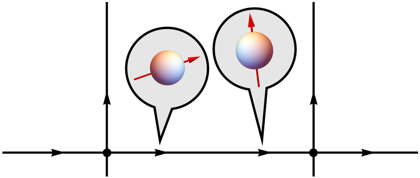
Qubitization and Compilation of Lattice Gauge Theories

Michael J. Cervia

Departments of Physics,
The George Washington University &
University of Maryland, College Park

Tuesday, November 22, 2022



Roadmap

- **Qubitization schemes**
  - Introduce and study proposals for Hamiltonian formulations of gauge theories
  - Learn some lessons about qubitization strategies
- **Quantum Compilation**
  - Minimize depth of quantum circuits simulating given Hamiltonians to study gauge theories
  - Compare efficiency of different models of $SU(2)$ gauge theory
- Considerations formulating gauge theories on quantum hardware

Articles

Basis for this talk:

- Edison M. Murairi, Michael J. Cervia, Hersh Kumar, Paulo F. Bedaque, and Andrei Alexandru
  "How many quantum gates do gauge theories require?"
  PRD **106**, 094504 (2022). arXiv:2208.11789

- Andrei Alexandru, Paulo F. Bedaque, Andrea Carosso, Michael J. Cervia, and Andy Sheng
  "Qubitization strategies for bosonic field theories," (2022). arXiv:2209.00098

- Michael J. Cervia & Edison M. Murairi
  manuscript(s) in preparation

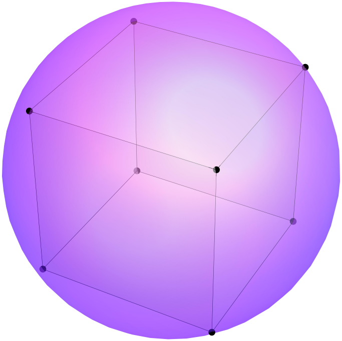
- Lattice: spatial volume $\mathbb{R}^d \to (a\mathbb{Z}_L)^d$ ("domain")
- Bosonic field's Hilbert space $\mathcal{H} \to \mathcal{H}_{\text{trunc}}$ ("target")

$\Big\}$ *Qubitization*

- Generically, need $\mathcal{H}_{\text{trunc}} \to \mathcal{H}$ as well as $L \to \infty$ & $a \to 0$. Not just inconvenient, but...

Each dim of $\mathcal{H}_{\text{trunc}}$ may be costly!

More on this later...



A Fuzzy $SU(2)$?

Motivating Truncations of the gauge group

- Standard gauge theory: U is a 2×2 unitary matrix

$$\left.\begin{aligned} H &= \sum_x \left[g^2 K(x) - \frac{1}{g^2} \sum_{\mu>\nu} W_{\mu\nu}(x) \right] \\ W_{\mu\nu} &= \mathrm{tr}\left[U(x,\mu) U(x+\hat{\mu},\nu) U(x+\hat{\nu},\mu)^\dagger U(x,\nu)^\dagger \right] \end{aligned}\right\} \text{Kogut-Susskind}$$

- *Promote* $U_{ab} \in \mathbb{C}$ to an operator on an infinite local Hilbert space:

$$U = g_4 \mathbb{1} + \mathrm{i} g_k \sigma_k \quad \longleftrightarrow \quad U = \mathbb{1} \otimes \Gamma_4 + \mathrm{i}\sigma_k \otimes \Gamma_k$$

- Left/right gauge group transformation symmetries:

$$\begin{aligned} U &\mapsto L U R^\dagger \\ L &= e^{\mathrm{i}\alpha_k \sigma_k/2}, \\ R &= e^{\mathrm{i}\beta_k \sigma_k/2} \end{aligned} \qquad \Longleftrightarrow \qquad \begin{aligned} U &\mapsto \mathcal{R}\mathcal{L} U \mathcal{L}^\dagger \mathcal{R}^\dagger \\ \mathcal{L} &= e^{\mathrm{i}\alpha_k J_k^L/2}, \\ \mathcal{R} &= e^{\mathrm{i}\beta_k J_k^R/2} \end{aligned}$$

Qubitizing $SU(2)$

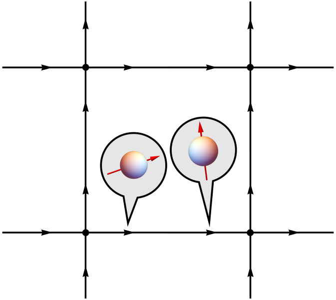
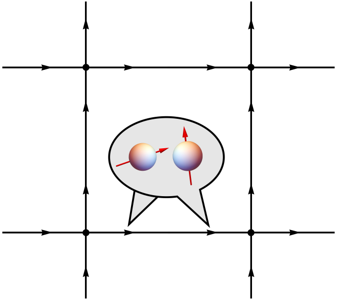
Options from $\mathrm{SO}(5)$ irreps

