Rodeo algorithm with controlled reversal gates

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Outline

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Controlled reversal gates

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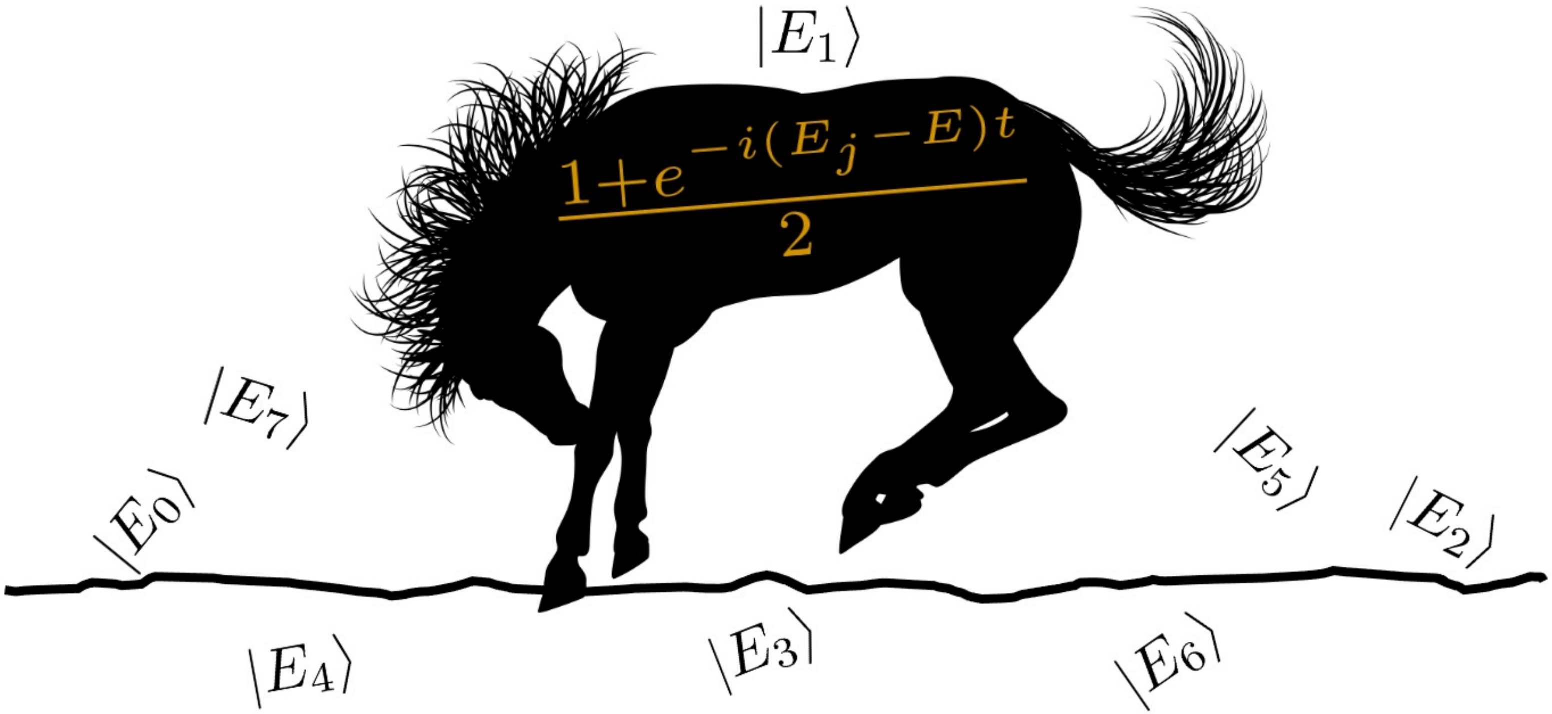
Towards the future

Summary

Rodeo algorithm



Choi, D.L., Bonitati, Qian, Watkins, PRL 127, 040505 (2021)

Consider a single qubit and a Hadamard gate

$$U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} = U^{\dagger} = U^{-1}$$

Consider another unitary operation that is a diagonal phase rotation

$$R(E_{\mathrm{obj}}, E, t) = \begin{bmatrix} 1 & 0 \\ 0 & e^{-it(E_{\mathrm{obj}} - E)} \end{bmatrix}$$

We then have

$$U^{\dagger} R(E_{\mathrm{obj}}, E, t) U = \begin{bmatrix} \frac{1}{2} + \frac{1}{2} e^{-it(E_{\mathrm{obj}} - E)} & \frac{1}{2} - \frac{1}{2} e^{-it(E_{\mathrm{obj}} - E)} \\ \frac{1}{2} - \frac{1}{2} e^{-it(E_{\mathrm{obj}} - E)} & \frac{1}{2} + \frac{1}{2} e^{-it(E_{\mathrm{obj}} - E)} \end{bmatrix}$$

Let us now start in the $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ state and perform these unitary operations

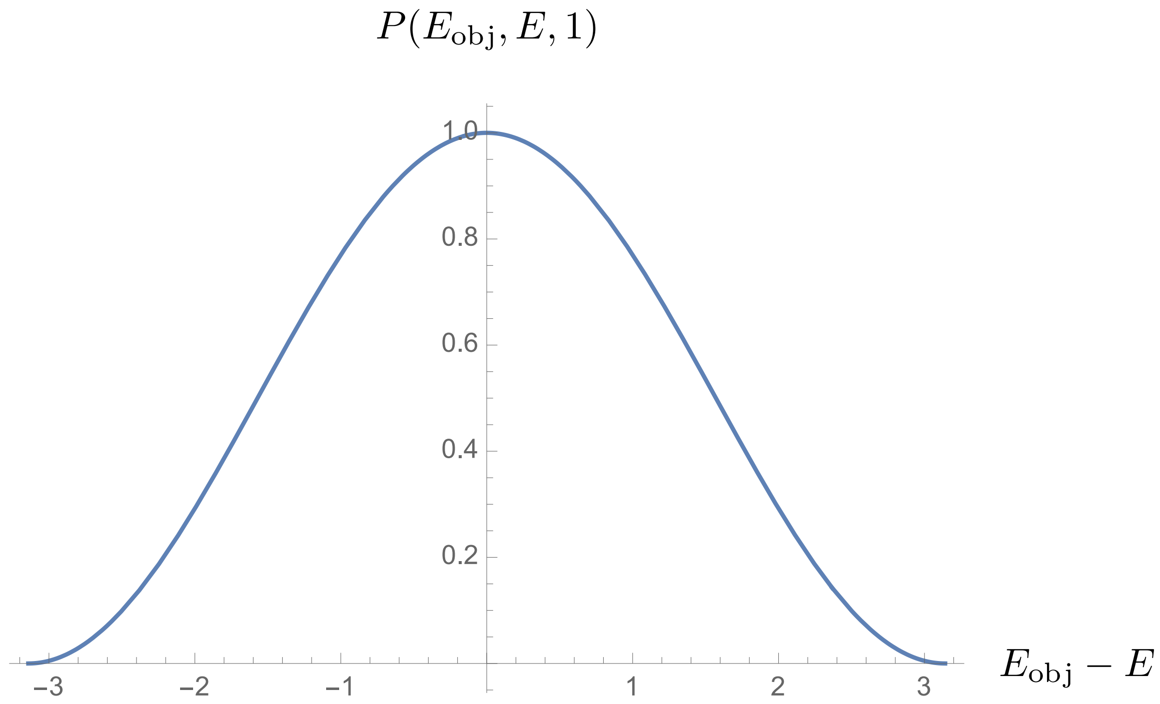
$$U^\dagger R(E_{\mathrm{obj}}, E, t) U \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} - \frac{1}{2} e^{-it(E_{\mathrm{obj}} - E)} \\ \frac{1}{2} + \frac{1}{2} e^{-it(E_{\mathrm{obj}} - E)} \end{bmatrix}$$

and then project back to the $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ state

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} U^\dagger R(E_{\mathrm{obj}}, E, t) U \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{2} + \frac{1}{2} e^{-it(E_{\mathrm{obj}} - E)} \end{bmatrix}$$

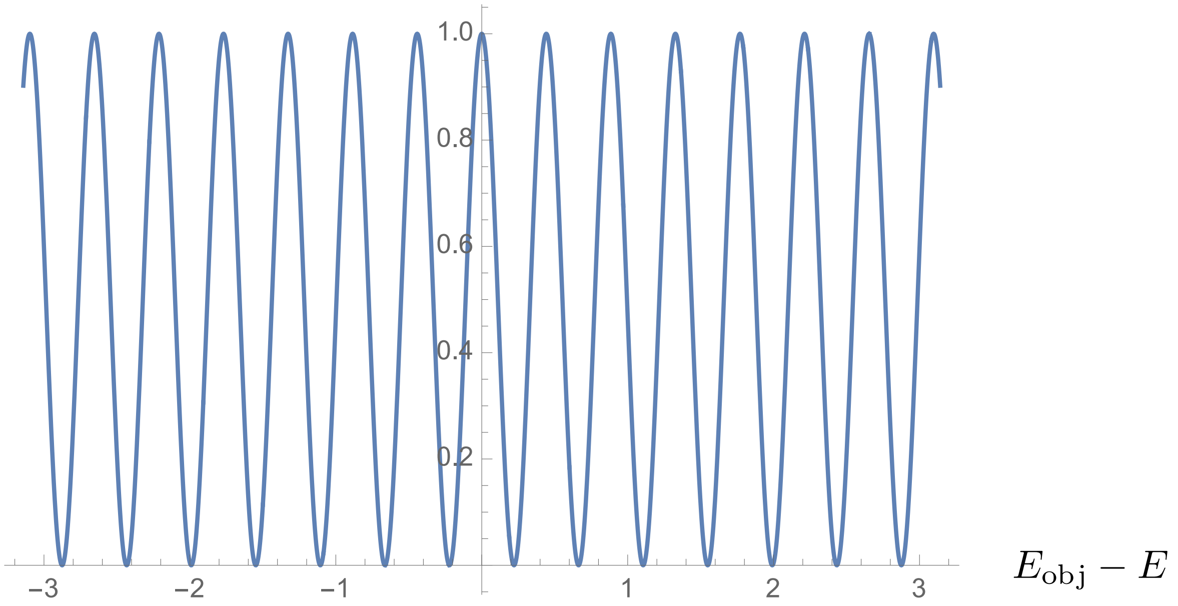
This projection is done via quantum measurement and the success probability is

$$P(E_{\mathrm{obj}}, E, t) = \left| \frac{1}{2} + \frac{1}{2} e^{-it(E_{\mathrm{obj}} - E)} \right|^2 = \cos^2 \left[\frac{t(E_{\mathrm{obj}} - E)}{2} \right]$$

