The unpredicted scaling of the 1D Kardar-Parisi-Zhang equation

C. Fontaine, F. Vercesi, M. Brachet, Léonie Canet

arXiv:2305.09358

The Burgers and Kardar-Parisi-Zhang equations

- Burgers equation for randomly stirred fluids Burgers (1948)

$$\partial_t \mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v} = \nu\nabla^2\mathbf{v} + \mathbf{f}$$

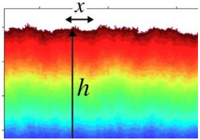
$\mathbf{f}$: stochastic Gaussian forcing at large scale

$\Longrightarrow$ exact mapping to:

- KPZ equation for stochastically growing interfaces

Kardar, Parisi and Zhang, PRL **56** (1986)

$$\partial_t h - \frac{\lambda}{2}\left(\nabla h\right)^2 = \nu\nabla^2 h + \sqrt{D}\eta$$



η: stochastic Gaussian noise with correlations

$$\left\langle \eta(t,\mathbf{x})\eta(t',\mathbf{x}')\right\rangle = 2\delta(t-t')\delta^d(\mathbf{x}-\mathbf{x}')$$

for $\mathbf{v} = -\lambda\nabla h$ with $\nabla\times\mathbf{v} = 0$ and $\mathbf{f} = -\lambda\sqrt{D}\nabla\eta$

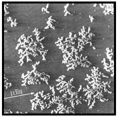
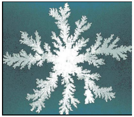
Stochastic interface growth and self-organised criticality

- KPZ equation describes kinetic roughening of growing interfaces
  - generic scale-invariance
  - universality

Halpin-Healy, Zhang, Phys. Rep. **254** (1995)

Krug, Adv. Phys. **46** (1997)



- correlation function takes Family-Vicsek scaling form

$$C(t,\mathbf{x}) = \left\langle (h(t,\mathbf{x}) - h(0,0))^2 \right\rangle \sim \begin{cases} |\mathbf{x}|^{2\chi} & t = 0 \\ t^{2\beta} & \mathbf{x} = 0 \end{cases}$$

$\longrightarrow$ collapse onto a universal scaling function

$$C(t,\mathbf{x}) \sim |\mathbf{x}|^{2\chi} F\left(t/|\mathbf{x}|^{z}\right), \quad z = \chi/\beta$$

- Galilean invariance

$\longrightarrow$ in all dimension

$$\chi + z = 2$$

- time-reversal symmetry

$\longrightarrow$ in one dimension

$$\chi = \tfrac{1}{2}\,, z = \tfrac{3}{2}$$

1D Kardar-Parisi-Zhang equation: exact results

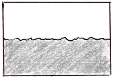
- universal height distribution for the KPZ equation

- curved geometry – droplet (TW-GUE)

  Sasamoto, Spohn, PRL **104** (2010)

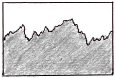
  Amir, Corwin and Quastel, Commun. Pure Appl. Math. **64** (2011)

  Calabrese, Le Doussal, Rosso, EPL **90** (2010)



- flat geometry (TW-GOE)

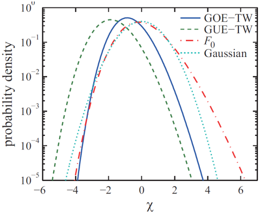
  Calabrese, Le Doussal, PRL **106** (2011),



- Brownian geometry (Baik-Rains)

  Imamura, Sasamoto, PRL (2012)

  Borodin, Corwin, Ferrari, Vetö, Math. Phys. Ann. Geom. **18** (2015)



- two-point correlation function: Airy processes

Prahöfer, Spohn, J. Stat. Phys. (2004), Sasamoto, J. Phys. A (2005), Imamura, Sasamoto, PRL (2012)

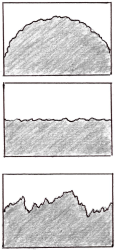
- large deviations for atypical large fluctuations

Le Doussal, Majumdar, Schehr, EPL **113** (2016)

- yet it still reserves its surprises !

