Freezing of nonlinear waves over an uneven bottom

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Nonlinear Schrödinger equation (NLS)

Focusing,
1D cubic

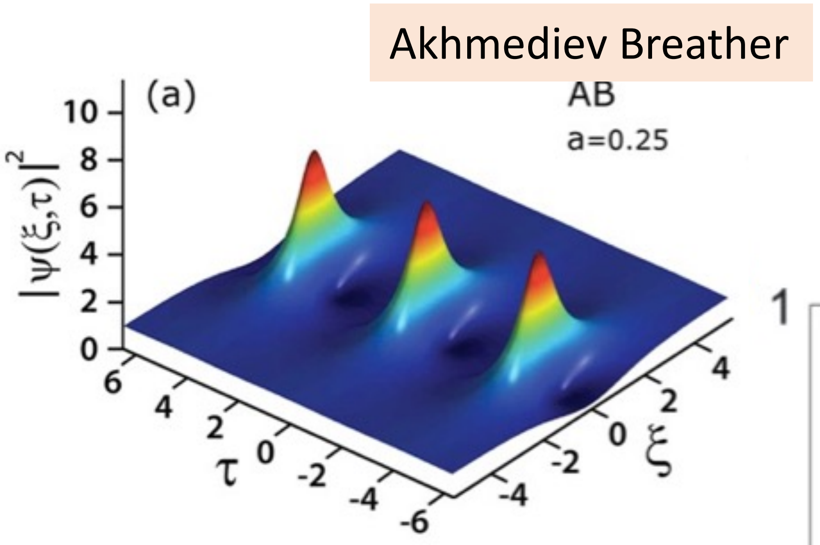
$$i\psi_z + \frac{1}{2}\psi_{tt} + |\psi|^2\psi = 0$$

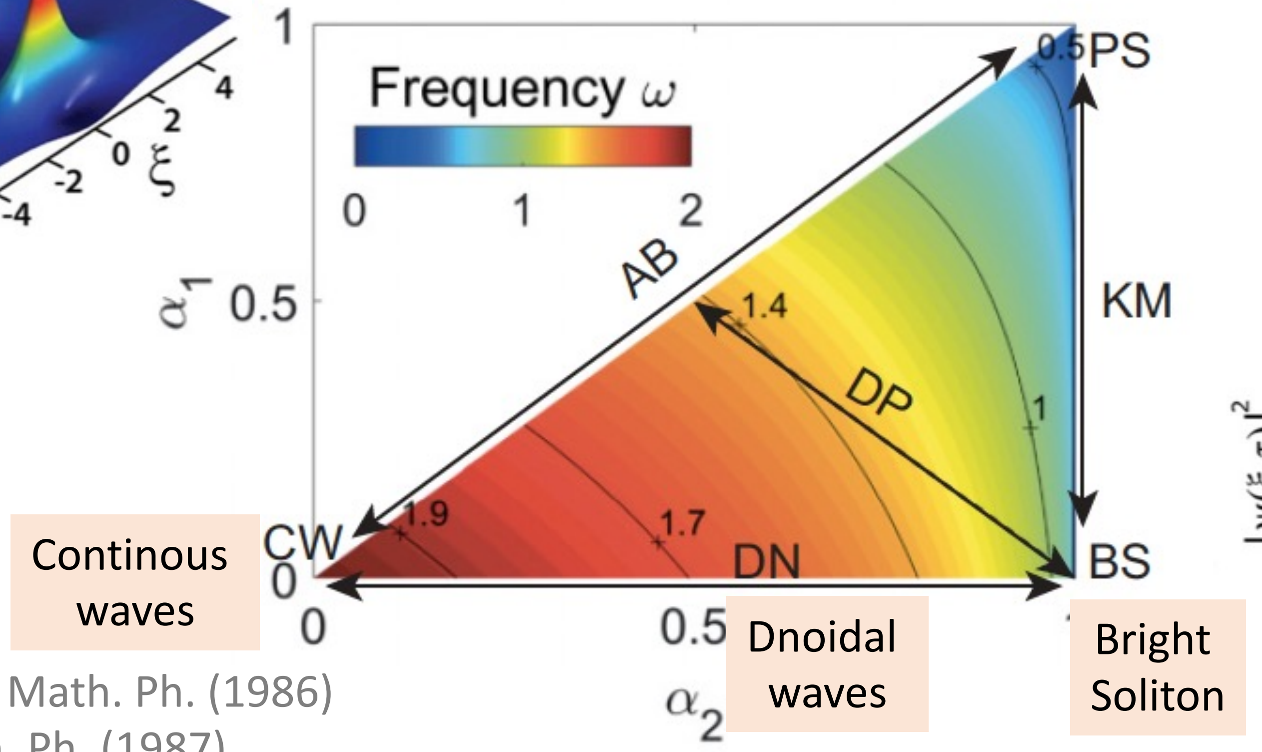
Dispersion

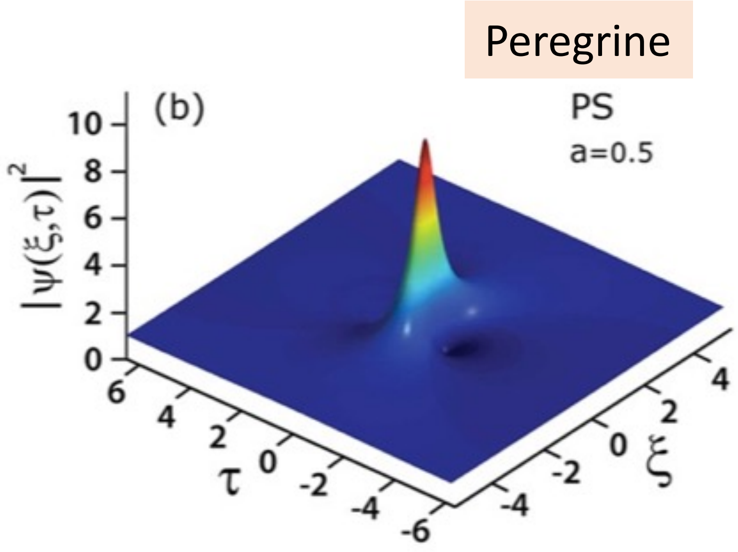
Nonlinearity

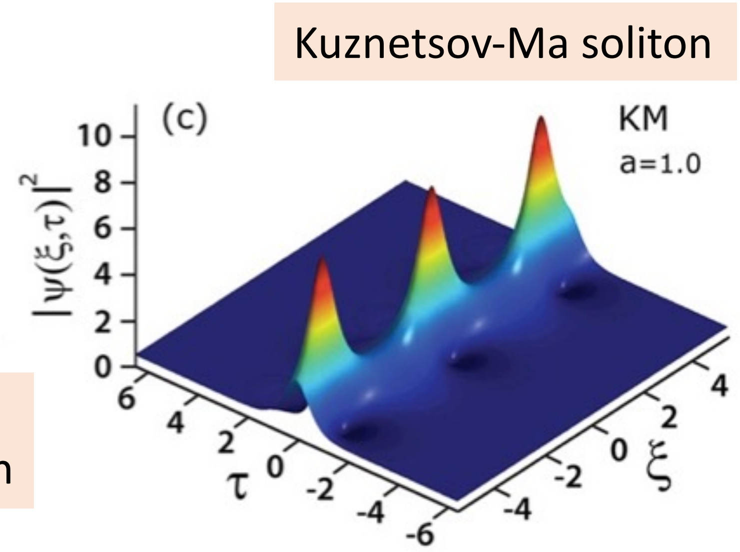
APPLICATIONS:
Optics
Hydrodynamics
Plasma
Cold atoms
Galaxies...

Nonlinear Schrödinger equation (NLS):
three-parameter family of solutions
(arbitrary amplitude and periods)







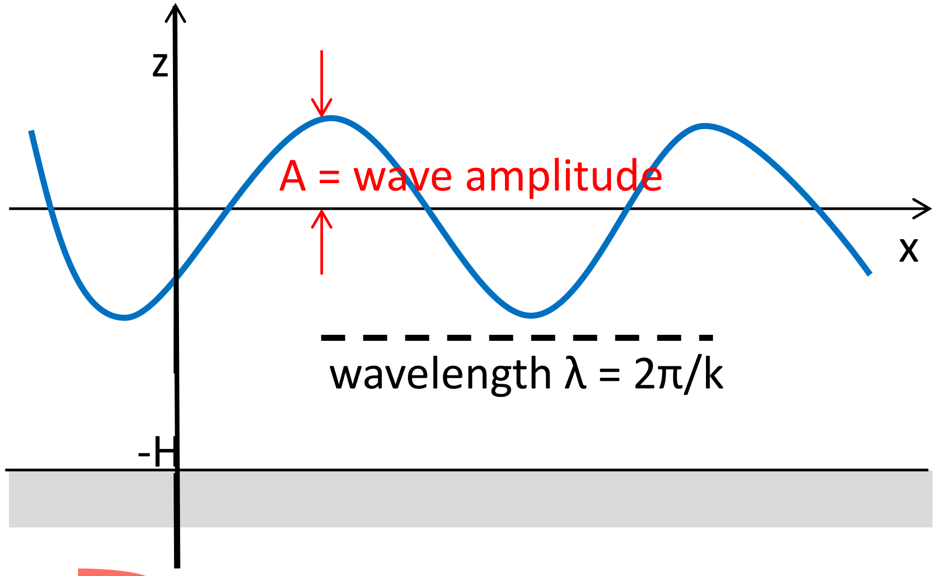


Akhmediev & Korneev Theor. Math. Ph. (1986)
Akhmediev *et al.* Theor. Math. Ph. (1987)
Toenger *et al.* Sc. Rep. (2015)
Conforti *et al*. PRA (2020)

From Navier-Stokes eq. to NLS

$$\frac{D\vec{v}}{dt} = -2\vec{\Omega} \times \vec{v} - \frac{\vec{\nabla} P}{\rho} + \vec{g} + \vec{F}$$