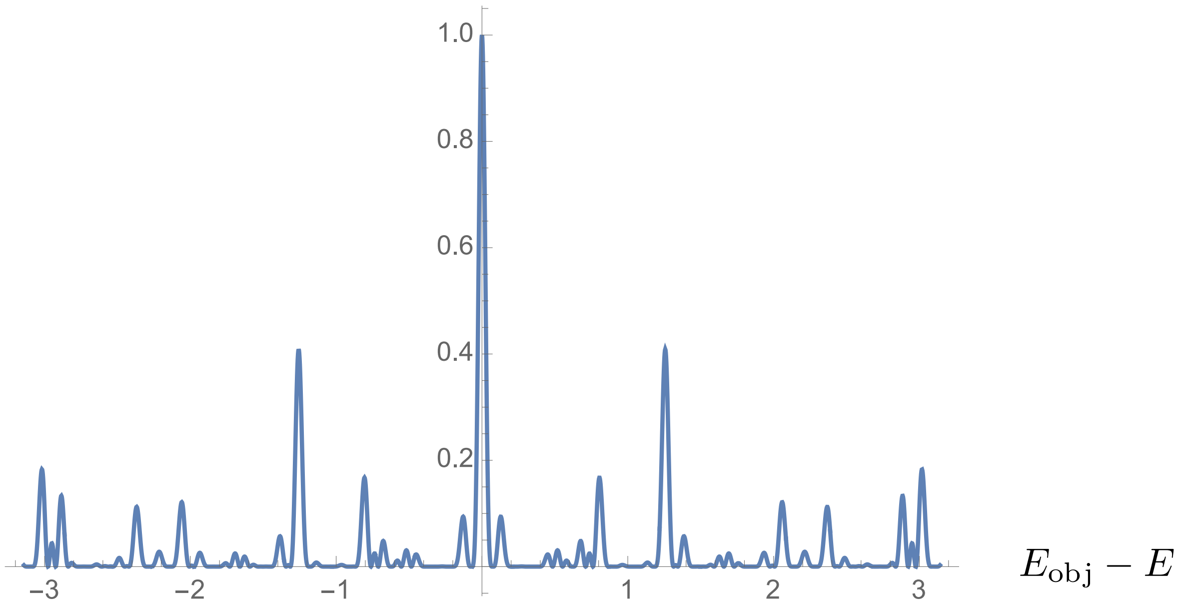


$P(E_{\mathrm{obj}}, E, 1)$
1.0
0.8
0.6
0.4
0.2
−3
−2
−1
1
2
3
$E_{\mathrm{obj}} - E$

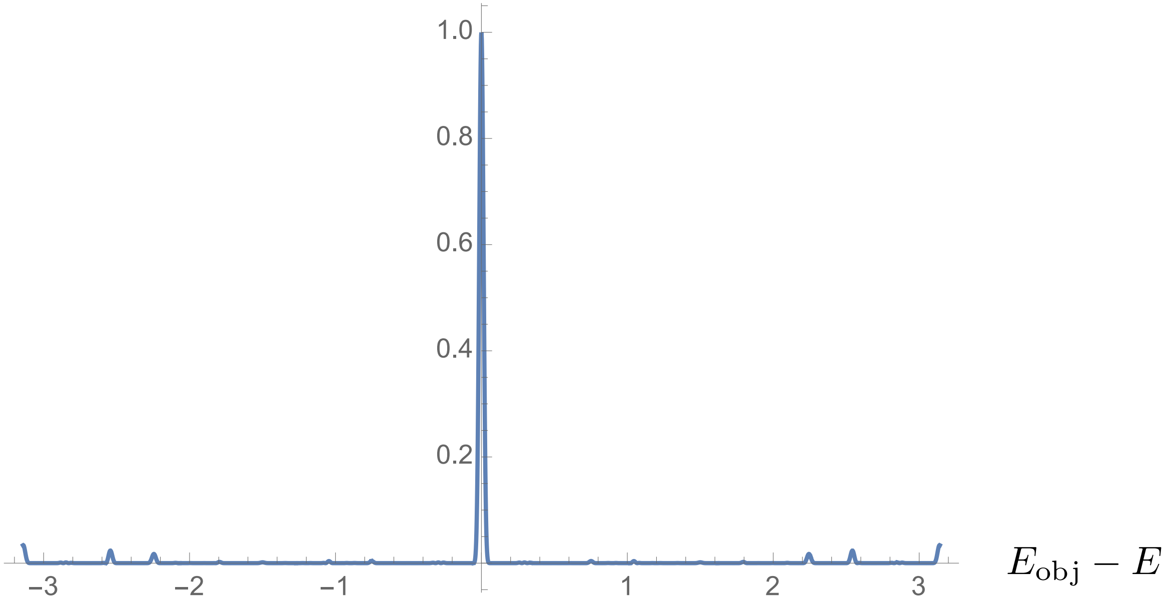
$$P(E_{\mathrm{obj}}, E, 14.2023)$$



$$\prod_{k=1}^{5} P(E_{\mathrm{obj}}, E, t_k) \qquad |t_k| < 50$$



$$\prod_{k=1}^{10} P(E_{\mathrm{obj}}, E, t_k) \qquad |t_k| < 50$$



Let us couple this qubit, which we call the “arena” or “ancilla” qubit, to another system that we call the “object”. We also promote the 2 x 2 matrices to become 2 x 2 matrices of operators acting on the object.

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \to \begin{bmatrix} \frac{\hat{I}}{\sqrt{2}} & \frac{\hat{I}}{\sqrt{2}} \\ \frac{\hat{I}}{\sqrt{2}} & -\frac{\hat{I}}{\sqrt{2}} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & e^{-it(E_{\mathrm{obj}}-E)} \end{bmatrix} \to \begin{bmatrix} \hat{I} & 0 \\ 0 & e^{-it(\hat{H}_{\mathrm{obj}}-E)} \end{bmatrix}$$

We then consider the same combination

$$\begin{bmatrix} \frac{\hat{I}}{\sqrt{2}} & \frac{\hat{I}}{\sqrt{2}} \\ \frac{\hat{I}}{\sqrt{2}} & -\frac{\hat{I}}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \hat{I} & 0 \\ 0 & e^{-it(\hat{H}_{\mathrm{obj}}-E)} \end{bmatrix} \begin{bmatrix} \frac{\hat{I}}{\sqrt{2}} & \frac{\hat{I}}{\sqrt{2}} \\ \frac{\hat{I}}{\sqrt{2}} & -\frac{\hat{I}}{\sqrt{2}} \end{bmatrix}$$

We start from the state $\begin{bmatrix} 0 \\ |\psi_{\text{init}}\rangle \end{bmatrix}$ and we perform the operations and then measure if the arena qubit is in the $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ state

$$\begin{bmatrix} 0 & 0 \\ 0 & \hat{I} \end{bmatrix} \begin{bmatrix} \frac{\hat{I}}{\sqrt{2}} & \frac{\hat{I}}{\sqrt{2}} \\ \frac{\hat{I}}{\sqrt{2}} & -\frac{\hat{I}}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \hat{I} & 0 \\ 0 & e^{-it(\hat{H}_{\text{obj}}-E)} \end{bmatrix} \begin{bmatrix} \frac{\hat{I}}{\sqrt{2}} & \frac{\hat{I}}{\sqrt{2}} \\ \frac{\hat{I}}{\sqrt{2}} & -\frac{\hat{I}}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 \\ |\psi_{\text{init}}\rangle \end{bmatrix} = \begin{bmatrix} 0 \\ \left[\frac{1}{2} + \frac{1}{2} e^{-it(\hat{H}_{\text{obj}}-E)}\right] |\psi_{\text{init}}\rangle \end{bmatrix}$$

By repeated successful measurements with random values of t, we reduce the spectral weight of eigenvectors with energies that do not match E.

The convergence is exponential. For N cycles of the rodeo algorithm, the suppression factor for undesired energy states is $1/4^N$.

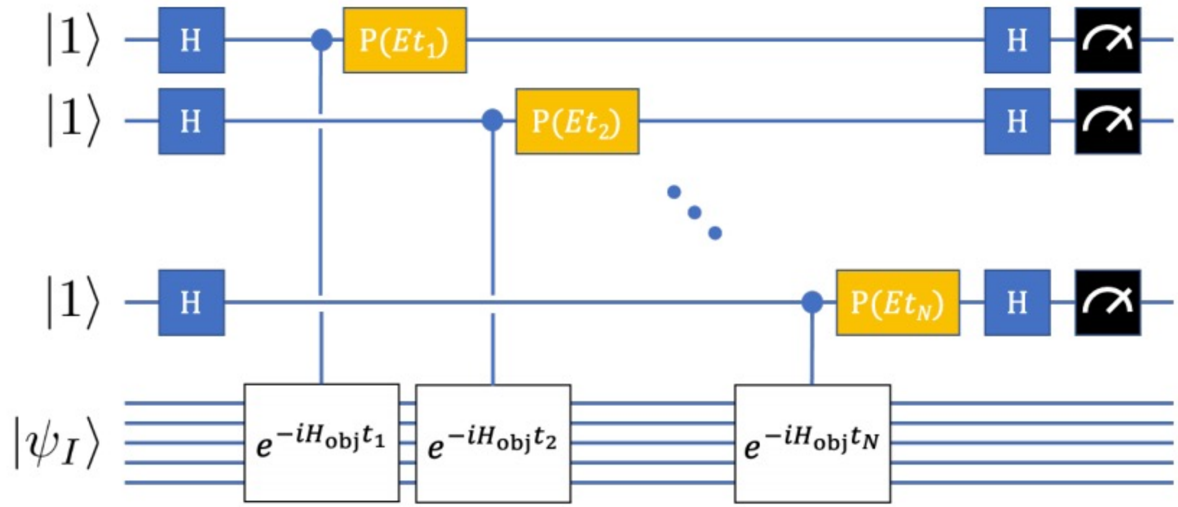


FIG. 1. (color online) **Circuit diagram for the rodeo algorithm.** The object system starts in an arbitrary state $|\psi_I\rangle$. Each of the arena qubits are initialized in the state $|1\rangle$ and operated on by a Hadamard gate H. We use each arena qubit $n = 1, \cdots, N$ for the controlled time evolution of the object Hamiltonian, H_{obj}, for time t_n. This is followed by a phase rotation $\mathrm{P}(Et_n)$ on arena qubit n, another Hadamard gate H, and then measurement.

