Gauss-Bonnet gravity, conformally coupled scalar fields and black holes

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Outline

Lovelock's Theorem

The only second-order, local gravitational field equations derivable from an action containing solely the 4-dimensional metric tensor (plus related tensors) are the Einstein field equations with a cosmological constant.

Our options for modifying gravity:

1. New field content
2. Higher dimensions
3. Derivatives of higher order in the field equations
4. Non-locality
5. Non-derivable from an action principle

Lovelock's Theorem

$$S_D = \int d^D x \sqrt{-g} \sum_j \alpha_j \mathcal{R}^j, \qquad \mathcal{R}^j \equiv \frac{1}{2^j} \delta^{\mu_1 \nu_1 \ldots \mu_j \nu_j}_{\alpha_1 \beta_1 \ldots \alpha_j \beta_j} \prod_{i=1}^{j} R^{\alpha_i \beta_i}{}_{\mu_i \nu_i}$$

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The above action changes depending on the number of spacetime dimensions

$$S_2 = \int d^2 x \sqrt{-g} \alpha_0 , \quad S_3 = \int d^3 x \sqrt{-g} \left(\alpha_0 + \alpha_1 R \right) , \quad S_4 = \int d^4 x \sqrt{-g} \left(\alpha_0 + \alpha_1 R \right) ,$$

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$$S_2 = \int d^2 x \sqrt{-g} \alpha_0, \quad S_3 = \int d^3 x \sqrt{-g} \left(\alpha_0 + \alpha_1 R \right), \quad S_4 = \int d^4 x \sqrt{-g} \left(\alpha_0 + \alpha_1 R \right),$$

$$S_5 = \int d^5 x \sqrt{-g} \left(\alpha_0 + \alpha_1 R + \alpha_2 \mathcal{G} \right), \qquad \mathcal{G} = R^2 - 4 R_{\mu\nu} R^{\mu\nu} + R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta}$$

Horndeski gravity and the Gauss-Bonnet term

Horndeski gravity is analogous to Lovelock gravity, but addresses the most general scalar-tensor theory with second-order equations of motion [Horndeski, 1974]

$$\mathcal{L} = G_2(\phi, X) - G_3(\phi, X)\Box\phi + G_4(\phi, X)R + G_{4X}\left[(\Box\phi)^2 - \phi^{\mu\nu}\phi_{\mu\nu}\right] + G_5(\phi, X)G^{\mu\nu}\phi_{\mu\nu} - \frac{G_{5X}}{6}\left[(\Box\phi)^3 - 3\Box\phi\phi^{\mu\nu}\phi_{\mu\nu} + 2\phi_{\mu\nu}\phi^{\nu\lambda}\phi^{\mu}_{\lambda}\right].$$

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In Horndeski's theory, scalar field couplings to curvature scalars are only allowed when the curvature scalar is either R or $\mathcal{G}$

$G_5 \propto \ln X$...

Allows for spontaneous black hole scalarization in some cases

String Theory and the Gauss-Bonnet term

String theory predicts that at the classical level the Einstein equations are subject to NLO corrections, typically described by higher-order curvature terms in the action.

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- M-theory compactified on a Calabi-Yau threefold down to $D = 5$ takes the form

$$S_{eff} = \int d^5x \sqrt{-g} \left(R + \frac{1}{16} c_2^{(I)} V_I \, \mathcal{G} \right).$$

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$$S_{eff} = \int d^5x \sqrt{-g} \left(R + \frac{1}{16} c_2^{(I)} V_I \, \mathcal{G} \right).$$
- The 1-loop 4D effective action for heterotic string theory in the Einstein frame is of the form
$$S_{eff} = \int d^4x \sqrt{-g} \left(R - \frac{1}{2} (\nabla\phi)^2 + \alpha' e^{\phi} \mathcal{G} \right).$$

The trace anomaly

One-loop quantum corrections to a classically conformally invariant theory result in a renormalized stress-energy tensor with expectation value $\langle T_{\mu\nu} \rangle$ and a non-zero trace

$$g^{\mu\nu} \langle T_{\mu\nu} \rangle = \frac{\beta}{2} C^2 - \frac{\alpha}{2} \mathcal{G}$$

This is a general feature of quantum theories in curved spacetimes

What happens if we consider the Gauss-Bonnet action in 4D?