Coriolis term — Gravity — Viscosity term



- Incompressible fluid: constant density
- Large Roosby number: Coriolis term can be neglected
- Viscosity effect neglected
- Constant bathymetry
- No wind forcing
- Velocity can be expressed in terms of a scalar potential function
- 1-dimensional propagation
- k A << 1: weak nonlinearities
- k H >> 1: deep-water limit

$$i\psi_z + \frac{1}{2}\psi_{tt} + |\psi|^2\psi = 0$$

From Navier-Stokes eq. to NLS

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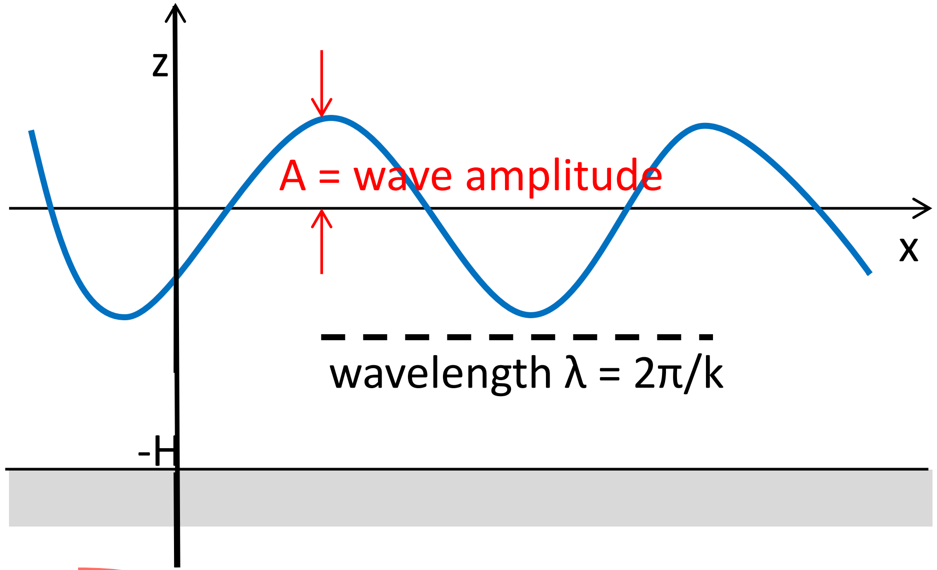
Coriolis term

Gravity

Viscosity term



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Outline

- **Bathymetry effect**

  - an abrupt bathymetry change can freeze an Akhmediev Breather at max amplitude
  - simplified model with only 3 waves
  - experiment in a water tank

- **Wind effect**

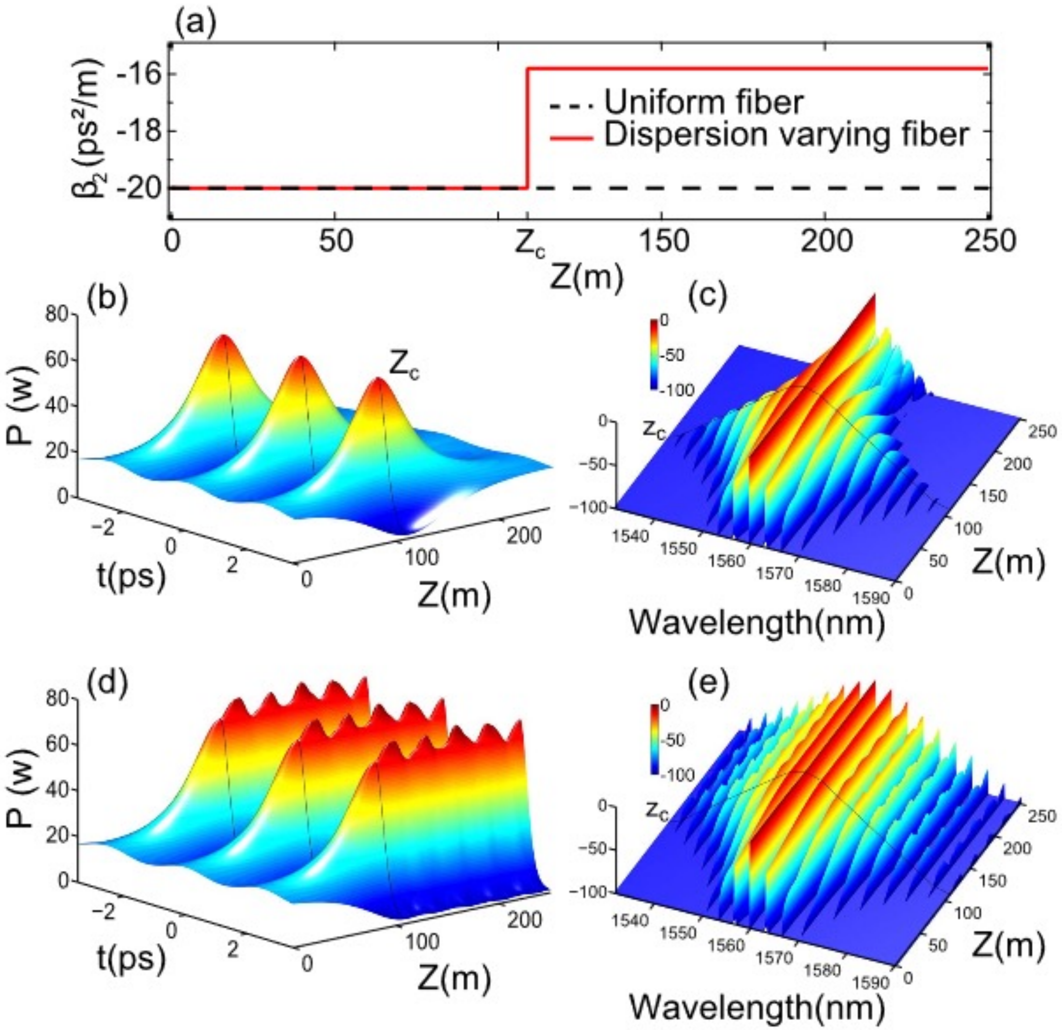
  - change between types of solutions
  - experiment in a water tank with wind turbine

Starting idea: experiment in an optical fiber with varying dispersion

Dispersion coefficient

Uniform fiber

Dispersion varying fiber

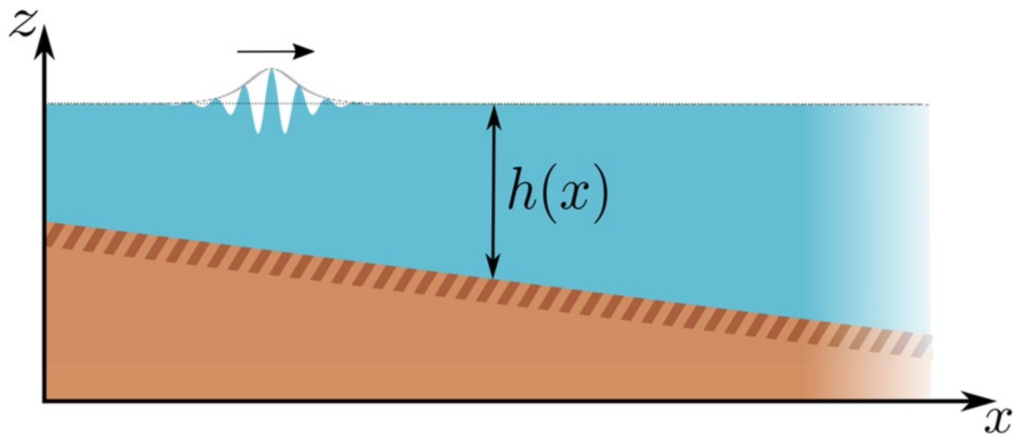


Akhmediev breather

Evolution can be quasi-stabilized in space at maximal compression

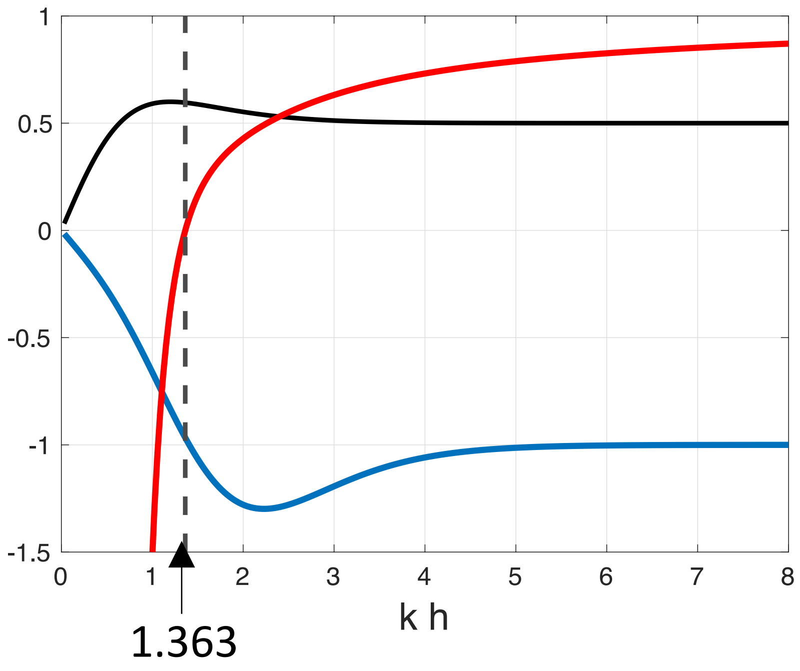
A. Bendahmane et al. Opt. Lett. (2014)

Can we reproduce the stabilization in surface water waves?



Uneven bottom

$$i\partial_\xi U + \underbrace{\alpha(kh)\partial_\tau^2 U}_{\text{Dispersion}} - \underbrace{\beta(kh)|U|^2 U}_{\text{Non linearity}} = -\underbrace{i\mu_0\partial_\xi(kh)U}_{\text{Shoaling}} - \underbrace{i\nu U}_{\text{Loss}}$$



Djordjevic & Redekopp, J. Phys. Oc. (1978)

Derivation of a NLSE-like equation

$$i\partial_\xi U + \underbrace{\alpha(kh)\partial_\tau^2 U}_{\text{Dispersion}} - \underbrace{\beta(kh)|U|^2 U}_{\text{Non linearity}} = -\underbrace{i\mu_0 \partial_\xi (kh) U}_{\text{Shoaling}} - \underbrace{i\nu U}_{\text{Loss}}$$

