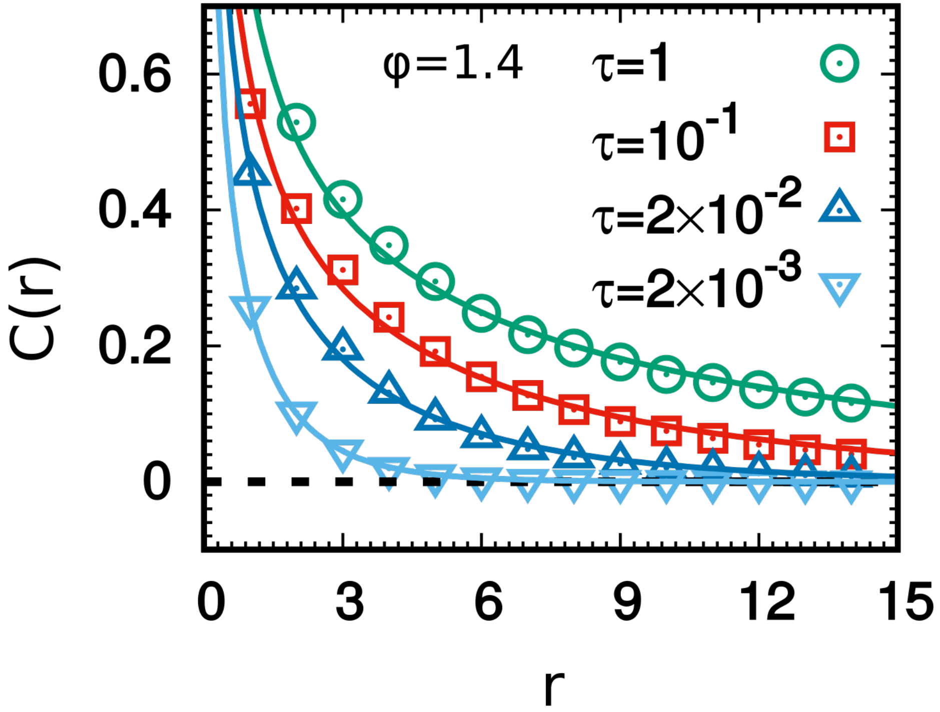
τ = 10



Velocity Correlations: solid-like starting point

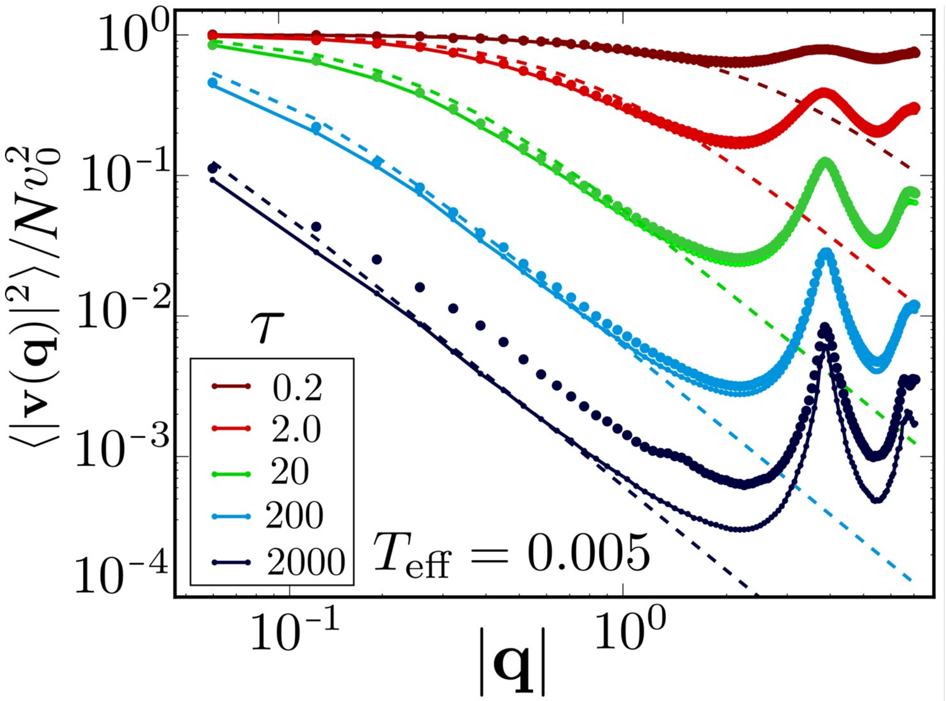
Equal time velocity correlation length increases with increasing persistence time.
Arrested active systems.
Lines are theoretical calculations.

Theory assumes vibrations around average positions.



Caprini *et al.* Phys. Rev. Res., **2**, 023321 (2020).

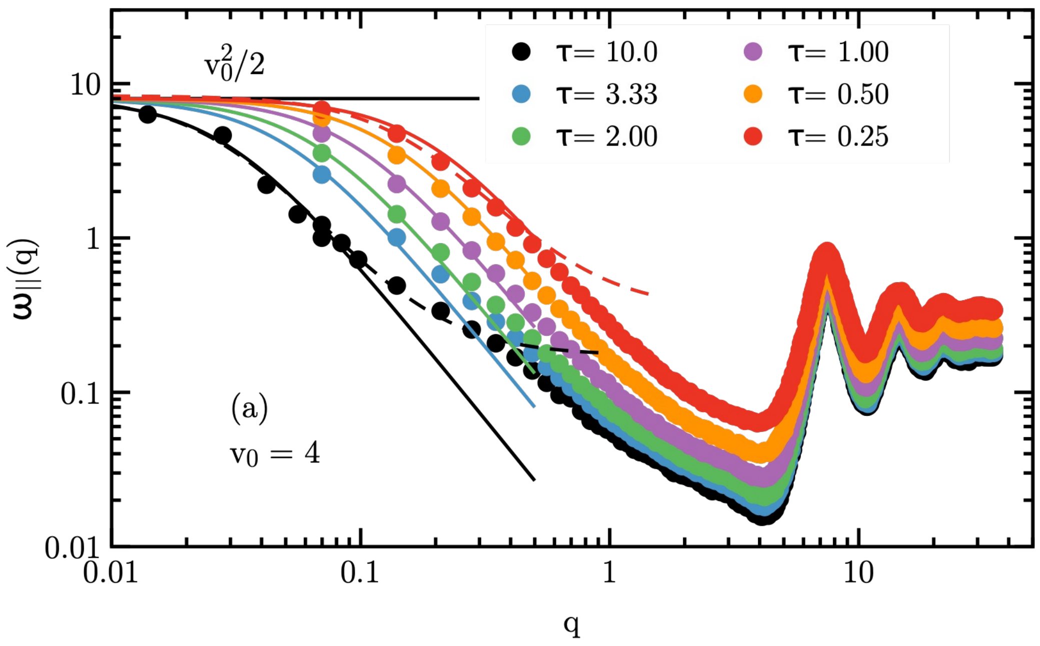
Theory approximates system as an amorphous elastic solid.

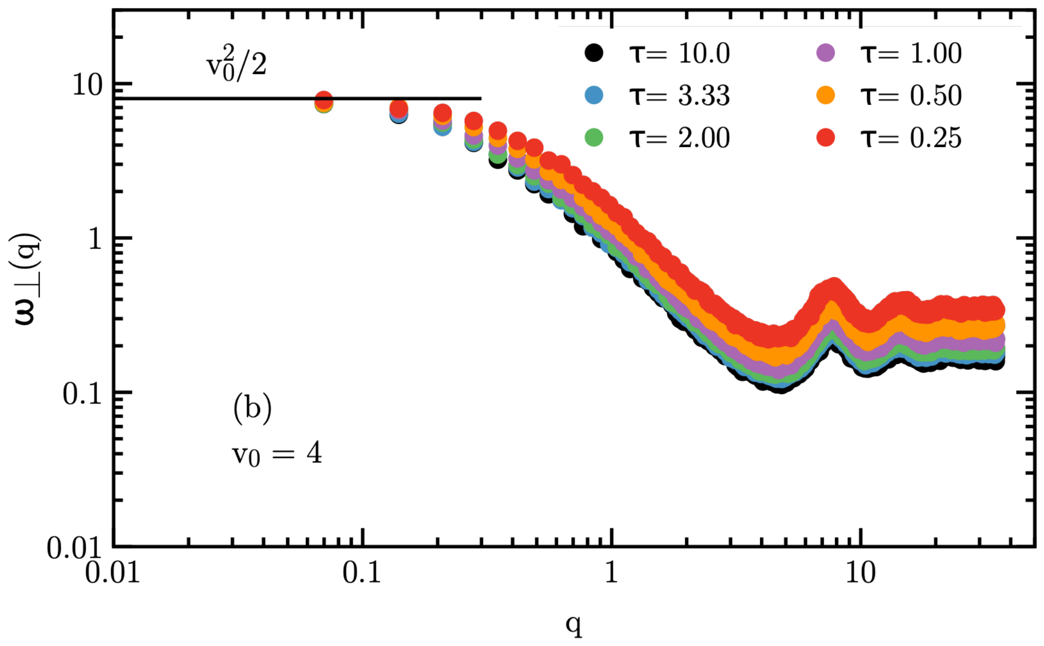


Henkes *et al.*, Nat. Commun., **11**, 1 (2020).

Velocity Correlations: liquid-like starting point

Equal time velocity correlations grow with increasing persistence time.
Dashed lines are numerical fits.
Solid lines are predictions of an approximate theory for longitudinal correlations.





$$\omega_\parallel(q) = \frac{1}{N}\langle\left|\hat{\mathbf{q}}\cdot\mathbf{v}(\mathbf{q})\right|^2\rangle$$

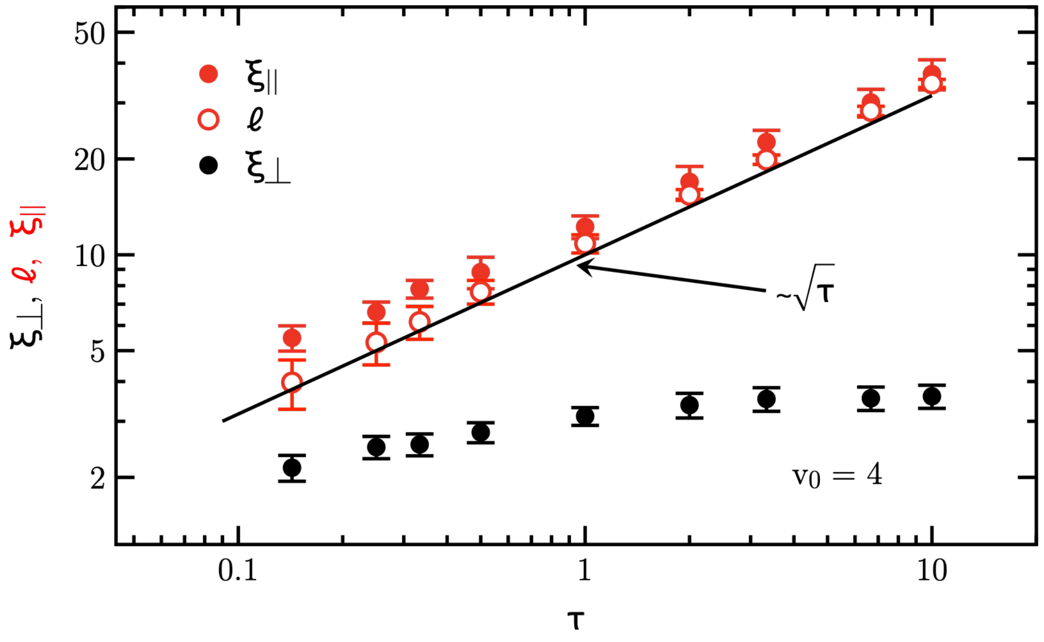
$$\omega_\perp(q) = \frac{1}{N}\langle\left|\mathbf{v}(\mathbf{q}) - \hat{\mathbf{q}}(\hat{\mathbf{q}}\cdot\mathbf{v}(\mathbf{q}))\right|^2\rangle$$

Grzegorz Szamel and Elijah Flenner, EPL 133, 60002 (2021)

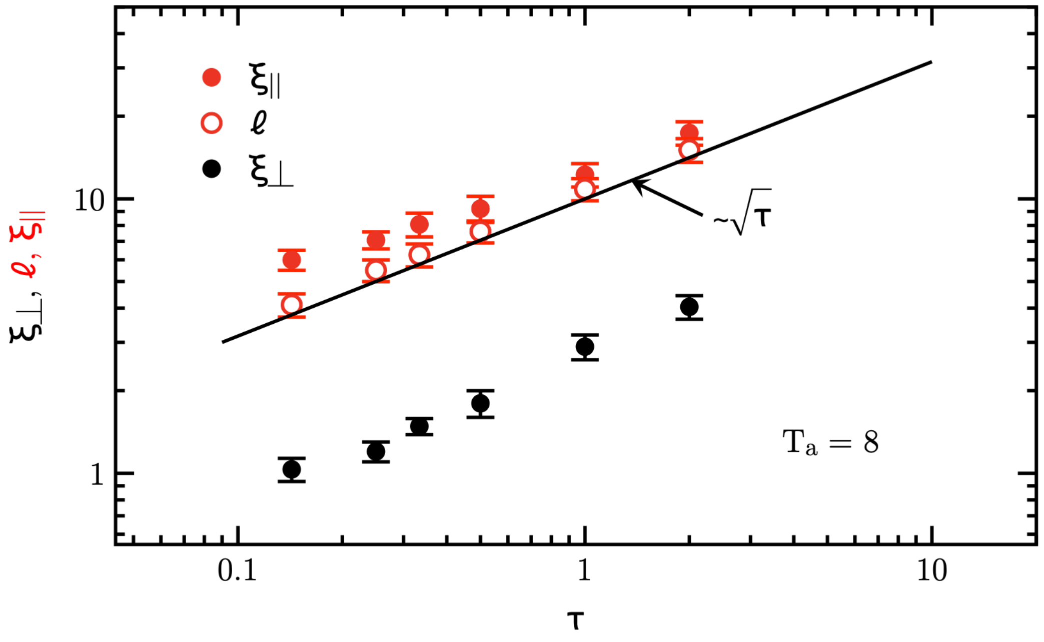
Comparison of scaling prediction with simulations

$$\ell = \sqrt{\tau B_v / (\gamma \rho)}$$

$\xi_{||}$ and $\xi_{\perp}$ obtained from fits to $\omega_{||}(q)$ and $\omega_{\perp}(q)$.
Open symbols are results of the theory.



Fixed magnitude of the velocity.



Fixed active temperature.

Grzegorz Szamel and Elijah Flenner, EPL 133, 60002 (2021)

References for derivations today:

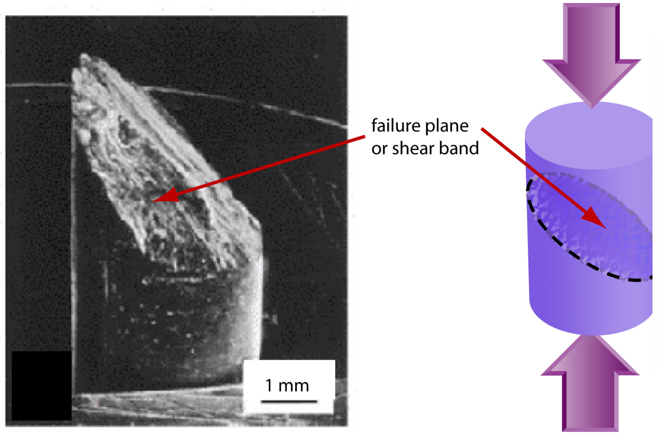
- Henkes, S. Kostanjevec, K., Collinson, J.M, Sknepnek, R and Bertin, E., Nat. Commun., **11**, 1 (2020).
  - see also Henkes, Fily, Marchetti PRE 2011
  - Bi, Yang, Marchetti, Manning PRX 2016
- Grzegorz Szamel and Elijah Flenner, EPL 133, 60002 (2021)

Day 2
Review: response of materials under shear

here, focus on zero temperature, limit of infinitely slow driving

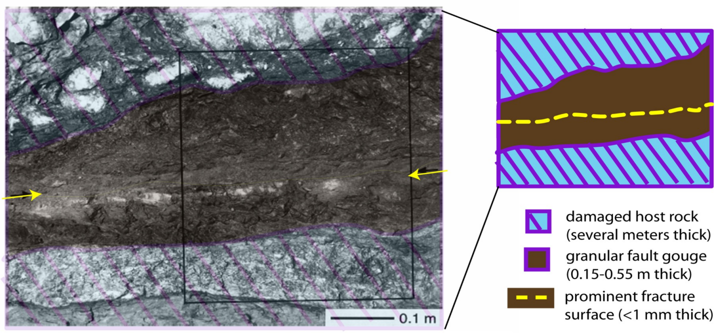
Sheared disordered materials are well-studied

bulk metallic glasses



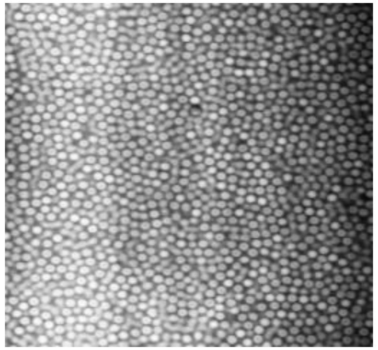
W. Johnson Group, Caltech

granular fault gouge



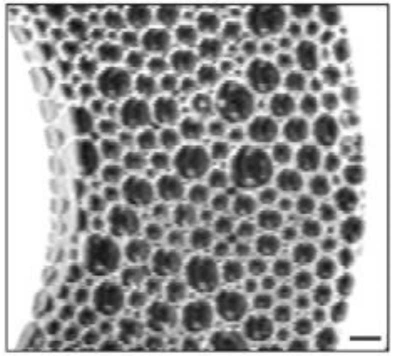
F. M. Chester and J. S. Chester, *Tectonophys.* 295, 1998.

colloids



Amann et al 2013

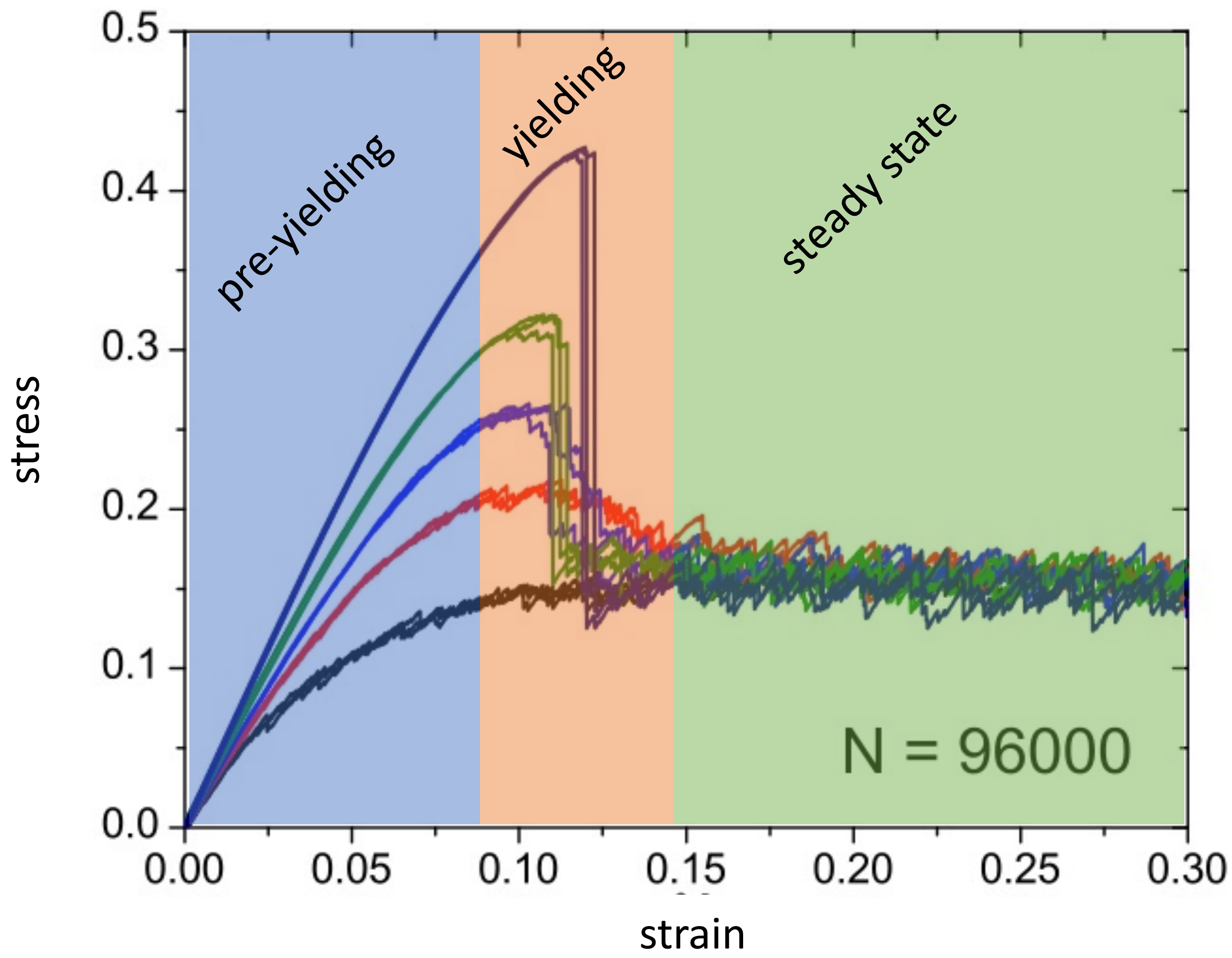
foams



Lauridsen et al 2002

Though perhaps still not well understood.