The Gauss-Bonnet Term in 4D

Consider the Einstein-Gauss-Bonnet theory in D-dimensions

$$S_D = \frac{1}{16\pi} \int d^D x \sqrt{-g} \left(-2\Lambda + R + \alpha \mathcal{G}\right) + S_M,$$

The Gauss-Bonnet Term in 4D

Consider the Einstein-Gauss-Bonnet theory in D-dimensions

$$S_D = \frac{1}{16\pi}\int d^D x\sqrt{-g}\,(-2\Lambda + R + \alpha\mathcal{G}) + S_M,$$

Extremization of the action gives the equations of motion

$$\Lambda g_{\mu\nu} + G_{\mu\nu} + \alpha H_{\mu\nu} = 8\pi T_{\mu\nu}$$

where

$$H_{\mu\nu} = 2R_{\mu}{}^{\alpha\rho\sigma}R_{\nu\alpha\rho\sigma} - 4R^{\rho\sigma}R_{\mu\rho\nu\sigma} - 4R_{\mu}{}^{\rho}R_{\nu\rho} + 2R\,R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{G}.$$

The Gauss-Bonnet Term in 4D

In four-dimensions $H_{\mu\nu}$ **vanishes** identically for all metrics. This is a consequence of Chern's theorem: in 4D, the Gauss-Bonnet invariant is the Euler density – its integral is a topological invariant χ called the *Euler characteristic* [Chern, 1945]

$$\chi \propto \int d^4x \sqrt{-g}\mathcal{G}.$$

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Trace of the field equations:

$$g^{\mu\nu}\left(\Lambda g_{\mu\nu} + G_{\mu\nu} + \alpha H_{\mu\nu}\right) = 8\pi g^{\mu\nu}T_{\mu\nu}$$

$$D\Lambda - \frac{(D-2)}{2}R - \alpha\frac{(D-4)}{2}\mathcal{G} = 8\pi T$$

What is the *most natural* generalization of Gauss-Bonnet gravity to four-dimensions?

Outline

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1. Motivating the Gauss-Bonnet term

2. Einstein-Gauss-Bonnet gravity in 4D

2.1 The rise and fall of (pure) 4DEGB

2.2 Well-defined 4DEGB theories

3. Black Holes

4. Conclusions

4D Einstein-Gauss-Bonnet Gravity – a regularization procedure

We will first consider a dimensional regularization procedure introduced by Glavan & Lin in **PRL 124 (2020) 8, 081301**

PHYSICAL REVIEW LETTERS **124,** 081301 (2020)

Einstein-Gauss-Bonnet Gravity in Four-Dimensional Spacetime

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In this Letter we present a general covariant modified theory of gravity in $D = 4$ spacetime dimensions which propagates only the massless graviton and bypasses Lovelock's theorem. The theory we present is formulated in $D > 4$ dimensions and its action consists of the Einstein-Hilbert term with a cosmological constant, and the Gauss-Bonnet term multiplied by a factor $1/(D-4)$. The four-dimensional theory is defined as the limit $D \to 4$. In this singular limit the Gauss-Bonnet invariant gives rise to nontrivial contributions to gravitational dynamics, while preserving the number of graviton degrees of freedom and being free from Ostrogradsky instability. We report several appealing new predictions of this theory, including the corrections to the dispersion relation of cosmological tensor and scalar modes, singularity resolution for spherically symmetric solutions, and others.

