Boltzmann or Bogoliubov?
A Case of Gravitational Particle Production

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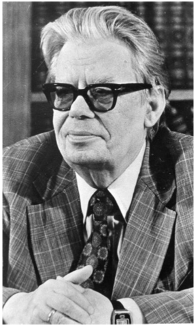
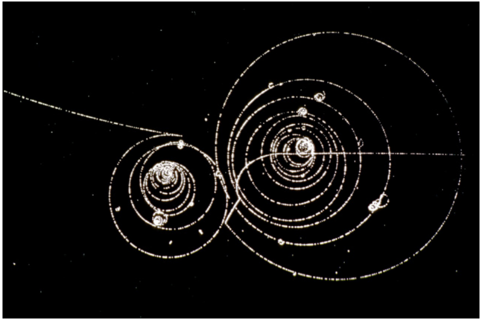


+ Sung Mook Lee (Yonsei U.) and Kin-ya Oda (TWCU) [JCAP09(2022)018]

AstroParticle Symposium 2022 at Paris-Saclay, November 2, 2022

Outline



[Image from CERN]

1. Introduction

Particle Physics and Cosmology

- Friendship of Particle Physics and Cosmology
- Big Bang Nucleosynthesis
- Nuclear reactions and cooling due to the expansion



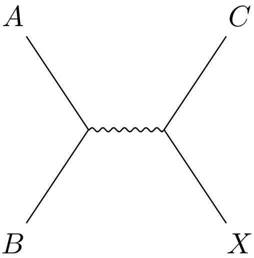
Particle Physics and Cosmology

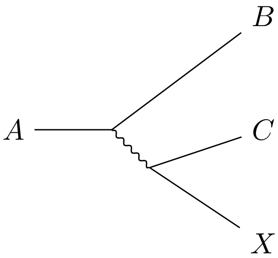
- Friendship of Particle Physics and Cosmology
- Big Bang Nucleosynthesis
- Nuclear reactions and cooling due to the expansion

But, sometimes different viewpoint

Particle Production in Particle Physics

- Based on QFT in Minkowski spacetime
- Elementary process: scattering or decay

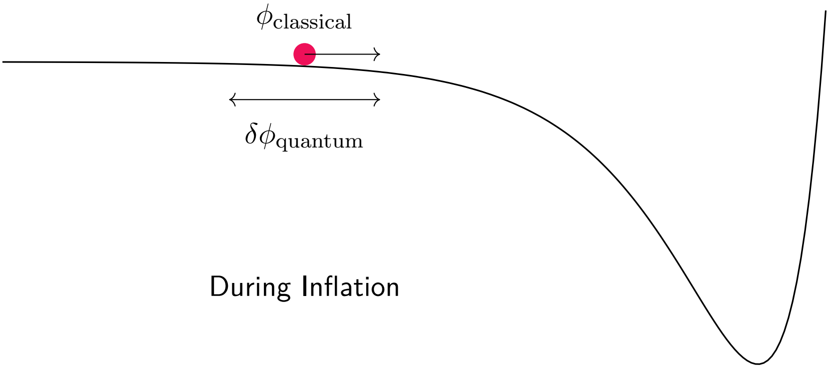




- Use the regular Feynman rules $\Rightarrow$ production/depletion rate of X
- Boltzmann equations works well: $\frac{dN_X}{dt} = (\text{production}) - (\text{depletion})$

Particle Production in Cosmology

- Based on QFT in curved spacetime ⇒ particle picture is not always available
- Elementary process: amplification of quantum fluctuations



- Frequency of $\delta\phi_{\text{quantum}}$ is not a constant of time
- Can be taken into account by Bogoliubov transformation

Example: Gravitational DM Production from Inflaton

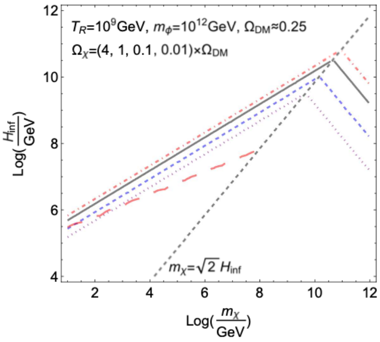
$\phi \to$ DM can efficiently produce DM via graviton exchange

- ϕ oscillates $\Rightarrow$ R oscillates

$$\begin{cases} \chi_k'' + \omega_k^2 \chi_k = 0 \\ \omega_k^2 = k^2 + a^2 m_\chi^2 + \frac{1}{6} a^2 R \end{cases}$$