Choi, D.L., Bonitati, Qian, Watkins, PRL 127, 040505 (2021)

Initial-state spectral function and state preparation. The example shown below is for a 1D Heisenberg chain with ten sites, antiferromagnetic interactions, and uniform magnetic field.

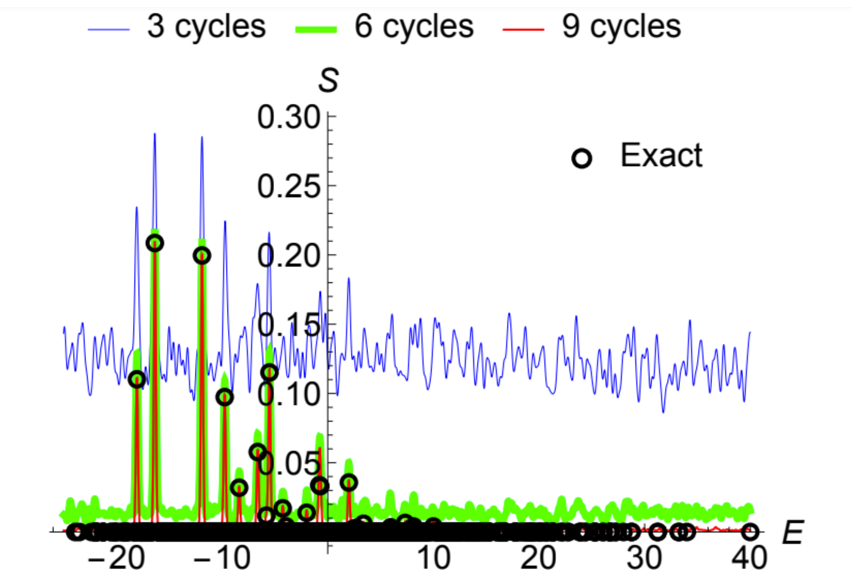


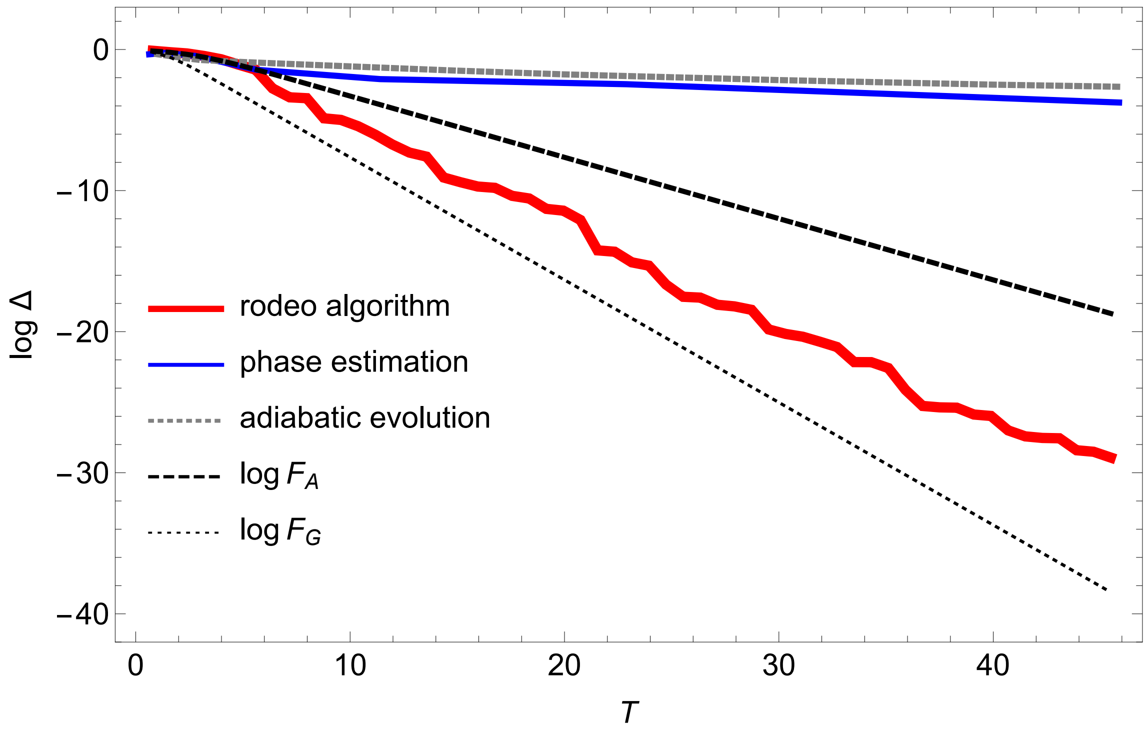
FIG. 4. (color online) **Initial-state spectral function for the Heisenberg model.** We plot the initial-state spectral function using the rodeo algorithm for the Heisenberg spin chain with 3 (thin blue line), 6 (thick green line), and 9 (medium red line) cycles. We have averaged over 20 sets of Gaussian random values for t_n with $t_{\mathrm{RMS}} = 5$. For comparison, we also show the exact initial-state spectral function with black open circles.

$$|\psi_{\mathrm{init}}\rangle = |0101010101\rangle$$

TABLE I. Overlap probability with energy eigenvector $|E_j\rangle$ after N cycles of the rodeo algorithm using Gaussian random values for t_n with $t_{\mathrm{RMS}} = 5$ and $E = E_j$.

E_j	$N = 0$	$N = 3$	$N = 6$	$N = 9$
−18.1	0.110	0.746	0.939	0.997
−16.4	0.209	0.841	0.993	1.000
−11.9	0.200	0.629	0.889	0.999
−9.76	0.0974	0.488	0.903	0.999
−8.38	0.0320	0.467	0.832	0.993
−6.63	0.0577	0.309	0.818	0.996
−5.81	0.0118	0.179	0.637	0.817
−5.52	0.115	0.456	0.766	0.997
−4.26	0.0171	0.144	0.696	0.995
−3.95	0.00401	0.0430	0.343	0.952
−2.00	0.0139	0.158	0.593	0.942
−0.802	0.0338	0.216	0.545	0.594
−0.704	0.0331	0.286	0.540	0.585
2.00	0.0357	0.371	0.925	0.994
2.42	0.00235	0.0122	0.0874	0.521
2.68	0.00291	0.0845	0.639	0.929
3.39	0.00592	0.0360	0.754	0.943
5.96	0.00336	0.0951	0.559	0.981
7.33	0.00650	0.184	0.792	0.978
8.13	0.00393	0.0832	0.665	0.841
8.24	0.00105	0.0275	0.142	0.289
10.0	0.00397	0.0128	0.295	0.902

Comparison with other well-known algorithms. Let Δ be the norm of the error in the wave function.



$$F_A \equiv \sqrt{2^{-N}(1-p)/[p+2^{-N}(1-p)]}$$

$$F_G \equiv \sqrt{4^{-N}(1-p)/[p+4^{-N}(1-p)]}$$

Preconditioning with adiabatic evolution

The computational effort needed for the rodeo algorithm is inversely proportional to the overlap probability between the initial state and the desired eigenvector.

We can use adiabatic evolution to increase this overlap probability.

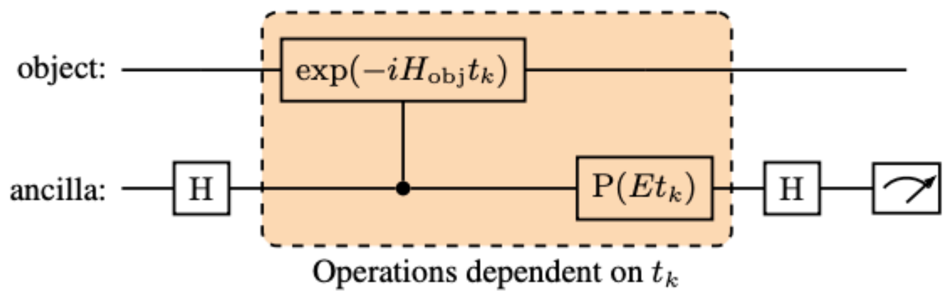
TABLE I. Overlap probability with energy eigenvector $|E_j\rangle$ with $E = E_j = -18.1$ after preconditioning with adiabatic evolution for time t_{AE} and the applying N cycles of the rodeo algorithm using Gaussian random values for t_n with $t_{\mathrm{rms}} = 5$.