4D Einstein Gauss-Bonnet Gravity – a regularization procedure

1. Start with the Einstein-Gauss-Bonnet action in D dimensions and introduce a divergent factor for the Gauss-Bonnet sector

$$S = \lim_{D\to 4}\left[\frac{1}{16\pi}\int d^D x\sqrt{-g}\left(R+\frac{\alpha}{D-4}\mathcal{G}\right)+S_M\right]$$

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2. Vary the action and obtain the equations of motion

$$\lim_{D\to 4}\left[G_{\mu\nu} + \frac{\alpha}{D-4}H_{\mu\nu} = 8\pi T_{\mu\nu}\right]$$

4D Einstein Gauss-Bonnet Gravity – a regularization procedure

3. The trace equation, for instance, takes the finite form

$$\lim_{D\to 4}\left[g^{\mu\nu}\left(G_{\mu\nu}+\frac{\alpha}{D-4}H_{\mu\nu}\right)=8\pi T\right]\Leftrightarrow$$

$$\lim_{D\to 4}\left[\frac{(D-2)}{2}R+\frac{\alpha}{(D-4)}\frac{(D-4)}{2}\mathcal{G}=-8\pi GT\right]\Leftrightarrow$$

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4. When computing the equations of motion for certain line elements, such as static spherically symmetric or FLRW, it has been observed that these equations of motion do not diverge in the 4D limit.

4D Einstein Gauss-Bonnet Gravity – Black holes and Friedmann equations

Gauss-Bonnet corrected black holes:

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right),$$

$$f(r) = 1 + \frac{r^2}{2\alpha}\left(1 - \sqrt{1 + \frac{8M\alpha}{r^3}}\right) = 1 - \frac{2}{r} \times \frac{2M}{1 + \sqrt{1 + \frac{8M\alpha}{r^3}}},$$

Gauss-Bonnet corrected Friedmann equations:

$$H^2 + \alpha H^4 = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3},$$

$$\left(1 + 2\alpha H^2\right)\dot{H} = -4\pi G\left(\rho + p\right).$$

Have we broke Lovelock's theorem?

4D Einstein Gauss-Bonnet Gravity – Shortcomings

- The process relies on the specification of the geometry of the extra dimensional space before taking the 4D limit. There are countless ways to do this.
- In spacetimes that lack explicit symmetries, the equations of motion (other than the trace equation) will not be well-defined in general in the 4D limit

$$\lim_{D\to 4}\frac{H_{\mu\nu}}{D-4} = \text{finite term} + \lim_{D\to 4}\frac{1}{D-4}\left(C_{\mu\alpha\beta\rho}C_{\nu}{}^{\alpha\beta\rho} - \frac{1}{4}g_{\mu\nu}C^2\right)$$

The finite term does not obey the Bianchi identities [Gurses et. al, 2004.03390].

Are there well-defined 4D theories that mimic the solutions obtained with this regularization?

Outline

Dimensional Regularization of the Gauss-Bonnet term (2004.08362)

PHYSICAL REVIEW D **102,** 024025 (2020)

Derivation of regularized field equations for the Einstein-Gauss-Bonnet theory in four dimensions

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We propose a regularization procedure for the novel Einstein-Gauss-Bonnet theory of gravity, which produces a set of field equations that can be written in closed form in four dimensions. Our method consists of introducing a counterterm into the action, and does not rely on the embedding or compactification of any higher-dimensional spaces. This counterterm is sufficient to cancel the divergence in the action that would otherwise occur, and exactly reproduces the trace of the field equations of the original formulation of the theory. All other field equations display an extra scalar gravitational degree of freedom in the gravitational sector, in keeping with the requirements of Lovelock's theorem. We discuss issues concerning the equivalence between our new regularized theory and the original.

DOI: 10.1103/PhysRevD.102.024025

See also [Hennigar et. al, 2004.09472]

Dimensional Regularization of the Gauss-Bonnet term (2004.08362)

- We will construct a well-defined 4DEGB theory by removing the divergent terms in the field equations.