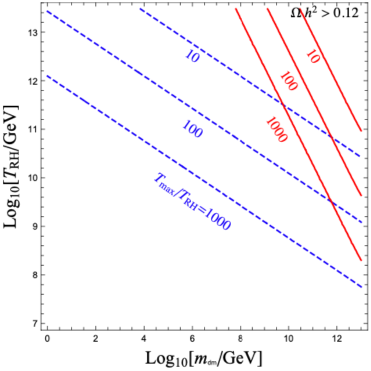
- Bogoliubov approach

[Ema, Nakayama, Tang '18]



- $\phi\phi \to$graviton$\to \chi\chi$
- Production rate $R_{\phi\to\chi} \sim \frac{\rho_\phi^2}{M_P^4}$
- Boltzmann approach

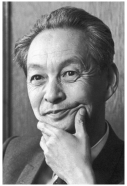
[Mambrini, Olive '21]



The Tower of Babel

Are these two approaches (Boltzmann and Bogoliubov) equivalent or not?

The Tower of Babel




On a Relativistically Invariant Formulation of the Quantum Theory of Wave Fields.*

By S. Tomonaga

(Received May 17, 1946)

Quantum Electrodynamics. I. A Covariant Formulation

Julian Schwinger
Harvard University, Cambridge, Massachusetts
(Received July 29, 1948)

Volume 20, Number 2 April, 1948

Space-Time Approach to Non-Relativistic Quantum Mechanics

R. P. Feynman
Cornell University, Ithaca, New York

The Radiation Theories of Tomonaga, Schwinger, and Feynman

F. J. Dyson
Institute for Advanced Study, Princeton, New Jersey
(Received October 6, 1948)

Are these two approaches (Boltzmann and Bogoliubov) equivalent or not?

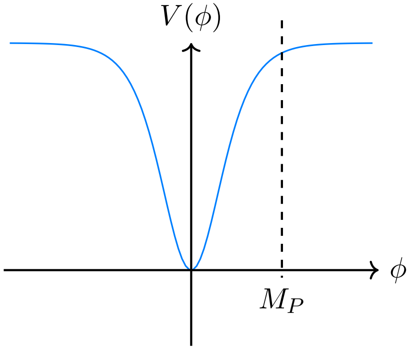
2. Boltzmann Approach

Setup

Consider the system with the Einstein gravity, inflaton (ϕ), and a real scalar (χ):

$$S = \int \sqrt{-g} d^4x \left[\mathcal{L}_{\rm EH} + \mathcal{L}_\phi + \mathcal{L}_\chi \right], \quad \begin{cases} \mathcal{L}_{\rm EH} = -\frac{M_P^2}{2} R \\ \mathcal{L}_\phi = \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \\ \mathcal{L}_\chi = \frac{1}{2} g^{\mu\nu} \partial_\mu \chi \partial_\nu \chi - \frac{1}{2} m_\chi^2 \chi^2 \end{cases}$$

Inflaton sector (example);



- T-model: $V(\phi) = 6\lambda M_P^4 \tanh^2\left(\frac{\phi}{\sqrt{6} M_P}\right)$
- CMB input: $\ln(10^{10} A_S) = 3.044$ for $k = 0.05~\mathrm{Mpc}^{-1}$

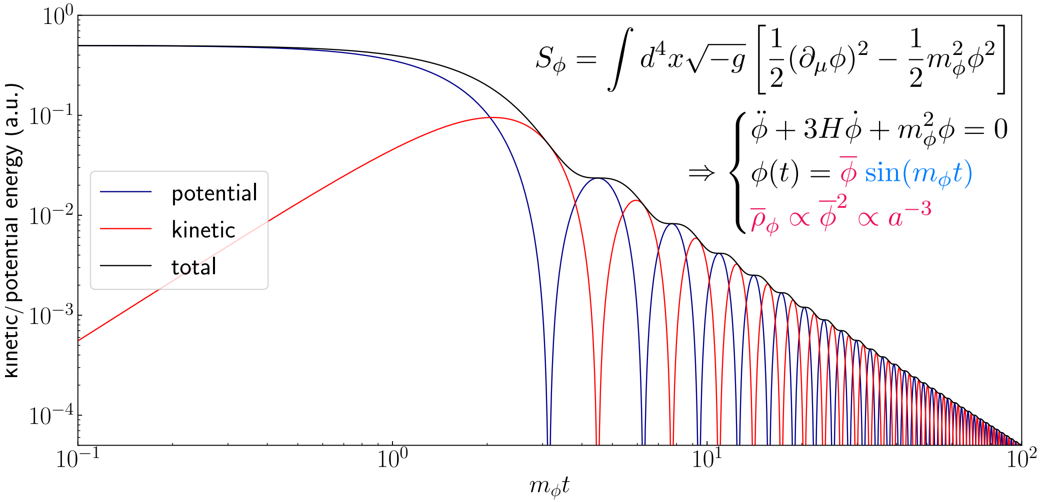
$$\lambda \simeq \frac{18\pi^2 A_S}{6N^2} \simeq 2.1 \times 10^{-11} \quad (N = 55)$$