E_j	t_{AE}	$N = 0$	$N = 3$	$N = 6$	$N = 9$
−18.1	0	0.110	0.746	0.939	0.997
−18.1	5	0.83074	0.99875	0.99988	0.99999

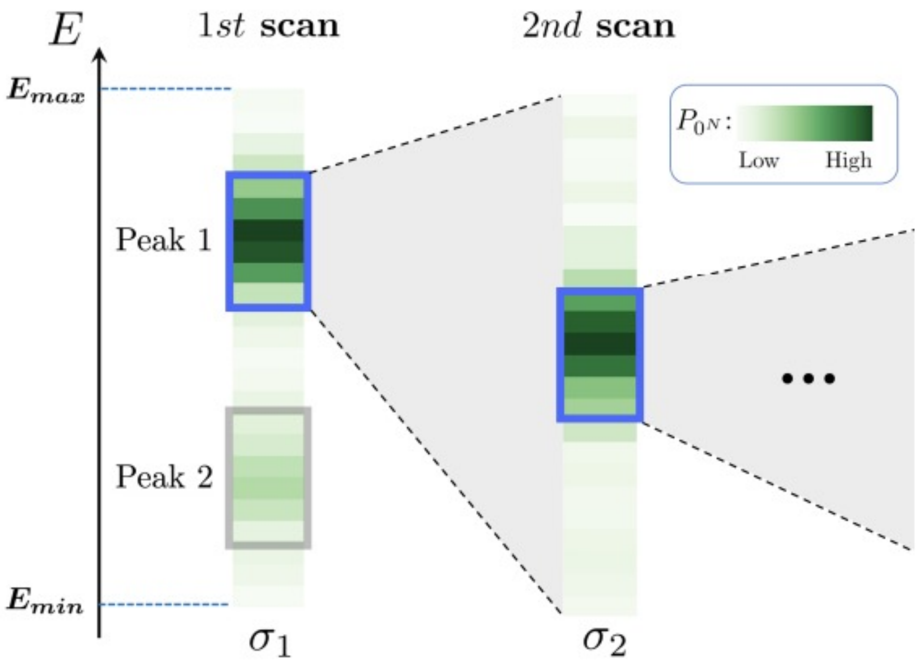
Using IBM Q devices, we implement the rodeo algorithm for a one qubit Hamiltonian. We consider a random Hamiltonian of the form

$$H_{\mathrm{obj}} = H^{(0)} = -0.08496I - 0.89134X + 0.26536Y + 0.57205Z$$

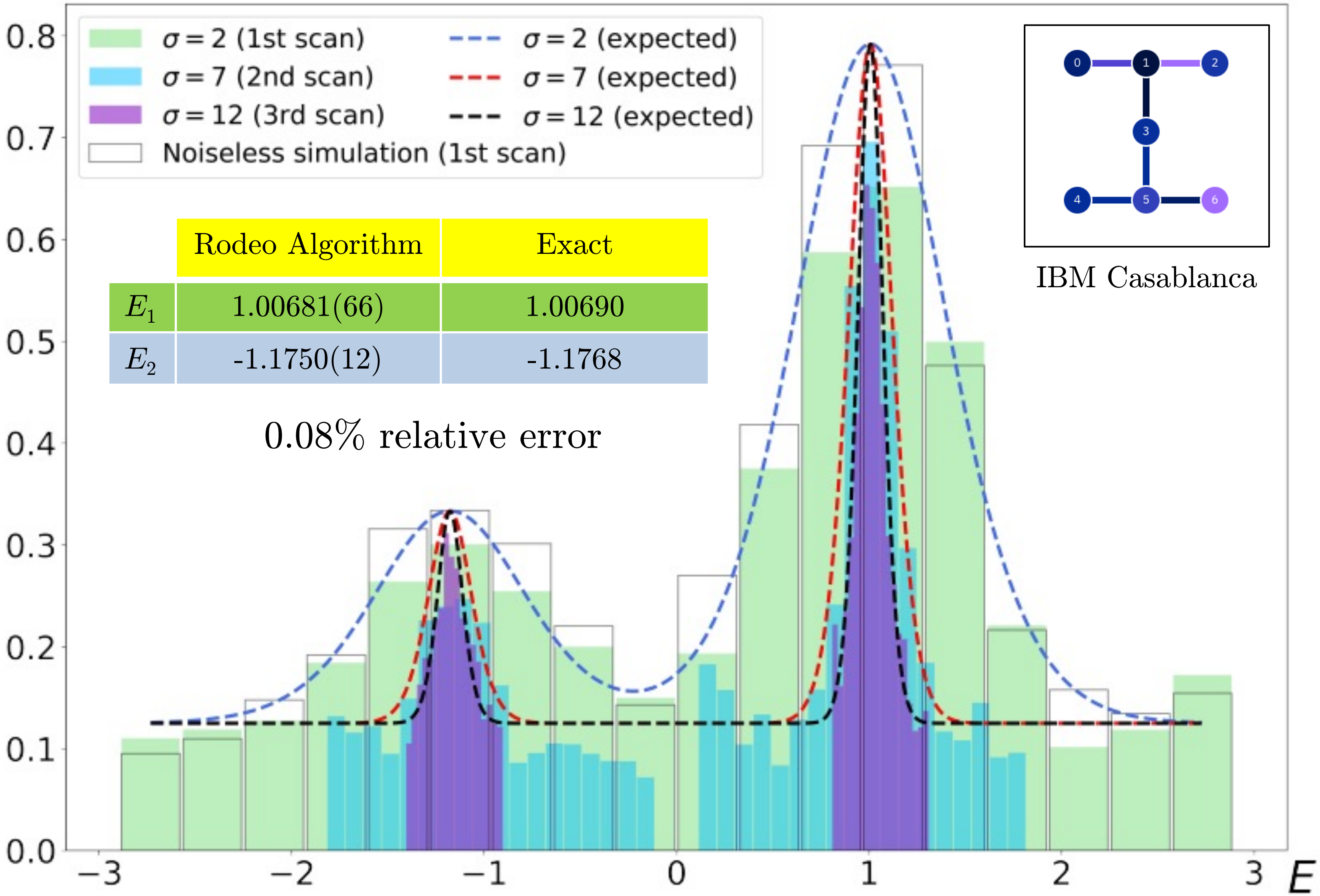
We use mid-circuit measurements without resets for the ancilla qubit



Each circuit consists of three cycles of the rodeo algorithm, corresponding to three controlled time evolutions and three ancilla qubit measurements. We sweep through the target energy E to perform an energy scan of the spectrum. We perform three separate scans of the energy, each time zooming in with more resolution.



Single qubit Hamiltonian



	Rodeo Algorithm	Exact
E_1	1.00681(66)	1.00690
E_2	-1.1750(12)	-1.1768

0.08% relative error

IBM Casablanca

Z. Qian, J. Watkins, G. Given, J. Bonitati, K. Choi, D.L., arXiv:2110.07747

Controlled reversal gates

A reversal gate, R, is a product of single qubit gates that anticommutes with some subset of the terms in a Hamiltonian.

$$RH = -HR$$

We note that

$$Re^{-iHt}R = e^{+iHt}$$

Let C_R be the controlled reversal gate that performs R if the ancilla qubit is in the 1 state and does nothing if if the ancilla qubit is in the 0 state.

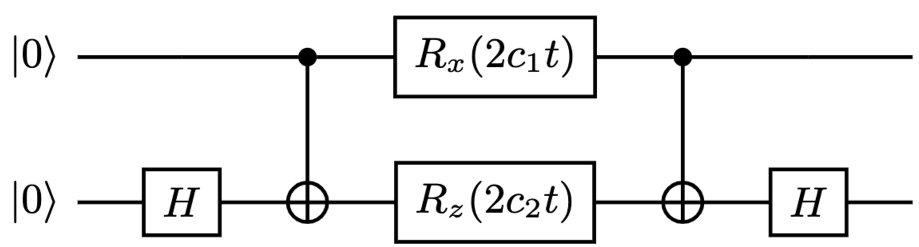
We note that C_R toggles the flow of time back and forth. Using C_R we can reduce the number of gates needed for state preparation using the rodeo algorithm or phase estimation by a factor of at least two.

Using IBM Q and Quantinuum devices, we implement the rodeo algorithm using controlled for a two qubit Hamiltonian. This work was done in collaboration with the Quantinuum Theory Group. Our Hamiltonian has the form

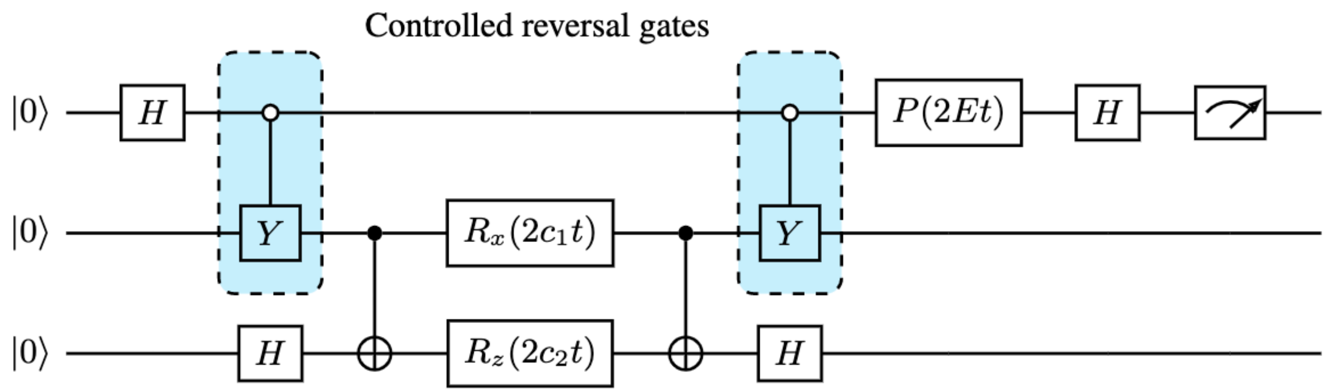
$$H_{\text{obj}} = c_1 X_1 \otimes Z_2 + c_2 Z_1 \otimes X_2$$