Dimensional Regularization of the Gauss-Bonnet term (2004.08362)

- We will construct a well-defined 4DEGB theory by removing the divergent terms in the field equations.
- The problematic term in the field equations is

$$\lim_{D\to 4} \frac{1}{D-4}\left(C_{\mu\alpha\beta\rho}C_{\nu}{}^{\alpha\beta\rho} - \frac{1}{4}g_{\mu\nu}C^2\right)$$

Dimensional Regularization of the Gauss-Bonnet term (2004.08362)

- Recall that the Weyl tensor is conformally invariant, i.e., $C^{\mu}{}_{\alpha\beta\rho} = \tilde{C}^{\mu}{}_{\alpha\beta\rho}$ given two conformally related metrics $g_{\mu\nu} = e^{-2\phi}\tilde{g}_{\mu\nu}$.
- We will then remove the divergences of the theory by adding to our action the counterterm

$$S = \lim_{D\to 4} \int d^D x \sqrt{-g} \left(R + \frac{\alpha}{D-4}\mathcal{G} \right) - \lim_{D\to 4} \int d^D x \frac{\alpha}{D-4} \sqrt{-\tilde{g}}\tilde{\mathcal{G}}$$

Dimensional Regularization of the Gauss-Bonnet term (2004.08362)

Writing $\sqrt{-\tilde{g}}\tilde{\mathcal{G}}$ in terms of $g_{\mu\nu}$

$$\sqrt{-\tilde{g}}\tilde{\mathcal{G}} = \sqrt{-g}e^{(D-4)\phi}\left[\mathcal{G} + (D-3)\nabla_\mu J^\mu + (D-3)(D-4)K\right],$$

where

$$J^\mu = 8G^{\mu\nu}\nabla_\nu\phi + 4(D-2)\left[\nabla^\mu\phi\Box\phi - \nabla^\mu\nabla^\nu\phi\nabla_\nu\phi + \nabla^\mu\phi\left(\nabla\phi\right)^2\right]$$

$$K = 4G^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi + (D-2)\left[4\left(\nabla\phi\right)^2\Box\phi + (d-1)\left(\nabla\phi\right)^4\right]$$

Dimensional Regularization of the Gauss-Bonnet term (2004.08362)

To obtain the 4D limit, we expand the exponential around $D = 4$

$$
\begin{aligned}
\sqrt{-\tilde{g}}\tilde{\mathcal{G}} &= \sqrt{-g}e^{(D-4)\phi}\left[\mathcal{G} + (D-3)\nabla_\mu J^\mu + (D-3)(D-4)K\right] \\
&= \sqrt{-g}\left[\mathcal{G} + \cancel{(D-3)\nabla_\mu J^\mu} + (D-3)(D-4)K + (D-4)\phi\left(\mathcal{G} + (D-3)\nabla_\mu J^\mu\right)\right] \\
&\quad + \mathcal{O}\left((D-4)^2\right)
\end{aligned}
$$

Rearranging we get

$$
\frac{\sqrt{-\tilde{g}}\tilde{\mathcal{G}} - \sqrt{-g}\mathcal{G}}{D-4} = (D-3)K + \phi\left(\mathcal{G} + (D-3)\nabla_\mu J^\mu\right) + \mathcal{O}\left((D-4)\right)
$$

Dimensional Regularization of the Gauss-Bonnet term (2004.08362)

Taking $D \to 4$ we get

$$S = \int d^4x \sqrt{-g} \left[R - \alpha \left(\phi \mathcal{G} - 4G^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - 4\Box\phi (\nabla\phi)^2 - 2(\nabla\phi)^4 \right) \right],$$

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The theory belongs to the shift-symmetric ($\phi \to \phi + c$) Horndeski class of theories

$$G_2 = 8\alpha X^2, \qquad G_3 = 8\alpha X, \qquad G_4 = 1 + 4\alpha X, \qquad G_5 = 4\alpha \log X.$$

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$$G_2 = 8\alpha X^2, \qquad G_3 = 8\alpha X, \qquad G_4 = 1 + 4\alpha X, \qquad G_5 = 4\alpha \log X.$$

Remarkably, a linear combination of the trace and scalar field equations leads to

$$R + \frac{\alpha}{2}\mathcal{G} \propto -T.$$

In this well-defined regularized theory we can recover the same static black holes and Friedmann equations as in the original regularization procedure!

Other known regularization procedures leading to scalar-tensor theories

Regularized Kaluza-Klein reduction [Lu and Pang, 2003.11552]

$$ds_D^2 = ds_4^2 + e^{-2\phi} d\Sigma_{D-4,\lambda}^2$$

where $d\Sigma_{D-4,\lambda}^2$ is the line element of a $(D-4)$-dimensional maximally symmetric space with curvature λ.

