Radiative leptonic decays of pseudoscalar mesons from lattice QCD

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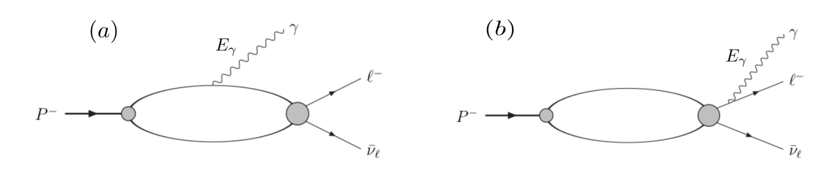
The $P \to \ell\ \nu\gamma$ decays

Feynman diagrams (all orders in QCD implicitly included): $[P = f\gamma^5 f']$



emission from the meson   emission from the lepton

- Diagram (b) can be computed in perturbation theory: the only lattice input required is the meson decay constant f_P $[\langle 0|A_\mu(0)|P(\boldsymbol{p})\rangle \equiv i p_\mu f_P]$.
- Diagram (a) can be reliably computed in a point-like approximation for the meson P only in the limit of soft-photon emission.
- We neglect the $\mathrm{SU}(3)-$vanishing diagram with photon emitted from sea-quarks.

Phenomenological relevance

Radiative leptonic decays $P \to \ell\nu\gamma$ are interesting!

- Important probes of the internal structure of pseudoscalar meson P.
- They allow for an independent extraction of $|V_{\mathrm{CKM}}|$ w.r.t. purely leptonic channels. Also relevant for V_{CKM} at $\mathcal{O}(\alpha_{em})$ from $P \to \ell\nu[\gamma]$.
- For heavy mesons $(D_s, D, B, \ldots)$, no model-independent results for the rate are available.
- The point-like contribution to the decay rate for $P \to \ell\nu\gamma$ is (helicity) suppressed w.r.t. SD contribution by a factor $r_\ell^2 = (m_\ell/m_P)^2$
- $\implies$ for heavy-meson decays, the electron mode is very sensitive to Structure-Dependent contributions.
- E.g. : for $P = D_s$ and $\ell = e$, $r_e^2 \simeq 6 \times 10^{-8}$.

Experimental results for $P \to \ell\nu\gamma$

For light-meson decays many accurate results available

- $\pi \to e\nu_e\gamma$ [PIBETA arXiv:0804.1815]
- $K \to e\nu_e\gamma$ [KLOE arXiv:0907.3594]
- $K \to \mu\nu_\mu\gamma$ [E787 arXiv:0003019, ISTRA+ arXiv:1005.3517, ...]

For heavy-mesons little is known, only upper-bounds available

- $\mathrm{Br}[D \to e\nu_e\gamma](E_\gamma > 10 \text{ MeV}) < 3 \times 10^{-5}$ [BESIII arXiv:1702.05837]
- $\mathrm{Br}[D_s \to e\nu_e\gamma](E_\gamma > 10 \text{ MeV}) < 1.3 \times 10^{-4}$ [BESIII arXiv:1902.03351]
- $\mathrm{Br}[B \to e\nu_e\gamma](E_\gamma > 1 \text{ GeV}) < 4.3 \times 10^{-6}$ [Belle arXiv:1810.12976]
- $\mathrm{Br}[B \to \mu\nu_\mu\gamma](E_\gamma > 1 \text{ GeV}) < 3.4 \times 10^{-6}$ [Belle arXiv:1810.12976]

Providing a first-principle determination of the branching fractions for heavy mesons may encourage further experimental searches!

Computing $P \to \ell\nu\gamma$ on the lattice

We need to compute the weak matrix element ($J_W = J_V - J_A$):

$$\langle \gamma(\boldsymbol{k},\epsilon)| J_W^\nu | P(\boldsymbol{p})\rangle \simeq -ie \int d^4y \; \langle \gamma(\boldsymbol{k},\epsilon)| A_\mu(y) |0\rangle \times \mathrm{T}\langle 0| J_{\rm em}^\mu(y) J_W^\nu(0) | P(\boldsymbol{p})\rangle_{\rm QCD}$$

$$= -ie\varepsilon_{*\mu} \times \int d^4y \; e^{iky} \; \mathrm{T}\langle 0| J_{\rm em}^\mu(y) J_W^\nu(0) | P(\boldsymbol{p})\rangle_{\rm QCD}$$

All non-perturbative information encoded in the hadronic tensor

$$H_W^{\mu\nu}(\boldsymbol{k},\boldsymbol{p}) \equiv i \int d^4y \; e^{iky} \; \mathrm{T}\langle 0| J_W^\nu(0) J_{\rm em}^\mu(y) | P(\boldsymbol{p})\rangle$$

- ...which for each pseudoscalar meson P must be computed for different photon energies $E_\gamma \equiv k_0 = |\boldsymbol{k}|$.