$$c_1 = 2.5,\ c_2 = 1.5$$

The time evolution of the Hamiltonian can be written as



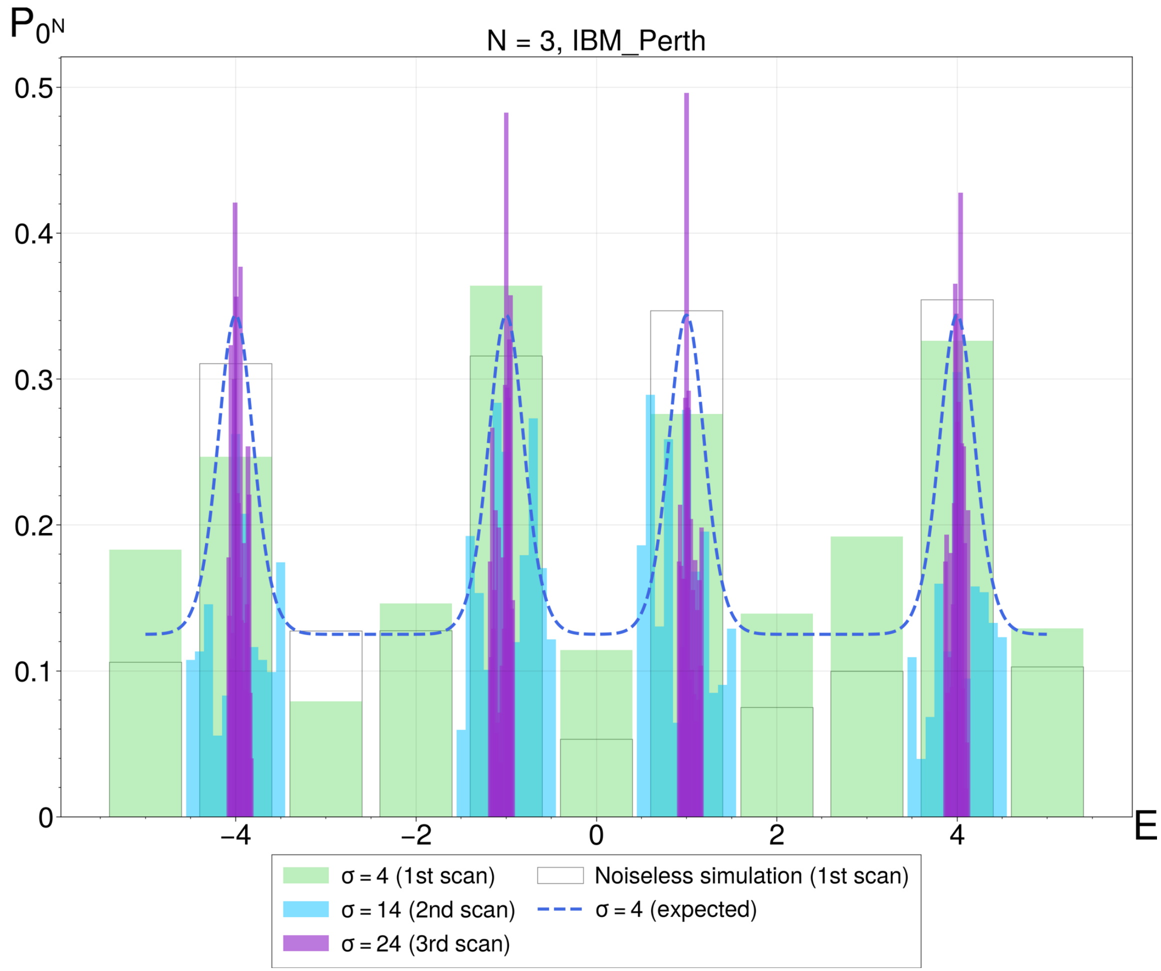
Using controlled reversal gates, one cycle of the rodeo algorithm is implemented as

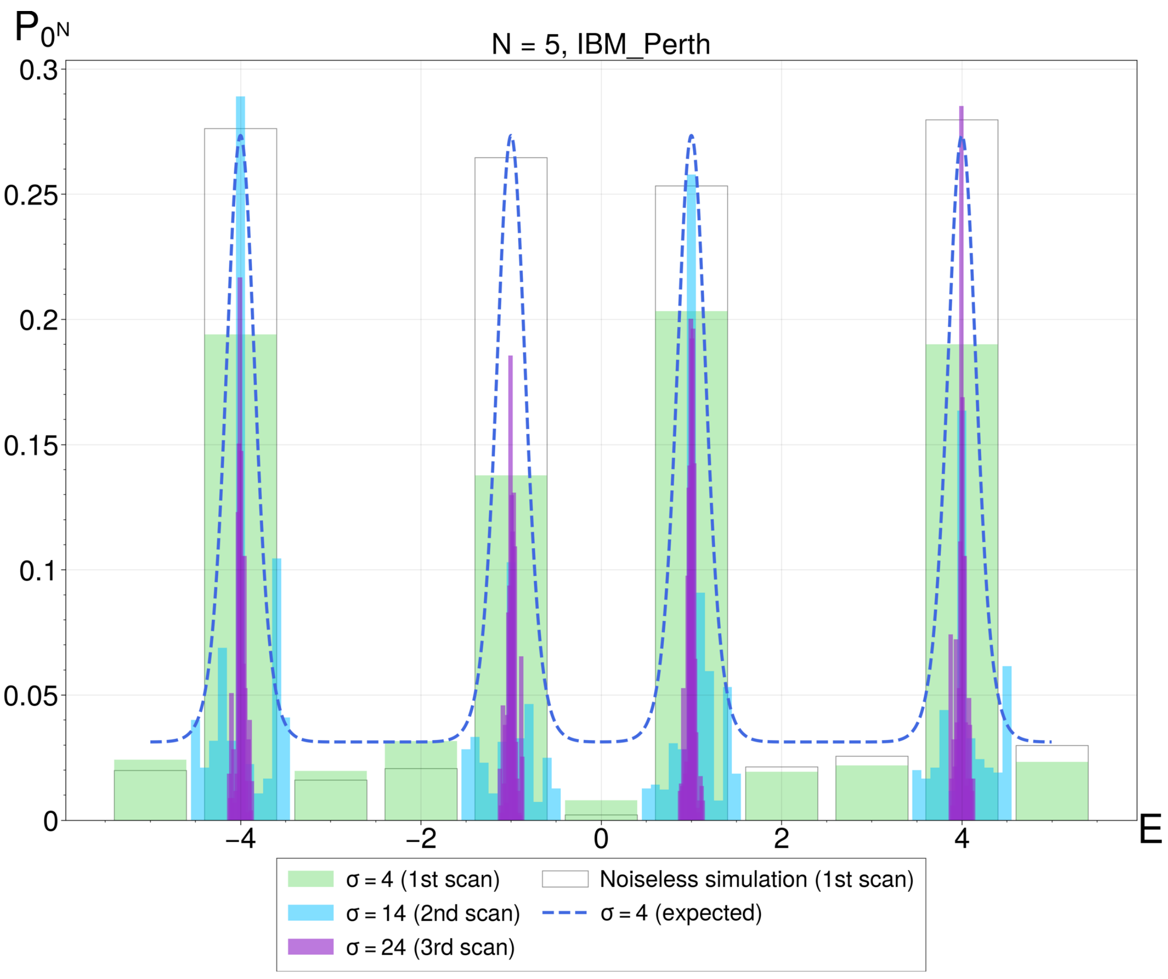


The controlled reversal gates provide a fivefold reduction in the number of gates. The comparison is made with respect to Qiskit-transpiled code without controlled reversal gates.

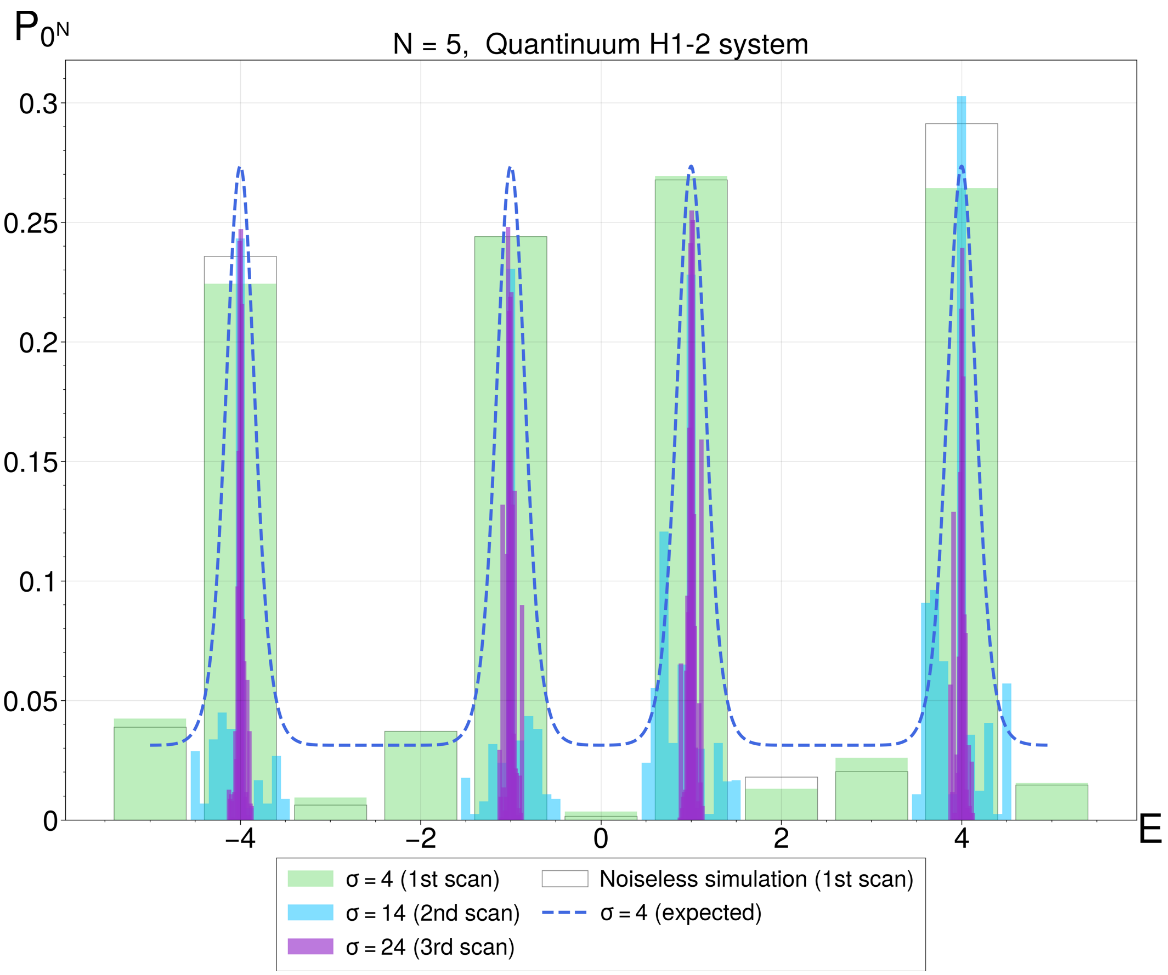
Bee-Lindgren, Qian, DeCross, Brown, Gilbreth, Watkins, Zhang, D.L., arXiv:2208.13557