Performing a KK reduction, $\alpha \to \alpha/(D-4)$, and removing divergent terms leaves

$$S = \int d^4x \sqrt{-g} \Big[R - \alpha \Big(\phi \mathcal{G} - 4 G^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - 4 \Box\phi (\nabla\phi)^2 - 2 (\nabla\phi)^4 + 2\lambda e^{2\phi} \left[R + 6 (\nabla\phi)^2 + 3\lambda e^{2\phi} \right] \Big) \Big]$$

The same combination of the trace and scalar field equations leads to the purely geometrical equation

- We observe that both regularized 4DEGB theories possess the same simple equation, $R + \frac{\alpha}{2}\mathcal{G} \propto T$, despite their complicated structure.
- One is then left to wonder about the relationship that connects these threads, and why it should be that a special combination of the field equations completely decouples from the scalar field.

Conformally coupled scalar field (2105.04687)

PHYSICAL REVIEW D **103,** 104065 (2021)

Gravity with a generalized conformal scalar field: Theory and solutions

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We naturally extend general relativity with a conformally coupled scalar field by only requiring conformal invariance of the scalar field equation of motion and not of the action. The classically extended theory incorporates a scalar-Gauss-Bonnet sector and has second-order equations of motion, belonging to the Horndeski class. Remarkably, the theory features a purely geometrical field equation that allows for closed-form black hole solutions and cosmologies to be easily found. These solutions permit investigations of in-vogue scalar-Gauss-Bonnet corrections to the gravitational action without the need of resorting to approximations or numerical methods.

DOI: 10.1103/PhysRevD.103.104065

Conformally coupled scalar field (2105.04687)

- The answer is related to the conformal symmetry of the scalar field equation of motion:

$$g_{\mu\nu} \to g_{\mu\nu} e^{2\sigma}, \quad \phi \to \phi - \sigma$$

- Usual conformally coupled scalar field theory:

$$S = \int d^4x \sqrt{-g} \left(R - e^{2\phi} \left[\frac{1}{6} R + (\nabla\phi)^2 \right] \right) = \int d^4x \sqrt{-g} \left(R - \frac{\Phi^2}{6} R - (\nabla\Phi)^2 \right)$$

Combination of the trace equation and scalar field equation leads to $R = -T$. Well-known analytical solution: BBMB black hole. Only the scalar field equation possesses conformal invariance.

Conformally coupled scalar field (2105.04687)

- Under a (infinitesimal) conformal transformation, a generic action depending on the metric and the scalar, $S[g_{\mu\nu}, \phi]$, varies by an amount $S \to S + \delta_\sigma S$ where

$$\delta_\sigma S = \int d^4x \left(\frac{\delta g_{\mu\nu}}{\delta\sigma} \frac{\delta S}{\delta g_{\mu\nu}} + \frac{\delta\phi}{\delta\sigma} \frac{\delta S}{\delta\phi} \right)$$
$$= \int d^4x \left(2g_{\mu\nu} \frac{\delta S}{\delta g_{\mu\nu}} - \frac{\delta S}{\delta\phi} \right) \sigma$$

Because $\frac{\delta}{\delta\sigma} = \frac{\delta g_{\mu\nu}}{\delta\sigma} \frac{\delta}{\delta g_{\mu\nu}} + \frac{\delta\phi}{\delta\sigma} \frac{\delta}{\delta\phi}$ and $g_{\mu\nu} \to g_{\mu\nu} e^{2\sigma}, \quad \phi \to \phi - \sigma$

Conformally coupled scalar field (2105.04687)

- If the scalar field equation $\frac{\delta S}{\delta \phi}$ is conformally invariant, it should be the same before and after the conformal transformation

$$\frac{\delta}{\delta \phi} S = \frac{\delta}{\delta \phi} \left(S + \delta_\sigma S \right) \Rightarrow \frac{\delta}{\delta \phi} \delta_\sigma S = 0$$