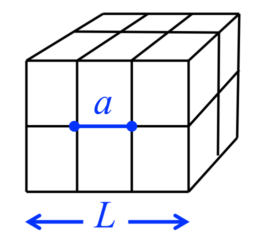
This is done by means of LATTICE QCD simulations

Basics of LQCD

The theoretical framework for lattice calculations is QFT in Euclidean time (obtained through Wick-rotation $t \to -i\tau$)

$$\langle \phi(x_1)\phi(x_2)\ldots\phi(x_n)\rangle = \frac{1}{\mathcal{Z}}\int [d\phi]\ \phi(x_1)\phi(x_2)\ldots\phi(x_n)\exp(-S_E[\phi])$$

The infinite-dimensional path integral is discretized on a 4-dimensional grid (the lattice) : $x_\mu \to n_\mu a$, which provides an UV ($1/a$) and IR ($1/L$) cut-off.



- We evaluate lattice path integral using MC methods.
- In QCD generate a stream of gauge configurations $\{U_1, \ldots, U_N\}$ distributed according to $e^{-S_E[U]}$, then…

$$\langle \bar{\mathcal{O}}\rangle = \frac{1}{N}\sum_{i=1}^{N}\mathcal{O}[U_i] \implies \sigma_{\bar{\mathcal{O}}} \propto \frac{1}{\sqrt{N}}$$

- Repeat the calculation for different L and lattice spacings a and extrapolate to $a, 1/L \to \infty$.

Generating gauge configurations

Generating state-of-the-art gauge-field configurations is an extremely expensive task, which requires massive HPC resources.

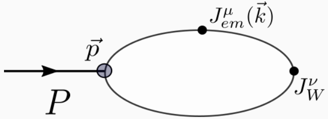
GPU-cluster Marconi100 at CINECA, Bologna.
It will cease its activities at the end of June. . .

. . . but will be "replaced" by LEONARDO, the 4th fastest supercomputer in the world.

- Within the LQCD community, it is customary for researchers to form collaborations where gauge configurations are produced and then shared among the members.
- Each collaboration has its own favoured lattice discretization: Wilson-clover, Twisted-mass, Staggered, Domain Wall, Overlap...
- $\Longrightarrow$ Important for checks of universality.

I am a member of the Extended Twisted-Mass Collaboration (ETMC), which has recently produced a "luxury" set of gauge configurations, corresponding to (four) lattice spacings $a \in [0.058, 0.09]$ fm, spatial volumes L^3 up to $L \simeq 7.7$ fm and $N_f = 2 + 1 + 1$ physical flavours. Let's do some physics with them!!

And back to physics...



$$H^{\mu\nu}_W(\boldsymbol{k},\boldsymbol{p}) = i\int d^4y\; e^{ik\cdot y}\; T\langle 0\big|J^{\nu}_W(0)J^{\mu}_{em}(y)\big|P(\boldsymbol{p})\rangle$$

Using Lorentz invariance, $H^{\mu\nu}_W$ decomposed in terms of scalar form factors:

$$H^{\mu\nu}_W(\boldsymbol{k},\boldsymbol{p}) = i\frac{F_V}{m_P}\epsilon^{\mu\nu\beta\gamma}k_\gamma p_\beta + \left[\frac{F_A}{m_P} + \frac{f_P}{p\cdot k}\right](p\cdot k\; g^{\mu\nu} - p^\mu k^\nu)$$
$$+ \underbrace{\frac{f_P}{p\cdot k}\,p^\mu p^\nu}_{\text{point-like contribution}} + H^{\mu\nu}_\perp(\boldsymbol{k},\boldsymbol{p})$$

- F_A and F_V functions of the invariant $x_\gamma = \frac{2p\cdot k}{m_P^2}$, $\quad 0 \leq x_\gamma \leq 1 - \frac{m_\ell^2}{m_P^2}$.
- Point-like contribution proportional to $P-$meson decay constant f_P.
- $H^{\mu\nu}_\perp$ term depends on two additional form factors H_1 and H_2, but does not contribute to the real photon emission process.

Canonical decomposition of $H_W^{\mu\nu}$

$$H_W^{\mu\nu}(\boldsymbol{k},\boldsymbol{0}) = i\int_{-\infty}^{0} dt_y\; e^{iE_\gamma t_y}\;\langle 0\big|J_W^{\nu}(0)J_{\mathrm{em}}^{\mu}(t_y,\boldsymbol{k})\big|P(\boldsymbol{0})\rangle$$
$$+\, i\int_{0}^{\infty} dt_y\; e^{iE_\gamma t_y}\;\langle 0\big|J_{\mathrm{em}}^{\mu}(t_y,\boldsymbol{k})J_W^{\nu}(0)\big|P(\boldsymbol{0})\rangle \equiv H_{W,1}^{\mu\nu}(\boldsymbol{k},\boldsymbol{0}) + H_{W,2}^{\mu\nu}(\boldsymbol{k},\boldsymbol{0})$$