- After the end of inflation ($\phi \ll M_P$):

$$V(\phi) \simeq \frac{1}{2} m_\phi^2 \phi^2, \quad m_\phi^2 = 2\lambda M_P^2 \simeq (10^{13}~\mathrm{GeV})^2$$

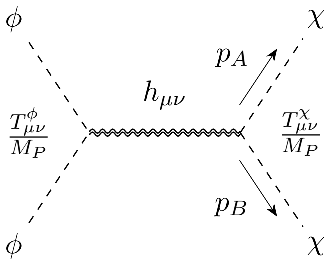
Inflaton Dynamics

Inflaton = coherent state (at leading order) ⇒ homogeneous in space



Rapid oscillation in the kinetic/potential energies is essential in gravitational production

Phase Space Distribution



- Boltzmann equation for $f_\chi(t, \vec{p}_A)$:

$$\frac{\partial f_\chi}{\partial t} - H|\vec{p}_A|\frac{\partial f_\chi}{\partial|\vec{p}_A|} = C[f_\chi]$$

- Collision term (when $\chi\chi \to \phi\phi$ is negligible):

$$C[f_\chi] = \frac{\pi}{64 m_\phi^2 \beta}\frac{\rho_\phi^2}{M_P^4}\left(1 + \frac{m_\chi^2}{2m_\phi^2}\right)^2 \delta(|\vec{p}_A| - \beta m_\phi)$$

$$\beta \equiv \sqrt{1 - \frac{m_\chi^2}{m_\phi^2}}$$

- Phase space distribution (comoving momentum $k = |\vec{p}_A|a(t)/a_e$):

$$f_\chi(t, |\vec{p}_A|) = \frac{9\pi}{64}\left(\frac{H_e}{m_\phi}\right)^3\left(\frac{m_\phi}{k}\right)^{9/2}\left(1 - \frac{m_\chi^2}{m_\phi^2}\right)^{5/4}\left(1 + \frac{m_\chi^2}{2m_\phi^2}\right)^2 \propto k^{-9/2}$$

What Was The Question Again?

Boltzmann approach

- Based on QFT in Minkowski spacetime
- Particle production is described by collision terms in the Boltzmann Eqs

Bogoliubov approach

- Based on QFT in curved spacetime
- Particle production is described by time dependence of effective frequencies

- Therefore, there seems no guarantee that both are equivalent
- $f_\chi^{\text{Boltzmann}}$ vs. $f_\chi^{\text{Bogoliubov}}$

3. Bogoliubov Approach

QFT in Curved Spacetime in a Nutshell

- Use $a d\eta = dt$ and $\widetilde{\chi} \equiv a^{-1}\chi \quad \Rightarrow \quad \begin{cases} \sqrt{-g}\mathcal{L}_\chi = \frac{1}{2}(\widetilde{\chi}')^2 - \frac{1}{2}\widetilde{\chi}\omega^2\widetilde{\chi} \\ \omega^2(t) = -\nabla^2 + a^2 m_\chi^2 + \frac{1}{6}a^2 R \end{cases}$
- Hamiltonian depends on time: $\widetilde{\chi}^{(\text{past})} \neq \widetilde{\chi}^{(\text{future})} \quad \Longleftrightarrow \quad |0^{(\text{past})}\rangle \neq |0^{(\text{future})}\rangle$
- $\widetilde{\chi}_k^{(1)} = \widetilde{\chi}_k(t_1)$ and $\widetilde{\chi}_k^{(2)} = \widetilde{\chi}_k(t_2)$ can be expaned using creation and annihilation operators:

$$\widetilde{\chi}_k^{(1)} = a_{\vec{k}} u_k^{(1)} + a^\dagger_{-\vec{k}} u_k^{(1)*}, \quad \widetilde{\chi}_k^{(2)} = b_{\vec{k}} u_k^{(2)} + b^\dagger_{-\vec{k}} u_k^{(2)*}$$