- Therefore, the combination

$$2 g_{\mu\nu} \frac{\delta S}{\delta g_{\mu\nu}} - \frac{\delta S}{\delta \phi}$$

is independent of ϕ

Conformally coupled scalar field (2105.04687)

- The most general theory with second-order equations of motion and whose scalar field equation is conformally invariant is given by

$$\begin{aligned}\mathcal{L} =& R - 2\Lambda - \beta e^{2\phi}\left[R + 6(\nabla\phi)^2\right] - 2\lambda e^{4\phi} \\ &- \alpha\left(\phi\mathcal{G} - 4G^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - 4\Box\phi(\nabla\phi)^2 - 2(\nabla\phi)^4\right)\end{aligned}$$

- For suitably chosen β and λ we can recover both the counterterm ($\beta = \lambda = 0$) and the KK ($\lambda = 3\beta^2/4\alpha$) regularized theories
- In terms of $\Phi = e^\phi$

$$\begin{aligned}\mathcal{L} =& R - 2\Lambda - 6\beta\left[\frac{\Phi^2}{6}R + (\nabla\Phi)^2\right] - 2\lambda\Phi^4 \\ &- \alpha\left[\log(\Phi)\mathcal{G} - \frac{4}{\Phi^2}G^{\mu\nu}\nabla_\mu\Phi\nabla_\nu\Phi - \frac{4}{\Phi^3}\Box\Phi(\nabla\Phi)^2 + \frac{2}{\Phi^4}(\nabla\Phi)^4\right]\end{aligned}$$

Conformally coupled scalar field (2105.04687)

- Generalizations to Lovelock theory [Babichev et. al, 2302.02920]
- Generalization by restricting only the SF equation to second-order [Ayón-Beato and Hassaine, 2305.09806]

Outline

1. Motivating the Gauss-Bonnet term

2. Einstein-Gauss-Bonnet gravity in 4D

2.1 The rise and fall of (pure) 4DEGB

2.2 Well-defined 4DEGB theories

3. Black Holes

4. Conclusions

Static black holes

First, we will consider vacuum static and spherically symmetric solutions to the generalized conformal scalar field theory

$$\mathcal{L} = R - 6\beta\left[\frac{\Phi^2}{6}R + (\nabla\Phi)^2\right] - 2\lambda\Phi^4$$
$$- \alpha\left[\log(\Phi)\mathcal{G} - \frac{4}{\Phi^2}G^{\mu\nu}\nabla_\mu\Phi\nabla_\nu\Phi - \frac{4}{\Phi^3}\Box\Phi(\nabla\Phi)^2 + \frac{2}{\Phi^4}(\nabla\Phi)^4\right]$$

of the form

$$ds^2 = -\left(1 - \frac{2\mathcal{M}(r)}{r}\right)dt^2 + \frac{dr^2}{1 - 2\mathcal{M}(r)/r} + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$
$$\equiv -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

Static black holes

- Counterterm regularized theory ($\beta = \lambda = 0$)

$$\mathcal{M}(r) = \frac{2M}{1 + \sqrt{1 + \frac{8M\alpha}{r^3}}}, \qquad \Phi = \exp\left(\int^r \frac{1 - \sqrt{f(r)}}{r\sqrt{f(r)}} \right)$$

- KK regularized theory with maximally symmetric internal space ($\lambda = 3\beta^2/4\alpha$)

$$\mathcal{M}(r) = \frac{2M}{1 + \sqrt{1 + \frac{8M\alpha}{r^3}}}, \qquad \Phi = \frac{\sqrt{-2\alpha/\beta}}{r\cosh\left(c_1 + \int \frac{1}{r\sqrt{f(r)}} dr\right)}$$

- Horizons located at $r_H = M + \sqrt{M^2 - \alpha}$

Static black holes

- KK regularized theory with product of two spheres as the internal space ($\lambda = \beta^2/4\alpha$)

$$\mathcal{M}(r) = \frac{2\left(M + \frac{\alpha}{r}\right)}{1 + \sqrt{1 + \frac{8\alpha}{r^3}\left(M + \frac{\alpha}{r}\right)}}, \qquad \Phi = \frac{\sqrt{-2\alpha/\beta}}{r}$$