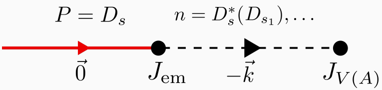
In the two time-orderings insert $\mathbb{1} = \sum_n |n\rangle \frac{1}{2E_n}\langle n|$ between the two currents and perform t_y-integral

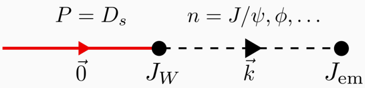
1st T.O. $t_y < 0$

$$H_{W,1}^{\mu\nu}(\boldsymbol{k},\boldsymbol{0}) = \sum_n \frac{B_W^{\mu\nu;n}(\boldsymbol{k})}{E_n(-\boldsymbol{k}) + E_\gamma - m_P - i\epsilon}$$

2nd T.O. $t_y > 0$

$$H_{W,2}^{\mu\nu}(\boldsymbol{k},\boldsymbol{0}) = \sum_n \frac{A_W^{\mu\nu;n}(\boldsymbol{k})}{E_n(\boldsymbol{k}) - E_\gamma - i\epsilon}$$





For real photon emission ($E_\gamma = |\boldsymbol{k}|$) always a positive mass gap $\implies$ no problems of analytic continuation from Minkowskian to Euclidean time.

Real photon on the lattice

Extracting the form factors from Euclidean lattice correlators

- We work in the meson rest frame $\boldsymbol{p}=0$, and compute the Euclidean VEV

$$C_W^{\mu\nu}(t,E_\gamma) = \int d^4y\ e^{t_y E_\gamma}\ e^{-i\boldsymbol{k}\cdot\boldsymbol{y}} \langle 0|\mathrm{T}\left\{J_W^\nu(t) J_{\mathrm{em}}^\mu(y)\right\}\phi_P^\dagger(0)|0\rangle\ , \quad \boldsymbol{k} = E_\gamma \hat{z}$$

- $\phi_P^\dagger(0)$ is an *interpolating operator* for $J^P=0^-$ hadronic states with $\boldsymbol{p}=0$, and same flavour content as the P meson.

- In the large (Euclidean) time limit t:

$$R_W^{\mu\nu}(t,E_\gamma) \equiv \underbrace{\frac{2m_P}{\langle 0|\phi_P|P\rangle} e^{t(E-E_\gamma)}}_{\text{amputating external states}} \times C_W^{\mu\nu}(t,E_\gamma) \to H_W^{\mu\nu}(\boldsymbol{k},0)$$

- Simple estimators can be built to extract the form factors:

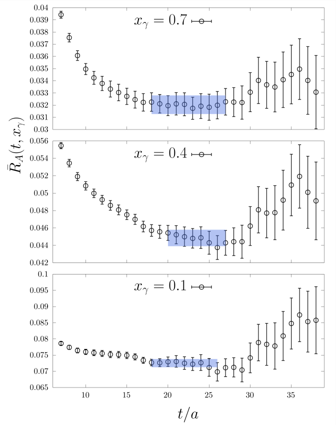
$$R_A^{11}(t,E_\gamma) - R_A^{11}(t,0) \underset{t\gg a}{\propto} F_A(E_\gamma)$$

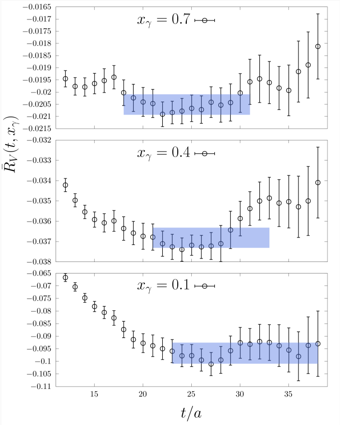
$$R_V^{12}(t,E_\gamma) \underset{t\gg a}{\propto} F_V(E_\gamma)$$

- We analyzed the case $P=D_s$.
- Simulated ten different values of $x_\gamma \equiv 2E_\gamma/M_P \simeq 0.1, 0.2, \ldots, 1$ using twisted b.c. [C.T. Sachrajda, G. Villadoro Phys.Lett.B 609].