Algebraic rules for qubitizing $SU(2)$ LGT:

- Promote $U = \mathbb{1} \otimes \Gamma_4 + \mathrm{i}\sigma_k \otimes \Gamma_k$
- Left & right $SU(2)$ generators $J_k^{L,R}$

$\Big\}$ $\mathrm{SO}(5)$ *algebra*

Crossroads: which irrep of $\mathrm{SO}(5)$ do we take?

$N = 4$ (spinor)	$N = 5$ (fundamental)
$[(\mathbf{1} \otimes \mathbf{2}) \oplus (\mathbf{2} \otimes \mathbf{1})]$	$[\mathbf{1} \oplus (\mathbf{2} \otimes \bar{\mathbf{2}})]$
	
(Orland & Rohrlich, 1990)	(Horn, 1981)

Goal: Break global time evolution operator $U(t) = \exp(-\mathrm{i}HT)$ into elementary quantum gates, minimizing noisy gates

1. Trotterization $e^{-\mathrm{i}HT} \approx \left(\prod_k e^{-\mathrm{i}h_k \delta t}\right)^{T/\delta t}$
2. Decomposition into Pauli strings $H = \sum_j c_j P_j$
3. Collection of commuting Pauli strings $[P, Q]_{\mp} = 0?$
4. Diagonalizing commuting strings $\{P_1', P_2', \ldots\} \subseteq \{I, Z\}^{\otimes n}$
5. Exponentiating Pauli strings $\exp(\mathrm{i}\phi P')$

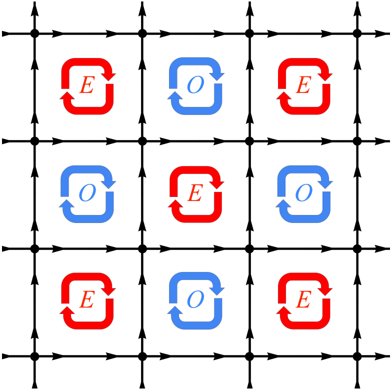
Guiding philosophy: make the terms in the Hamiltonian "share" as many CNOTs as possible!

Trotter-Suzuki Methods

Applications to LGT, in three stages

1. Break long-time evolution into many shorter ones $T \mapsto N_t \delta t$
2. Bipartition plaquettes into "Even" and "Odd" commuting sets

$$e^{-\mathrm{i}Ht} = \left(\prod_E e^{-\mathrm{i}H_E t}\right)\left(\prod_O e^{-\mathrm{i}H_O t}\right) + O\big(\|[H_E, H_O]\| t^2\big)$$



All evens (odds) run in parallel with no Trotter error

$\Longrightarrow$ Problem reduces to compiling a circuit for one plaquette

3. Need to trotterize each plaquette itself...

Trotterizing the Plaquette

Sharing CNOTs between commuting terms

Remaining Problem: Still need to decompose the single plaquette evolution operator $\exp(-\mathrm{i}W)$

$$W = \sum_{i,j,k,l} c_{ijkl} \Gamma_i \otimes \Gamma_j \otimes \Gamma_k \otimes \Gamma_l$$

1. Embed Γ in $2^n \times 2^n$ matrix (n qubits/link)
2. Write $\Gamma = \sum c_j P_j$ with "Pauli strings" P_j from $\{I, X, Y, Z\}^{\otimes n}$
3. Gather the commuting P_j, *diagonalize together*: graph theory (color graph: vertices $V \equiv \{P_i\}$, edges $E \equiv \{[P_i, P_j]_+ = 0\}$)
4. Clifford group theory: $O(n^2)$ CNOTs to diagonalize each cluster

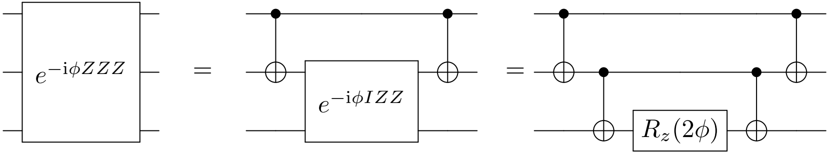
$\implies$ Just need to find the circuit for a *diagonal* unitary now...