- Horizons located at $r_H = M + \sqrt{M^2 + \alpha}$
- Also, the BBMB black hole is formally a solution for $\lambda = \alpha = 0$

Similar static black holes in other theories

- Recently in Babichev et. al [2303.04126], these static and spherically symmetric solutions were generalized by including new terms in the action, without any apparent symmetries in 4D
- These terms were chosen as to guarantee integrability of the field equations

$$\begin{aligned}\mathcal{L} =& R - \beta_4 e^{2\phi}\left[R + 6(\partial\Phi)^2\right] - 2\lambda_4 e^{4\phi} \\ & - \alpha_4\left(\phi\mathcal{G} - 4G^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - 4\Box\phi(\nabla\phi)^2 - 2(\nabla\phi)^4\right) \\ & -\beta_5 e^{3\phi}\left[R + 12(\partial\Phi)^2\right] - 2\lambda_5 e^{5\phi} \\ & -\alpha_5 e^{\phi}\left(\mathcal{G} - 8G^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - 12\Box\phi(\nabla\phi)^2 - 12(\nabla\phi)^4\right)\end{aligned}$$

Static black holes

- By choosing the coupling constants in the Lagrangian appropriately, we get the solution

$$\mathcal{M}(r) = \frac{2\left(M + \frac{\alpha_4}{r} + \frac{4\eta\alpha_5}{5r^2}\right)}{1 + \frac{4\eta\alpha_5}{3r^3} + \sqrt{\left(1 + \frac{4\eta\alpha_5}{3r^3}\right)^2 + \frac{8\alpha_4}{r^3}\left(M + \frac{\alpha_4}{r} + \frac{4\eta\alpha_5}{5r^2}\right)}},$$

$$\Phi \equiv e^{\phi} = \frac{\eta}{r},$$

$$\beta_4 = -\frac{2\alpha_4}{\eta^2}, \quad \lambda_4 = \frac{\alpha_4}{\eta^4} \quad \beta_5 = -\frac{4\alpha_5}{3\eta^2}, \quad \lambda_5 = \frac{4\alpha_5}{5\eta^4}$$

Rotating black holes (based on 2305.10382)

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Letter

Rotating black holes in semiclassical gravity

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We present analytic stationary and axially symmetric black hole solutions to the semiclassical Einstein equations that are sourced by the trace anomaly. We also find evidence that the same spacetime geometry satisfies the field equations of a subset of Horndeski theories featuring a conformally coupled scalar field. We explore various properties of these solutions and determine the domain of existence of black holes. These black holes display distinctive features, such as noncircularity, a non-spherically-symmetric event horizon, and violations of the Kerr bound.

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- When considering a stationary and axially-symmetric setting, the field equations tend to get quite complicated in modified gravity scenarios
- Not many analytic solutions are known
- In theories with Gauss-Bonnet terms, only numerical solutions have been constructed so far

- Recall the trace equation of the theory:

$$R - \frac{\alpha}{2}\mathcal{G} = 0$$

- In the static case we could easily obtain a black hole solution because the metric depended on a single function, and the trace equation is integrable.
- Motivated by the regularized/conformal scalar theories, and by the trace anomaly, can we find an analytical candidate rotating solution that solves the trace equation?
- In the context of semiclassical gravity, the expectation value of the stress-energy tensor is typically unknown, making exact solutions very difficult to find

Rotating black holes (based on 2305.10382)

- The simplest class of stationary and axially symmetric spacetimes one can consider follows from a Kerr-Schild ansatz:

$$ds^2 = ds^2_{\text{flat}} + H(\mathbf{x}) \left(l_\mu dx^\mu\right)^2$$

- In Kerr-type coordinates (with $H(\mathbf{x}) = 2r\mathcal{M}(r,\theta)/\left(r^2 + a^2\cos^2\theta\right)$):

$$\begin{aligned} ds^2 = &- \left(1 - \frac{2\mathcal{M}(r,\theta)r}{r^2 + a^2\cos^2\theta}\right)\left(dv - a\sin^2\theta d\varphi\right)^2 \\ &+ 2\left(dv - a\sin^2\theta d\varphi\right)\left(dr - a\sin^2\theta d\varphi\right) \\ &+ \left(r^2 + a^2\cos^2\theta\right)\left(d\theta^2 + \sin^2\theta d\varphi^2\right), \end{aligned}$$