μ

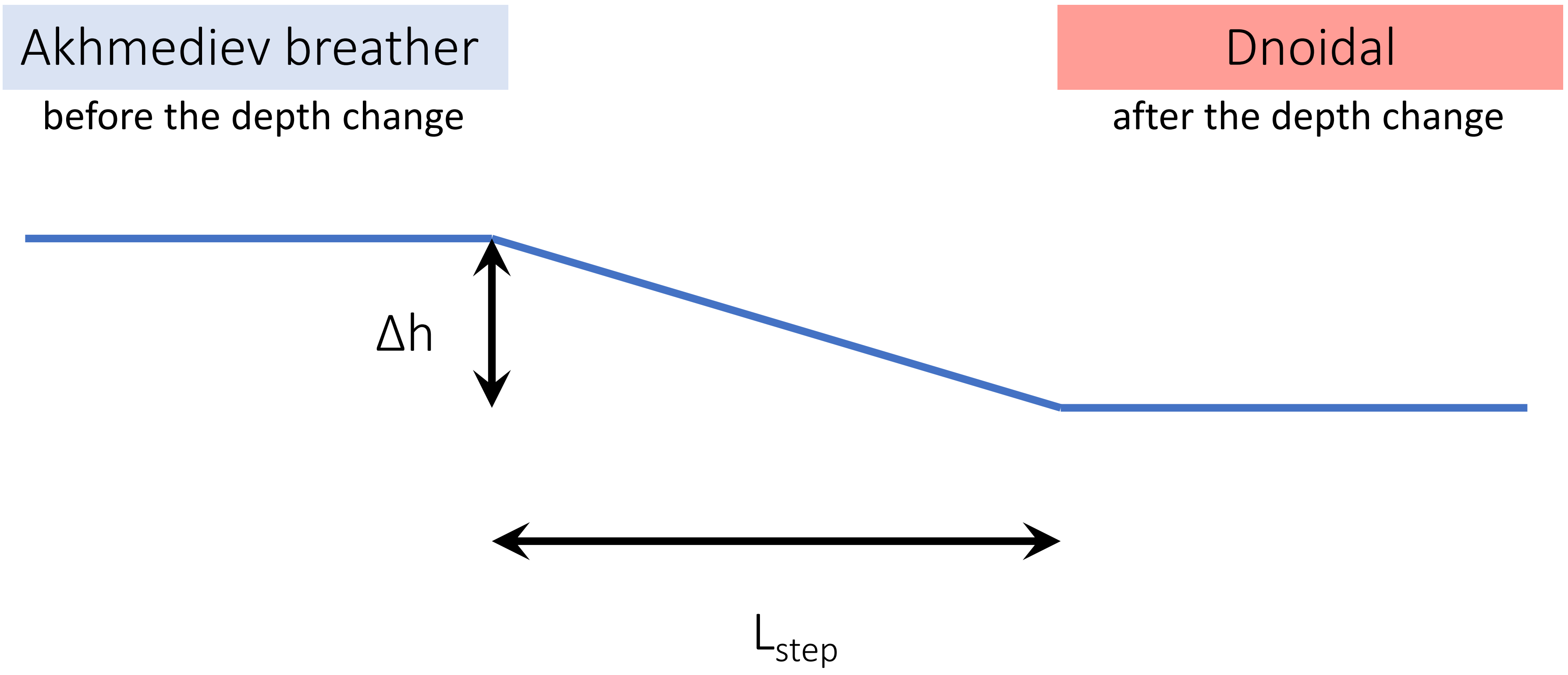
$$U = V \exp\left[-\int_0^\xi \mu(y)dy - \nu\xi\right]$$

$$\tilde{\beta} = \beta(\xi)\frac{c_g(0)}{c_g(\xi)}\exp(-2\nu\xi)$$

$$i\partial_\xi V + \underbrace{\alpha(kh)\partial_\tau^2 V}_{\text{Dispersion}} - \underbrace{\tilde{\beta}(kh)|V|^2 V}_{\text{Non linearity}} = 0$$

Djordjevic & Redekopp J. Phys. Oc. (1978)

Uneven bottom



$\Delta h \ll L_{step} \ll L_{nonlinear}$

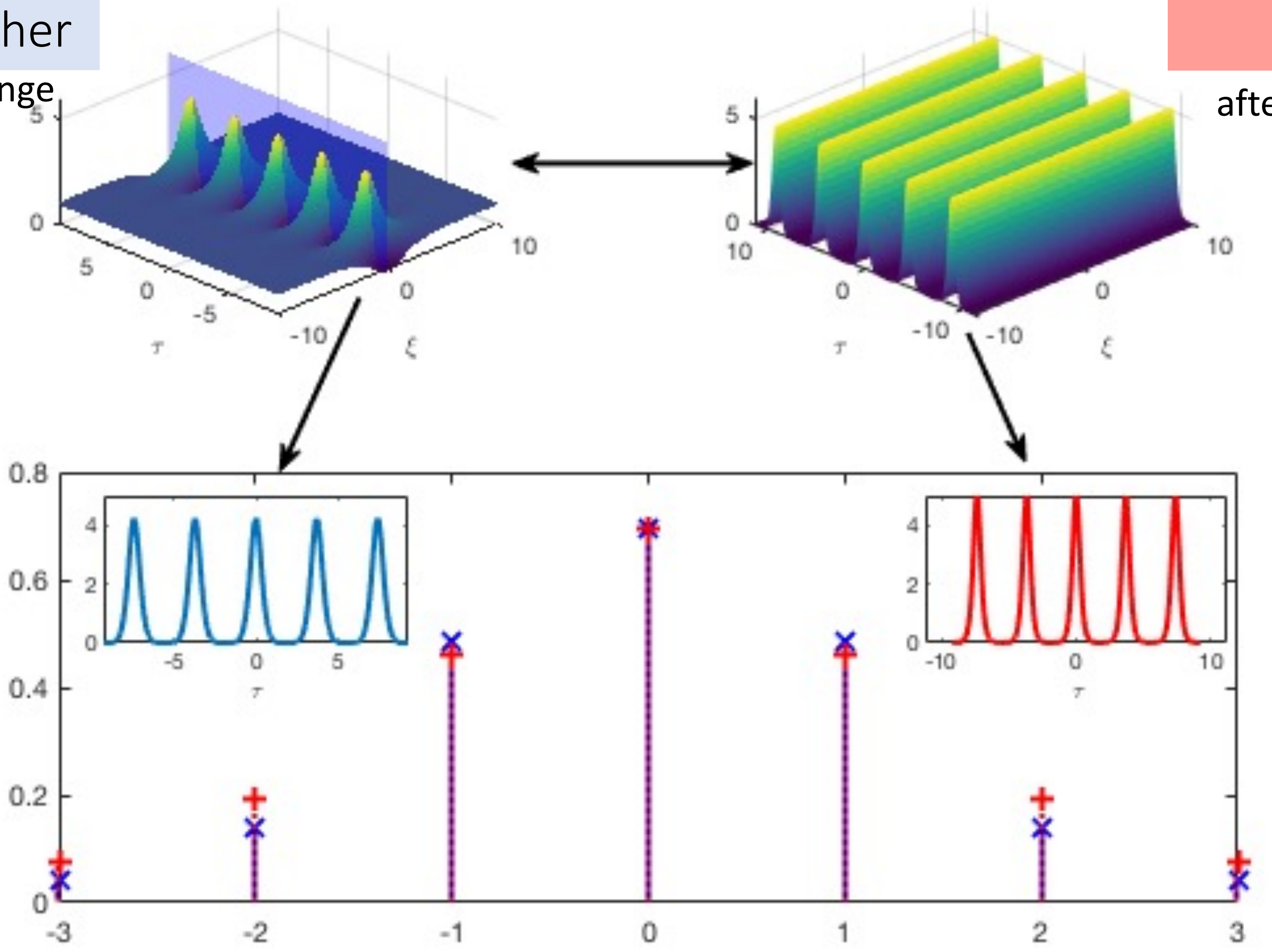
Matching NLS solutions

Akhmediev breather

before the depth change

Dnoidal

after the depth change



Spectrum

Gomel et al. PRL (2021)

Matching NLS solutions

Akhmediev breather

before the depth change

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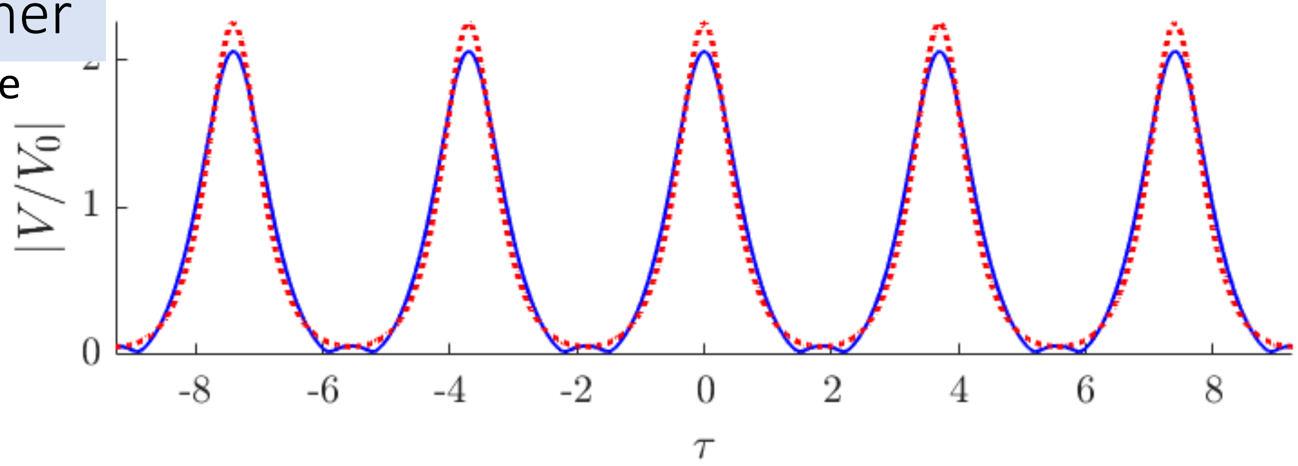
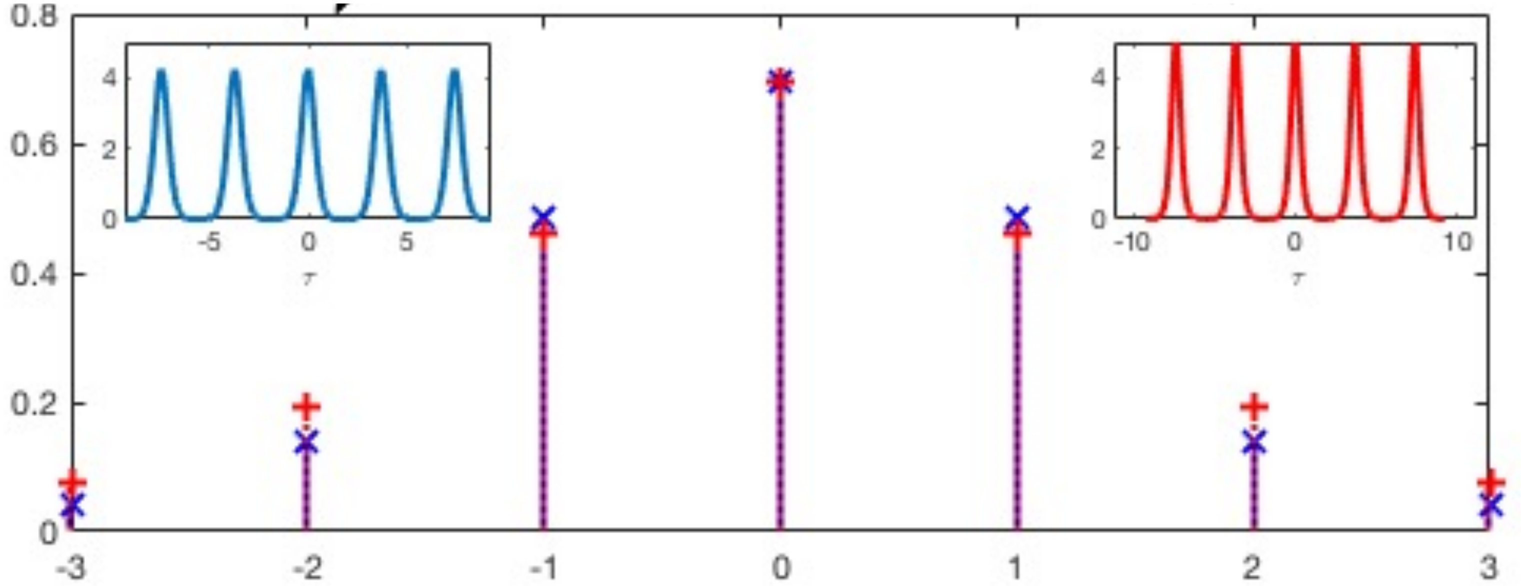
after the depth change