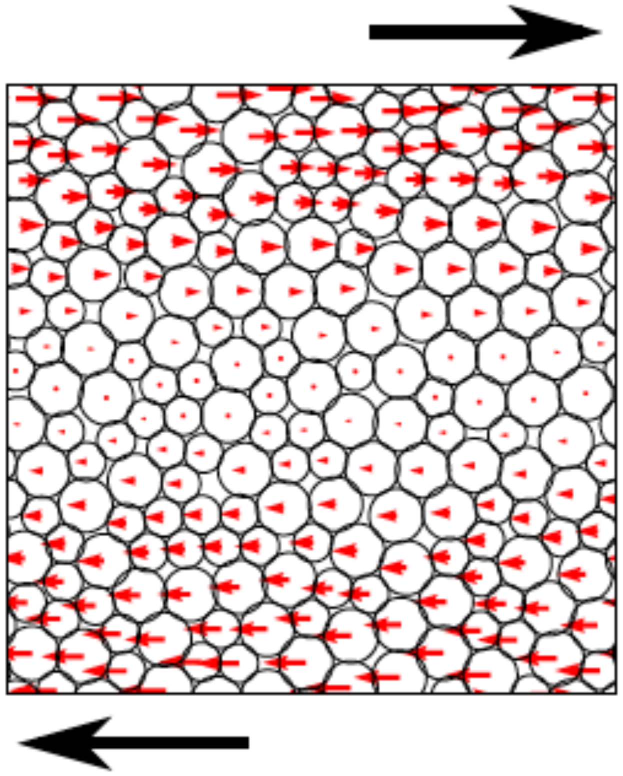
Plastic rearrangements and avalanches are well-studied in three regimes:

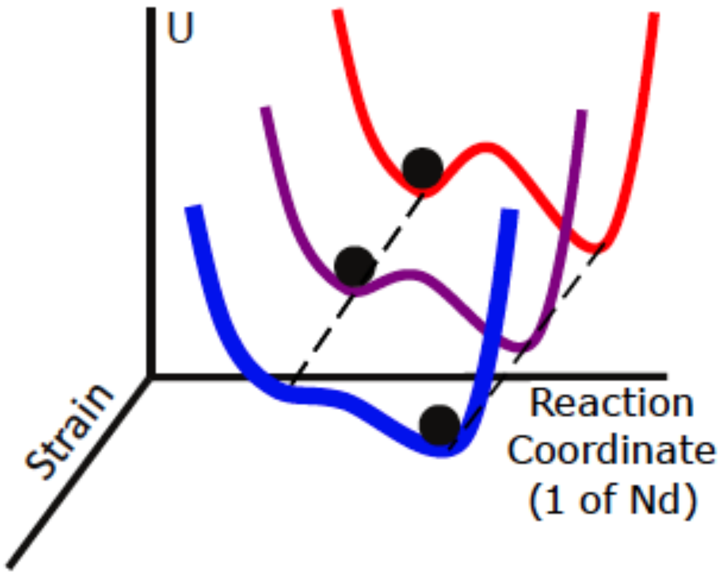


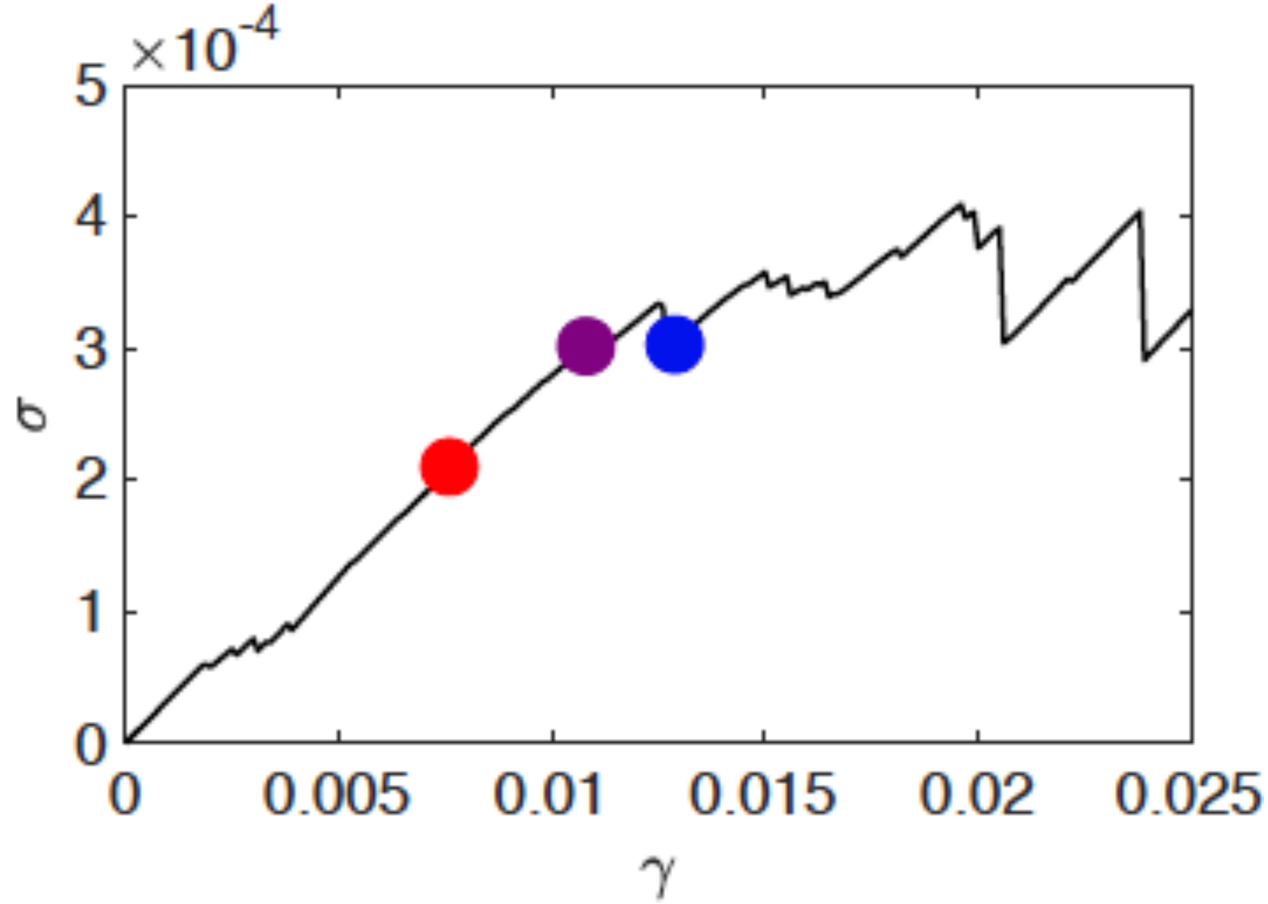
c.f. talk by Kirsten Martens yesterday

Ozawa PNAS 2018

Athermal, Quasistatic Shear (AQS)



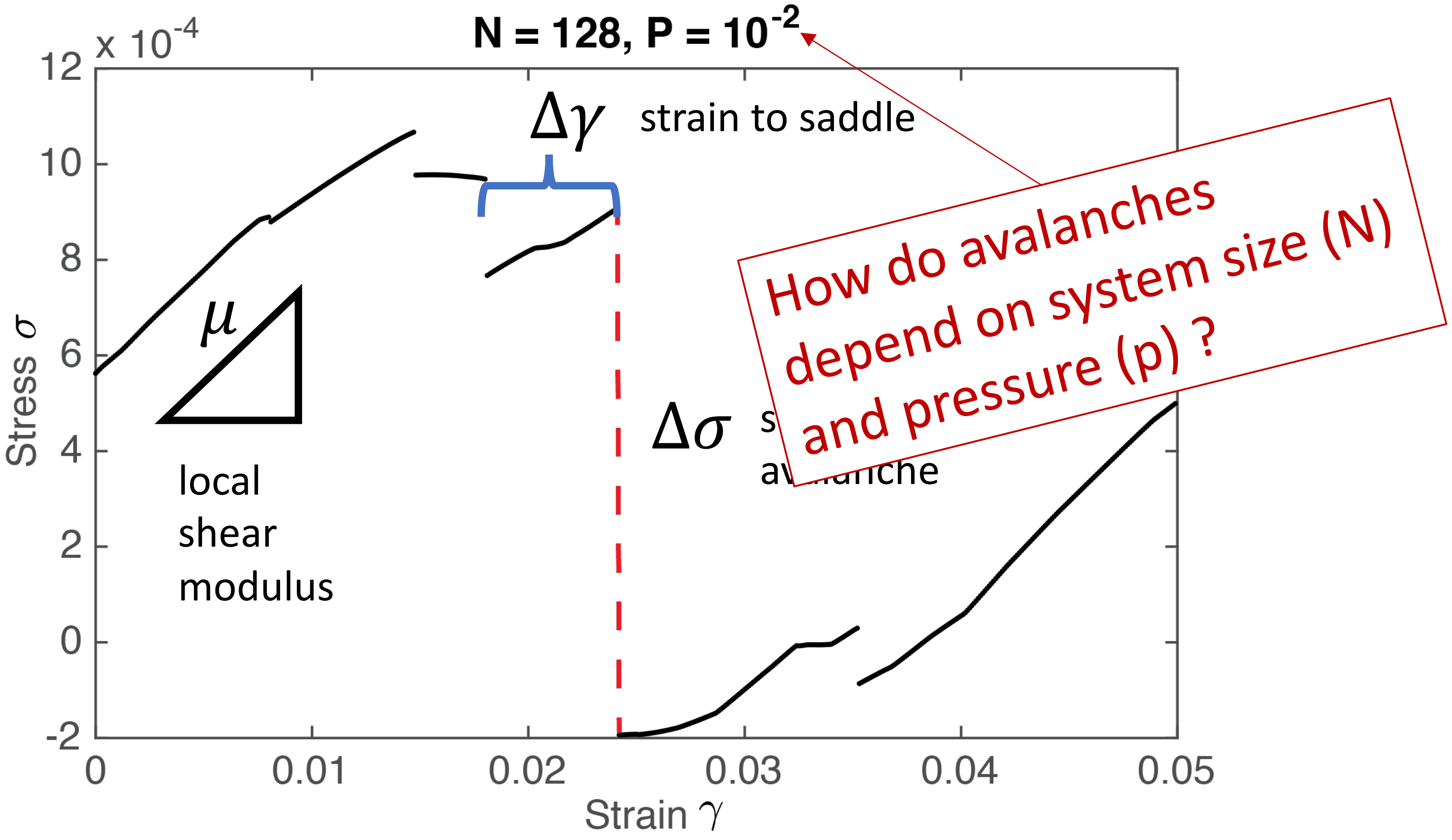




Maloney and Lemaitre PRE 2006

Quantifying linear response and avalanches in the pre-yielding regime

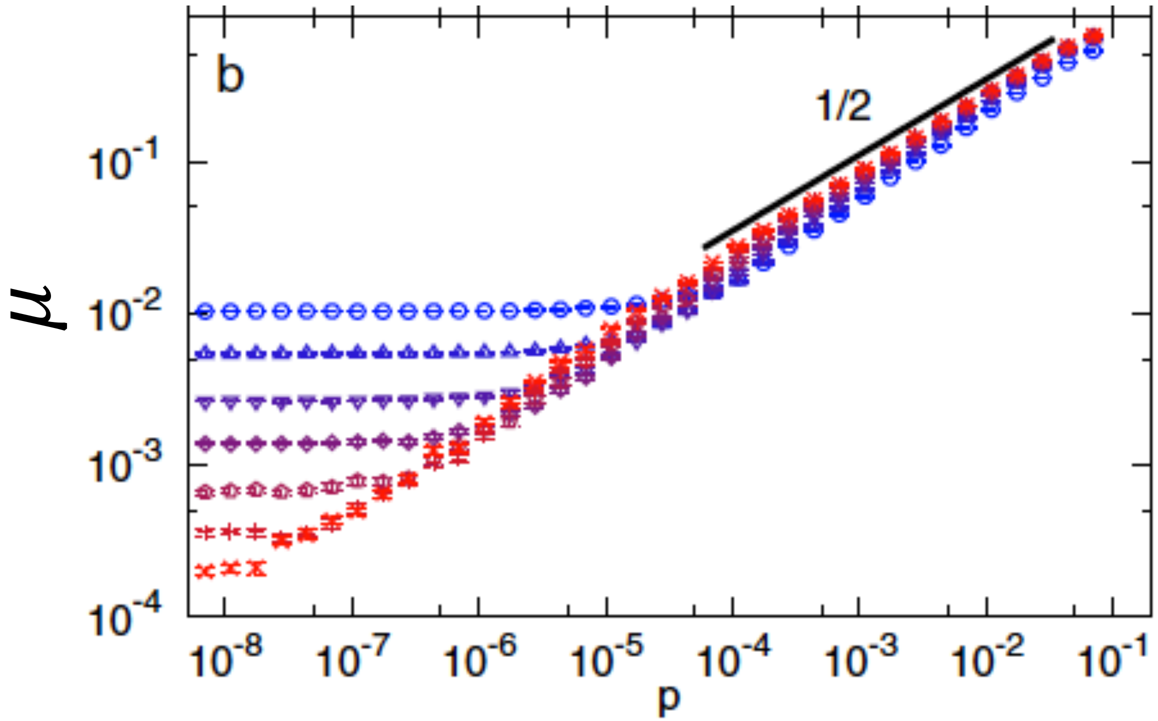
Peter Morse

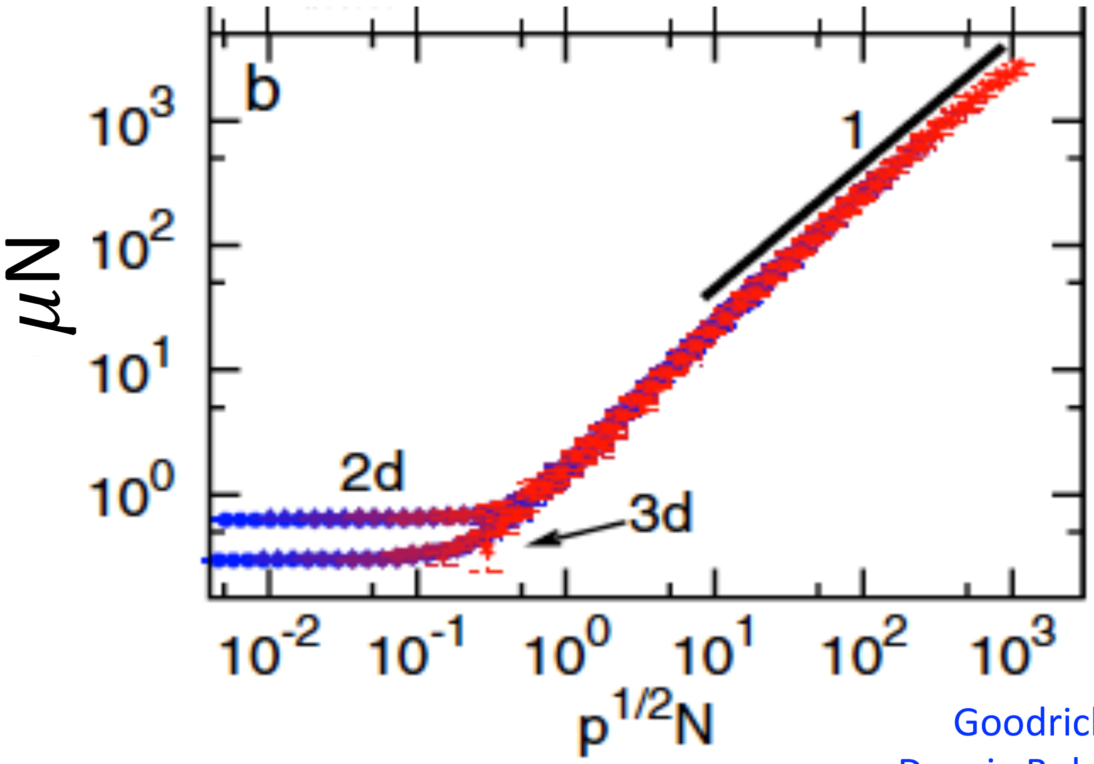


Morse et al Physical Review Research (2020

Shear modulus

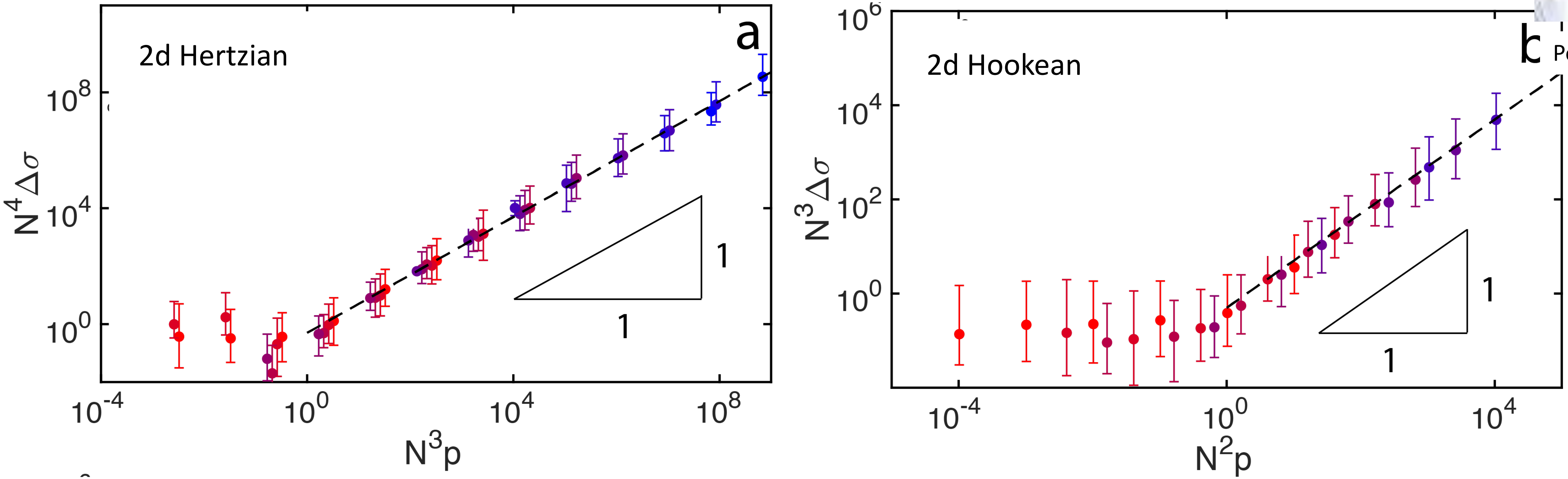
- Calculated directly from dynamical matrix (curvatures of landscape)
- Observe finite-size scaling: $\mu = N^y F(p^\eta N), y = -1; \eta = 1/2$
- Isostatic system is singular in linear response, while at any finite pressure the system is analytic around $p \to 0^+$ (hard sphere limit)





Goodrich PRL 2012
Dagois-Bohy PRL 2012

Nonlinear stress drops show same finite-size scaling as shear modulus(!)