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- The Kerr geometry in these coordinates corresponds to the choice $\mathcal{M}(r, \theta) = M \equiv \text{constant}$

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- The Kerr geometry in these coordinates corresponds to the choice $\mathcal{M}(r,\theta) = M \equiv \text{constant}$
- The key to find a rotating solution for the trace (anomaly) equation is to notice that for this line element we have

$$\left(r^2 + a^2\cos^2\theta\right) R = 2\partial_r^2 \left(r\mathcal{M}\right)$$

$$\left(r^2 + a^2\cos^2\theta\right) \mathcal{G} = 8\partial_r^2 \left(\frac{r^2\mathcal{M}^2\left(r^2 - 3a^2\cos^2\theta\right)}{\left(r^2 + a^2\cos^2\theta\right)^3}\right)$$

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- Substituting these expressions in the trace (anomaly) equation we get

$$\partial_r^2 \left(r\mathcal{M} - 2\alpha \frac{r^2 \mathcal{M}^2 \left(r^2 - 3a^2\cos^2\theta\right)}{\left(r^2 + a^2\cos^2\theta\right)^3} \right) = 0$$

- We get the following solution

$$\mathcal{M}(r,\theta) = \frac{2M}{1 + \sqrt{1 - \frac{8M\alpha r\left(r^2 - 3a^2\cos^2\theta\right)}{\left(r^2 + a^2\cos^2\theta\right)^3}}}$$

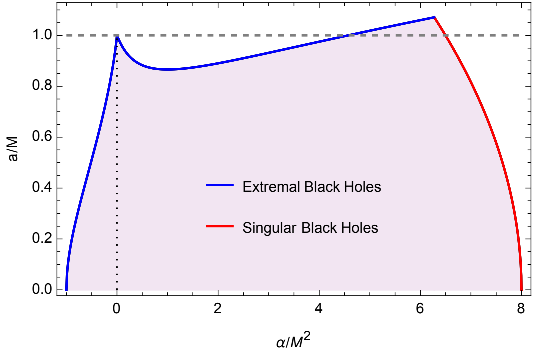
- The remaining field equations of the conformally coupled theory can be *formally* solved if $\Phi^2 = 6$

Rotating black holes (based on 2305.10382)

- We recover the static solution in the $a \to 0$ limit
- The Newman-Janis algorithm applied to the static solution does not yield a solution
- Algebraically special – Petrov Type type II
- Does not satisfy the circularity conditions

Rotating black holes (based on 2305.10382)

- Domain of existence for black hole solutions:



- Violations of the Kerr bound ($a/M \leq 1$) for $4.5 \lesssim \alpha/M^2 \lesssim 6.5$, with a maximum of $a/M \approx 1.07$.

Outline

1. Motivating the Gauss-Bonnet term

2. Einstein-Gauss-Bonnet gravity in 4D

2.1 The rise and fall of (pure) 4DEGB

2.2 Well-defined 4DEGB theories

3. Black Holes

4. Conclusions

Conclusions

- We started by motivating the Gauss-Bonnet term on theoretical grounds

Conclusions

- We started by motivating the Gauss-Bonnet term on theoretical grounds
- Described and discussed a regularization procedures that enable non-trivial Gauss-Bonnet effects in 4D
- Showed how these different procedures are all related by conformal properties of the scalar field

Conclusions

- Discussed the black holes of the conformally coupled and related theories.
- Derived an analytic rotating black hole which solves the trace equation $R + \frac{\alpha}{2}\mathcal{G} = 0$, motivated by these theories and by the trace anomaly
- Phenomenology of the rotating black holes is largely unexplored so far
- Non-constant Φ solution?

Conclusions

If you're interested in learning more please check the review "The 4D Einstein–Gauss–Bonnet theory of gravity: a review", arXiv:2202.13908

Thank you for your attention!