Spectrum

Gomel et al. PRL (2021)

Simplified model: three-wave truncation

$$V(\xi,\tau) = A_0(\xi) + A_1(\xi)e^{i\Omega\tau} + A_{-1}(\xi)e^{-i\Omega\tau}$$

Conversion rate to sidebands:

$$\eta = \frac{|A_1|^2 + |A_{-1}|^2}{E}$$

$$E = |A_0|^2 + |A_1|^2 + |A_{-1}|^2$$

Relative phase:

$$\psi = \frac{\phi_1 + \phi_{-1}}{2} - \phi_0$$

$$H(\psi,\eta) = \eta(\eta-1)\cos(2\psi) + \frac{3\eta^2}{4} + \gamma\eta$$

with: $\gamma = -\left[\frac{\alpha\Omega^2}{\tilde{\beta}E} + 1\right]$

Trillo & Wabnitz Opt. Lett. (1991)
Armaroli, Brunetti, Kasparian PRE (2017)

Simplified model: three-wave truncation

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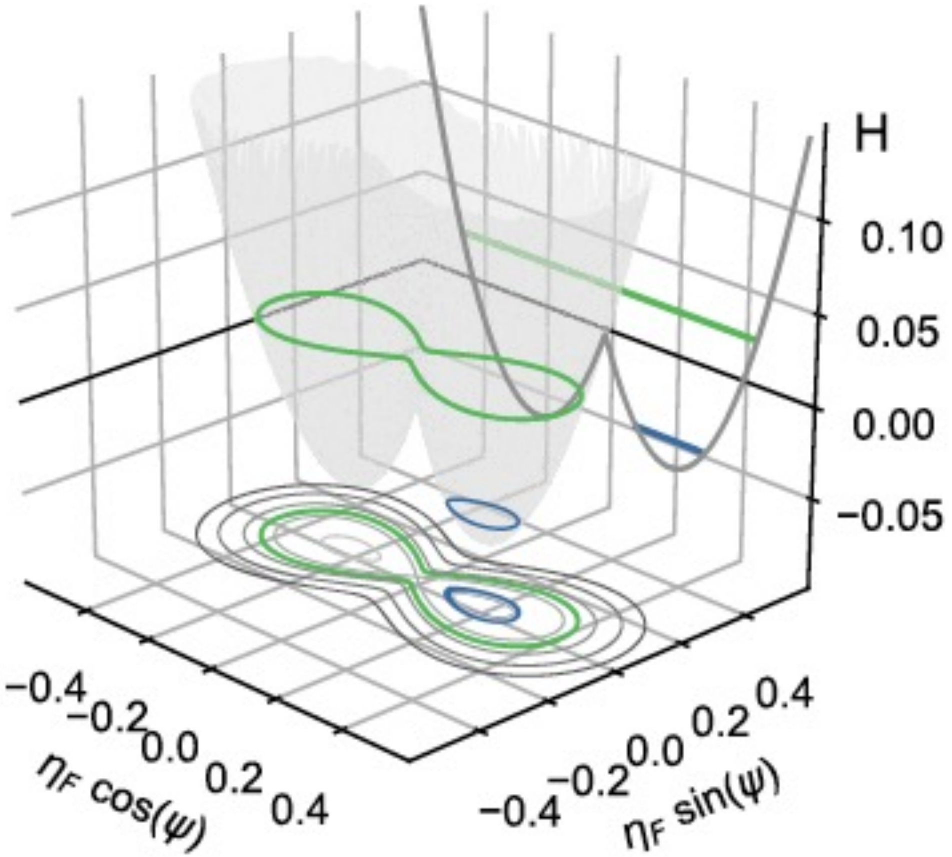

Trillo & Wabnitz Opt. Lett. (1991)
Armaroli, Brunetti, Kasparian PRE (2017)

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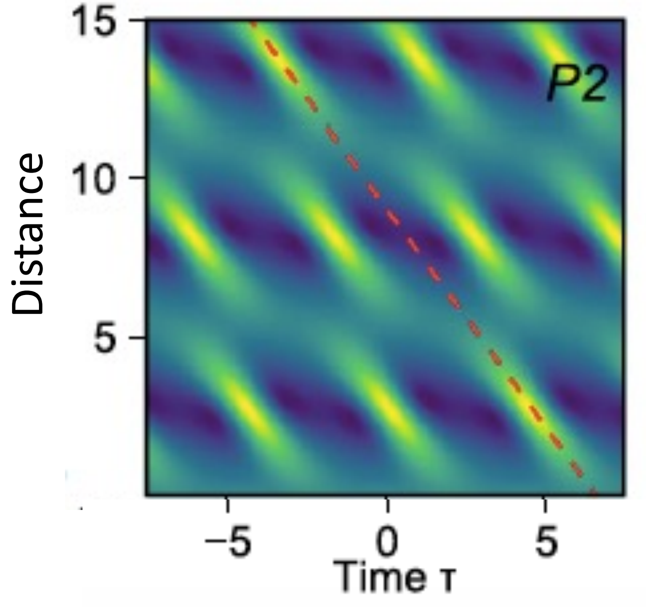
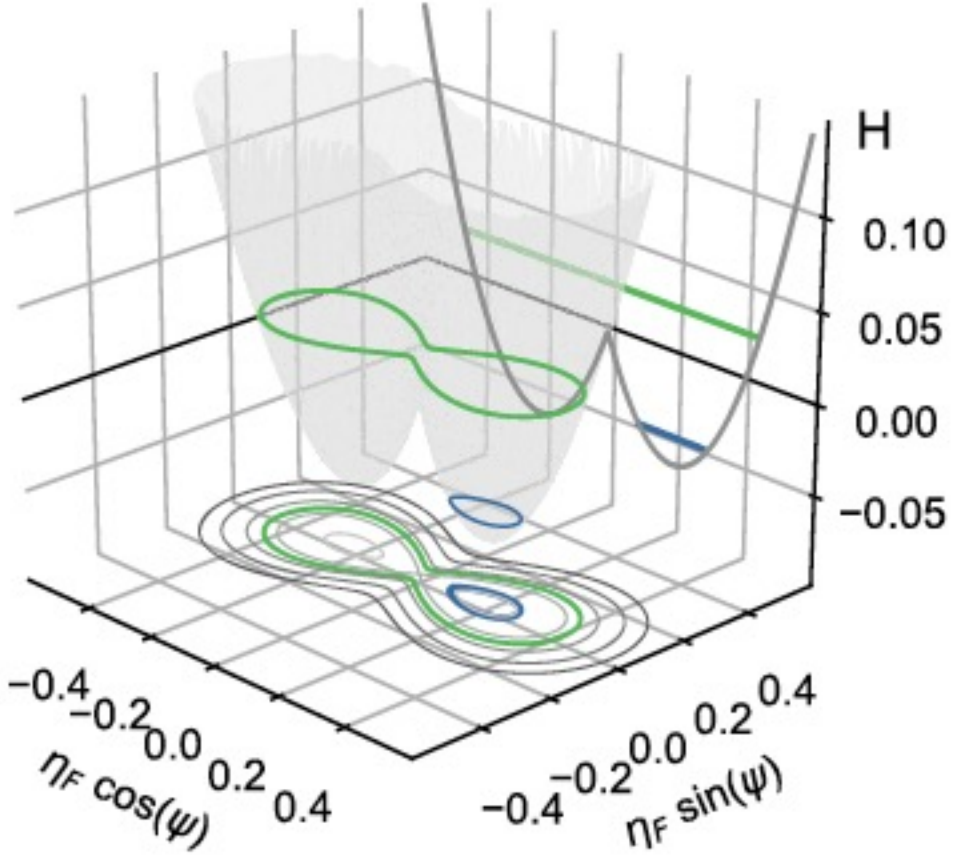
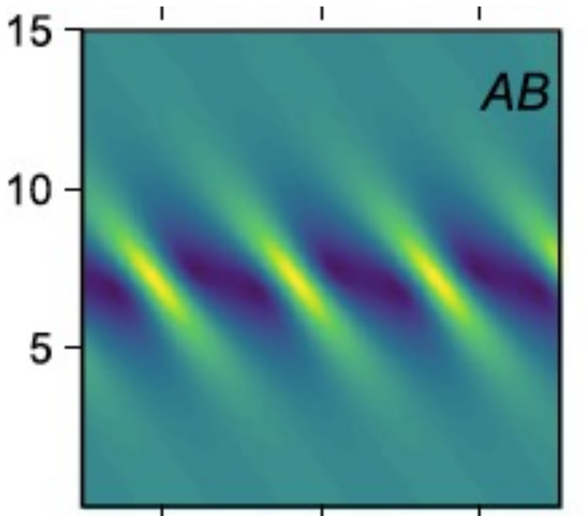
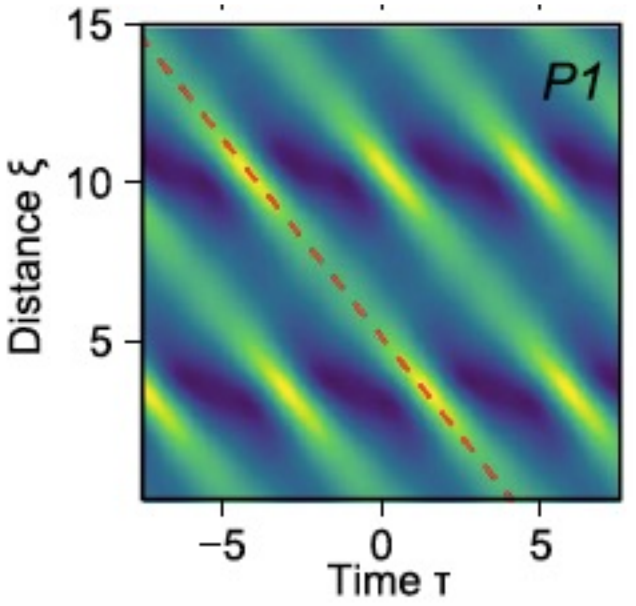