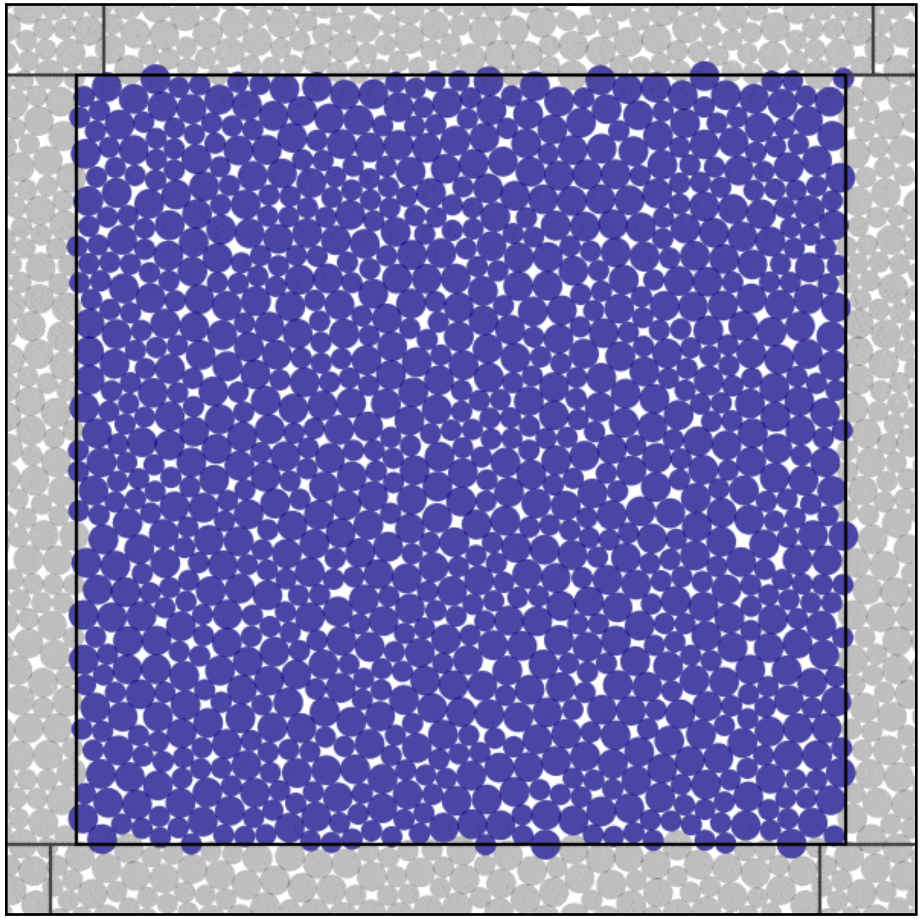


- Extend scaling argument: $z - z_{iso} = N^y W\left(N^\beta p\right); \beta = 2(\alpha - 1)$
  - α=5/2 for Hertzian and α=2 for Hookean O'Hern PRL 2002,Goodrich PRL 2012
- For systems in the high N and p regime: $\langle\Delta\sigma\rangle \sim \frac{p}{N}$

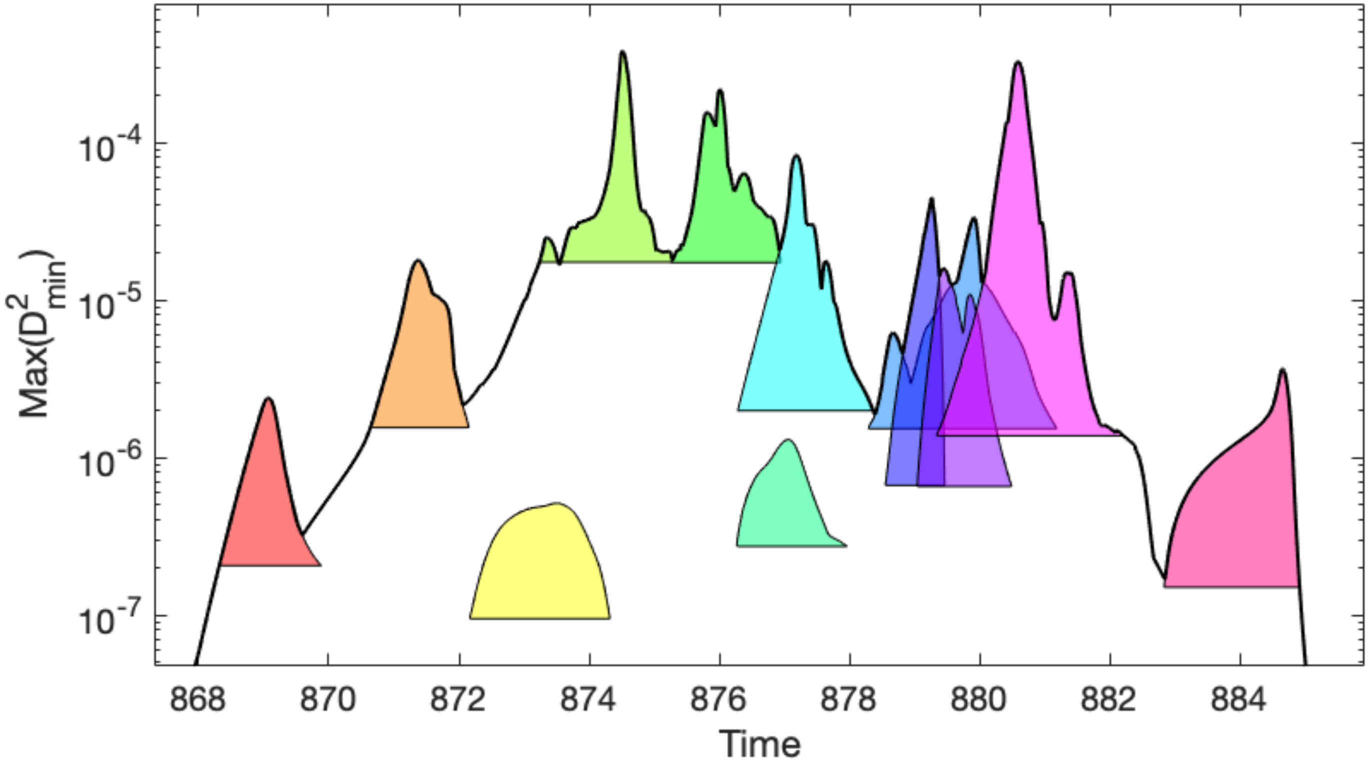
Morse et al Physical Review Research (2020)

Stress drops associated with bursts of localized deformation

Ethan Stanifer



persistent homology clustering of nonaffine displacements

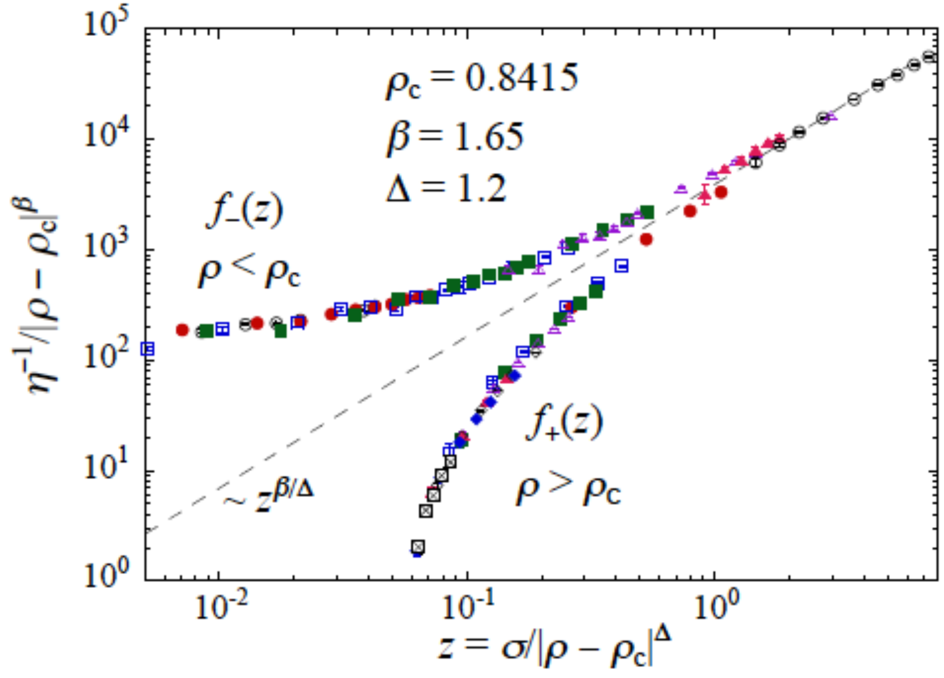


Stanifer and Manning, Soft Matter 2022

What happens in active matter?

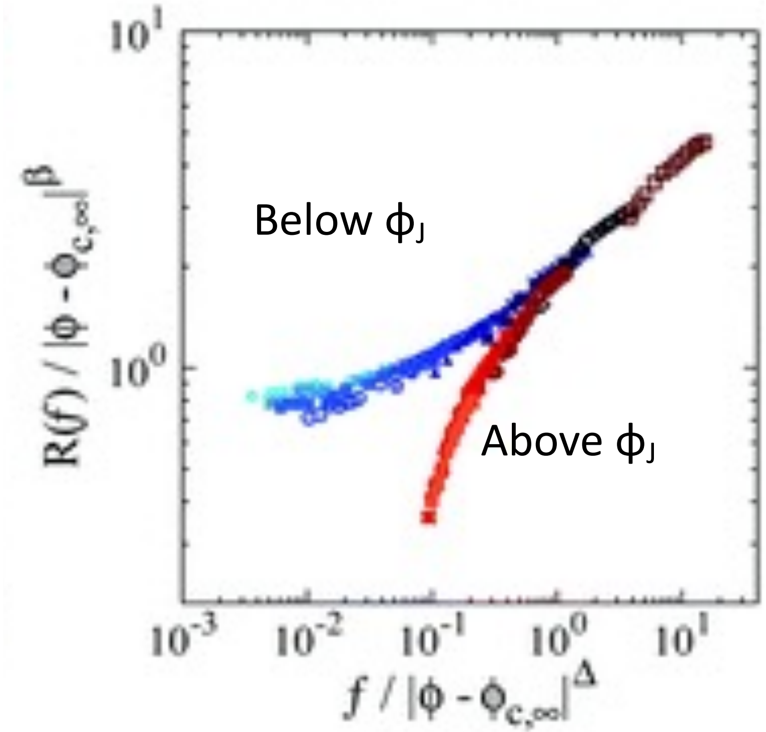
Lots of suggestive results at finite rates of driving

sheared granular matter



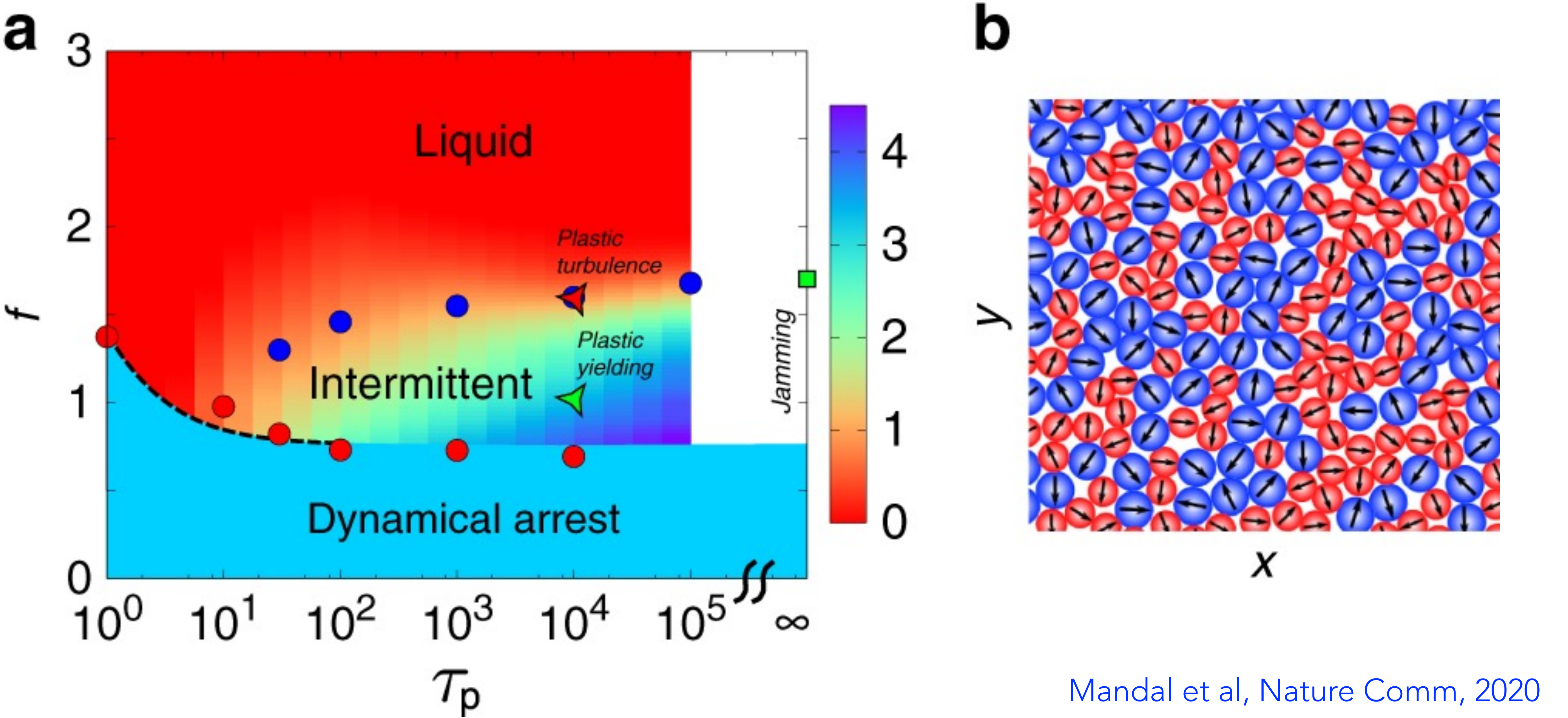
Olsson, Teitel, PRL (2007)

Self propelled particles



Liao, Xu, Soft Matter (2018)

dynamics share some features with fluids, some with solids (c.f. fluid turbulence talks last week)



Mandal et al, Nature Comm, 2020

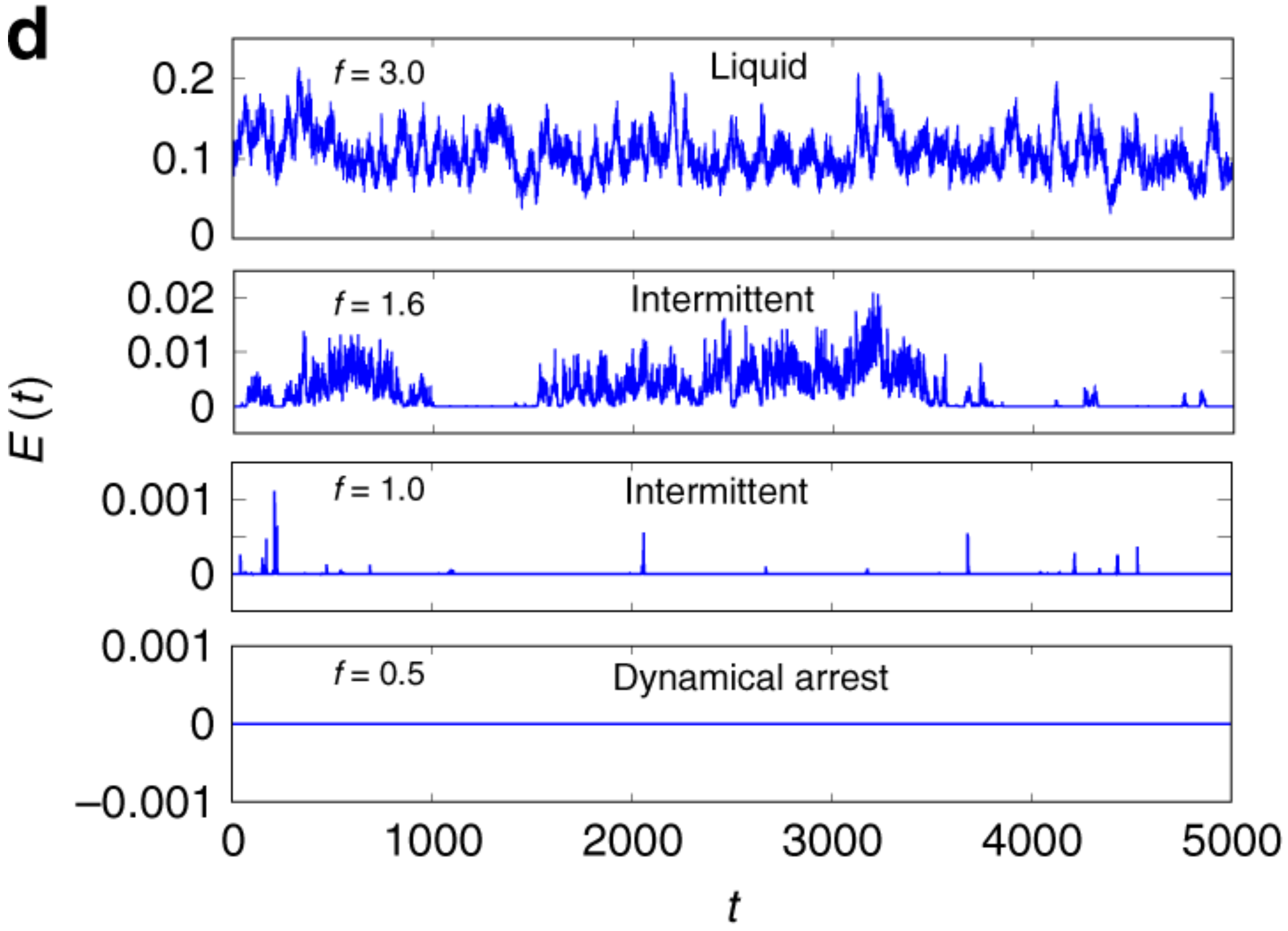
What happens in active matter?