Two qubit Hamiltonian





P_{0^N}
N = 5, IBM_Perth
0.3
0.25
0.2
0.15
0.1
0.05
0
−4
−2
0
2
4
E
σ = 4 (1st scan)
σ = 14 (2nd scan)
σ = 24 (3rd scan)
Noiseless simulation (1st scan)
σ = 4 (expected)

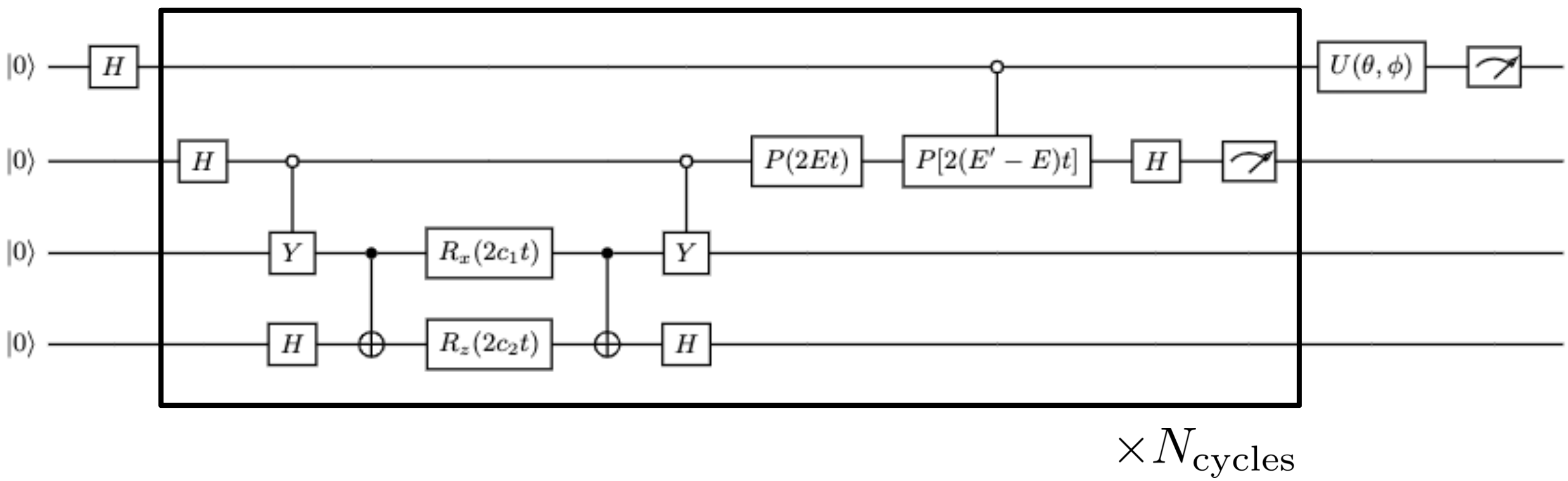


P_{0^N}
N = 5, Quantinuum H1-2 system
0.3
0.25
0.2
0.15
0.1
0.05
0
−4
−2
0
2
4
E
σ = 4 (1st scan)
σ = 14 (2nd scan)
σ = 24 (3rd scan)
Noiseless simulation (1st scan)
σ = 4 (expected)

	Exact	IBM Perth Three Cycles	IBM Perth Five Cycles	Quantinuum H1-2 Five Cycles
$\lvert\psi_0\rangle$	-4.0000	-4.0022(49)	-4.0006(42)	-3.9982(21)
$\lvert\psi_1\rangle$	-1.0000	-0.9829(56)	-0.9927(40)	-1.0083(39)
$\lvert\psi_2\rangle$	1.0000	1.0007(26)	1.0008(19)	1.0028(22)
$\lvert\psi_3\rangle$	4.0000	4.0093(84)	3.9982(25)	4.0036(17)

Multi-state rodeo algorithm

We can prepare an arbitrary linear combination of two eigenvectors with two different energies.



Bee-Lindgren, Bonitati, Given, Qian, Watkins, D. L., *et al.*, work in progress

This allows us to create the general superposition state

$$|\theta, \phi\rangle = \cos(\theta/2)\,|E\rangle + e^{i\phi}\sin(\theta/2)\,|E'\rangle$$

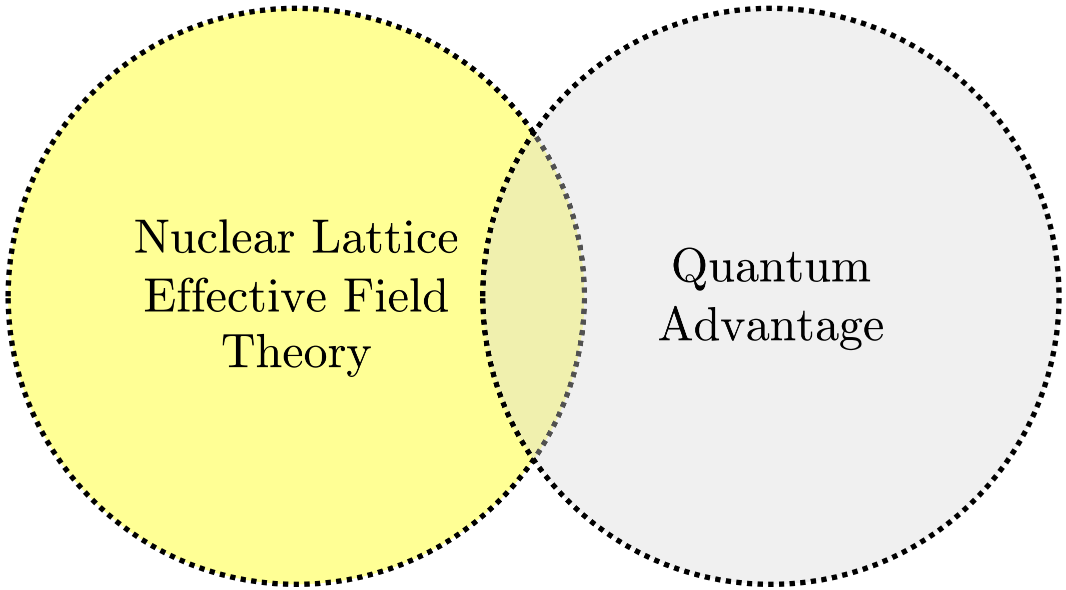
We can now measure the expectation value of any observable O

$$\begin{aligned}\langle\theta, \phi|O|\theta, \phi\rangle = \cos^2(\theta/2)\,\langle E|O|E\rangle + \sin^2(\theta/2)\,\langle E'|O|E'\rangle \\ + \Re[e^{i\phi}\sin(\theta)\,\langle E|O|E'\rangle]\end{aligned}$$