Outside the separatrix:
P2 orbits





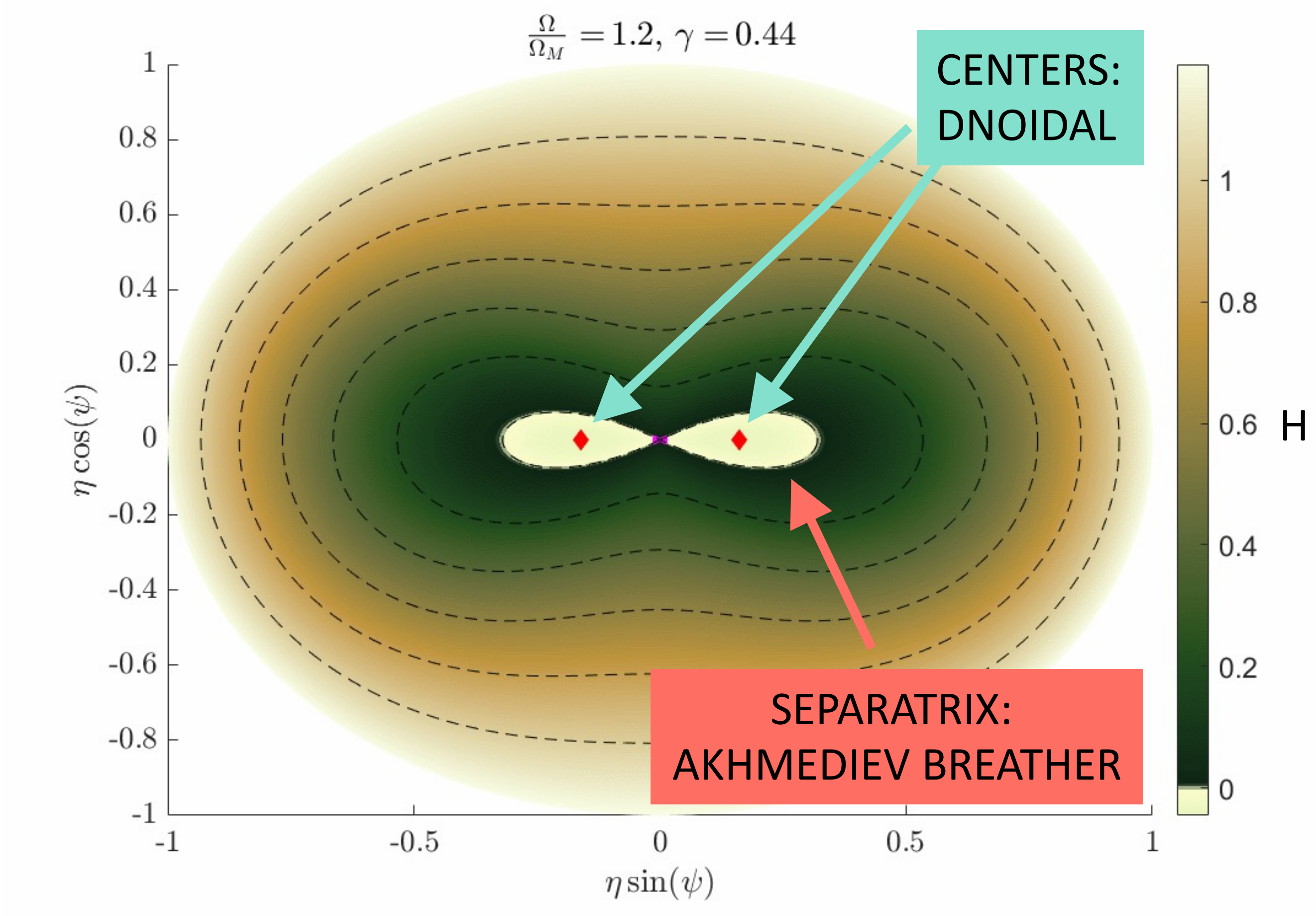
Separatrix: Akhmediev Breather



Inside the separatrix: P1 orbits

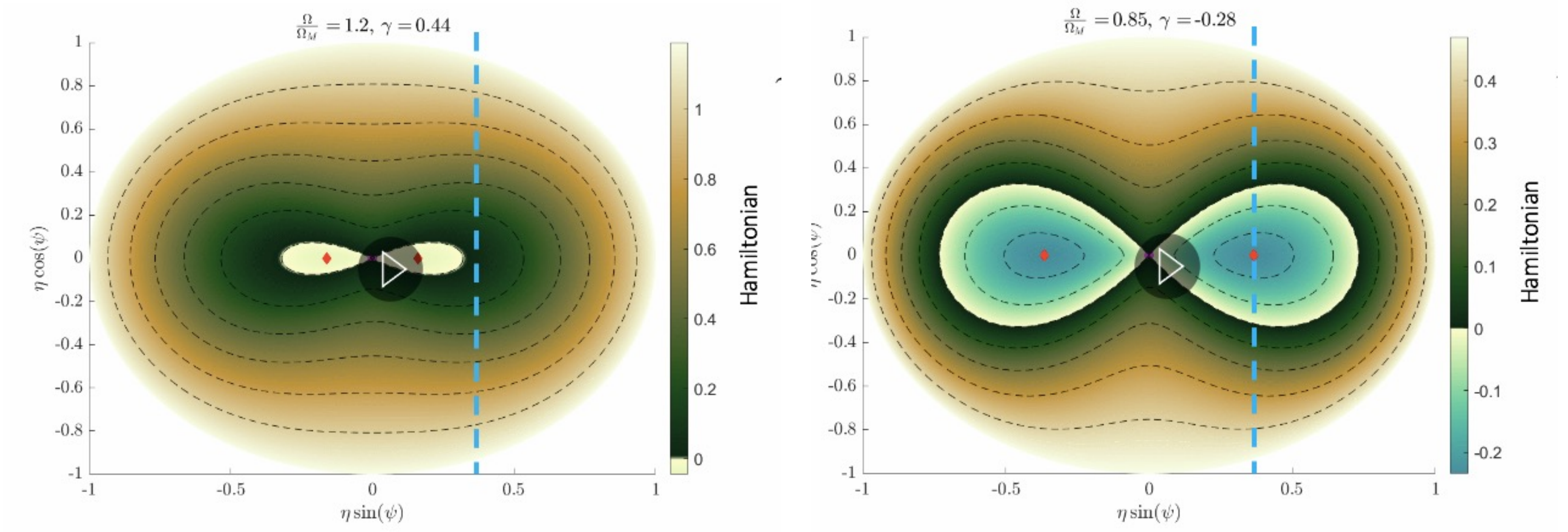
Trillo & Wabnitz Opt. Lett. (1991)
Armaroli, Brunetti, Kasparian PRE (2017)
Eeltink et al. Nonlin. Dyn. (2020)

Simplified model: three-wave truncation



Trillo & Wabnitz Opt. Lett. (1991)
Armaroli, Brunetti, Kasparian PRE (2017)

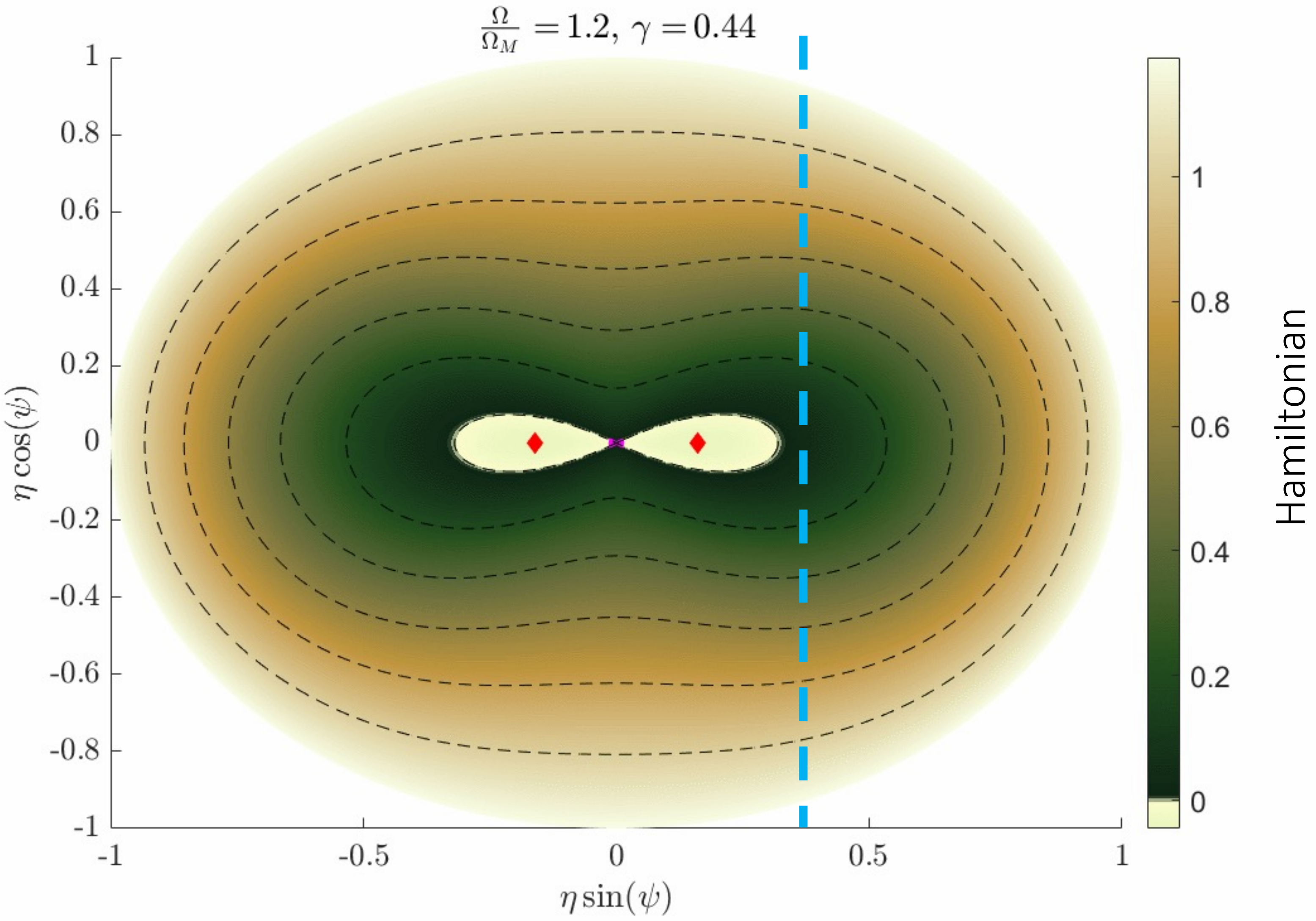
Phase space evolution



Matching condition between AB and dnoidal detuning: $\Omega_{dn} = 2(\Omega_{AB} - 1)$