1D Kardar-Parisi-Zhang equation: exact results

- universal height distribution for the KPZ equation



  - curved geometry – droplet (TW-GUE)
    Sasamoto, Spohn, PRL **104** (2010)
    Amir, Corwin and Quastel, Commun. Pure Appl. Math. **64** (2011)
    Calabrese, Le Doussal, Rosso, EPL **90** (2010)
  - flat geometry (TW-GOE)
    Calabrese, Le Doussal, PRL **106** (2011),
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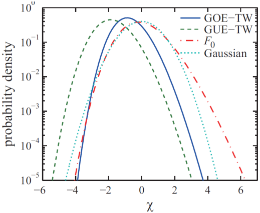
    Borodin, Corwin, Ferrari, Vetö, Math. Phys. Ann. Geom. **18** (2015)



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- **yet it still reserves its surprises !**

The KPZ - Burgers equation in the inviscid limit

The Galerkin-truncated Burgers equation: Crossover from inviscid-thermalised to Kardar-Parisi-Zhang scaling

C. Cartes[1], E. Tirapegui[2], R. Pandit[3] and M. Brachet[4]

Phil. Trans. A **380** (2022)

observation of an unpredicted scaling $z = 1$ in the limit $\nu \to 0$

Anomalous ballistic scaling in the tensionless or inviscid Kardar-Parisi-Zhang equation

Enrique Rodríguez-Fernández,[1,*] Silvia N. Santalla,[2,†] Mario Castro,[3,‡] and Rodolfo Cuerno[1,§]

Phys. Rev. E **106** (2022)

Family-Vicsek Scaling of Roughness Growth in a Strongly Interacting Bose Gas

Kazuya Fujimoto,[1,2] Ryusuke Hamazaki,[3,4] and Yuki Kawaguchi[2]

Phys. Rev. Lett. **124** (2020)

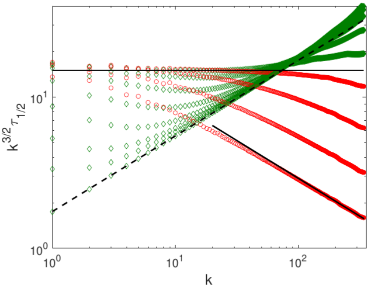
(c)

Model	α	β	z
KPZ	$1/2$	$1/3$	$3/2$
EW	$1/2$	$1/4$	2
BHM ($\nu \gg 1$)	0.517 ± 0.030	0.255 ± 0.012	2.07 ± 0.20
BHM ($\nu \simeq 1/2$)	0.500 ± 0.003	0.489 ± 0.004	1.00 ± 0.01

this Letter

The KPZ - Burgers equation in the inviscid limit

- Decorrelation time from the two-point function $C(t, k)$



Inviscid Burgers ($\nu = 0$): $z = 1$

Kardar-Parisi-Zhang: $z = 3/2$

Edwards-Wilkinson ($\lambda = 0$): $z = 2$

Cartes, Tirapegui, Pandit, Brachet, Phil. Trans. A **380** (2022)

- note: $z = 1$ scaling also predicted in $d \to \infty$, Re $\to \infty$ in Burgers

Bouchaud, Mézard, Parisi, PRE **52** (1995)

Renormalisation group for the KPZ equation

- Wilson's renormalisation group (RG)
  - progressive averaging of fluctuations modes with $|\mathbf{p}| \leq \kappa$
  - flow of κ-dependent effective models

scale invariance $\Longleftrightarrow$ fixed-point of the RG flow

- Kardar-Parisi-Zhang action

$$\mathcal{S}[h,\tilde{h}]=\int_{t,\mathbf{x}} \left\{ \tilde{h} \left(\partial_t h - \frac{\lambda}{2} (\nabla h)^2 - \nu \nabla^2 h \right) - D\, \tilde{h}^2 \right\}$$

$\longrightarrow$ only one independent coupling for $g = \lambda^2 D / \nu^3$

- perturbative renormalisation group
  - resummed to all-order Wiese, J. Stat. Phys. **93** (1998)

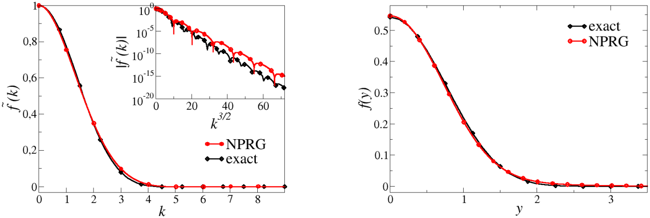
fails to find the KPZ fixed-point in $d > 1$!