From this we can extract the transition matrix element

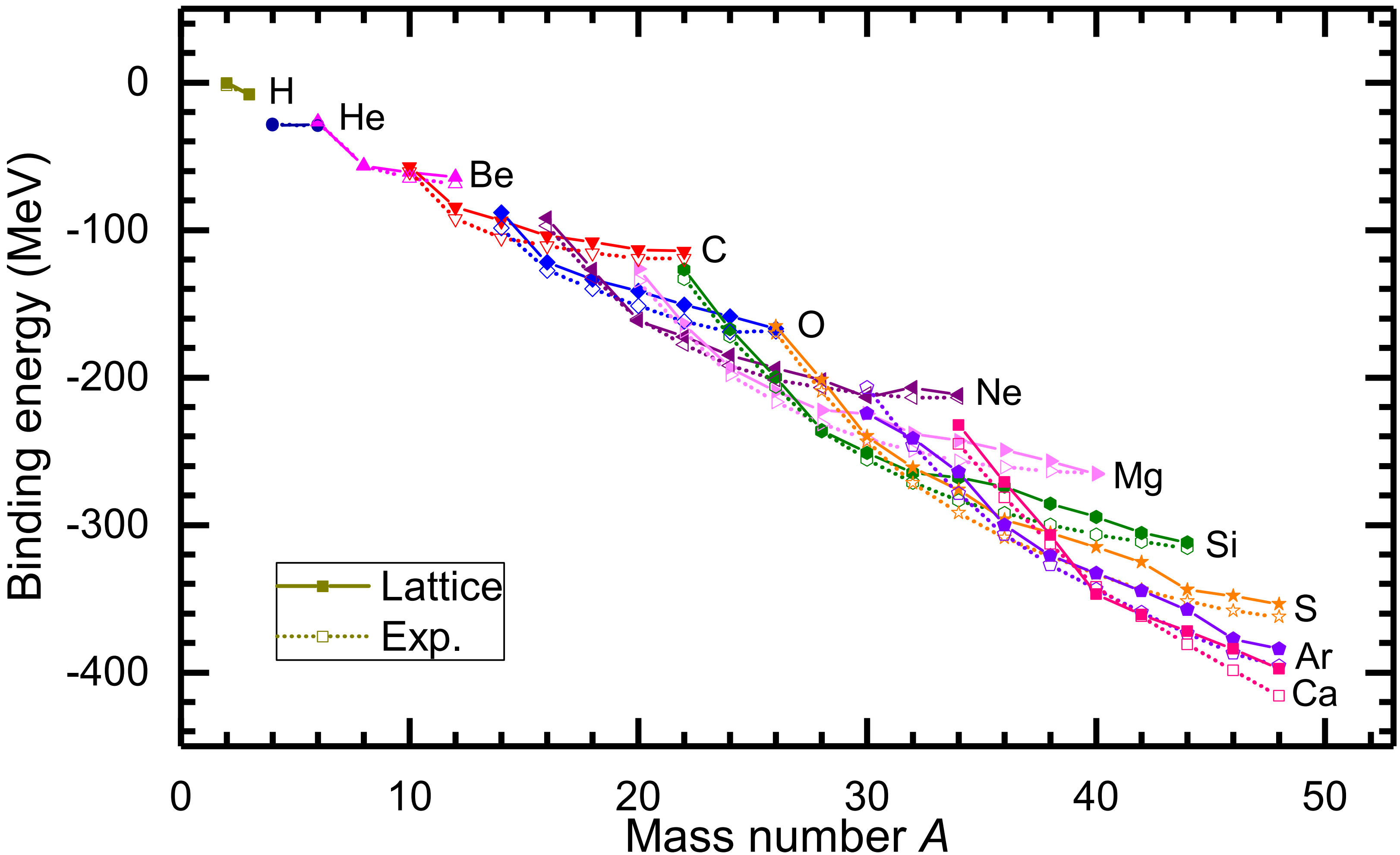
$$\langle E|O|E'\rangle$$

Towards the future

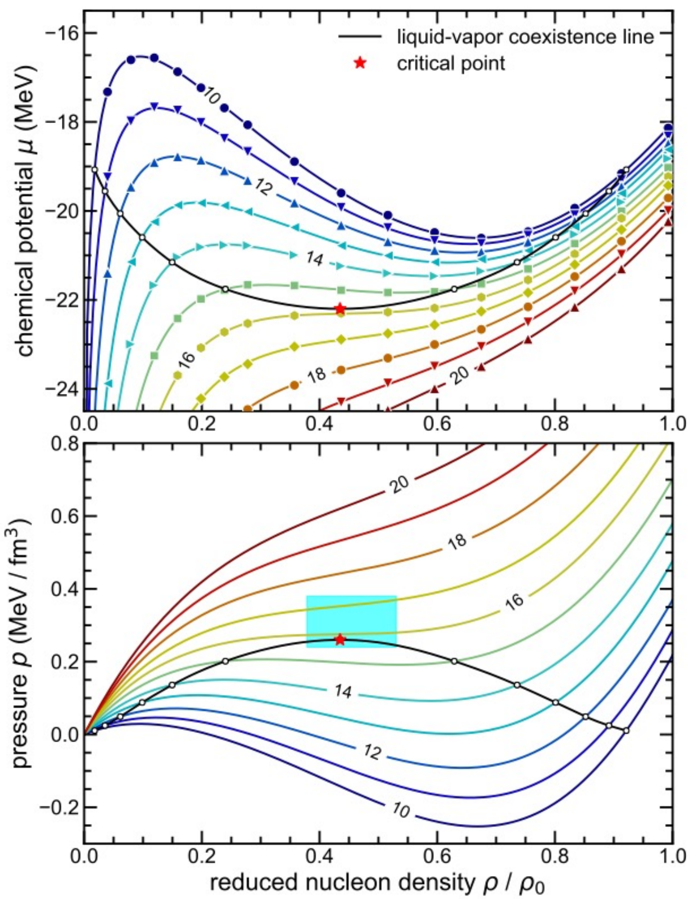


state preparation, spectral functions, real time dynamics

$$H = H_{\text{free}} + \frac{1}{2!}C_2 \sum_{\mathbf{n}} \tilde{\rho}(\mathbf{n})^2 + \frac{1}{3!}C_3 \sum_{\mathbf{n}} \tilde{\rho}(\mathbf{n})^3$$



Lu, Li, Elhatisari, D.L., Epelbaum, Meißner, PLB 797, 134863 (2019)



$$T_c = 15.80(0.32)(1.60)\ \mathrm{MeV}$$

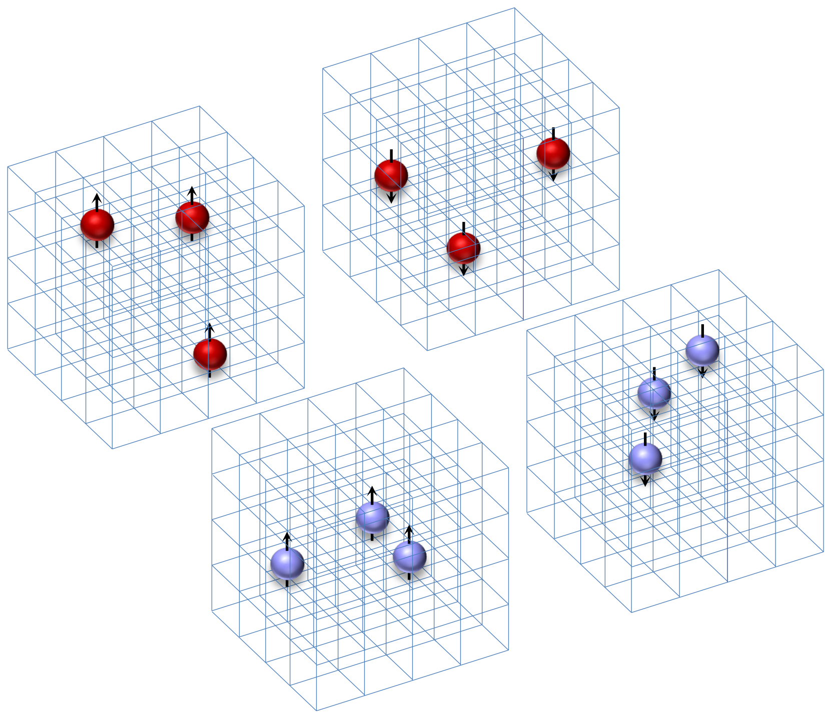
$$\rho_c = 0.089(04)(18)\ \mathrm{fm}^{-3}$$

$$\mu_c = -22.20(0.44)(2.20)\ \mathrm{MeV}$$

$$P_c = 0.260(05)(30)\ \mathrm{MeV\ fm}^{-3}$$

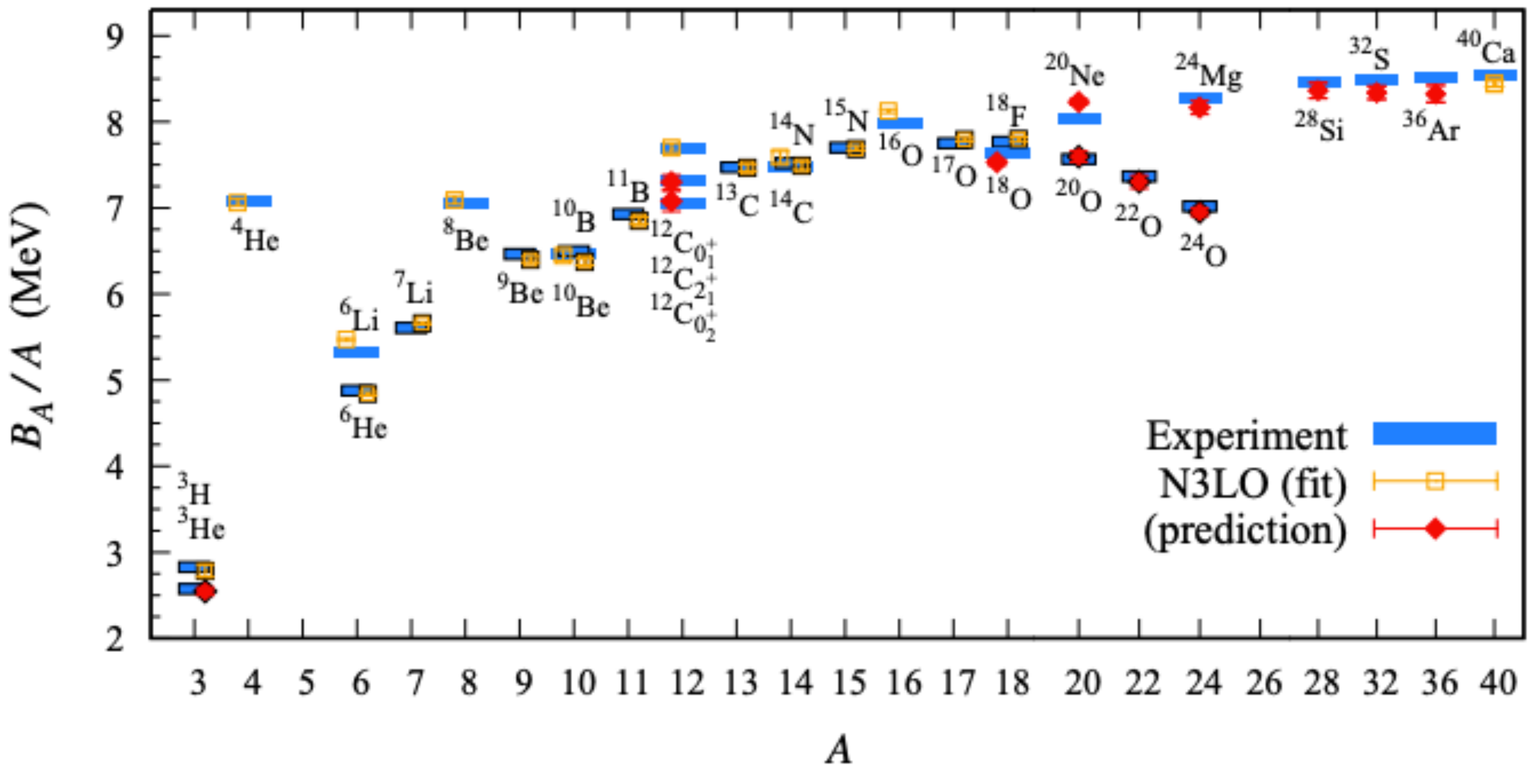
Lu, Li, Elhatisari, D.L., Drut, Lähde, Epelbaum, Meißner, PRL 125, 192502 (2020)

$$H = H_{\mathrm{free}} + \frac{1}{2!} C_2 \sum_{\mathbf{n}} \tilde{\rho}(\mathbf{n})^2$$