Extracting F_V and F_A

Example of extraction, on a selected gauge ensemble, of F_A and F_V from the large-time behavior of the estimators $\bar{R}_{V/A}(t) \propto R_{V/A}(t)$

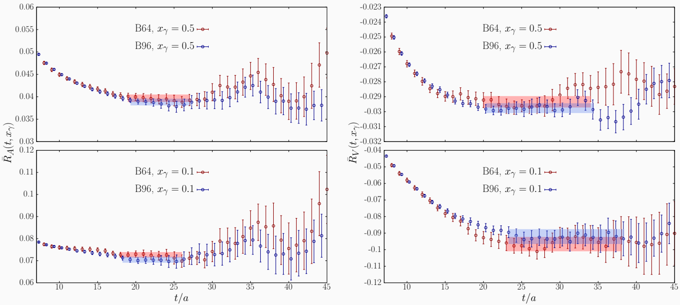




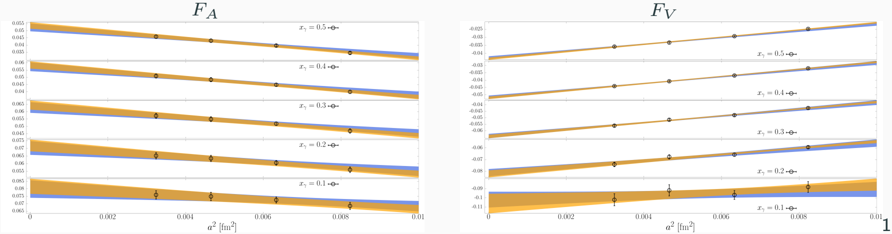
Ensemble parameters: $a \simeq 0.08 \text{ fm}, L \simeq 5 \text{ fm}, T = 2L$ (temporal lattice extent).

Finite-volume effects and continuum-limit extrapolation

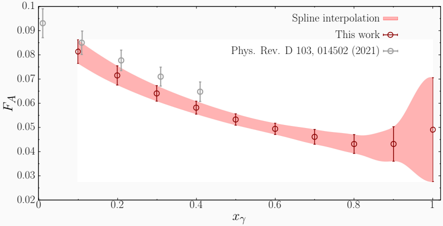
Finite-volume effects controlled by simulating on $L \simeq 5$ fm (B64) and $L \simeq 7.7$ fm (B96) boxes.

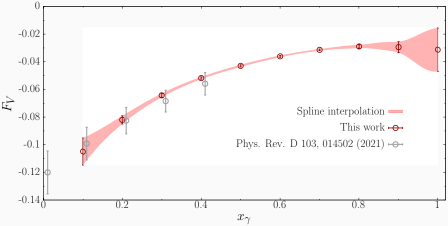


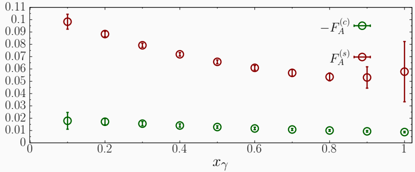
Continuum-limit extrapolation performed using four lattice spacing. $O(a^2)$ scaling observed, coarsest lattice used to estimate systematics.

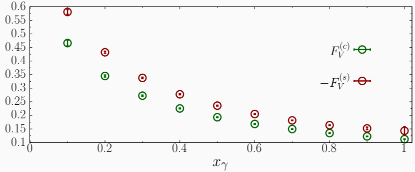


Final results and comparison with previous calculations









- $\Longleftarrow$ Single-flavour contributions (photon emitted from strange or charm quark-line).
- $F_{V/A} \equiv F_{V/A}^{(s)} + F_{V/A}^{(c)}$.
- Strong cancellation between strange- and charm-quark contributions to vector form factor F_V [Firstly observed by R. Zwicky].

From the form factors to the decay rate

The photon-energy differential decay rate for $D_s \to \ell\nu\gamma$ is written as a sum of three contributions

$$\frac{d\Gamma(D_s \to \ell\nu\gamma)}{dx_\gamma} = \frac{\alpha_{\rm em}}{4\pi}\Gamma^{(0)}\left\{\frac{dR^{\rm pt}}{dx_\gamma} + \underbrace{\frac{dR^{\rm int}}{dx_\gamma}}_{\propto F_A, F_V} + \underbrace{\frac{dR^{\rm SD}}{dx_\gamma}}_{\propto F_A^2, F_V^2}\right\}$$

- Γ^0 is leptonic decay rate without QED

$$\Gamma^{(0)} = \frac{G_F^2|V_{cs}|^2 f_{D_s}^2}{8\pi} m_{D_s}^3 r_\ell^2 (1 - r_\ell^2)^2, \qquad r_\ell = m_\ell / m_{D_s}$$

- At small x_γ:

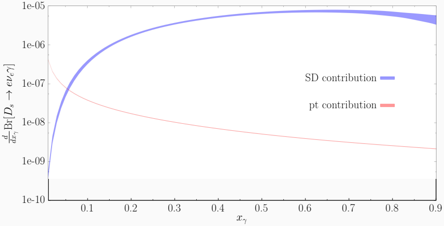
$$\frac{dR^{\rm pt}}{dx_\gamma} \propto x_\gamma^{-1}, \qquad \frac{dR^{\rm int}}{dx_\gamma} \propto x_\gamma, \qquad \frac{dR^{\rm SD}}{dx_\gamma} \propto x_\gamma^3$$

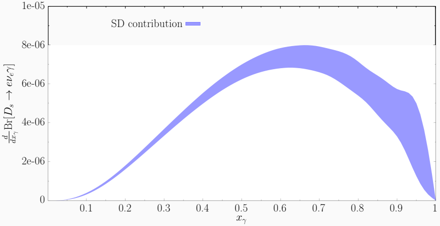
- But... $R^{SD} \propto r_\ell^{-2} \implies$ structure-dependent (SD) contribution enhanced by lepton/meson squared mass ratio.