- Suppose each of $u_k^{(1)}$ and $u_k^{(2)}$ a complete set, so

$$u_k^{(2)} = \alpha_k u_k^{(1)} + \beta_k u_k^{(1)*} \quad \Leftrightarrow \quad \begin{pmatrix} b_{\vec{k}} \\ b^\dagger_{-\vec{k}} \end{pmatrix} = \begin{pmatrix} \alpha_k^* & -\beta_k^* \\ -\beta_k & \alpha_k \end{pmatrix} \begin{pmatrix} a_{\vec{k}} \\ a^\dagger_{-\vec{k}} \end{pmatrix}$$

- α_k and β_k are called Bogoliubov coefficients
- Define the initial vacuum by $a_{\vec{k}}|0^{(1)}\rangle = 0$, then $n_k^{(2)} = \langle 0^{(1)}|b^\dagger_{\vec{k}} b_{\vec{k}}|0^{(1)}\rangle = |\beta_k|^2 \neq 0$

Bogoliubov Coefficients in Perturbative Regime

- The Bogoliubov coefficients α_k and β_k follow $\begin{cases} \alpha_k' = -i\omega_k\alpha_k + \dfrac{\omega_k'}{2\omega_k}\beta_k \\ \beta_k' = i\omega_k\beta_k + \dfrac{\omega_k'}{2\omega_k}\alpha_k \end{cases}$
- If $|\beta_k|^2 \ll 1$, then $\beta_k(\eta) \simeq \int_{\eta_i}^{\eta} d\eta' \left[\frac{\omega_k'}{2\omega_k}\right] e^{-2i\int_{\eta_i}^{\eta'} d\eta''\omega_k(\eta'')}$
- η-integration with $\phi(t) = \overline{\phi}\,\sin(m_\phi t)$
- For $k \gg a\overline{H} \sim \sqrt{\Delta}$ (otherwise, breaks adiabaticity), expand the integrand in $\overline{H}/m_\phi$:

$$\beta_k \simeq \frac{1}{2}\int_\eta \left[\frac{a^3\overline{H}m_\chi^2}{k^2+a^2m_\chi^2} + \frac{3}{2}\frac{(a\overline{H})^3(m_\phi/\overline{H})}{k^2+a^2m_\chi^2}\left(1+\frac{m_\chi^2}{2m_\phi^2}\right)\sin(2m_\phi t)\right] e^{-2i\int_\eta \omega_k}$$

- Use stationary phase approximation to find

$$|\beta_{k\gg a\overline{H}}|^2 \simeq \frac{9\pi}{64}\left(\frac{H_e}{m_\phi}\right)^3\left(\frac{m_\phi}{k}\right)^{9/2}\left(1-\frac{m_\chi^2}{m_\phi^2}\right)^{5/4}\left(1+\frac{m_\chi^2}{2m_\phi^2}\right)^2$$

which is **identical** to f_χ obtained in the Boltzmann approach

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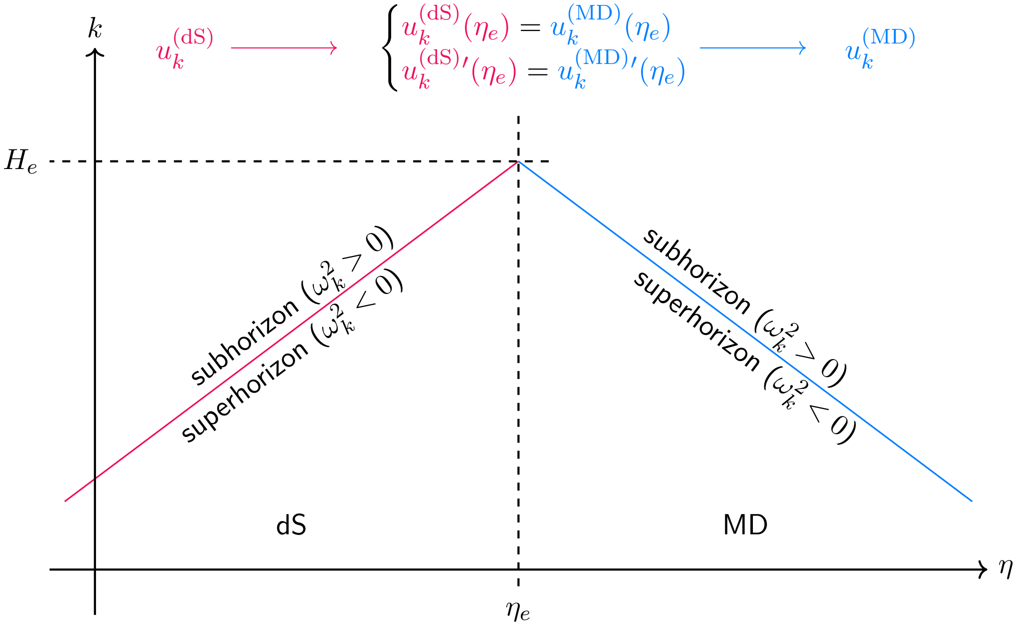
Particle Production through Phase Transition