Gomel et al. PRL (2021)

Phase space evolution



$$\gamma = -\left[\frac{\alpha\Omega^2}{\tilde{\beta}E} + 1\right] = \left(\frac{\Omega}{\Omega_M}\right)^2 - 1$$

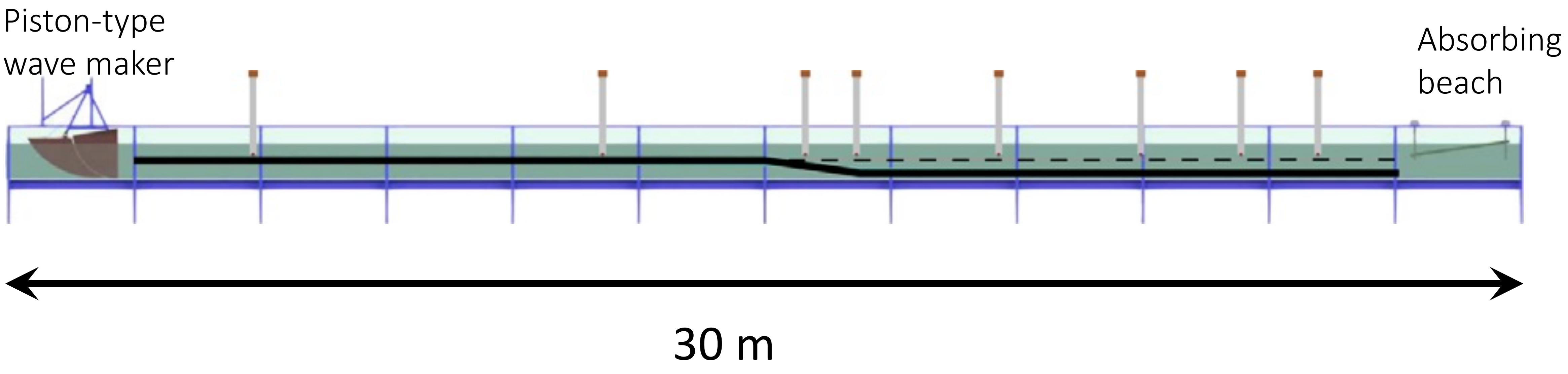
Matching condition between AB and dnoidal detuning:

$$\Omega_{dn} = 2(\Omega_{AB} - 1)$$

Gomel et al. PRL (2021)

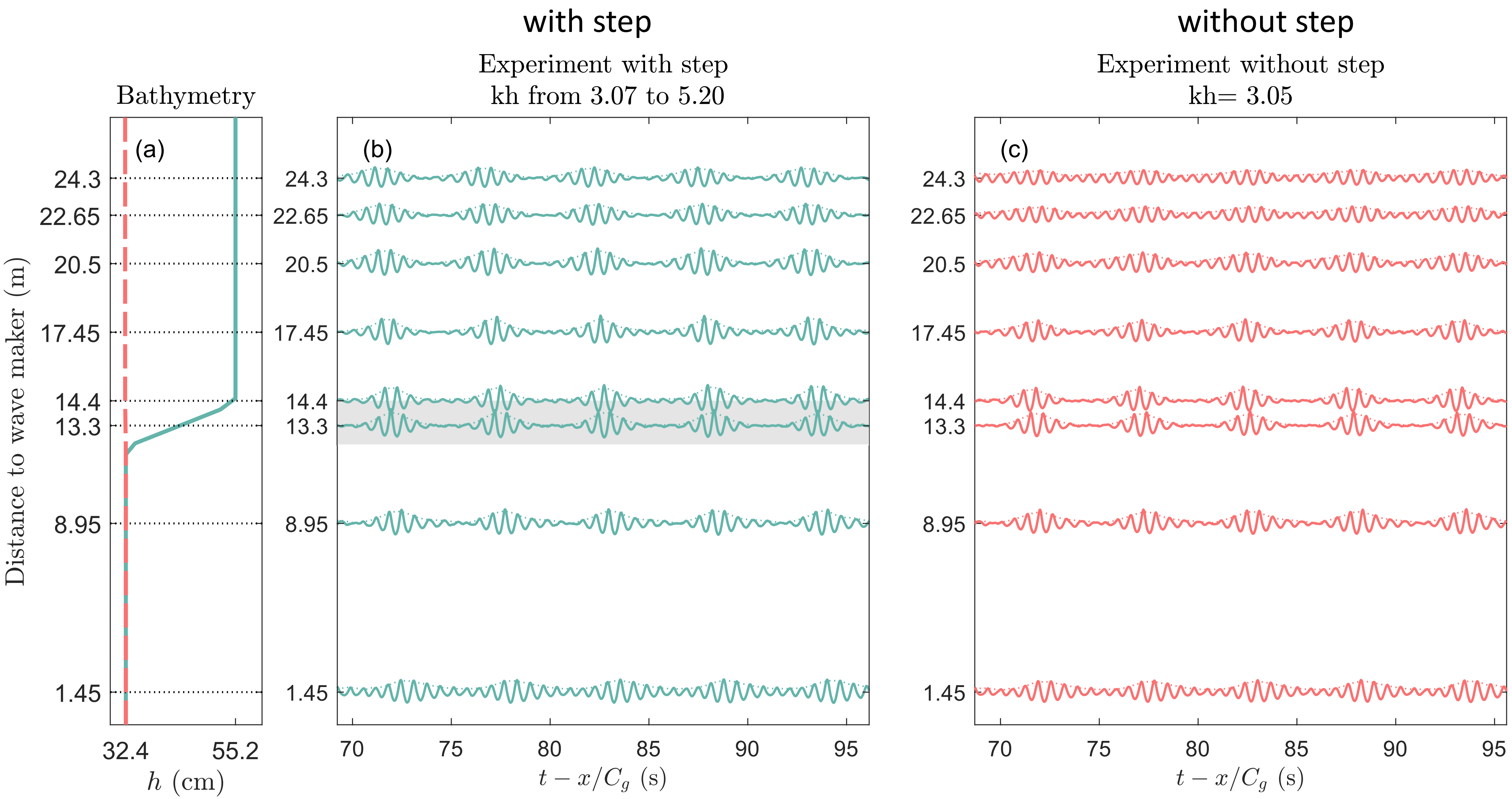
Experiments in the water tank at The University of Sydney

by Alexis Gomel & Amin Chabchoub



- Water wave flume with artificial floor setup
- 8 wave gauges
- Water depth varied from h = 32.4 cm to 55.2 cm
- $\Delta h << L_{step} << L_{nonlinear}$ to prevent spurious reflections