Exponentiating a Pauli String

The Final Step, Naïvely

Remaining Problem: Still need to exponentiate a collection diagonal Pauli strings $P_j \in \{I, Z\}^{\otimes 4n}$

- Easy for one Pauli string: e.g., ZZZ



- Problem: We want to make the strings “share” CNOTs

Exponentiating Many Pauli Strings

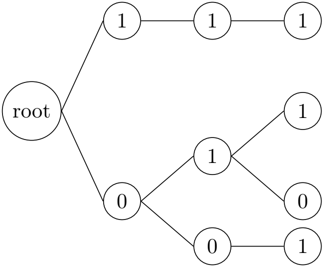
The Final Step: Tree method

Goal: Make Pauli strings share CNOTs on the circuit when possible

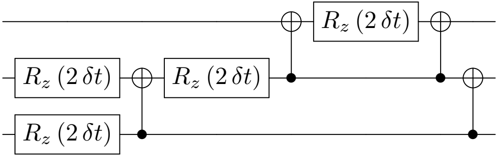
- Example: $H = IIZ + IZI + IZZ + ZZZ$

  $$001, \quad 010, \quad 011, \quad 111$$

- Traverse a tree representing H, depth-first search (DFS)



$\Longrightarrow$



- *Formalism reveals linear lower bound on CNOT cost per string*

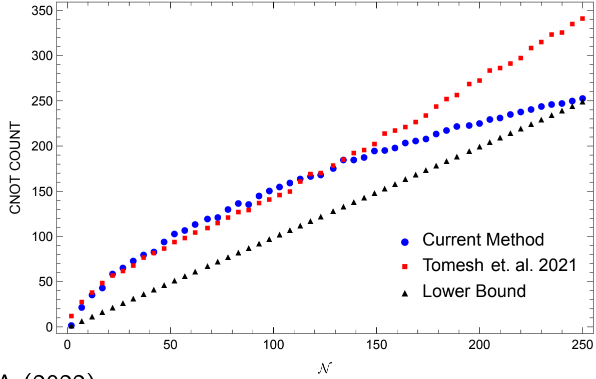
Results for Basic Examples

Proof of Concept

How does our CNOT cost fare?

- Optimal for a general, *diagonal* $2^n \times 2^n$ unitary: $2^n - 2$ CNOTs
- Competitive with state-of-the-art on various random, diagonal Hamiltonians of $\mathcal{N}$ Pauli strings:

e.g., $n = 8$ qubits $\Longrightarrow$



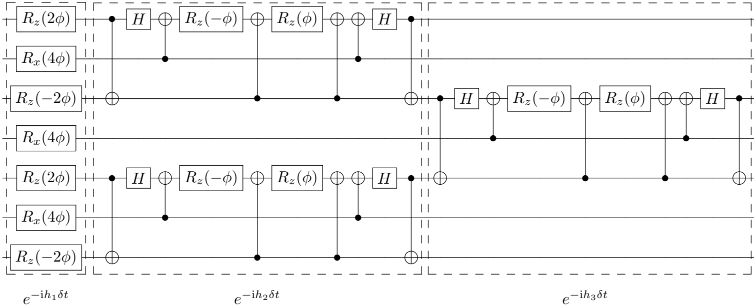
EMM, **MJC**, HK, PFB, AA (2022)

Resources for $\mathbb{Z}_2$ theory

A circuit to simulate $\mathbb{Z}_2$ theory with fermions

Consider a Hamiltonian for $\mathbb{Z}_2$ theory with staggered fermions

- E.g., 4-site chain with open boundary
- Existing methods: 8 qubits, 36 CNOTs
- Our circuit: 7 qubits, 18 CNOTs



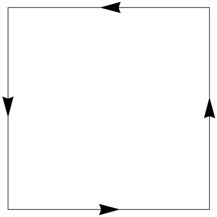
EMM, **MJC**, HK, PFB, AA (2022)

Resource Estimates for Truncated Models

CNOT costs for qubitized $SU(2)$ models' time step

Resources to simulate one time step of $SU(2)$ plaquette

- Spinor rep: 2 qubits/link
  $\implies$ 64 Pauli strings
  $\implies$ 180 CNOTs
- Fundamental rep: 3 qubits/link
  $\implies$ $64 \times 4^4 = 16,384$ strings
  $\implies$ 16,384+128 CNOTs



Adding *one* qubit for a link
$\implies$ order(s) of magnitude cost!!!

Summary of compilation

What can our quantum circuits tell us?

- Assembly of state-of-the-art circuit compilation methods, with our own augmentations
- *Automatic* procedure — stay tuned for code *you* can use!
- Foundation for resource estimates of simulations
- Cautionary tale about the importance of *small* qubitization schemes!