For sufficiently large x_γ and small r_ℓ the SD contribution is the dominant.

The differential branching fraction

The most interesting decay-channel to probe the internal structure of D_s meson is $D_s \to e\nu_e\gamma$ $[r_e^2 \simeq 6 \times 10^{-8}]$



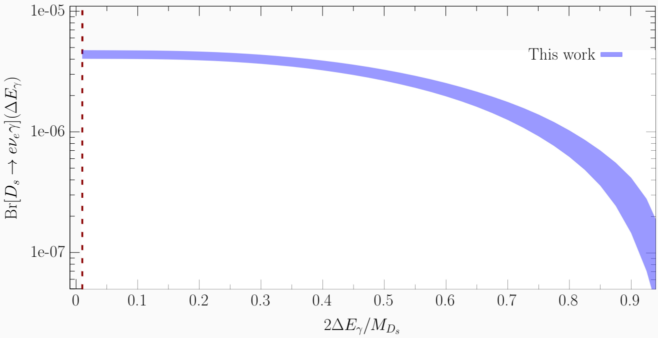


- The pt contribution to the differential branching is suppressed w.r.t. SD one for $x_\gamma \gtrsim 0.06$. SD contribution maximum at $x_\gamma \simeq 0.6 - 0.7$.
- The total decay rate

$$\Gamma_e(\Delta E_\gamma) \equiv \int_{\frac{2\Delta E_\gamma}{m_{D_s}}}^{1-r_e^2} dx_\gamma \, \frac{d\Gamma(D_s \to e\nu_e\gamma)}{dx_\gamma}$$

turns out to be dominated by SD contribution for photon-energy cuts ΔE_γ as low as 10 MeV.

The branching $\Gamma_e(\Delta E_\gamma)$



- BESIII has recently "measured" the branching fraction for $D_s \to e\nu_e\gamma$ employing a lower-cut $\Delta E_\gamma = 10$ MeV finding

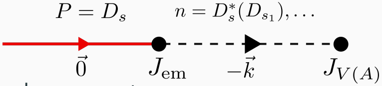
$$\mathrm{Br}[D_s \to e\nu_e\gamma](10 \text{ MeV}) \equiv \frac{\Gamma_e(10 \text{ MeV})}{\Gamma_{\mathrm{tot}}} < 1.3 \times 10^{-4} \quad \text{at } 90\% \text{ C.L.}$$

- Quark-model and HQET+pQCD calculations predicted a branching fraction of order $10^{-4} - 10^{-5}$ and 10^{-3} respectively.
- Our value $\mathrm{Br}[D_s \to e\nu_e\gamma](10 \text{ MeV}) \simeq 4.4(3) \times 10^{-6}$ is lower and well within the BESIII bound. Boring, but it's life. . .

Testing pole models (I)

Assuming dominance of lowest-lying hadronic-state:

$$F_W = \frac{1}{2E_W(-\boldsymbol{k})} \frac{B_W}{E_W(-\boldsymbol{k}) + E_\gamma - m_{D_s}}$$

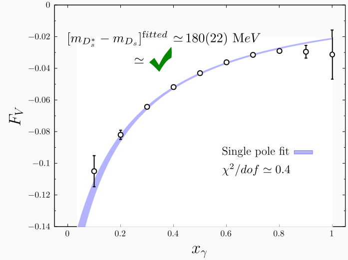


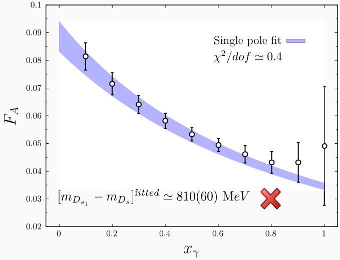
In vector and axial channel one expects

$$E_V(-\boldsymbol{k}) = \sqrt{m^2_{D_s^*} + |\boldsymbol{k}|^2} \qquad E_A(-\boldsymbol{k}) = \sqrt{m^2_{D_{s1}} + |\boldsymbol{k}|^2}$$

$$m_{D_s^*} - m_{D_s} \simeq 150 \text{ MeV} \qquad m_{D_{s1}} - m_{D_s} \simeq 500 \text{ MeV}$$