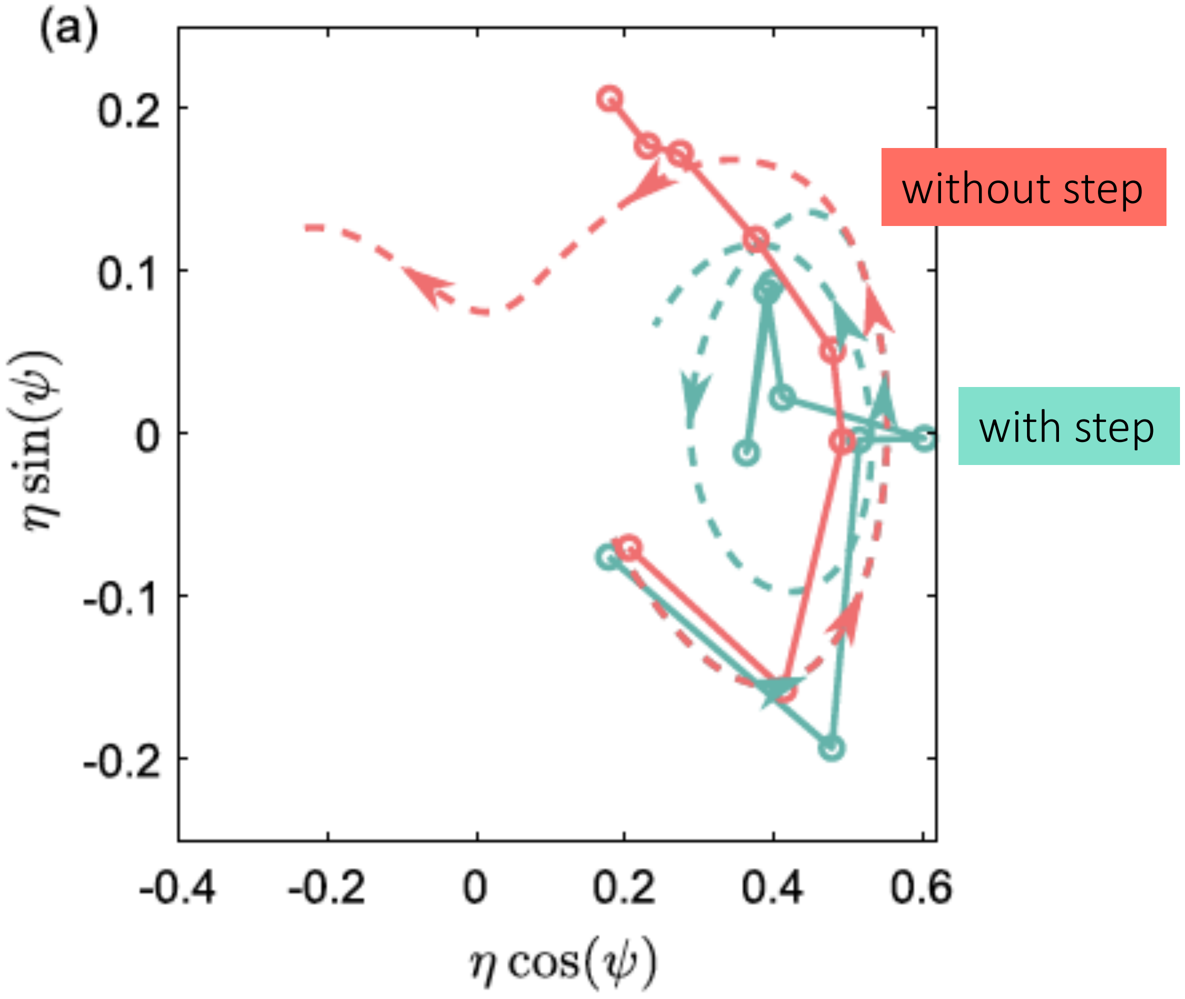
Experiments in the water tank

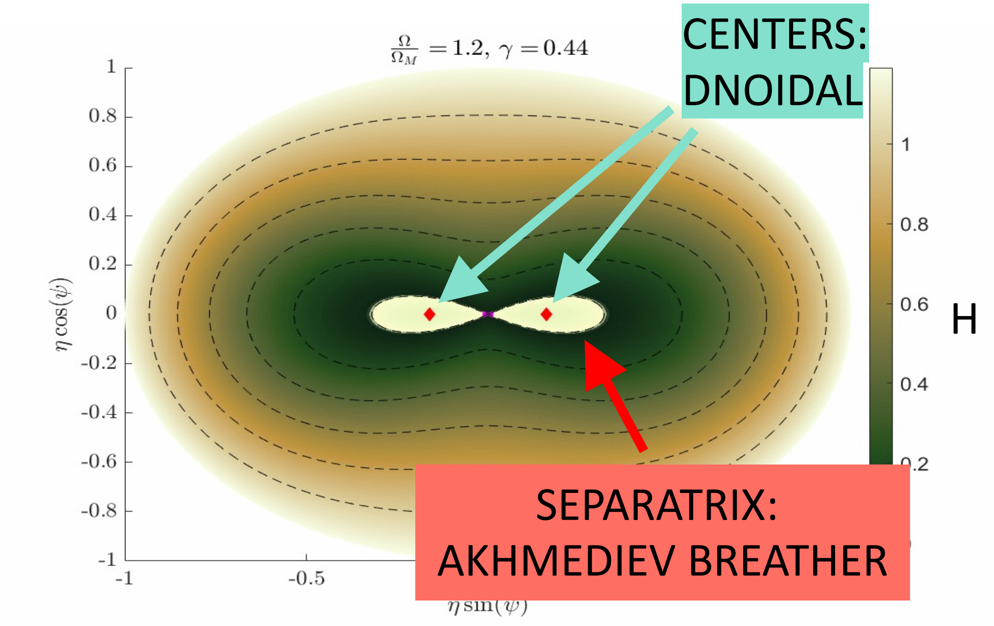


Gomel et al. PRL (2021)

Experiments in Sydney *vs* numerical simulations

Evolution in 3-wave phase space

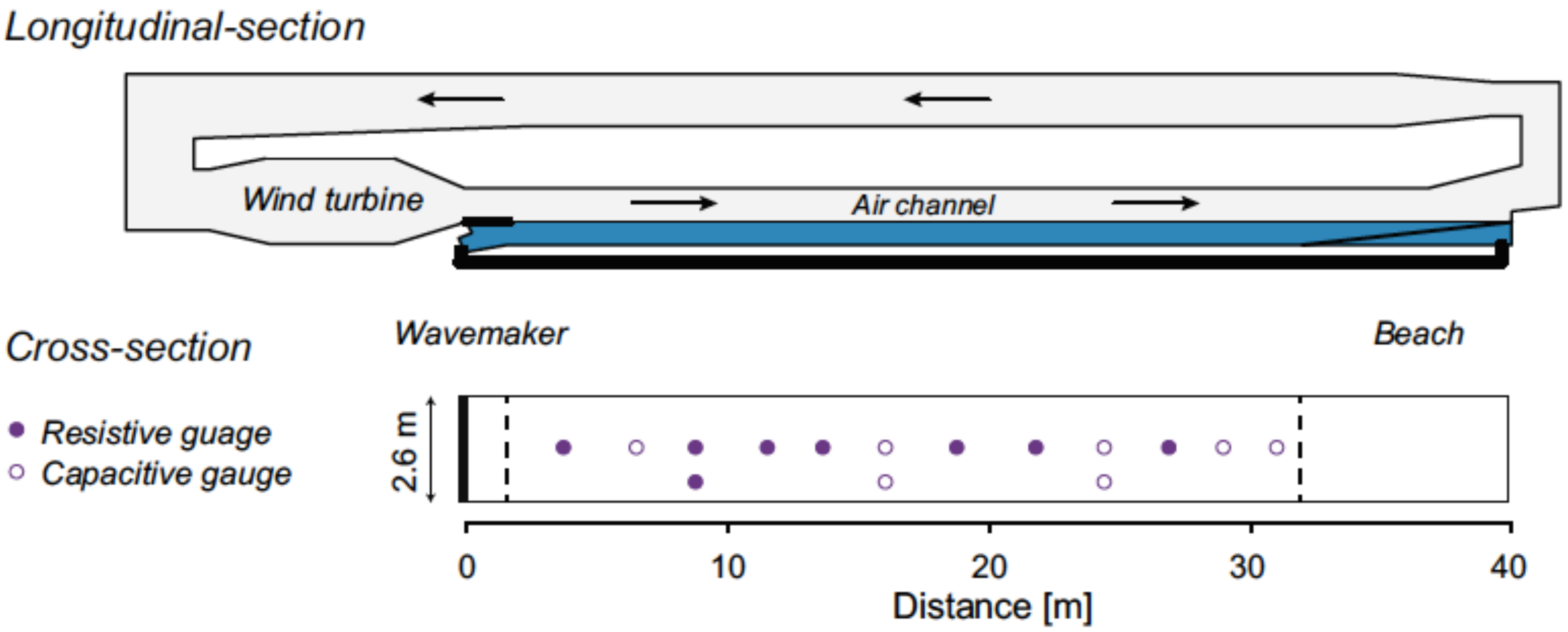




Gomel et al. PRL (2021)

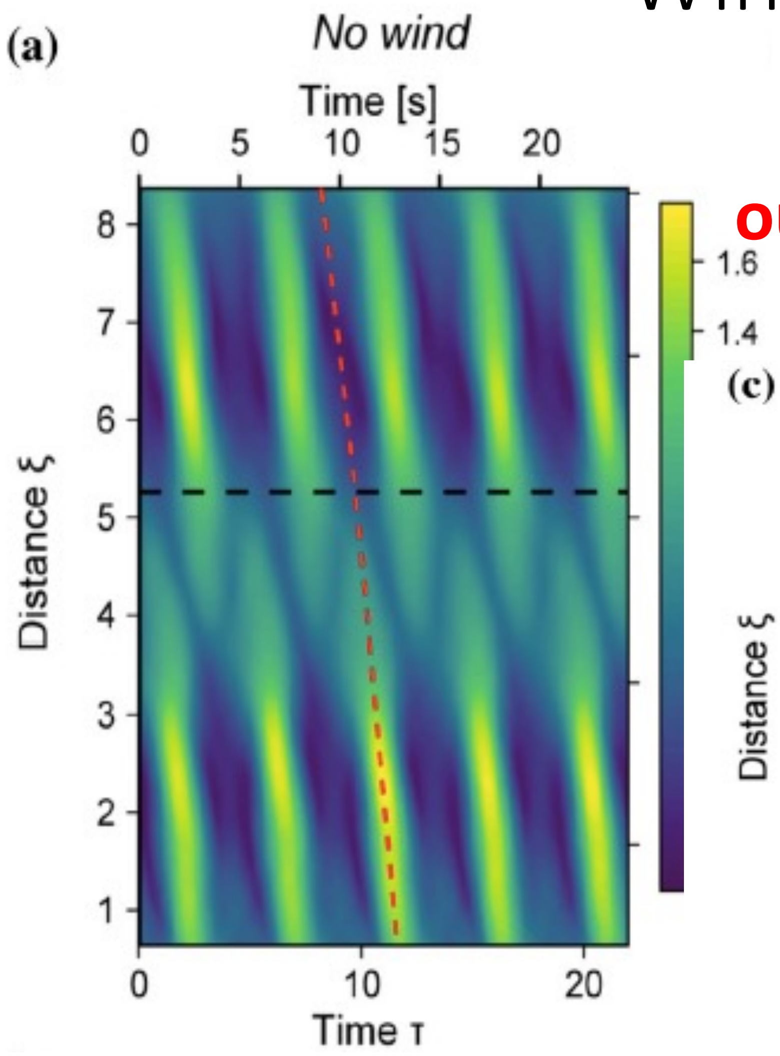
Experiments in the wind-wave facility at IRPHE/PYTHEAS (Luminy) Aix Marseille University