Renormalisation group for the KPZ equation

- Functional (non-perturbative) Renormalisation Group
  - KPZ fixed-point in all d

LC, Chaté, Delamotte, Wschebor, PRL **104**, (2010)
Kloss, LC, Wschebor, PRE **86** (2012), Kloss, Delamotte, LC, Wschebor, PRE **89** (2014)

  - very accurate results in $d = 1$



LC, Chaté, Delamotte, Wschebor, PRE **84**, (2011), Prähofer and Spohn, J. Stat. Phys. **115** (2004)

  - precise agreement with numerical results in $d = 2$ and in $d = 3$

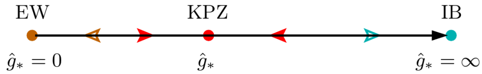
Halpin-Healy, PRE **88** (2013), K. Deligiannis, *et al*, PRR **4** (2022)

Renormalisation group analysis

- KPZ-Burgers equation: one coupling constant $g = \lambda^2 D/\nu^3$ (or Re)

$$\text{inviscid limit } \nu \to 0 \iff \text{infinite coupling limit } g \to \infty$$

- possible scenario :



$\Longrightarrow$ scaling $z = 1$ controlled by UV fixed point IB

Functional Renormalisation Group approach

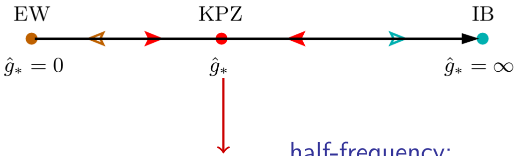
- within the NLO approximation:

Kloss, LC, Wschebor PRE **86** (2012)

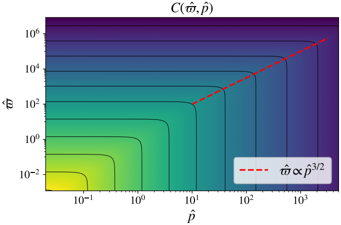
  - effective functions $\nu_\kappa(\omega, p)$, $D_\kappa(\omega, p)$, λ
  - numerical integration of the FRG flow equations

Functional Renormalisation Group: Numerical solution in the IR

- Integration of the FRG flow from the UV to the IR

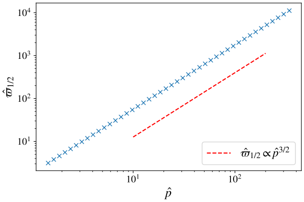


correlation function



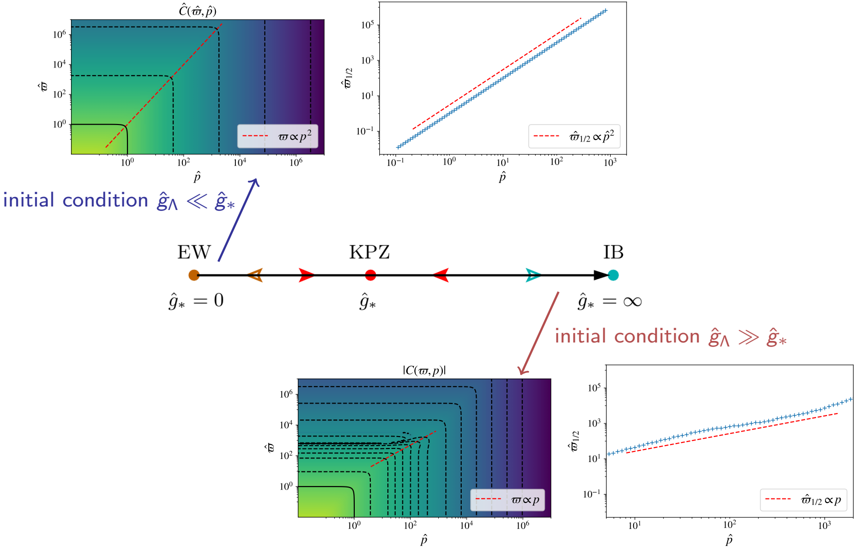
half-frequency:

$C_\kappa(\varpi_{1/2}(p), p) = \frac{C_\kappa(0,p)}{2}$



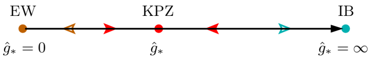
KPZ fixed-point in the IR for any initial condition g_Λ

Functional Renormalisation Group: Numerical solution probing the UV

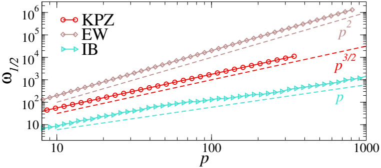


Functional Renormalisation Group: Summary of numerical results

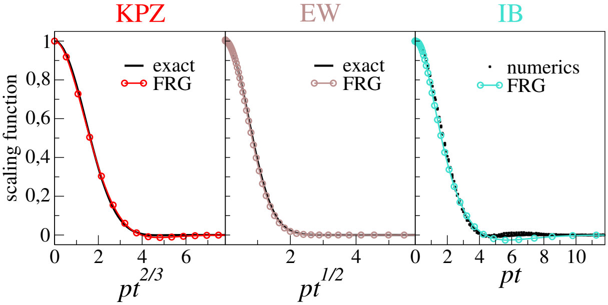
- 3 fixed-points with different z
  - KPZ: stable, controls the IR
  - EW, IB: unstable, control the UV



Fontaine, Vercesi, Brachet, LC, arXiv:2305.09358



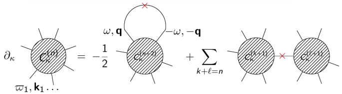
and different scaling functions



Can we rigorously demonstrate that $z = 1$?

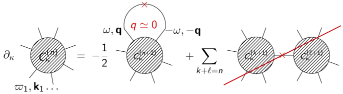
Analytical solution with Functional Renormalisation Group: Exact closure in the large wavenumber limit

- flow for $C^{(n)}_{\alpha_1\dots\alpha_n}(\{t_i,\mathbf{x}_i\}) \equiv \Big\langle v_{\alpha_1}(t_1,\mathbf{x}_1)\cdots v_{\alpha_n}(t_n,\mathbf{x}_n)\Big\rangle_c$



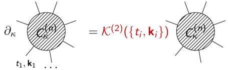
exact (but infinite hierarchy of) flow

large p →



asymptotic flow at large wavenumber

extended symmetries



closed flow at large wavenumber

for Navier-Stokes: exact asymptotic form of space-time correlations

Tarpin, LC, Wschebor, Phys. Fluids **30** (2018), Pagani, LC, Phys. Fluids **33** (2021)
Gorbunova, Balarac, LC, Eyink, Rossetto, Phys. Fluids **33** (2021), LC, J. Fluid Mech. *Perspectives* **950** (2022)

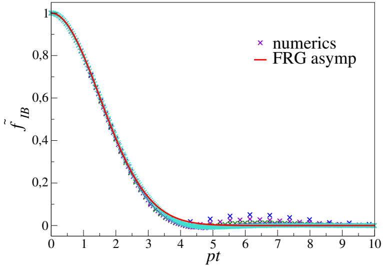
- extended symmetries for Burgers
  - gauged Galilean invariance: $\begin{cases} \mathbf{x} \to \mathbf{x}' = \mathbf{x} + \epsilon(t) \\ \mathbf{v}(t,\mathbf{x}) \to \mathbf{v}(t,\mathbf{x}') - \dot{\epsilon}(t) \end{cases}$
  - gauged shift symmetry in 1D: $\bar{v}(t,\mathbf{x}) = \bar{v}(t,\mathbf{x}) + \bar{\epsilon}(t)$