Fitting F_V and F_A with a single effective-pole Ansatz having free params. $B_{V/A}$ and the resonance mass we get





Testing pole models (II)

- In the vector channel the fitted value of the lowest-lying resonance mass compatible within 1.5σ with the D_s^* mass.
- Failure in the axial channel can be attributed to the larger gap $m_{D_{s_1}} - m_{D_s} \simeq 500$ MeV...
- ...and to the small (76 MeV) mass difference between the first- ($D_{s_1}(2460)$) and second-lightest ($D_{s_1}(2536)$) intermediate states.
- The *residue* B_V determined in pole-fit to F_V related to $g_{D_s^* D_s \gamma}$ coupling via

$$g_{D_s^* D_s \gamma} = -\frac{A_V}{m_{D_s} m_{D_s^*} f_{D_s^*}}$$

- Our determination of $g_{D_s^* D_s \gamma}$ can be compared with existing results

	LCSR [*]	HPQCD [**]	This work
$g_{D_s^* D_s \gamma}$ [GeV^{-1}]	✗ 0.60(19)	0.10(2)	0.118(13)

[*] light-cone sum rule prediction, B. Pullin & R. Zwicky *JHEP 09 (2021) 023*

[**] Direct lattice QCD calculation, HPQCD collaboration *Phys. Rev. Lett. 112*

Conclusions on leptonic decays with real photon emission

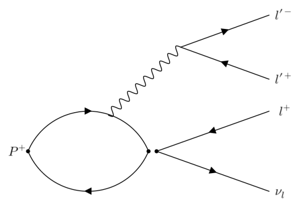
- I hope to have conveyed the message that high-precision calculations of $P \to \ell\nu\gamma$ decays, even when the meson P is heavy, can be nowadays performed on the lattice.
- Such decays are important, especially for heavy mesons, where the point-like contribution is suppressed, and the decay-rate is dominated by Structure-Dependent contributions, which can be sensitive to NP effects.
- As a part of our program to determine F_A and F_V for all heavy mesons, we considered the case $P = D_s$ and computed the two form factors over the whole phase space.
- Experimentally only an upper bound for $D_s \to e\nu_e\gamma$ exists (BESIII), and our predictions are well within the bound. We hope that our high-precision calculations may trigger further experimental searches.
- For $P = D_s$ we find disagreement with existing model-dependent calculations (quark-model, LCSR, HQET+pQCD).

Leptonic decays with virtual photon emission

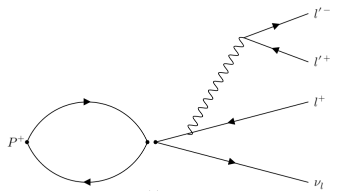
In principle, we can evaluate the hadronic tensor [From now on I drop the $\boldsymbol{p} = \mathbf{0}$]

$$H_W^{\mu\nu}(E_\gamma, \boldsymbol{k}) = i \int dt \; e^{iE_\gamma t} \; \mathrm{T}\langle 0 \big| J_W^\nu(0) J_{em}^\mu(t, \boldsymbol{k}) \big| P \rangle$$

also for off-shell photons $E_\gamma \neq |\boldsymbol{k}|$



(a)



(b)

- Through the convertion of the virtual photon γ^* into a dilepton we can study the rare decays

$$P \to \ell \nu_\ell \gamma^* \to \ell \nu_\ell \bar{\ell}' \ell'$$

- At first sight it looks as a trivial generalization of the real-photon case. Actually, it is not!

The problem of analytic continuation

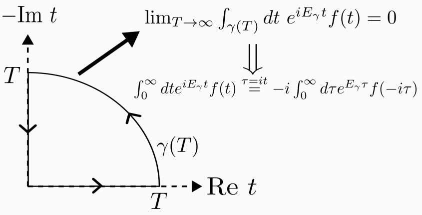
The hadronic tensor $H^{\mu\nu}(E_\gamma, \boldsymbol{k})$ is defined as the Fourier time-transform with argument E_γ of the Minkowskian $\langle 0|\mathrm{T}\left\{J^\mu_{\mathrm{em}}(t,\boldsymbol{k})J^\nu_W(0)\right\}|P\rangle$

On the lattice we compute everything in Euclidean time $\implies$ the connection between Euclidean and (physical) Minkowskian amplitudes has to be established case by case, and not always possible [Maiani & Testa no-go].