Future Work

Strategies to make LGT simulations viable in NISQ era

- Algorithms exploiting particular initial states $|\Psi(t=0)\rangle$
- Derive Hamiltonian models that already incorporate gauge constraints (i.e., Gauss's Law)
- Determine which qubitizations are in the correct universality class
  - Valid truncations in reps of the symmetry group $\mathcal{G}$
  - Models derived from subgroups $\mathcal{S} < \mathcal{G}$

- THANK YOU -

Gauss's Law

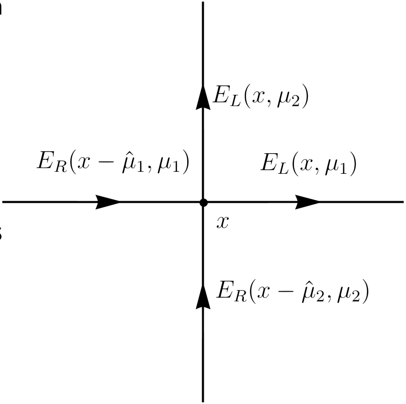
Non-local gauge constraint

Gauss's law op defined at each vertex of a square lattice, $x \in \Lambda$:

$$G^i(x) \equiv \sum_{\mu=1}^{d} E_L^i(x,\mu) + E_R^i(x-\hat{\mu},\mu)$$

is conserved. $E^i_{L/R}$ are "left/right" copies of group generators on a D-truncated gauge link V;

$$[E_L^i, V] = -\frac{1}{2}(\sigma_i \otimes 1_{D\times D})V$$
$$[E_R^i, V] = +\frac{1}{2}V(\sigma_i \otimes 1_{D\times D}).$$



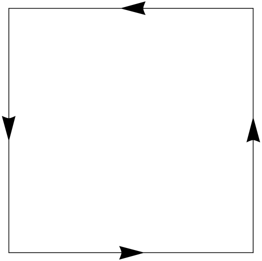
$\boxed{\text{The Physical Hilbert space is smaller! i.e., } G^i(x)\,|\text{phys}\rangle = 0}$

Trotterizing the Plaquette

Sharing CNOTs between commuting terms

Remaining Problem: Still need to decompose the single plaquette evolution operator $\exp(-\mathrm{i}W)$

$$W = \sum_{i,j,k,l=1}^{4} c_{ijkl}\Gamma_i \otimes \Gamma_j \otimes \Gamma_k \otimes \Gamma_l$$



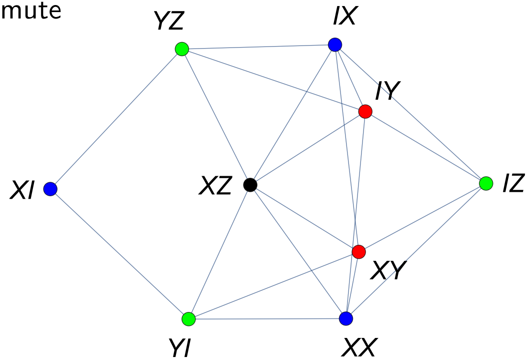
1. Embed Γ in $2^n \times 2^n$ matrix (n qubits/link)
2. Write $\Gamma = \sum c_j P_j$ with "Pauli strings" $P_j \in \{I, X, Y, Z\}^{\otimes n}$

N.B.: Qubitized Hamiltonian is $4n$-local...

Problem: $H = \sum_{j=1}^{N} c_j P_j$. Gather commuting P_j, *diagonalize together*

Solution: Graph $G = (V, E)$ coloring

- Vertices $V \equiv \{P_i\}$
- Edges $E \equiv \{[P_i, P_j]_+ = 0\}$
- Same color $\implies$ commute

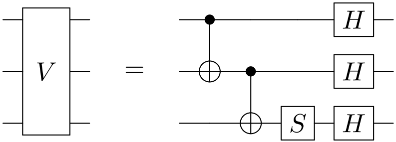


Diagonalizing Pauli strings together

Diagonalizing w/ CNOTs together, Part I

- Need unitary matrices diagonalizing $\{P_j\} \iff$ *Clifford* gates
- Generated by Hadamard (H), phase (S, $S^\dagger$), and CNOT gates

e.g., $H = IXX + ZYZ + XXI \to V\,H\,V^\dagger = IZI - IZZ + ZII$



$\implies$ Induce sharing of CNOTs by simultaneously diagonalizing

Clifford group theory: $O(n^2)$ CNOTs to diagonalize each cluster