Lots of suggestive results at finite rates of driving

Plus additional work by:

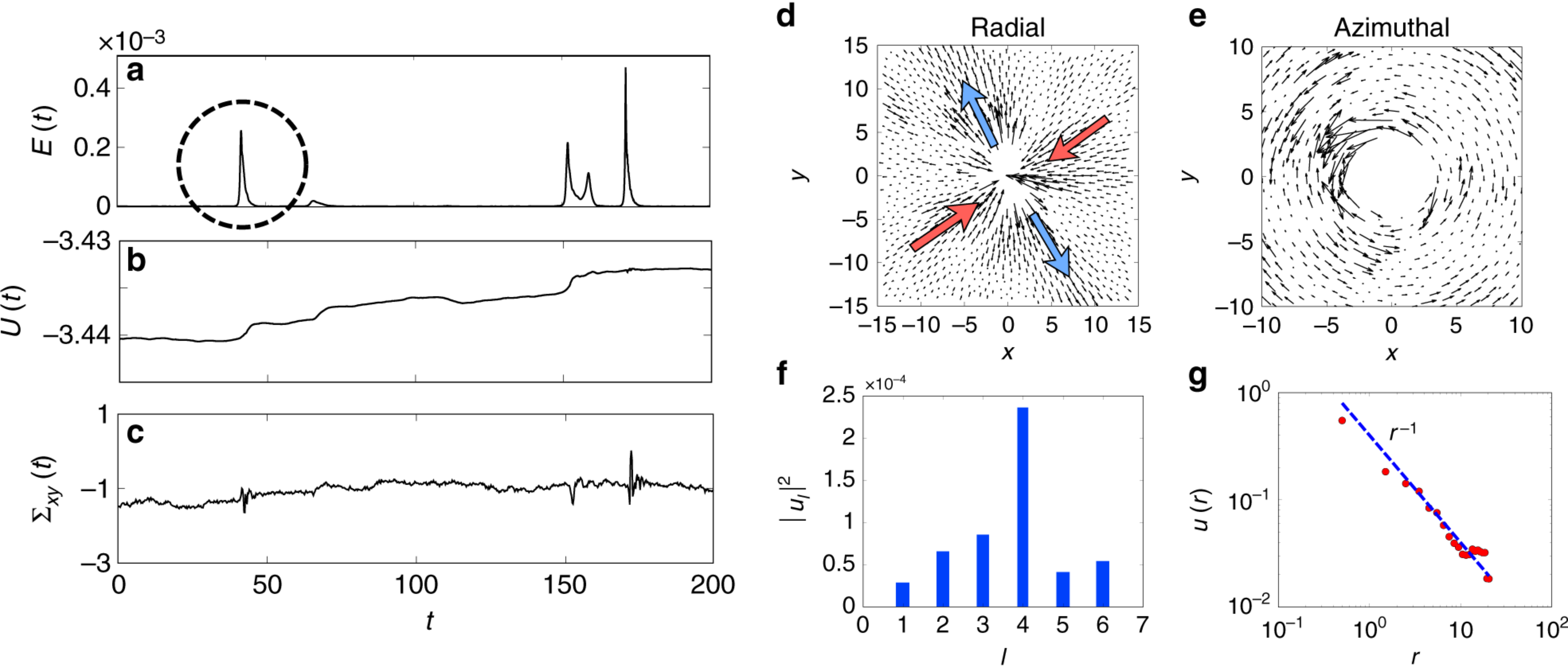
Nandi and Gov (Nandi EPJE 2018, 2019)
Mandal and Sollich (arXiv:1911.04558, 2020)
Silke Henkes and co-workers (Nature Communications, 2020)

At low forces and large persistence times, dynamics becomes highly intermittent



Mandal et al, Nature Comm, 2020

Intermittency driven by local plastic rearrangements



Mandal et al, Nature Comm, 2020

What happens in active matter?

Let's try to first look at the analogous limit to AQS:

zero temperature, in the limit that driving is infinitely persistent and infinitely slow

focus on initial response (pre-yielding regime)


Peter Morse


Sudeshna Roy

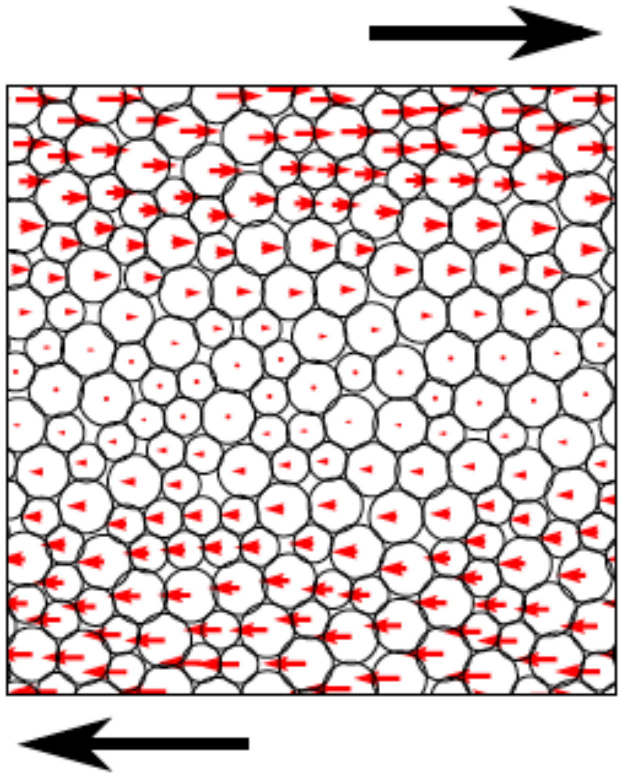

Elisabeth Agoritsas

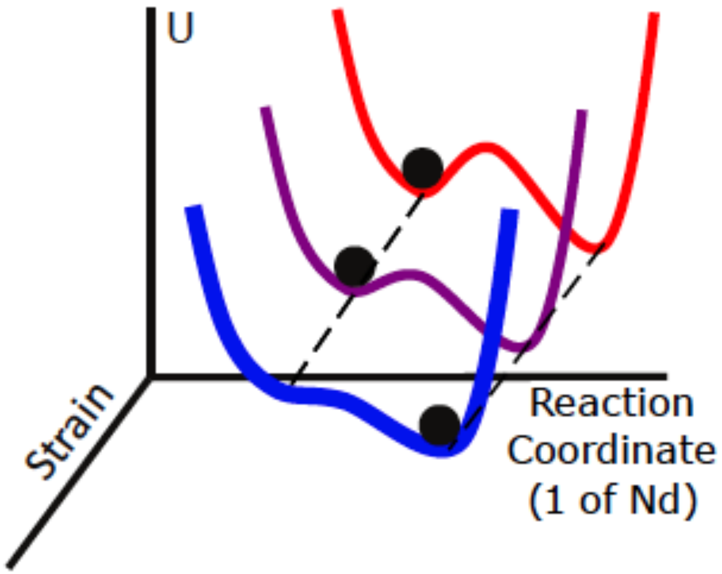

Ethan Stanifer

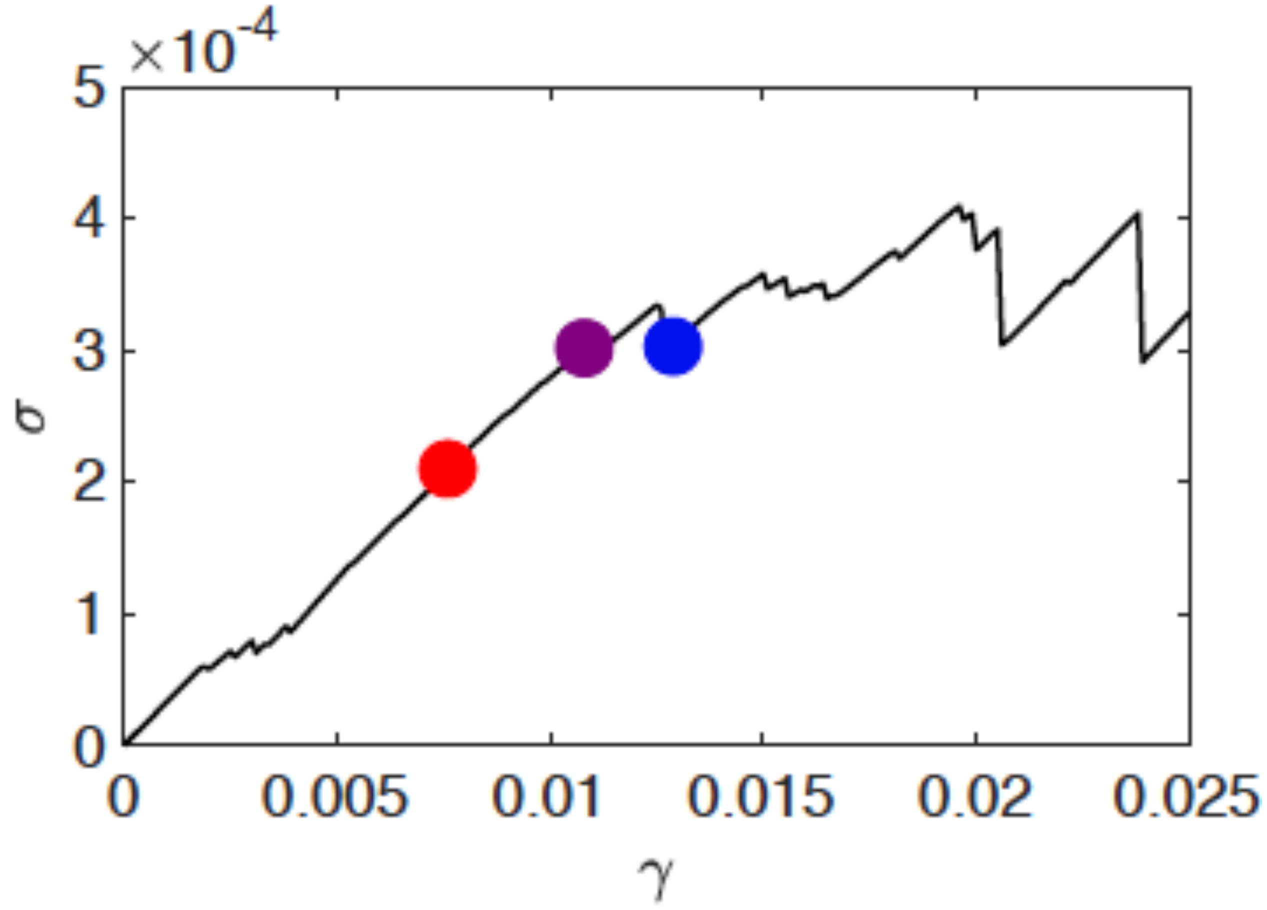

Eric Corwin

Morse et al, PNAS, 2021

Recall:
Athermal, Quasistatic Shear (AQS)

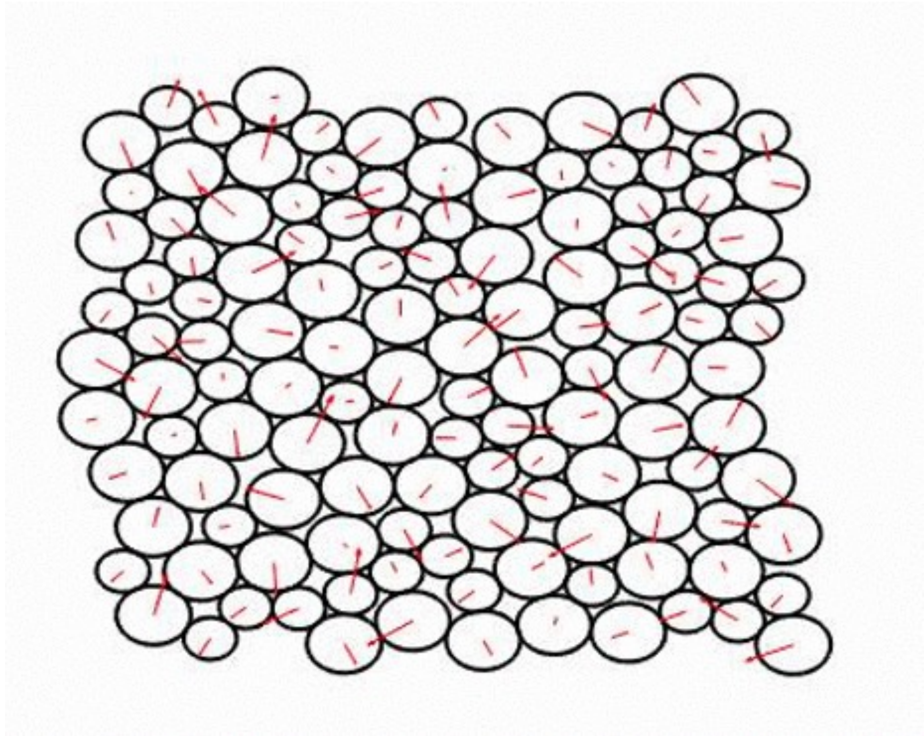


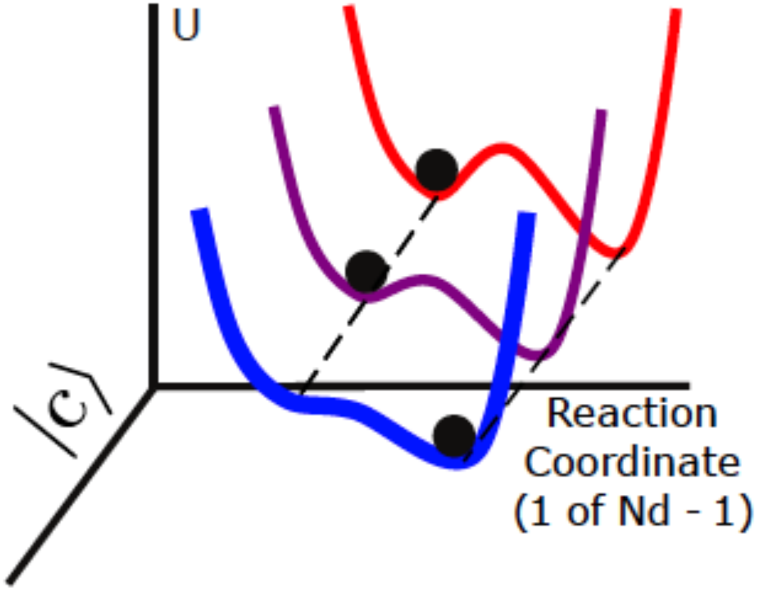


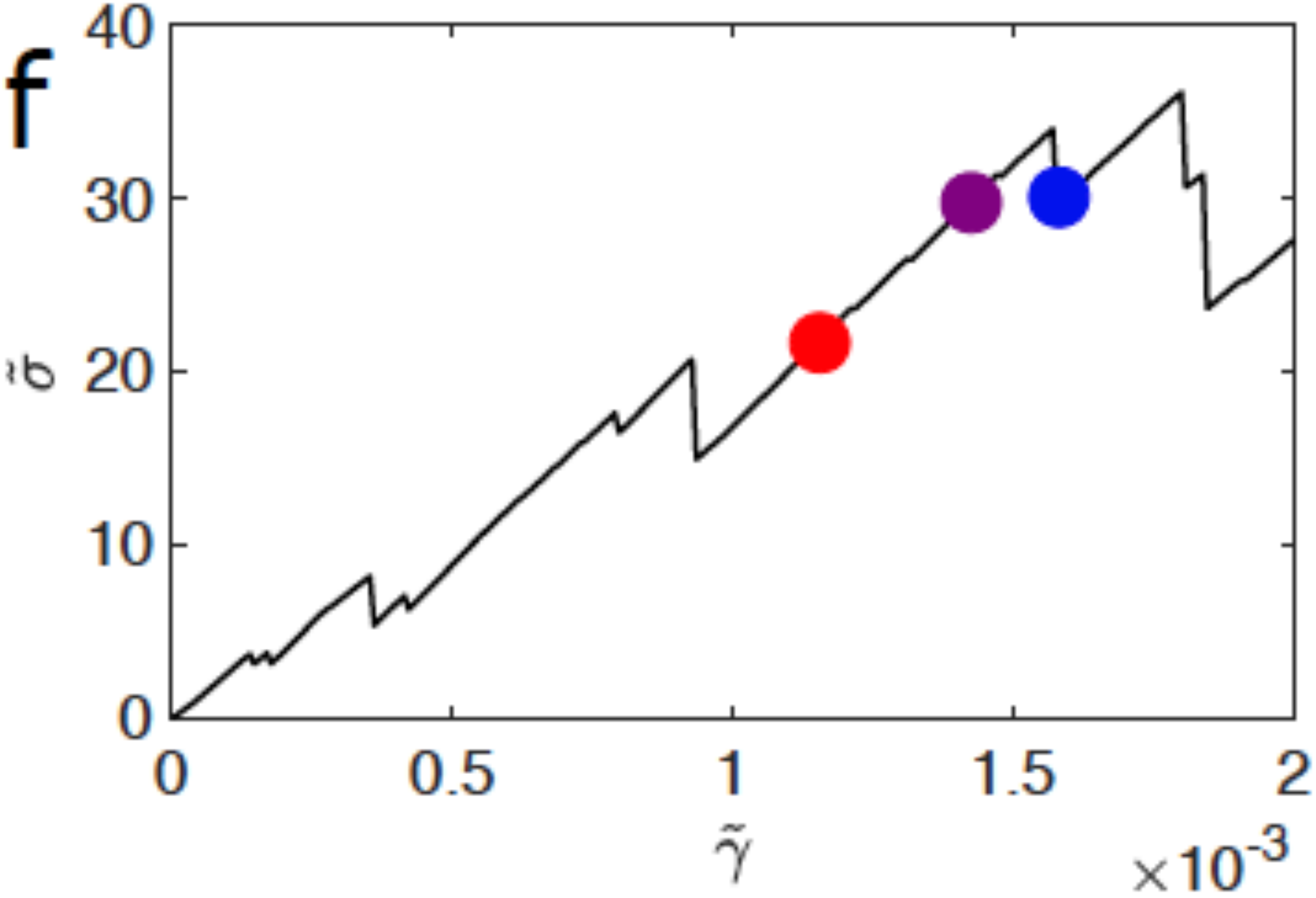


Maloney and Lemaitre PRE 2006

Athermal, Quasistatic Random Displacement (AQRD)