In our case:
$$H^{\mu\nu}_W(E_\gamma,\boldsymbol{k}) = i\int_{-\infty}^{0} dt\ \langle 0|J^\nu_W(0)\ e^{i(\hat{\mathcal{H}}+E_\gamma-m_p)t}\ J^\mu_{\mathrm{em}}(0,\boldsymbol{k})|P\rangle$$
$$+\, i\int_{0}^{\infty} dt\ \langle 0|J^\mu_{\mathrm{em}}(0,\boldsymbol{k})\ e^{-i(\hat{\mathcal{H}}-E_\gamma)t}\ J^\nu_W(0)|P\rangle$$

Analytic continuation $t \to -it$ possible if all contributing eigenstates E_n of $\hat{\mathcal{H}}$ satisfy:

(i) $E_n + E_\gamma > m_p$ $\qquad$ (ii) $E_n > E_\gamma$

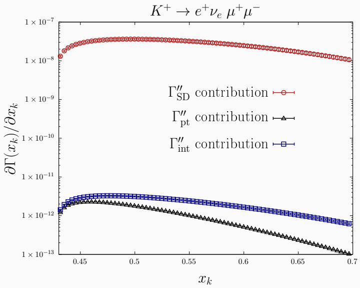


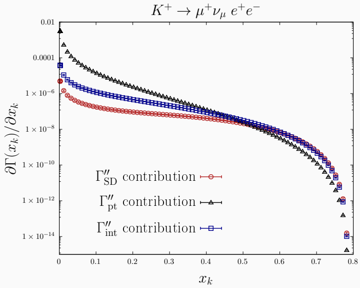
Condition (ii) not satisfied for general E_γ. Problems when photon *offshellness* $k^2 \equiv E_\gamma^2 - |\boldsymbol{k}|^2 >$ mass of lightest unflavoured $J^P = 1^-$ state.

The case $P = K$

In *arXiv:2202.03833* we evaded the problem by considering unphysical light-quark masses such that $2M_\pi > M_K$ and the condition $k^2 > 4M_\pi^2$ is never verified.

For $K \to \ell \nu_\ell \bar{\ell}' \ell'$ we computed the differential and total rate for different decay-channels ($x_k = \sqrt{k^2}/M_K$):





This work	point-like	E865 exp.
$\mathrm{Br}\,[K^+ \to e^+\,\nu_e\,\mu^+\,\mu^-]$		
$0.762(49) \times 10^{-8}$	3.0×10^{-13}	$1.72(45) \times 10^{-8}$
$\mathrm{Br}\,[K^+ \to \mu^+\,\nu_\mu\,e^+\,e^-]$ for $x_k > 0.284$		
$8.26(13) \times 10^{-8}$	4.8×10^{-8}	$7.93(33) \times 10^{-8}$

- Having unphysical quark-masses is OK for semi-quantitative predictions.
- However, for accurate predictions the analytic continuation problem must be tackled!

Finding the root of all evils...

[I drop now irrelevant indices and consider 2nd T.O. only]

Minkowskian correlator, Hadronic amplitude, Euclidean correlator (input)

$$C(t) \equiv \langle 0|J_{\rm em}(t)J_W(0)|P\rangle \ , \quad H(E) \equiv i\int_0^\infty dt e^{iEt}C(t) \ , \quad C_E(t) \equiv \langle 0|J_{\rm em}(-it)J_W(0)|P\rangle$$

Spectral density $\rho(E')$ defined as: $\rho(E') = \left\langle 0 \middle| J_{\rm em}(0)\, \delta(\mathcal{H} - E')\, J_W(0) \middle| P \right\rangle$

- $\mathcal{H}$ is the QCD Hamiltonian. One has ($[E^*, \infty)$ is support of ρ):

$$C(t) \overset{t\geq 0}{=} \int_{E^*}^\infty dE'\, \rho(E')\, e^{-iE't}, \qquad C_E(t) \overset{t\geq 0}{=} \int_{E^*}^\infty dE'\, \rho(E')\, e^{-E't}$$

- The hadronic amplitude $H(E)$ can be computed as

$$H(E) = \lim_{\epsilon\to 0} i\int_{E^*}^\infty dE'\, \rho(E') \int_0^\infty dt\, e^{-i(E'-E-i\varepsilon)t} = \lim_{\epsilon\to 0}\int_{E^*}^\infty dE' \frac{\rho(E')}{E'-E-i\varepsilon}$$

If $E > E^*$ CANNOT set $\varepsilon = 0$ before integrating. Instead if $E < E^*$