Analytical solution with Functional Renormalisation Group: Exact asymptotic solution for inviscid Burgers

- solution at the fixed-point at large p (UV):

$$C(t,p) \propto \begin{cases} \exp\Big(-\alpha_0\,(pt)^2 + \mathcal{O}(pL)\Big) & t \ll \tau_0 \\ \exp\Big(-\alpha_\infty\,p^2|t| + \mathcal{O}(pL)\Big) & t \gg \tau_0 \end{cases}$$

  - proof of $z=1$ scaling at small t
  - analytical form of the scaling function
  - crossover at large t ?



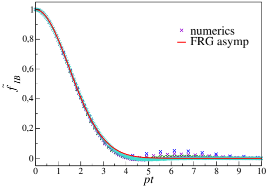
Fontaine, Vercesi, Brachet, LC, arXiv:2305.09358

Summary and Perspectives

Unpredicted scaling $z = 1$ for inviscid KPZ-Burgers from FRG

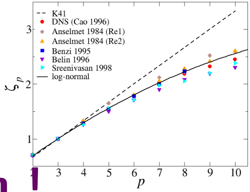
- numerical evidence probing the UV
- exact asymptotic solution:

  $z = 1$ and scaling function



Towards intermittency corrections in turbulence

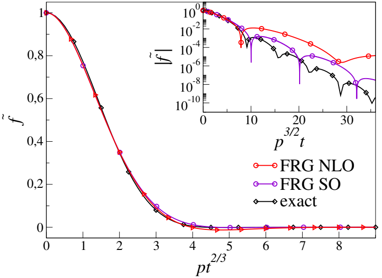
- in shell models
- in Burgers
- in Navier-Stokes ...



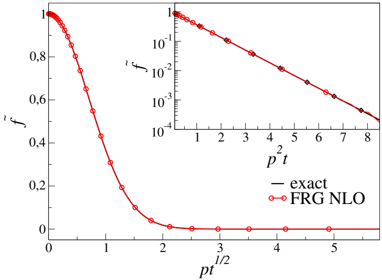
Thank you for attention !

Functional Renormalisation Group: Zooming in the tails of the scaling functions

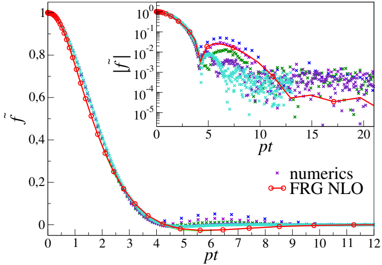
KPZ



EW



IB



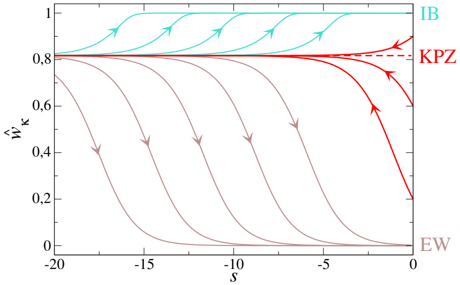
Functional Renormalisation Group: Existence of the Inviscid-Burgers fixed-point

- FRG flow equation for $\hat{w}_\kappa = \hat{g}_\kappa/(1+\hat{g}_\kappa) \in [0,1]$

$$\partial_s \hat{w}_\kappa = \hat{w}_\kappa \left(1 - \hat{w}_\kappa\right)\left(2\chi_\kappa - 1\right)$$

$$\text{with} \quad \chi_\kappa = -\partial_s \ln \nu_\kappa, \qquad s = \ln(\kappa/\Lambda) \qquad \text{(RG ‘time’)}$$

- 3 fixed point solutions
  - KPZ: $0 < \hat{w}_* < 1$
    $\chi_* = 1/2$, $z_{\rm KPZ} = 3/2$
  - EW: $\hat{w}_* = 0$
    $\chi_* = 0$, $z_{\rm EW} = 2$
  - IB: $\hat{w}_* = 1$
    χ_* to be determined,
    $z_{\rm IB} = ?$



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