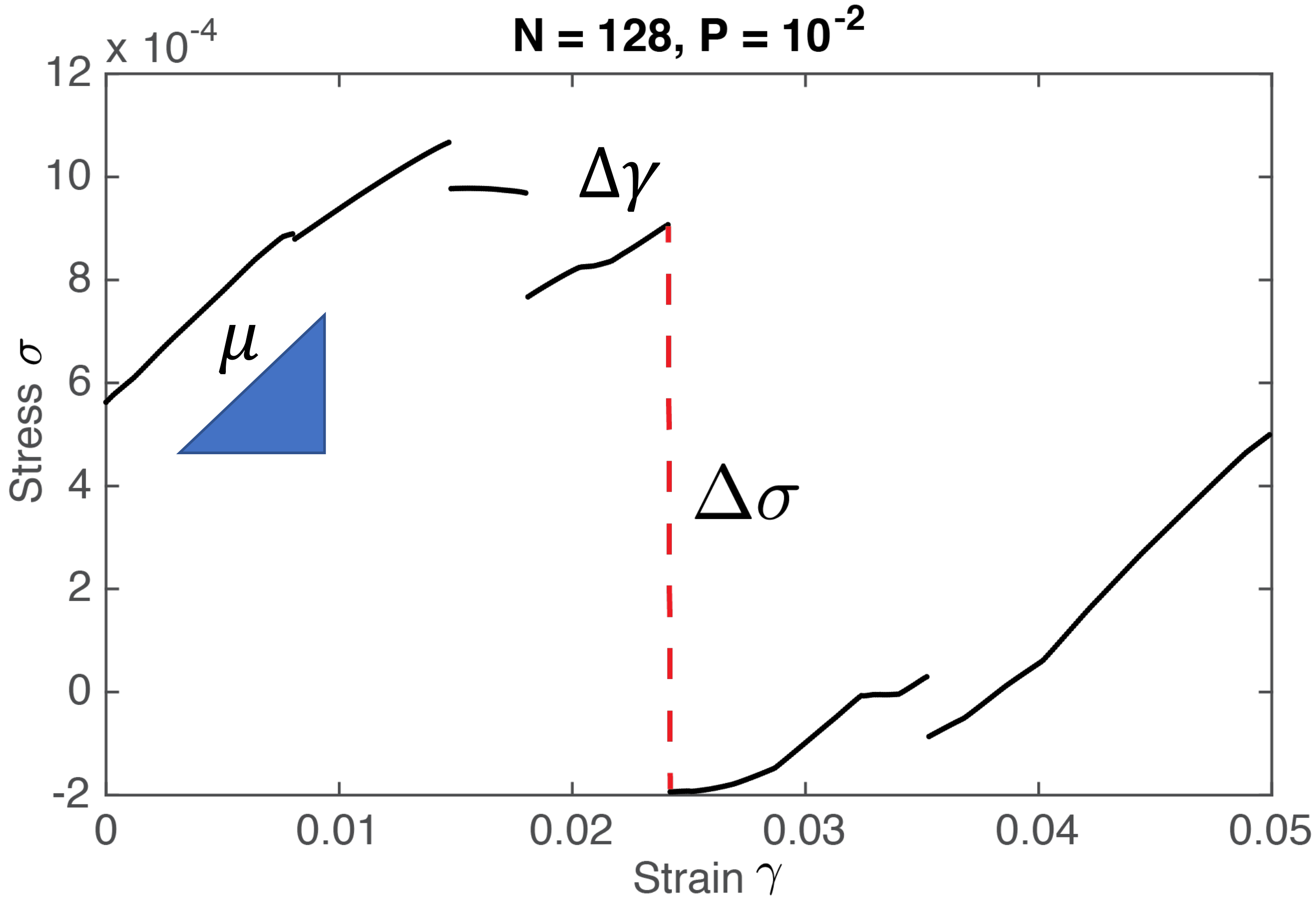




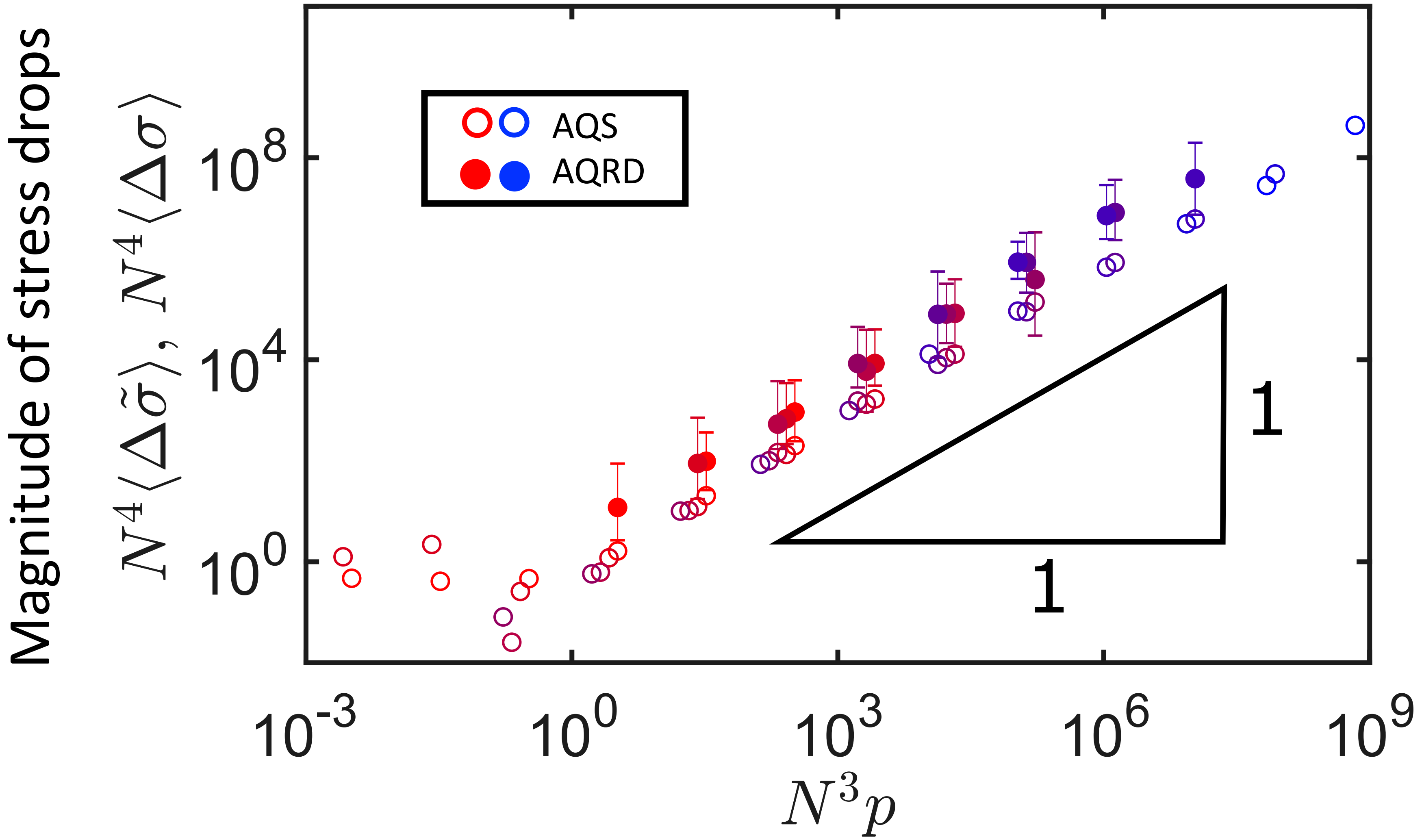
Fix the displacement (fixed strain experiment), not the force (creep experiment)

Morse et al, PNAS, 2021

Recall: linear, nonlinear response



AQRD and AQS exhibit exactly the same scaling with N,p



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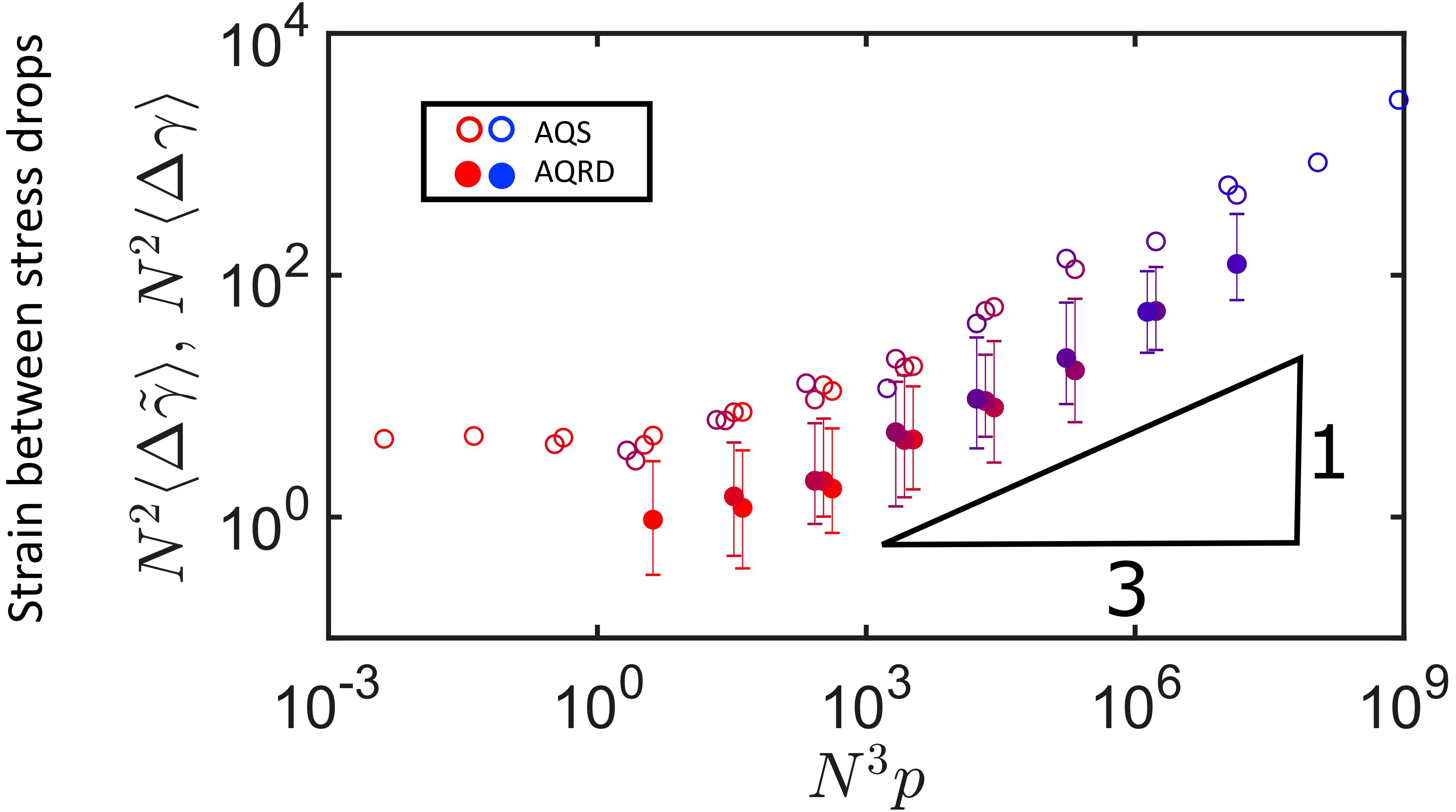
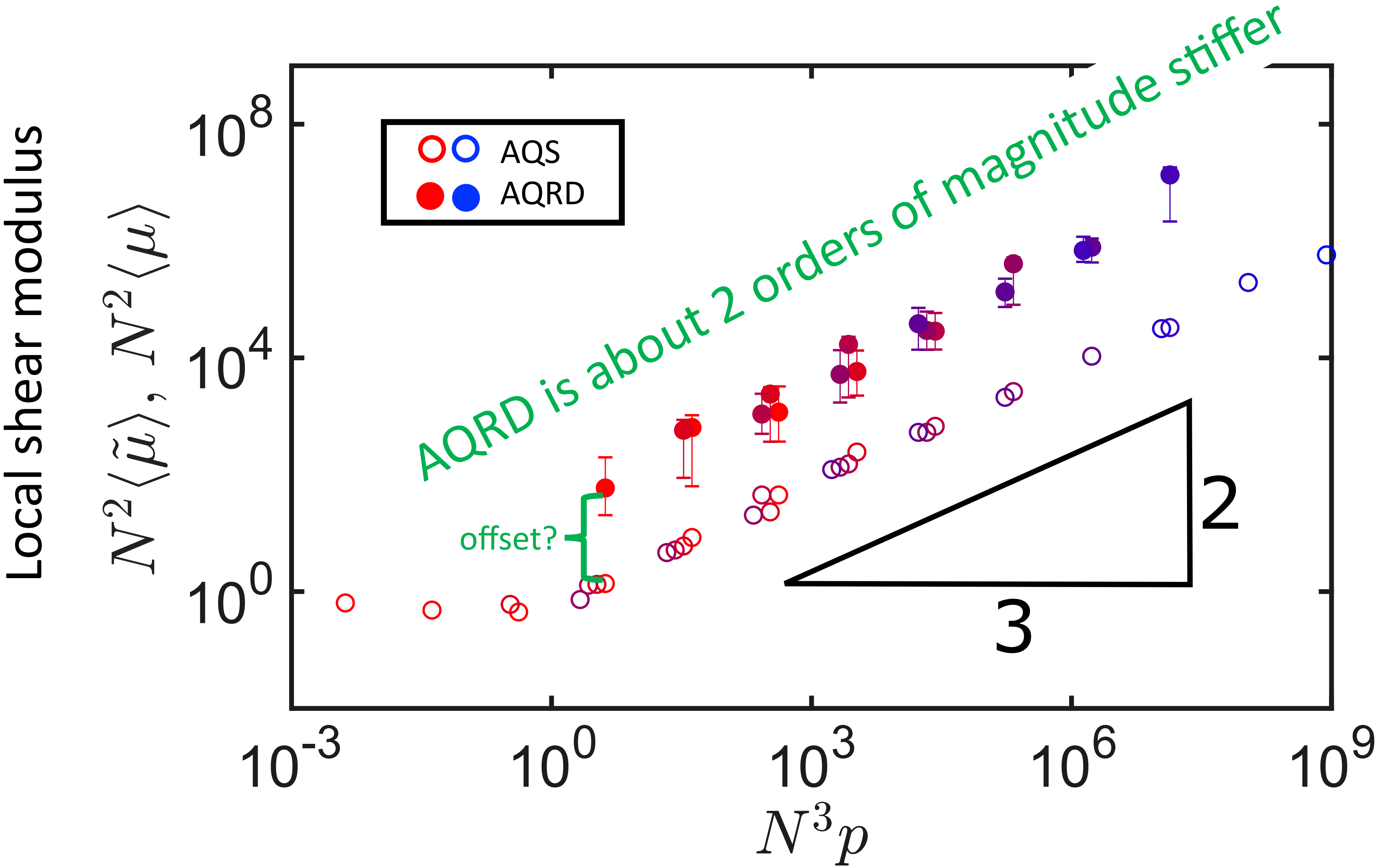
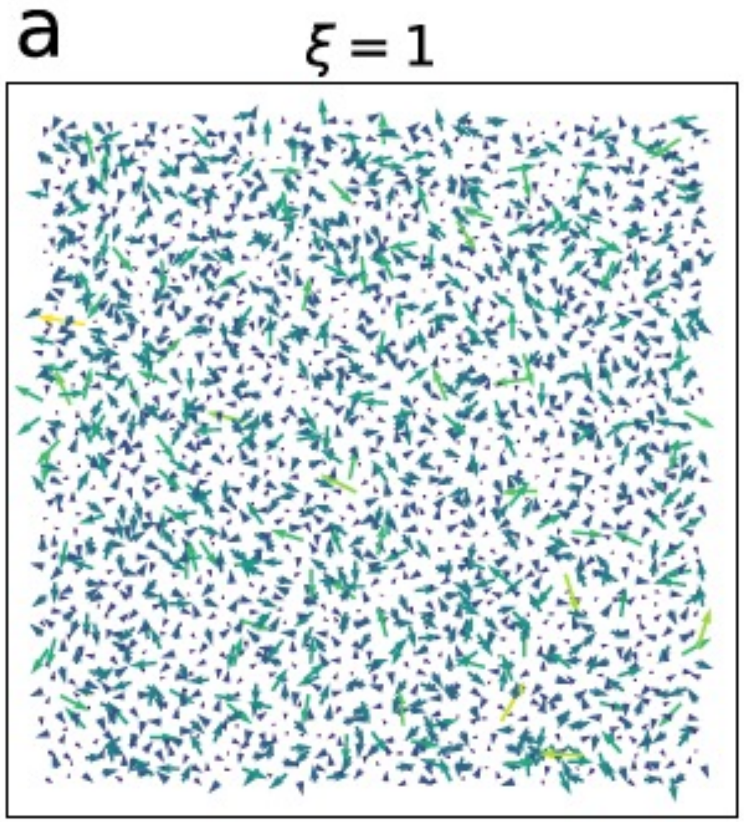


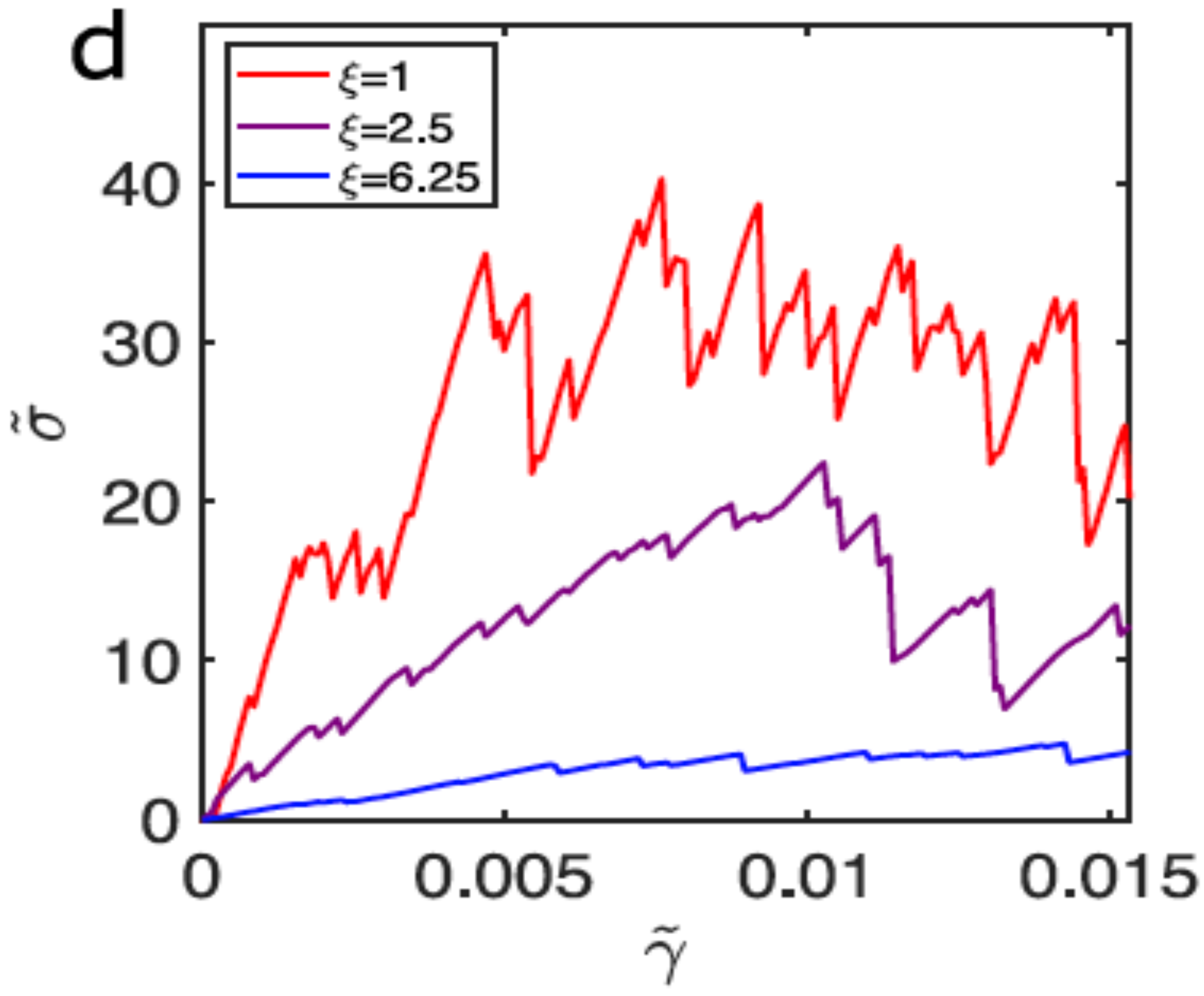
AQRD and AQS exhibit exactly the same scaling with N,p