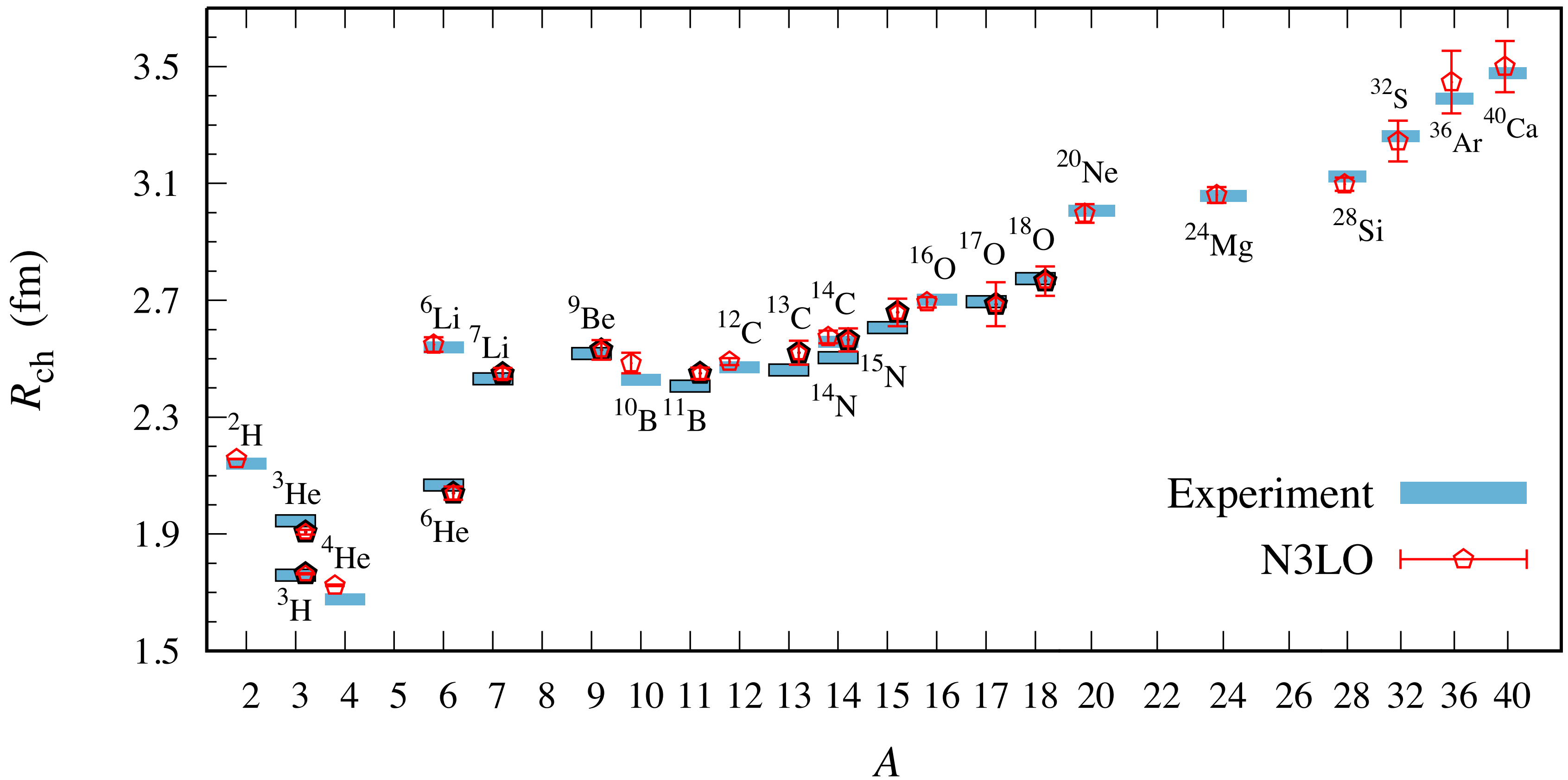
$\#$ qubits $= 4L^3$

Binding energy per nucleon



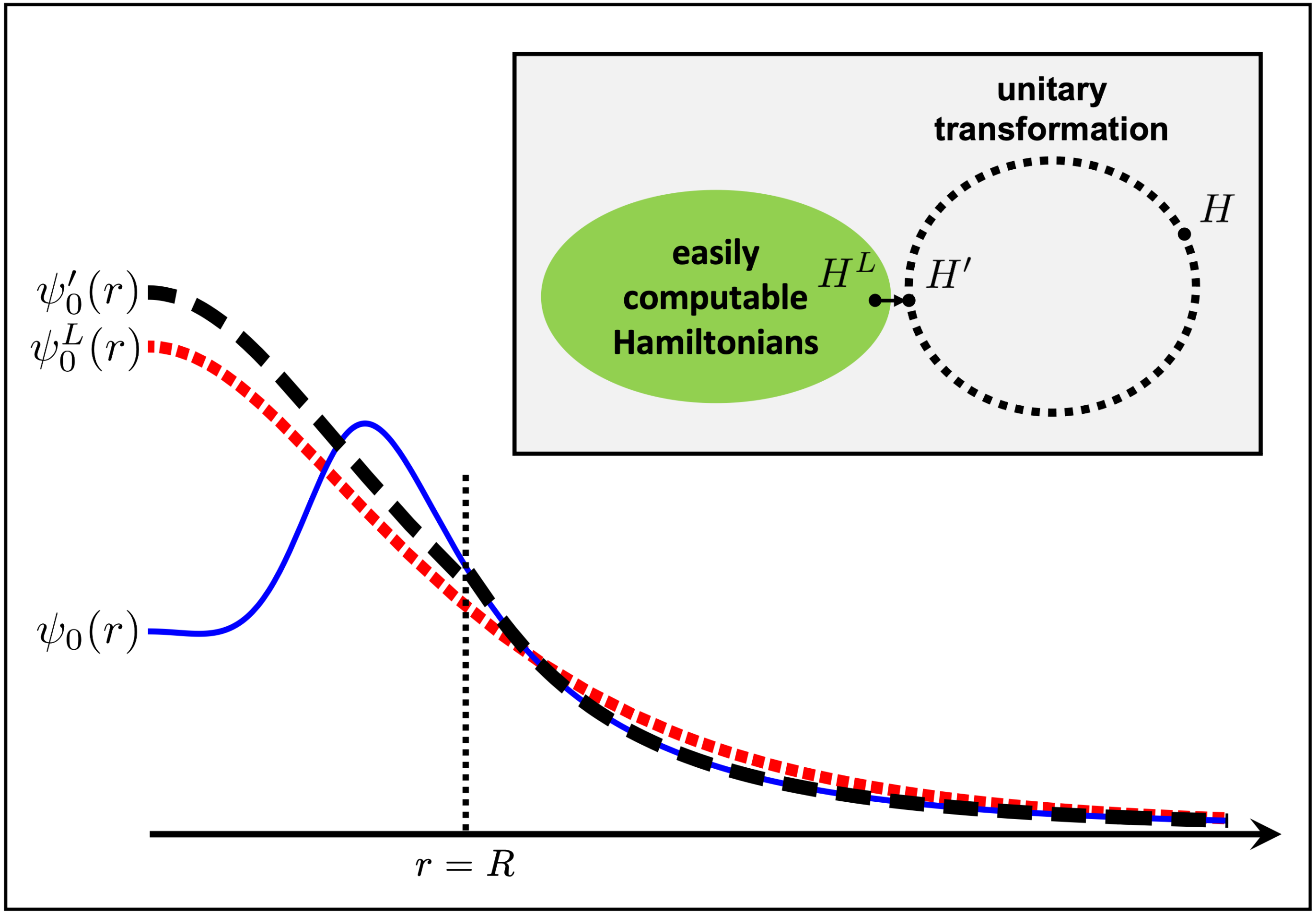
Elhatisari, Bovermann, Epelbaum, Frame, Hildenbrand, Krebs, Lähde, D.L., Li, Lu, M. Kim, Y. Kim, Ma, Meißner, Rupak, Shen, Song, Stellin, arXiv: 2210.17488

Charge radius



Elhatisari, Bovermann, Epelbaum, Frame, Hildenbrand, Krebs, Lähde, D.L., Li, Lu, M. Kim, Y. Kim, Ma, Meißner, Rupak, Shen, Song, Stellin, arXiv: 2210.17488

Wave function matching



Elhatisari, Bovermann, Epelbaum, Frame, Hildenbrand, Krebs, Lähde, D.L., Li, Lu, M. Kim, Y. Kim, Ma, Meißner, Rupak, Shen, Song, Stellin, arXiv: 2210.17488

Summary

We considered the rodeo algorithm. It is exponentially faster than other well-known algorithms for quantum state preparation. It is accurate and resilient for determining the energy spectrum in the presence of noise. We then discussed the concept of controlled reversal gates for reducing circuit depth and showed some applications of the rodeo algorithm on real quantum devices. We then considered the multi-state rodeo algorithm for preparing arbitrary linear combinations of energy eigenstates in order to compute transition matrix elements. We concluded with some comments on future nuclear lattice simulations with quantum computers.