by Debbie Eeltink, Huber Branger & Christopher Luneau

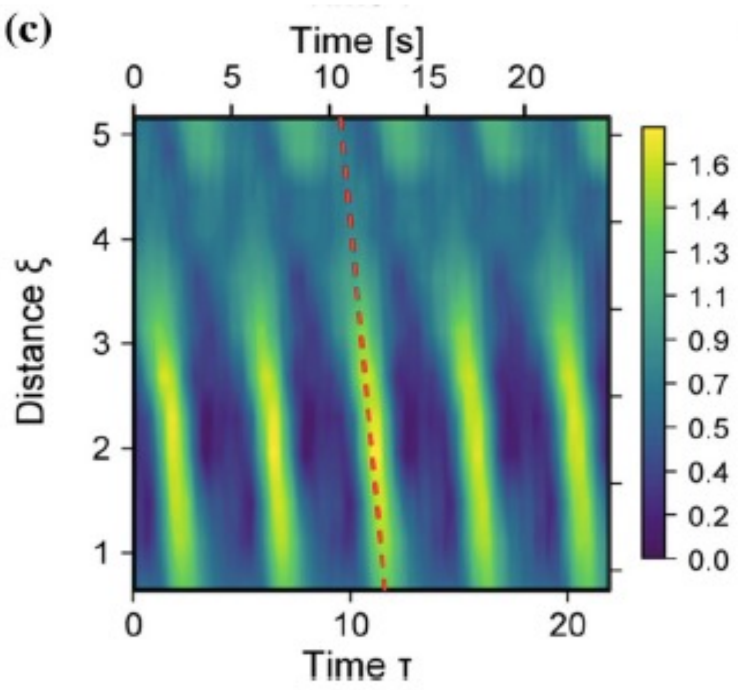


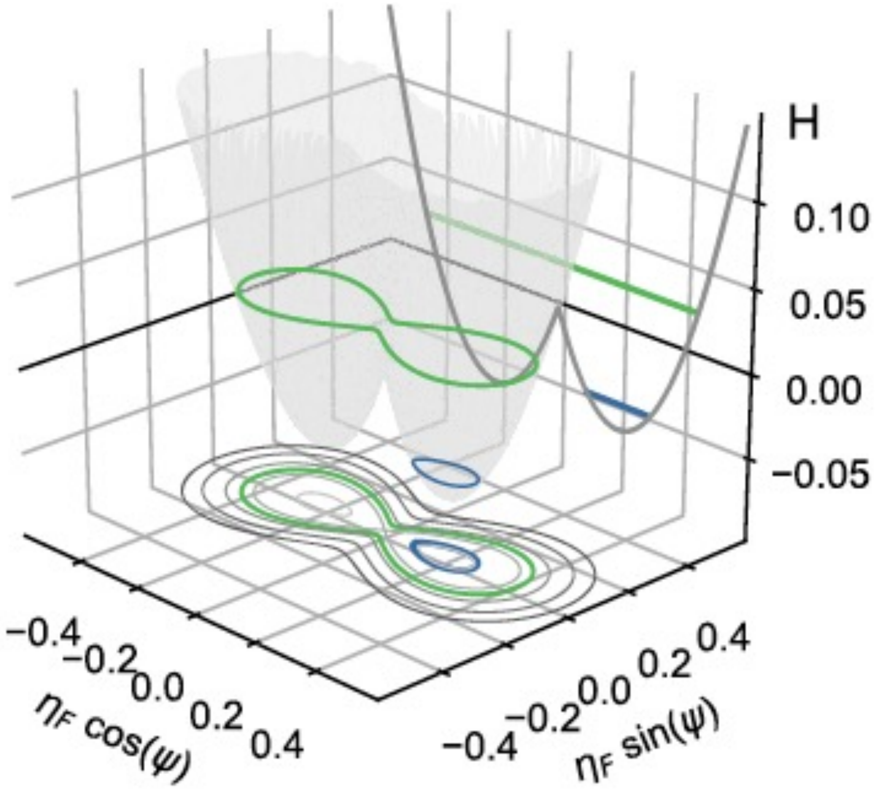
Eeltink et al. Nonlin. Dyn. (2020)

Wind effect: separatrix crossing



WIND OFF:
outside the separatrix
P2 orbit



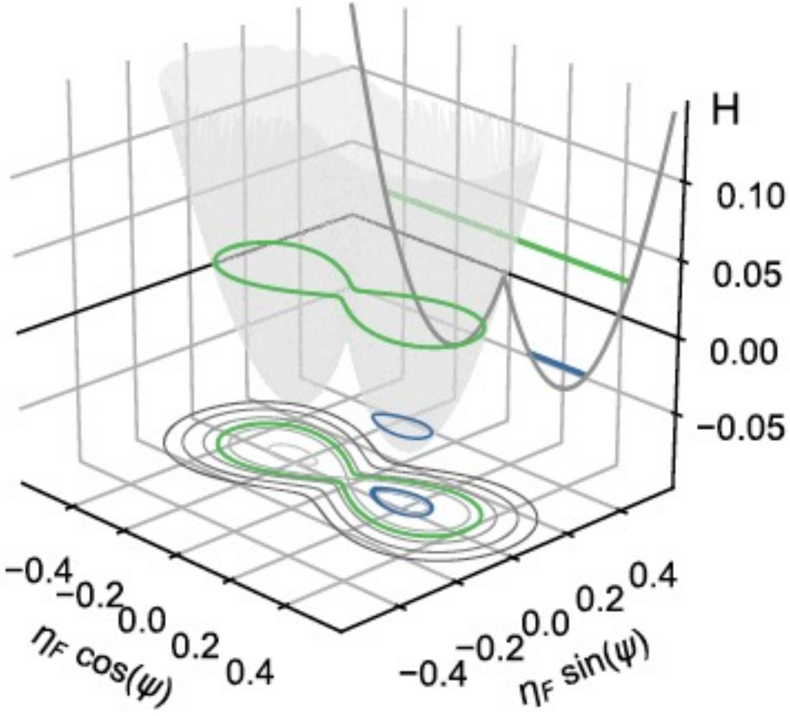


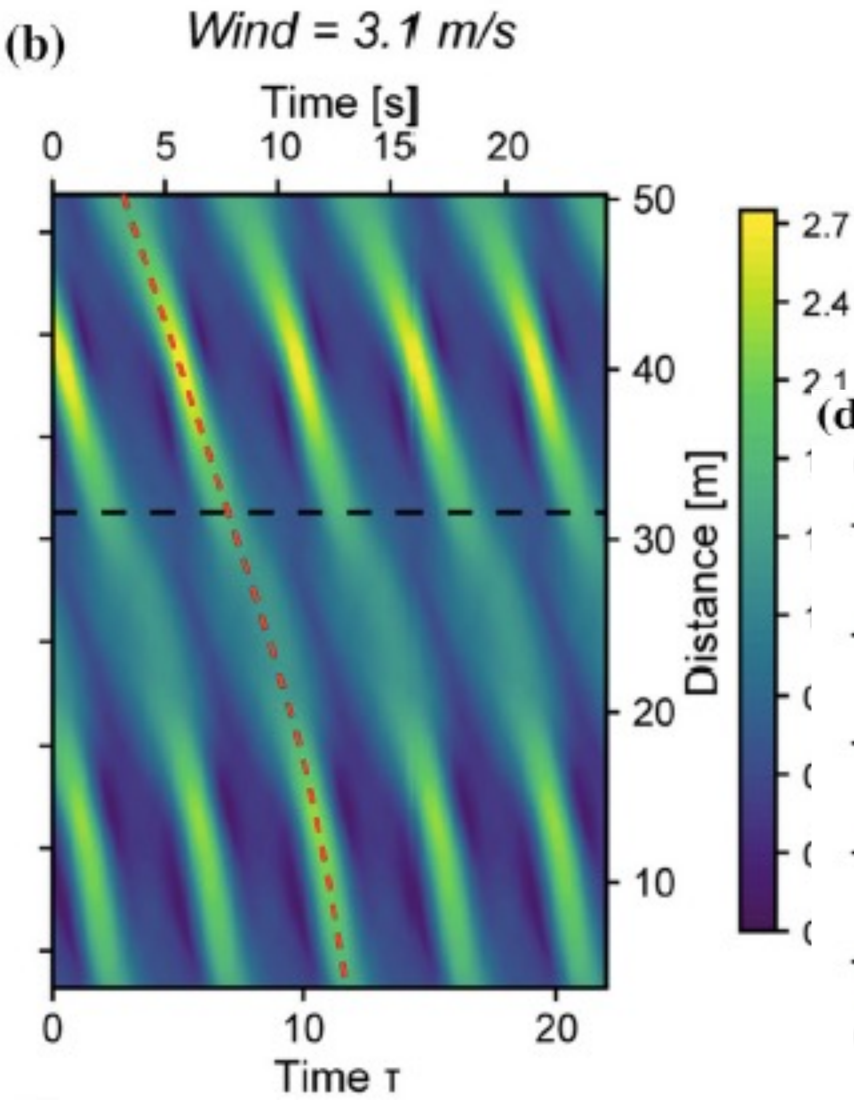
SIMULATION

EXPERIMENT

Eeltink et al. Nonlin. Dyn. (2020)

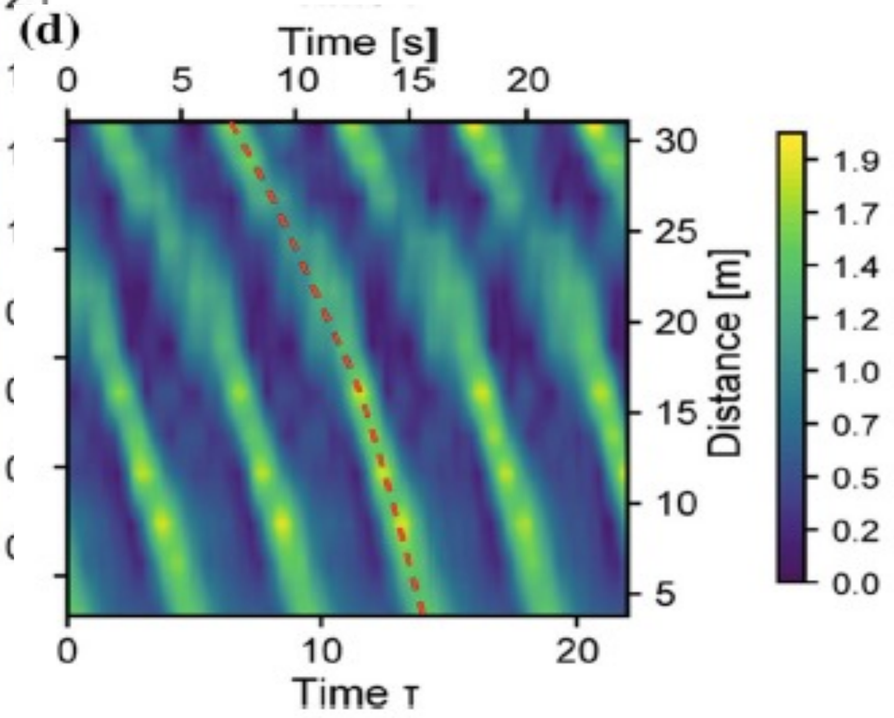
Wind effect: separatrix crossing





SIMULATION

WIND ON:
inside the separatrix
P1 orbit



EXPERIMENT

Eeltink et al. Nonlin. Dyn. (2020)

Conclusions

1. Phase space manipolation:
   bathymetry is changed to obtain a shift toward a stable state

2. The experiment is described sufficiently well by the NLS framework:
   - dissipation & high-order terms are small in one recurrence cycle
   - reflections can be controlled by choosing a mild slope
     $\Delta h \ll L_{step} \ll L_{nonlinear}$

3. Separatrix crossing: example of wind forcing

4. Main physical processes are described by NLS and three-wave approximation:
   integrable equation, construction of whole families of solutions,
   many possibilities for matching

5. The same procedure can be applied to other systems
   (for example KdV, see Binder, Vanden-Broeck & Dias, Chaos (2005))

Thanks!

Fonds national suisse

THE UNIVERSITY OF SYDNEY

Faculté des Sciences
Aix∗Marseille Université

Gomel, Chabchoub, Brunetti, Trillo, Kasparian, Armaroli
Stabilization of Unsteady Nonlinear Waves by Phase-Space Manipulation
Physical Review Letters 126, 174501 (2021)

Armaroli, Gomel, Chabchoub, Brunetti, Kasparian
Stabilization of uni-directional water wave trains over an uneven bottom
Nonlinear Dynamics 101 , 1131-1145 (2021)

Eeltink, Armaroli, Luneau, Branger, Brunetti, Kasparian
Separatrix crossing and symmetry breaking in NLSE-like systems due to forcing and damping
Nonlinear Dynamics 102, 2385-2398 (2020)