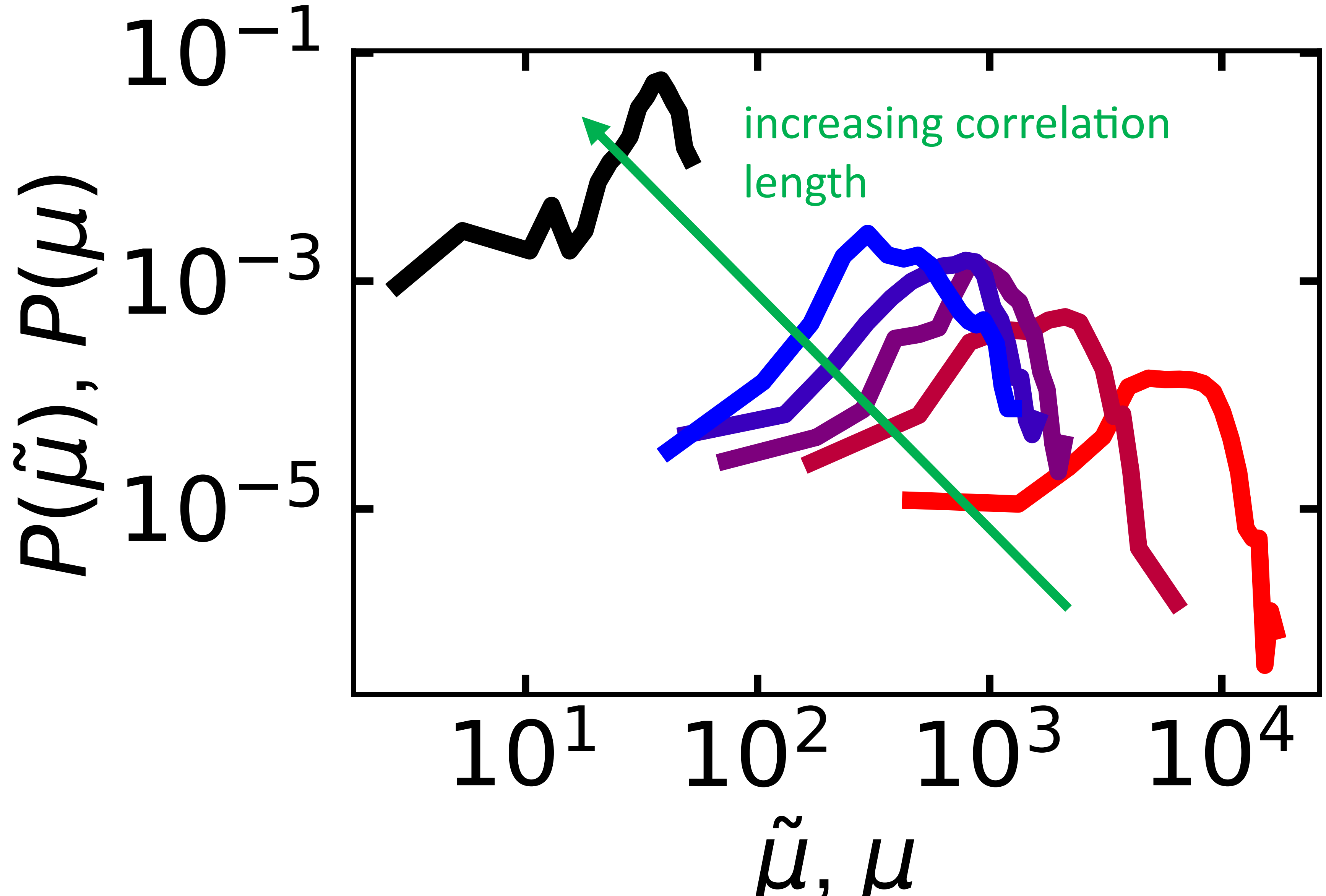
Random Gaussian displacement fields



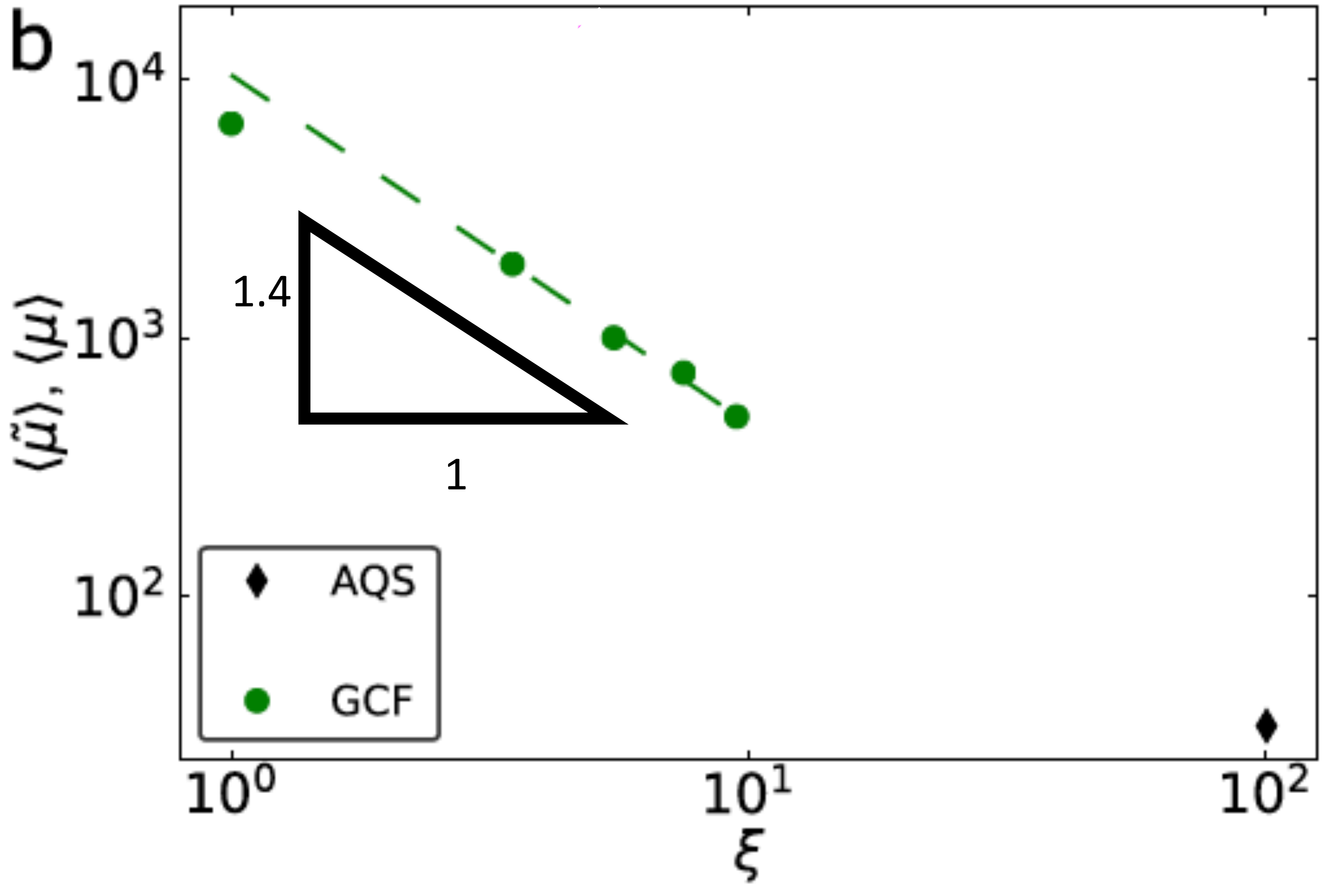
Stiffness changes with correlation length

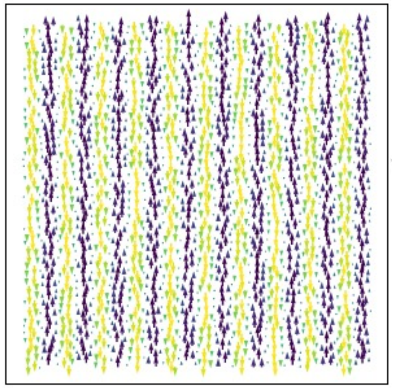


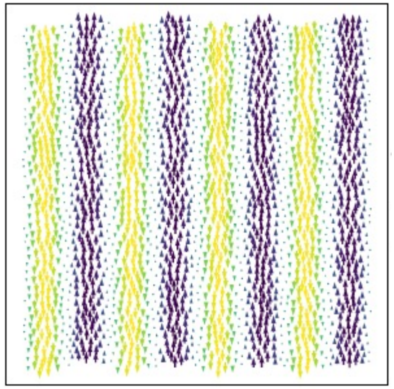
Stiffness changes with correlation length

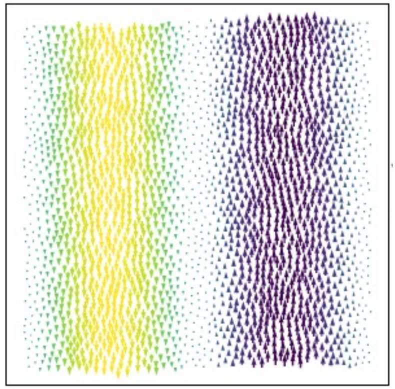


The shear modulus appears to be a power law function of the correlation length









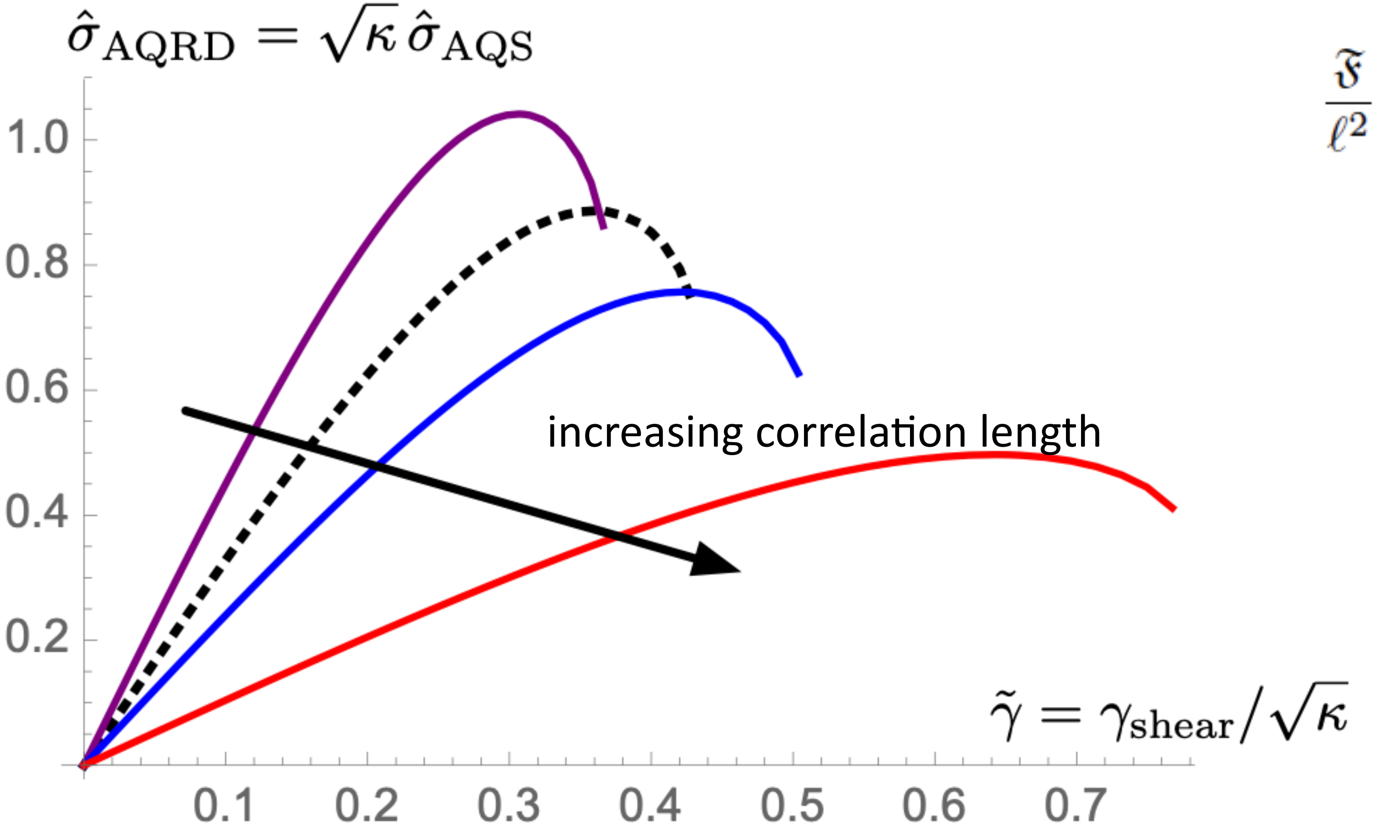
Morse et al, arXiv:2009.07706, 2020

Why?

I will work a tiny bit of this out on the board: prediction for infinite-dimensional solution for AQRD dynamics

Elisabeth Agoritsas

$$\kappa \equiv \frac{\mu_{\mathrm{AQRD}}}{\mu_{\mathrm{AQS}}} = \frac{\mathfrak{F}}{\ell^2}\,, \quad \gamma_{\mathrm{shear}} = \tilde{\gamma}\sqrt{\kappa}\,, \quad \sigma_{\mathrm{AQS}} = \sigma_{\mathrm{AQRD}}(\tilde{\gamma})/\sqrt{\kappa}\,.$$



$$\frac{\mathfrak{F}}{\ell^2} = \frac{d}{N_c}\sum_{\langle i,j\rangle}(\mathbf{c}_i - \mathbf{c}_j)^2,$$

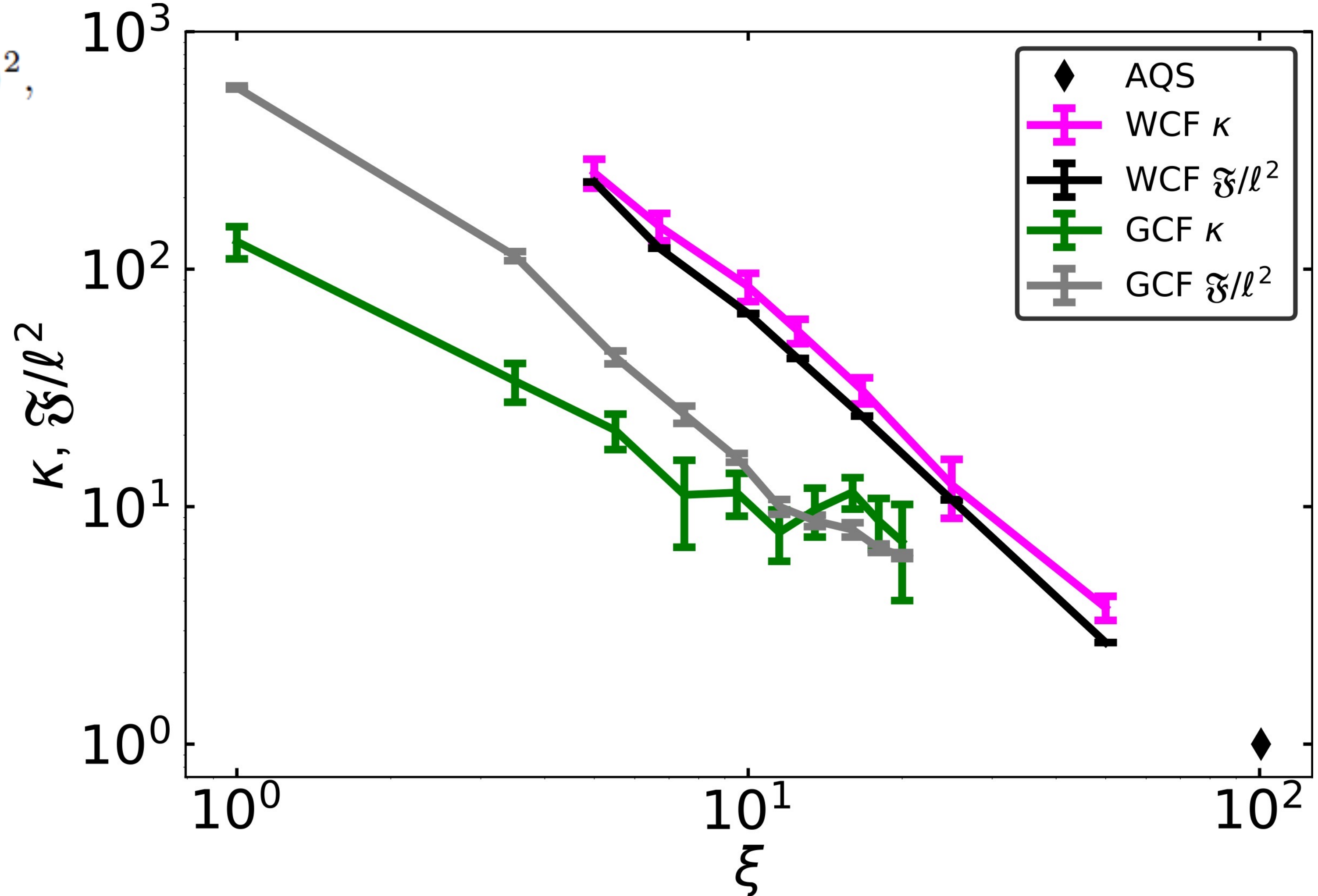
local strain induced by active displacement field |c >

E. Agoritsas, J. Stat. Mech (2021).

Test: calculate MF expression for $\frac{\mathfrak{F}}{\ell^2}$, compare $\kappa \equiv \frac{\mu_{\text{AQRD}}}{\mu_{\text{AQS}}}$

Neglects all higher order correlations between particles displacements in lower dimensions.

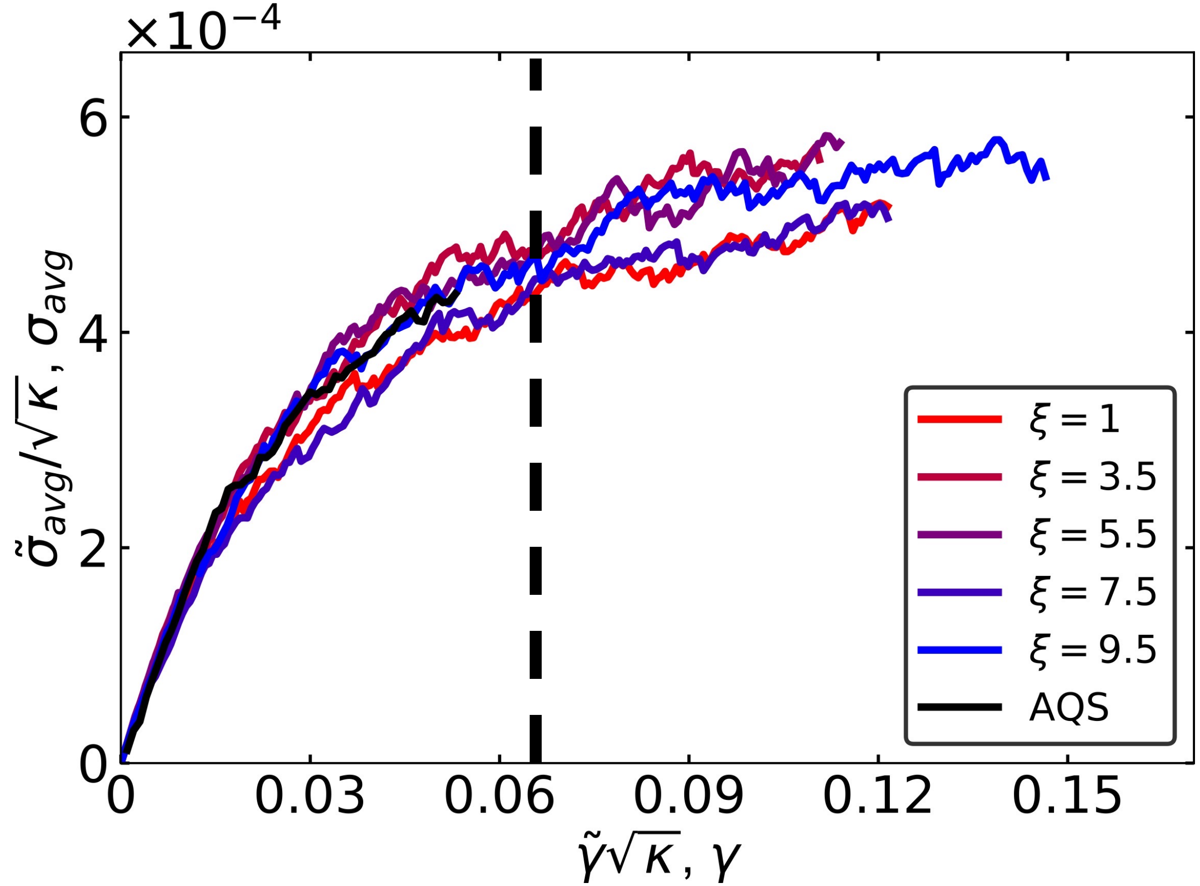
$$\frac{\mathfrak{F}}{\ell^2} = \frac{d}{N_c} \sum_{\langle i,j \rangle} (\mathbf{c}_i - \mathbf{c}_j)^2,$$



Morse et al, PNAS, 2021

Let's treat $\kappa \equiv \frac{\mu_{\mathrm{AQRD}}}{\mu_{\mathrm{AQS}}}$ as the rescaling parameter in low d:

$$\gamma_{\mathrm{shear}} = \tilde{\gamma}\sqrt{\kappa}\,, \quad \sigma_{\mathrm{AQS}} = \sigma_{\mathrm{AQRD}}(\tilde{\gamma})/\sqrt{\kappa}\,.$$