$$H(E) = \int_{E^*}^\infty C_E(t) e^{Et}$$

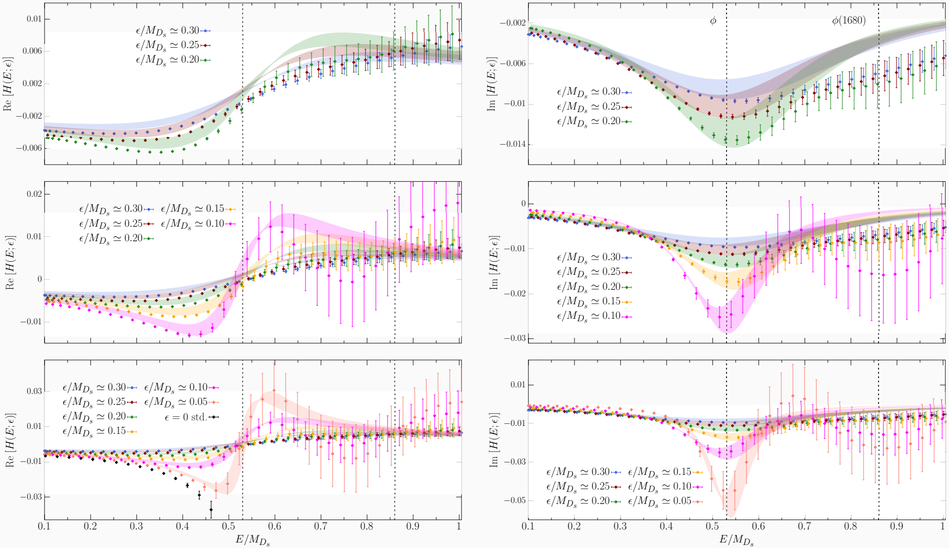
Solving the problem through the spectral representation

We propose to use ε as a regulator and evaluate the smeared amplitude $H(E;\varepsilon)$ at finite ε, and then take $\lim \varepsilon \to 0$.

$$H(E;\varepsilon) \equiv \int_{E^*}^{\infty} dE' \, \frac{\rho(E')}{E' - E - i\varepsilon} \overset{\text{trust me}}{=} \int_{-\infty}^{\infty} dE' \, \frac{1}{\pi} \frac{\varepsilon}{(E'-E)^2 + \varepsilon^2} H(E')$$

- The regulator exactly smears the hadronic amplitude over an energy-interval ε.
- While evaluating $\rho(E')$ from $C_E(t)$ (our input) is an ill-posed problem, the convolution between $\rho(E')$ and $(E' - E - i\varepsilon)^{-1}$ can be evaluated with controlled errors using the recently developed HLT method [M. Hansen et al. arXiv:1903.06476].
- Proof-of-principle calculation on a single ensemble with $L \simeq 5$ fm for $P = D_s$ where threshold $E^* \simeq E_\phi(\boldsymbol{k}) \simeq 1.02$ GeV with $|\boldsymbol{k}| \simeq 0.2$ GeV.

The smeared amplitude $H(E;\varepsilon)$



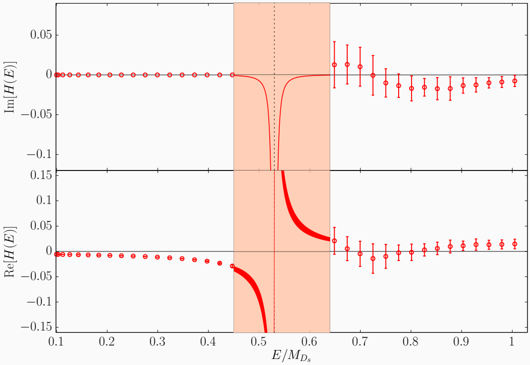
For $E > E^*$ statistical errors increase by decreasing ε.

Extrapolation to vanishing ε

- The criterion for a smooth $\varepsilon-$converge of $H(E;\varepsilon)$ is:

$$1/L \ll \varepsilon \ll \Delta(E) \quad \text{(I)}$$

- $\Delta(E)$ is the typical size of the logarithmic derivative of $H(E)$ in $E \pm \varepsilon$.
- E.g. if $\rho(E>0) \propto \frac{1}{(E-M)^2+\Gamma^2} \implies \Delta(E) = \sqrt{(E-M)^2+\Gamma^2}$.
- When (I) is satisfied $H(E;\varepsilon) - H(E)$ is $\mathcal{O}(\varepsilon)$ (scaling regime).



- Close to a sharp resonance (like the ϕ) extremely small ε and large volumes needed to be in the scaling region.
- In the orange region (around the ϕ) we extrapolated in a model-dependent way assuming a Breit-Wigner of $\Gamma \simeq 5$ MeV.

Conclusions on virtual photon emissions

- We presented a new strategy to evaluate complex electroweak amplitudes from Euclidean correlators also when intermediate states with energy larger than that of the external states are present.

- We circumvent the problem of analytic continuation, by evaluating, via spectral reconstruction methods, the hadronic amplitude $H(E;\varepsilon)$ smeared over an energy radius ε, and by then taking afterwards the $\varepsilon \to 0$ limit.

- We performed a pilot-study on a single ETMC ensemble, computing the hadronic tensor relevant to $D_s \to \ell\nu_\ell\bar{\ell}'\ell'$ using the HLT method.

- We are ready to apply the method to kaon decays, where the ε extrapolation will be probably smoother due to the presence of the broad ρ-resonance in place of the sharp ϕ resonance of the D_s decays.

Thank you for your attention!