Simplest Example: Massless Particle

- Suppose $m_\chi = 0$ and set the initial condition for u_k:

$$\begin{cases} u_k(\eta \to -\infty) = \dfrac{e^{-ik\eta}}{\sqrt{2k}} & \text{(BD vacuum)} \\ \dfrac{d^2 u_k}{dy^2} + \left(1 - \dfrac{2}{y^2}\right) u_k = 0 & (y \equiv k(\eta - \overline{\eta}_i)) \end{cases} \Rightarrow \quad u_k^{(\mathrm{dS})}(y) = \sqrt{\frac{\pi}{4k}} \sqrt{y} H_{3/2}^{(2)}(y)$$

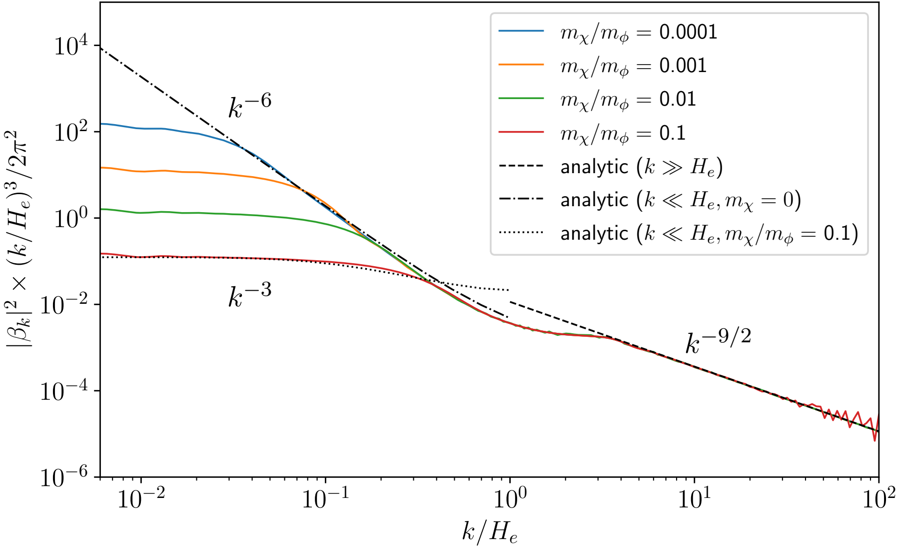
- For $y \ll 1$ ("super-horizon modes"):

$$u_k \simeq -\frac{i}{\sqrt{2k}} \frac{e^{-ik\eta}}{k\eta} \quad \Longrightarrow \quad \mathcal{P}_\chi = \frac{k^3}{2\pi^2} |a\widetilde{\chi}_k|^2 \sim \left(\frac{H_k}{2\pi}\right)^2$$

- Later time (MD) solution: $u_k^{(\mathrm{MD})}(x) = \sqrt{\frac{\pi}{4k}} \sqrt{x} H_{3/2}^{(2)}(x)$ with $x \equiv k(\eta - \overline{\eta}_e)$
- Insisting $u_k^{(\mathrm{dS})}$ and $u_k^{(\mathrm{MD})}$ are smooth at $\eta = \eta_e$ to obtain

$$\beta_k = \frac{\pi x_e}{4\sqrt{2}} \left[H_{-\frac{3}{2}}^{(2)}(y_e) H_{\frac{5}{2}}^{(2)}(x_e) - H_{\frac{3}{2}}^{(2)}(x_e) H_{-\frac{1}{2}}^{(2)}(y_e) \right] \Rightarrow |\beta_{k \ll H_e}|^2 \simeq \frac{9}{64} \left(\frac{H_e}{k}\right)^6 \propto k^{-6}$$

Quantitative Comparison



Summary

1. Boltzmann and Bogoliubov are equivalent for $k > m_\phi$
2. Bogoliubov can take care of $k < m_\phi$, while Boltzmann can not
3. Boltzmann approach: easy to compute, but only perturbative processes
4. Bogoliubov approach: hard to compute, but can deal with non-pert. processes