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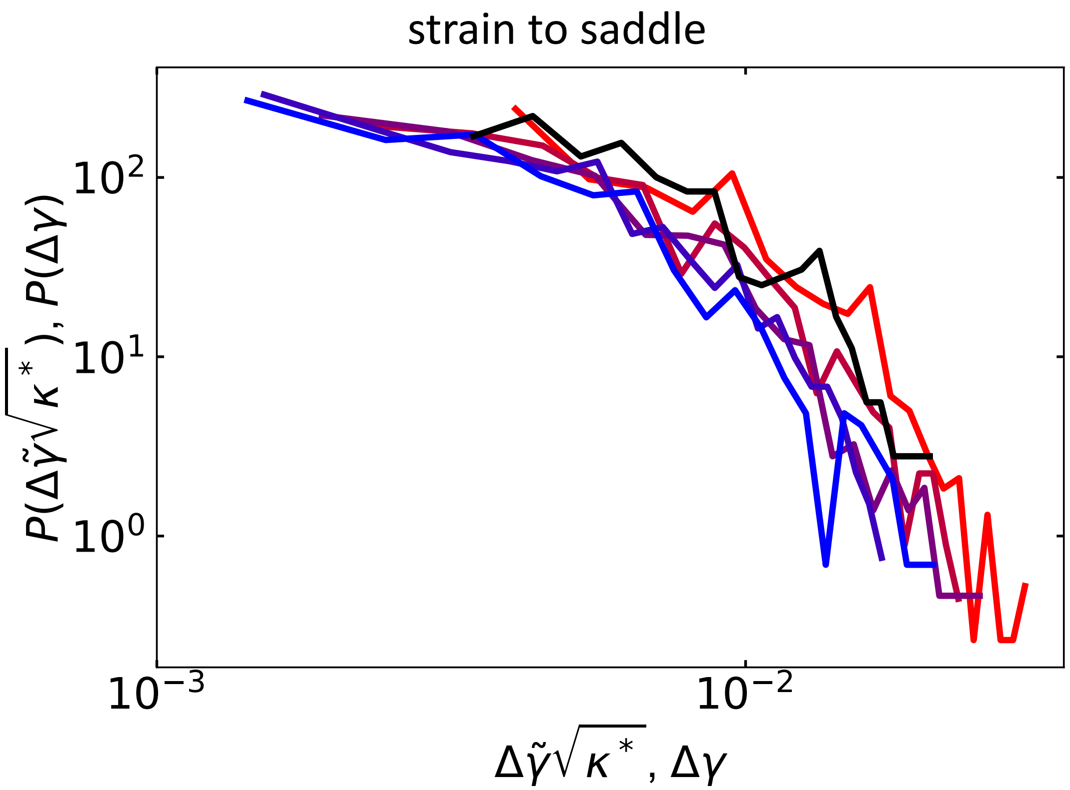
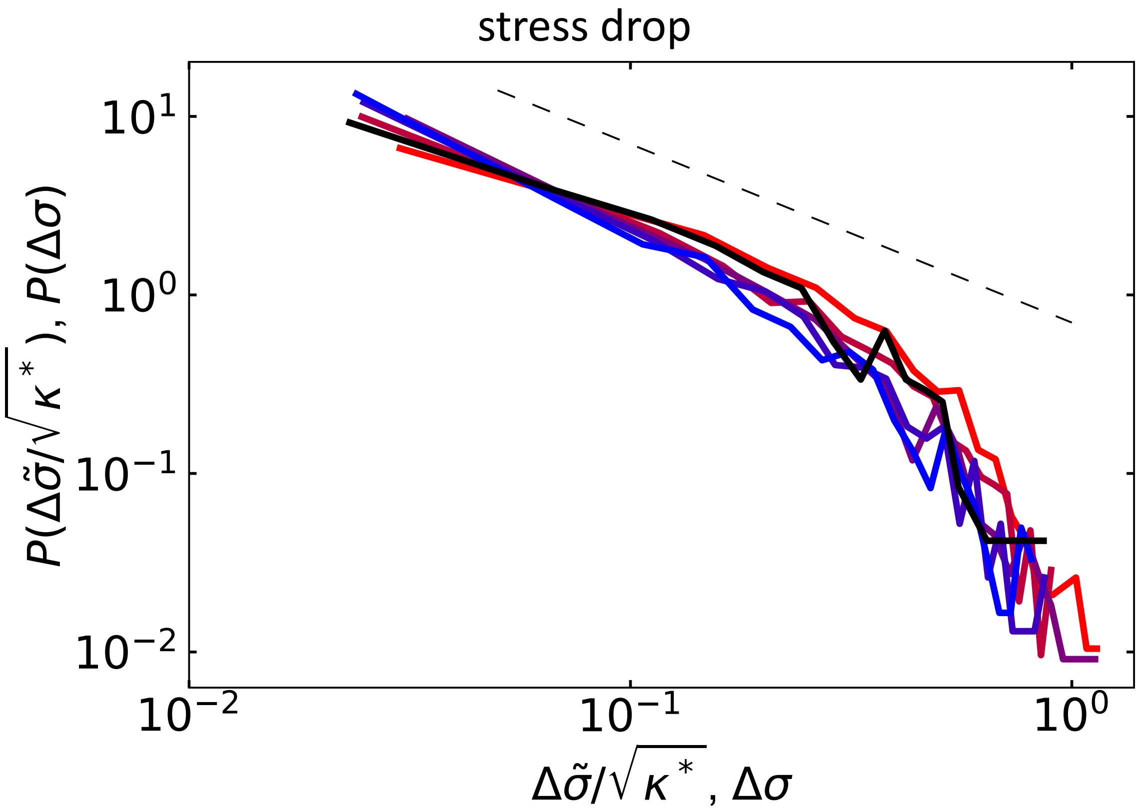
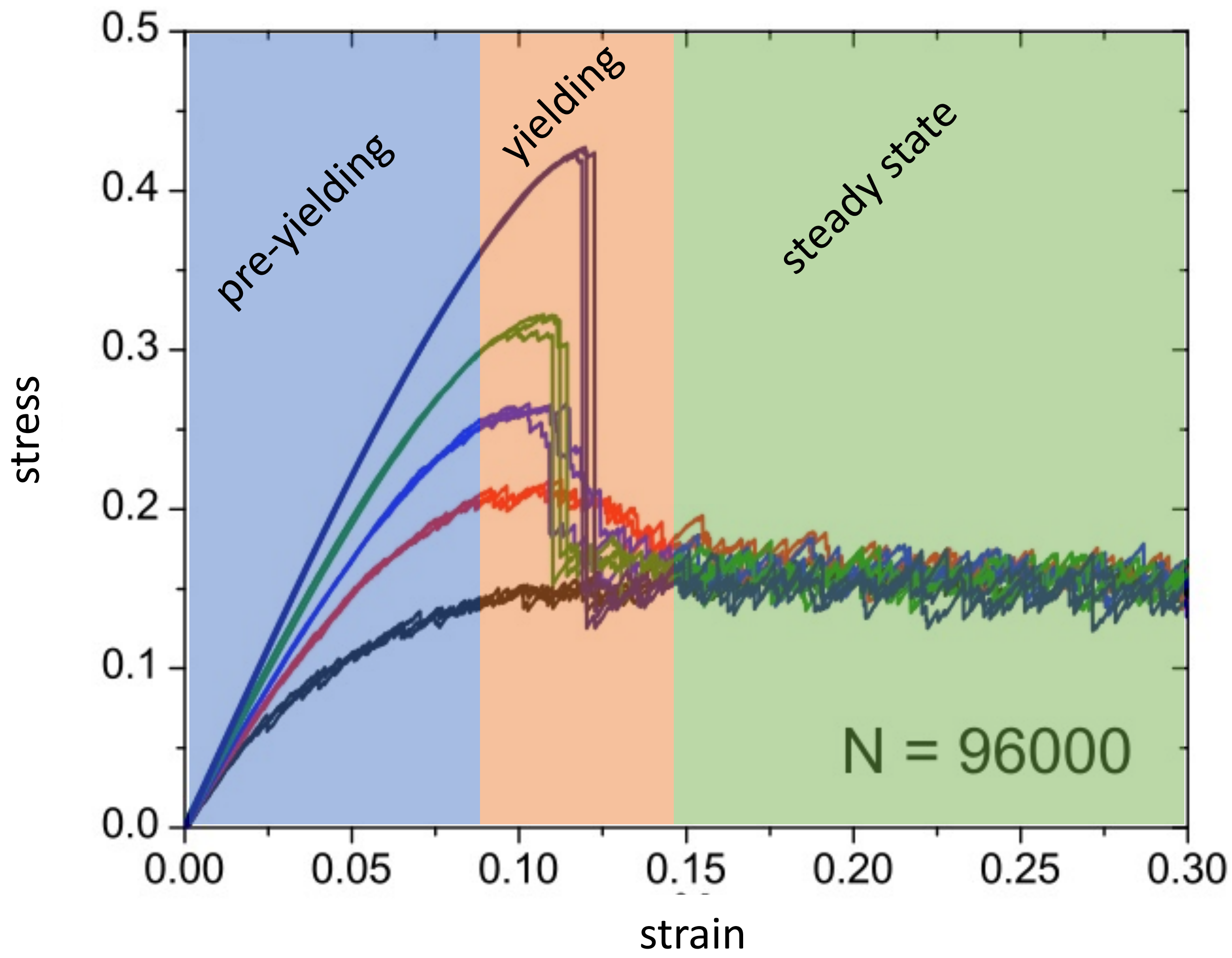




Morse et al, arXiv:2009.07706, 2020

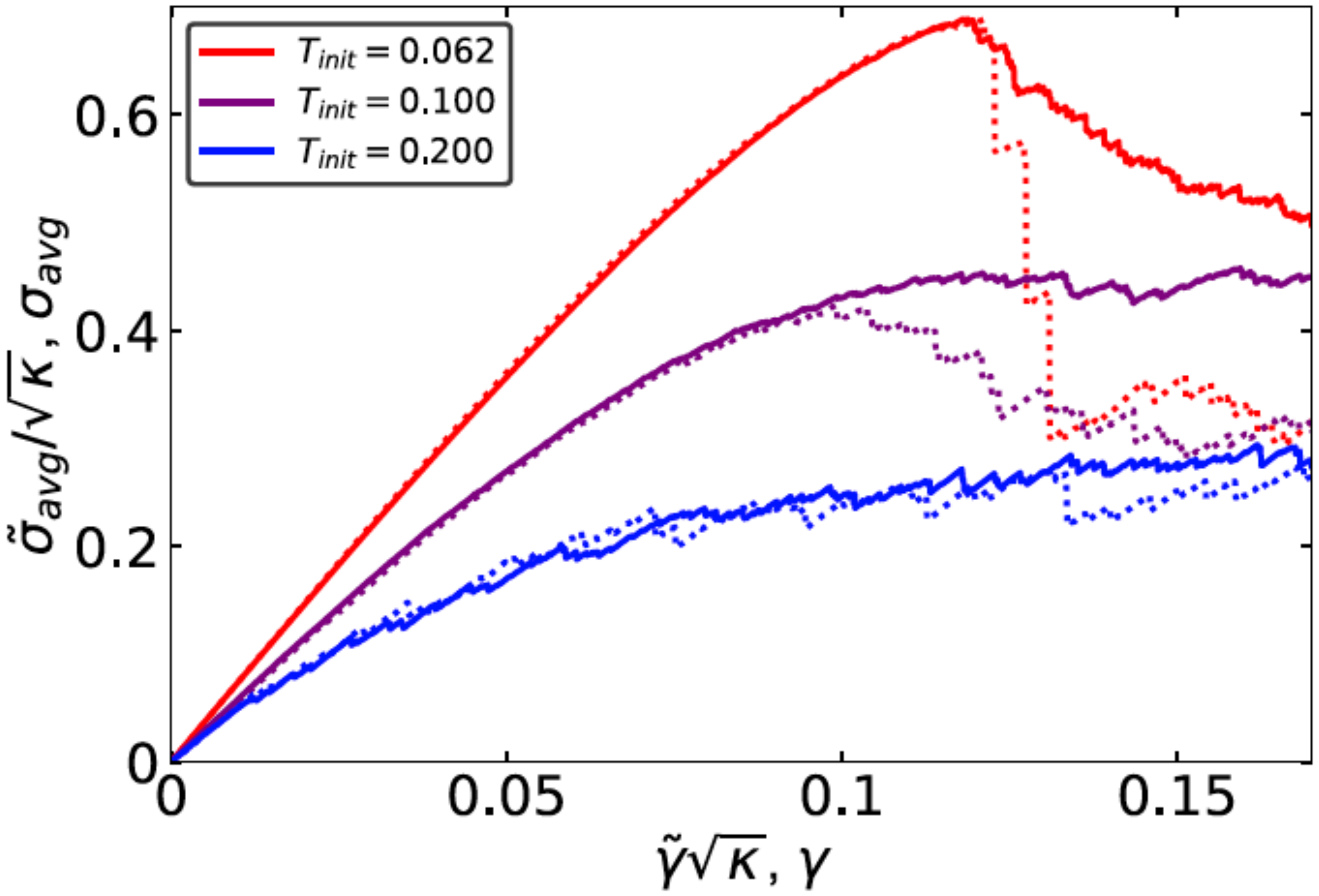
What about beyond "pre-yielding" regime



c.f. talk by Kirsten Martens yesterday

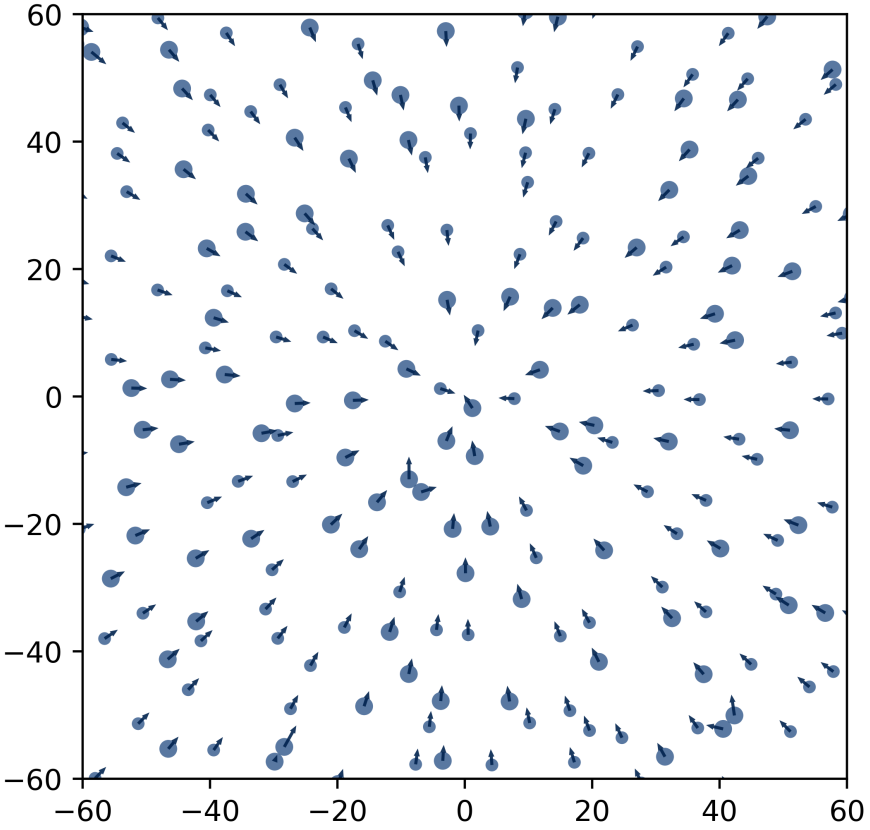
Ozawa PNAS 2018

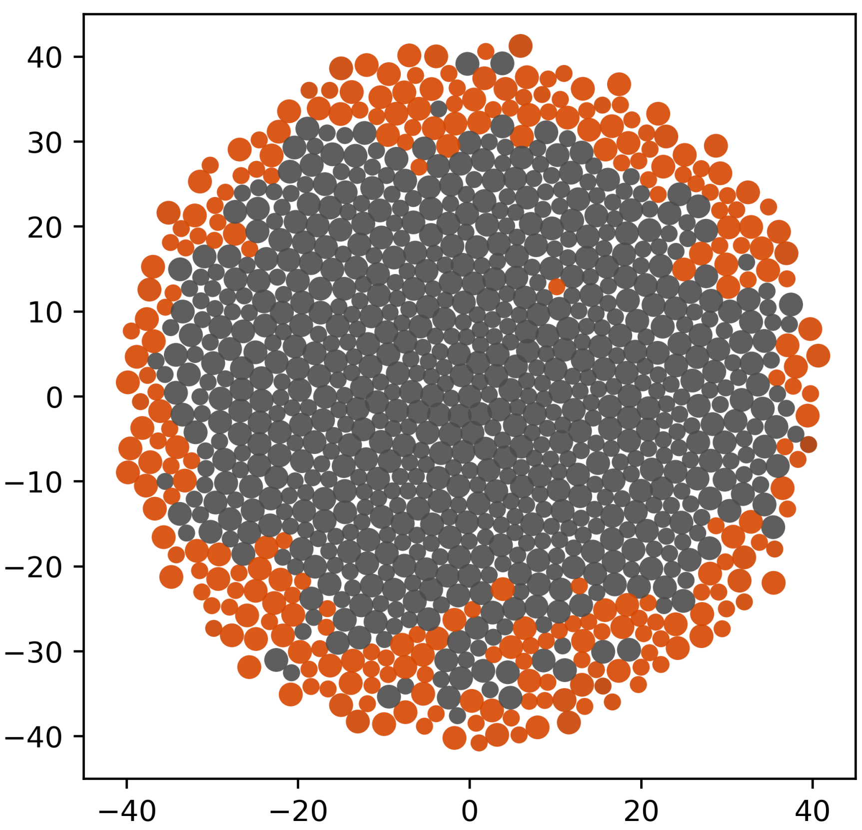
Outlook: changing material stability:



Can we break this direct link between shear and active matter?

Julia Giannini





Giannini, Stanifer, Manning, Soft Matter 2022

Not so different from real crowds with dangerous crushing events

https://www.youtube.com/watch?v=SYO5kJm6JNM

Conclusions and Outlook

- In the limit of slow driving, shear strain is simply a special case of infinitely persistent active driving.
  - The linear (shear modulus) and nonlinear (stress drop, strain to saddle) response in the pre-yielding regime possess identical scaling.
  - All the data can be collapsed using the effective parameter $\kappa \equiv \frac{\mu_{\mathrm{AQRD}}}{\mu_{\mathrm{AQS}}}$ as predicted by mean-field
  - The mean-field prediction for the exact value of $\kappa \equiv \frac{\mu_{\mathrm{AQRD}}}{\mu_{\mathrm{AQS}}} = \frac{\mathfrak{F}}{\ell^2}$ is close, but not quite right.
  - The macroscopic shear modulus and stress overshoot changes with material stability as expected, though the nature of the yielding transition may change.
- What happens to shear bands in brittle materials? Are the same defects excited?
- What happens at finite strain rates, persistence times? (c.f. Kirsten Martens)

Infinite-dimensional solution for AQRD dynamics

Global shear

$$\mathbf{x}_i(t) = \underbrace{\mathbf{x}_i(0) + \gamma(t)\, x_{i,2}(0)\, \hat{\mathbf{x}}_1} + \mathbf{u}_i(t)$$

$$\mathbf{r}_{ij}(t) = \underbrace{\mathbf{r}_{ij}(0) + \gamma(t)\, r_{ij,2}(0)\, \hat{\mathbf{x}}_1} + \mathbf{w}_{ij}(t)$$

Random local forcing

$$\mathbf{x}_i(t) = \underbrace{\mathbf{x}_i(0) + \gamma(t)\, \mathbf{c}_i} + \mathbf{u}_i(t)$$

$$\mathbf{r}_{ij}(t) = \underbrace{\mathbf{r}_{ij}(0) + \gamma(t)\, (\mathbf{c}_i - \mathbf{c}_j)} + \mathbf{w}_{ij}(t)$$

distance between pairs

non-affine motion

$$r_{ij}(t) = \ell \left(1 + \frac{h_{ij}(t)}{d} \right)$$

gap between pairs

E. Agoritsas, J. Stat. Mech (2021).

Infinite-dimensional solution for AQRD dynamics

Elisabeth Agoritsas

Global shear

$$\mathbf{x}_i(t) = \underbrace{\mathbf{x}_i(0) + \gamma(t)\, x_{i,2}(0)\, \hat{\mathbf{x}}_1} + \mathbf{u}_i(t)$$

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Random local forcing

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$$\mathbf{r}_{ij}(t) = \underbrace{\mathbf{r}_{ij}(0) + \gamma(t)\, (\mathbf{c}_i - \mathbf{c}_j)} + \mathbf{w}_{ij}(t)$$

$$\mathbf{c}_{ij} = \mathbf{c}_i - \mathbf{c}_j$$

In infinite dimensions, global shear is a special case of random local forcing: $\mathbf{c}_i = x_{i,2}(0)\, \hat{\mathbf{x}}_1 \quad \mathbf{c}_{ij} = r_{ij,2}(0)\, \hat{\mathbf{x}}_1$

Generally then:

$$\mathbf{c}_i = \mathcal{C}(\mathbf{x}_i(0))$$

$$\overline{\mathcal{C}(\mathbf{x})} = 0$$

$$\overline{\mathcal{C}(\mathbf{x}) \cdot \mathcal{C}(\mathbf{x}')} = \ell^2 \underbrace{\Xi\, f_\xi(|\mathbf{x} - \mathbf{x}'|)}_{\text{(unitless)}}$$

$$f_\xi(x) = \frac{e^{-x^2/(2\xi^2)}}{\sqrt{2\pi}\xi}$$

For individual local displacements: $\overline{\mathbf{c}_i} = 0$

$$\overline{\mathbf{c}_i \cdot \mathbf{c}_j} = \ell^2\, \Xi\, f_\xi(r_{ij}(0))$$

For relative local displacements: $\overline{\mathbf{c}_{ij}} = 0$

$$d\, \overline{\mathbf{c}_{ij} \cdot \mathbf{c}_{i'j'}} \xrightarrow{(d\to\infty)} 0 \; : \; (ij) \neq (i'j')$$

$$d\, \overline{\mathbf{c}_{ij}^2} \xrightarrow{(d\to\infty)} = 2\ell^2\, \Xi \left[f_\xi(0) - f_\xi(\ell) \right] \equiv \mathfrak{F}(\Xi, \ell, \xi)$$

E. Agoritsas, J. Stat. Mech (2021).

Infinite-dimensional solution for AQRD dynamics

Elisabeth Agoritsas

Global shear

$$\mathbf{x}_i(t) = \underbrace{\mathbf{x}_i(0) + \gamma(t)\, x_{i,2}(0)\, \hat{\mathbf{x}}_1}\, + \mathbf{u}_i(t)$$

$$\mathbf{r}_{ij}(t) = \underbrace{\mathbf{r}_{ij}(0) + \gamma(t)\, r_{ij,2}(0)\, \hat{\mathbf{x}}_1}\, + \mathbf{w}_{ij}(t)$$

Random local forcing

$$\mathbf{x}_i(t) = \underbrace{\mathbf{x}_i(0) + \gamma(t)\, \mathbf{c}_i}\, + \mathbf{u}_i(t)$$

$$\mathbf{r}_{ij}(t) = \underbrace{\mathbf{r}_{ij}(0) + \gamma(t)\, \left(\mathbf{c}_i - \mathbf{c}_j\right)}\, + \mathbf{w}_{ij}(t)$$

Many-body dynamics: $\zeta\left[\dot{\mathbf{x}}_i(t) - \dot{\gamma}(t)\,\mathbf{c}_i\right] = \boldsymbol{F}_i(t) + \boldsymbol{\xi}_i(t)\,, \quad \text{with} \quad \boldsymbol{F}_i(t) = -\sum_{j(\neq i)} \nabla v\left(|\mathbf{x}_i(t) - \mathbf{x}_j(t)|\right)$

Large-dimension assumptions:

$$|\mathbf{u}_i(t)| \sim \mathcal{O}(1/d)\,, \quad |\mathbf{w}_{ij}(t)| \sim \mathcal{O}(1/d)$$

⇒ Self-consistent <u>scalar</u> stochastic process for the gap, with three kernels:

$$r_{ij}(t) = \ell\left(1 + \frac{h_{ij}(t)}{d}\right)$$

$$k^{\mu\nu}(t) \sim \langle \nabla_\mu \nabla_\nu v \rangle\,,$$

$$M_C^{\mu\nu}(t,s) \sim \langle \nabla_\mu v \nabla_\nu v \rangle$$

$$M_R^{\mu\nu}(t,s) \sim \delta \langle \nabla_\mu v \rangle / \delta P_\nu$$

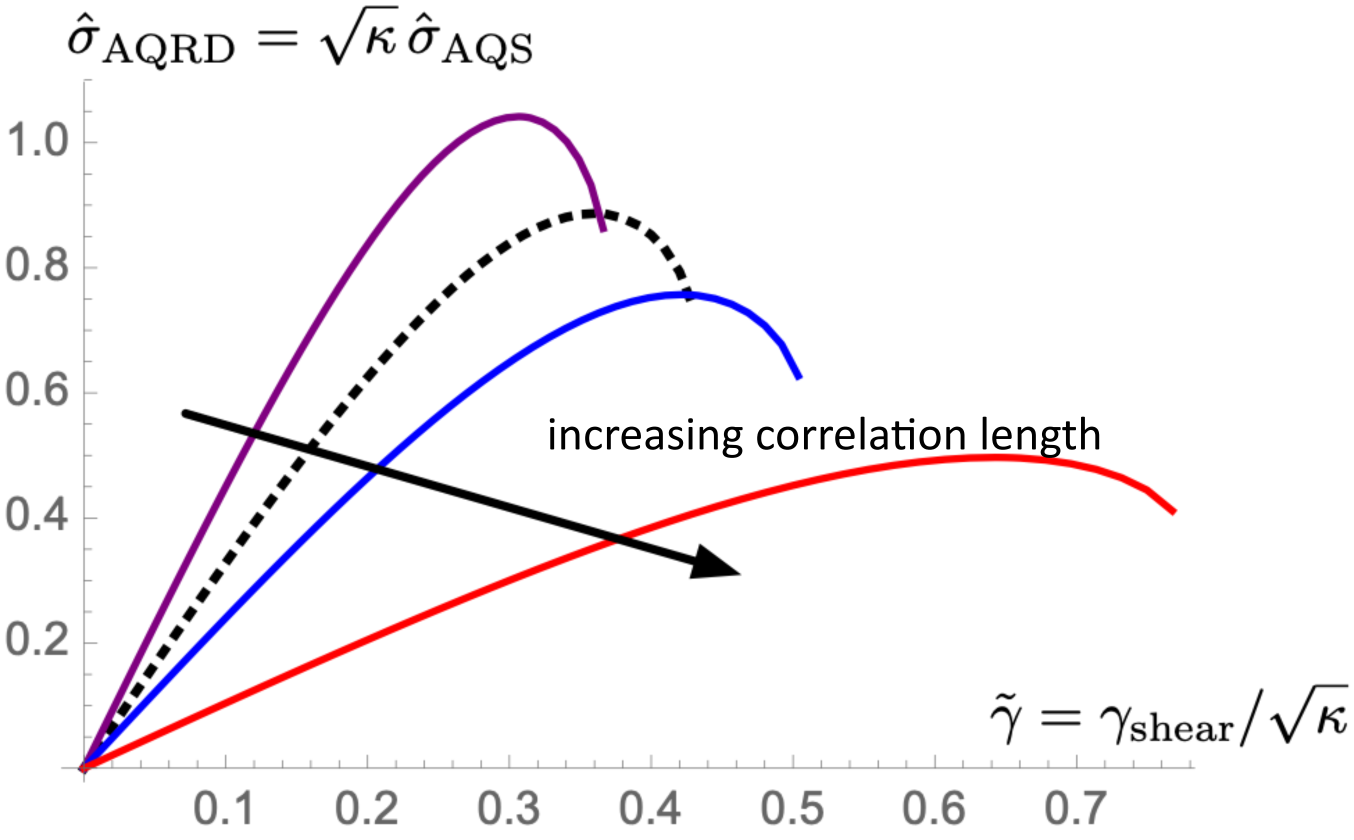
E. Agoritsas, J. Stat. Mech (2021).

Infinite-dimensional solution for AQRD dynamics

Elisabeth Agoritsas

$$\mathfrak{F}(\Xi, \ell, \xi) = d\,\overline{\mathbf{c}_{ij}^2} = 2\ell^2 \Xi \left[f_\xi(0) - f_\xi(\ell) \right]$$

$$\kappa \equiv \frac{\mu_{\mathrm{AQRD}}}{\mu_{\mathrm{AQS}}} = \frac{\mathfrak{F}}{\ell^2}, \quad \gamma_{\mathrm{shear}} = \tilde{\gamma}\sqrt{\kappa}, \quad \sigma_{\mathrm{AQS}} = \sigma_{\mathrm{AQRD}}(\tilde{\gamma})/\sqrt{\kappa}.$$



E. Agoritsas, J. Stat. Mech (2021).

Extra slides

2D Active Brownian Particles

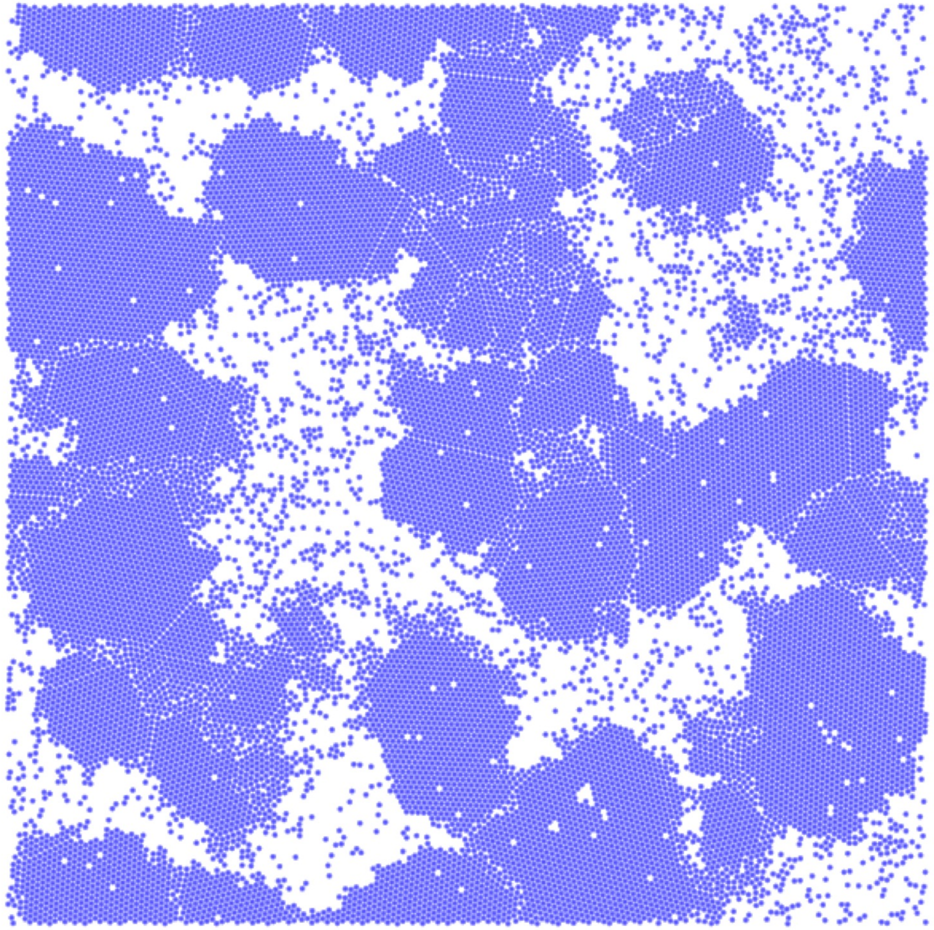
$$\gamma\dot{\mathbf{r}}_i = -\nabla\sum_j V(r_{ij}) + \gamma v_0\mathbf{n}_i$$

$$\dot{\phi} = \eta_i$$

$$\mathbf{n}_i = (\cos(\phi_i), \sin(\phi_i))$$

$$\langle\eta_o(t)\eta_j(t')\rangle = 2D_r\delta_{ij}\delta(t-t')$$

- Persistence time τ = 1/D_r .
- Long time MSD of an isolated particle grows as $2v_0^2 t/D_r$.
- Define an active temperature $T_a = v_0^2\gamma/(2D_r)$.



Approximate Theory

Start from the equation of motion.

$$\gamma \dot{\mathbf{r}}_i = -\nabla_i \sum_j V(r_{ij}) + \gamma v_0 \mathbf{n}_i$$

Derive an expression relating velocity polarization and force fields.

$$\gamma \mathbf{v}(\mathbf{q};t) = \sum_j \sum_{k \neq j} \mathbf{F}_{jk} e^{-i\mathbf{q}\cdot\mathbf{r}_j} + \gamma v_0 \mathbf{n}(\mathbf{q};t)$$

$$\mathbf{v}(\mathbf{q};t) = \sum_j \dot{\mathbf{r}}_j e^{-i\mathbf{q}\cdot\mathbf{r}_j(t)} \qquad \mathbf{n}(\mathbf{q};t) = \sum_j \mathbf{n}_j e^{-i\mathbf{q}\cdot\mathbf{r}_j(t)}$$

Rewrite first term on the right hand side.

$$i\mathbf{q} \cdot \sum_j \sum_{k \neq j} \mathbf{r}_{jk} \frac{\mathbf{r}_{jk}}{2 r_{jk}} V'(r_{jk}) \left[\frac{e^{i\mathbf{q}\cdot\mathbf{r}_{jk}} - 1}{i\mathbf{q}\cdot\mathbf{r}_{jk}} \right] e^{-i\mathbf{q}\cdot\mathbf{r}_j} = -i\mathbf{q} \cdot \mathbf{\Pi}_v(\mathbf{q};t)$$

Interaction part of the pressure tensor.

Approximate Theory

Assume in direct space $\mathbf{\Pi}_v(\mathbf{r};t)$ can be expressed in terms of the deviation of the microscopic density ρ.

$$\mathbf{\Pi}_v(\mathbf{r};t) \approx \langle \mathbf{\Pi}_v(\mathbf{r};t)\rangle + \mathbf{I}\left(\partial_\rho P_v\right)(\rho(\mathbf{r};t) - \rho)$$

After some manipulations we arrive at the following expression.

$$\langle \left|\hat{\mathbf{q}} \cdot \mathbf{v}(\mathbf{q})\right|^2 \rangle = \frac{N v_0^2}{2} \frac{1}{1 + q^2 \tau B_v/(\gamma\rho)}$$

$B_v = \rho \partial_\rho P_v$ is the interaction part of the bulk modulus.

We can identify a longitudinal correlation length.

$$\ell = \sqrt{\tau B_v/(\gamma\rho)}$$

Grzegorz Szamel and Elijah Flenner, EPL 133, 60002 (2021)

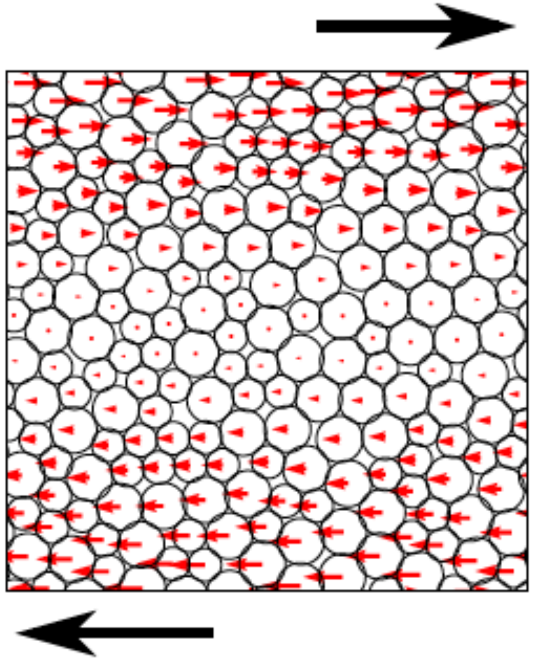
Difference between strain and unit vector in coordinate space

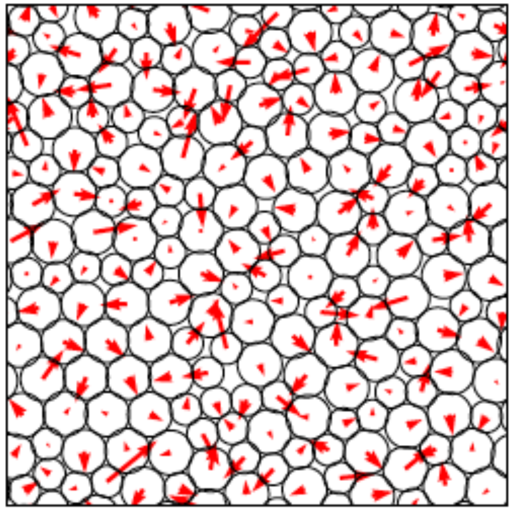
$$u_i^\alpha = \gamma\delta^{\alpha x}(y_i - L_y/2)$$

$$|u(\gamma)| = \gamma\left[\sum_i (y_i - L_y/2)^2\right]^{1/2}$$

$$|u(\gamma)| \approx \gamma L_y\sqrt{N/12}$$

$$\tilde{\gamma} = \frac{\tilde{u}}{L_y\sqrt{\frac{N}{12}}}$$





Algorithm for AQRD

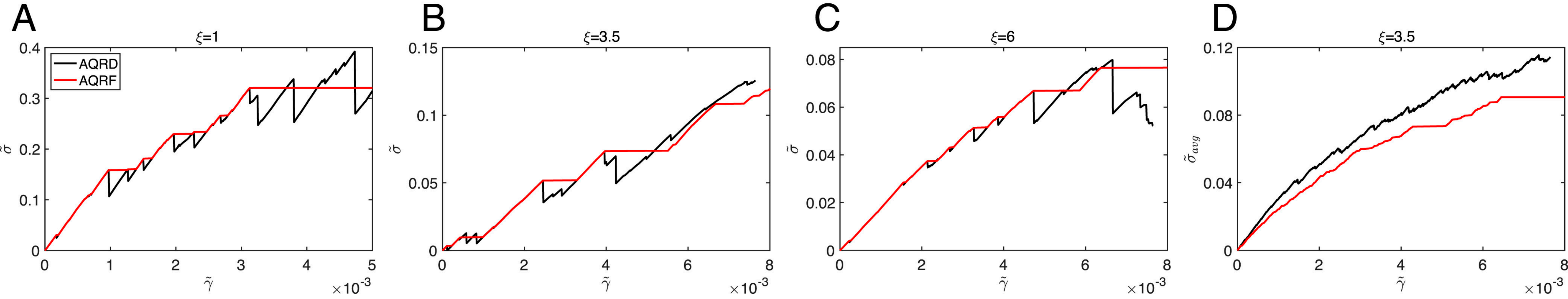
$$|x\rangle = \left|x^{min}\right\rangle + \tilde{u}\,|c\rangle$$

$$\left|F^{ext}\right\rangle = -\lambda\,|c\rangle \qquad\qquad |F\rangle - \langle c|F\rangle\,|c\rangle$$

$$\tilde{\sigma} = \frac{dU}{d\tilde{\gamma}} = \sum_{i=1}^{N}\left(\frac{\partial U}{\partial x_i^{\parallel}}\frac{dx_i^{\parallel}}{d\tilde{\gamma}} + \frac{\partial U}{\partial x_i^{\perp}}\frac{dx_i^{\perp}}{d\tilde{\gamma}}\right)$$

$$\tilde{\sigma} = -\sum_{i=1}^{N}\mathbf{F}_i\cdot\mathbf{c}_i L_y\sqrt{\frac{N}{12}} = -\langle F|c\rangle\,L_y\sqrt{\frac{N}{12}}$$

Random displacement vs. Random force



Generating Gaussian random field

$$\tilde{\psi}(\mathbf{k}_{nm}) = A(\mathbf{k})\exp\{(iB(\mathbf{k}))\} \qquad \mathbf{k}_{nm} = \left(\frac{2\pi n}{L_x}, \frac{2\pi n}{L_x}\right)$$

$$\tilde{f}(|\mathbf{k}|) = \exp\left[\left(-\frac{|\mathbf{k}|^2\xi^2}{8}\right)\right]$$

$$\tilde{\Psi}(\mathbf{k}) = \tilde{f}(|\mathbf{k}|)\,\tilde{\psi}(\mathbf{k})$$

$$\Psi(\mathbf{x}) = \sum_{n,m=1}^{Q} A_{nm} e^{-|\mathbf{k}_{nm}|^2\xi^2/8}\cos\left(B_{nm} + \mathbf{k}_{nm}\cdot\mathbf{x}\right) \qquad \mathbf{c}_i = \Psi(\mathbf{x}_i)$$