Discussion: Black Hole Perturbations

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Outline

1. Perturbation framework in GR: recovering a **master equation**
2. Generalisation to modified gravity: what is known and what is not
3. Quasinormal modes: definition, computation, properties
4. Discussion

Perturbation framework in GR

General setup

Perturbations of the metric

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$$
$$\bar{g}_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = -A(r)\,\mathrm{d}t^2 + \mathrm{d}r^2\,/B(r) + C(r)\,\mathrm{d}\Omega^2$$

- Focus on gravitational perturbations: main conclusions still valid for scalar/spinor/vector perturbations
- Work initially done in [Regge+57; Zerilli+70]
- Decompose $h_{\mu\nu}$ as SVT on the 2-sphere

Metric components

Axial modes: odd-parity perturbations

$$h_{\mu\nu} = \begin{pmatrix} & & \frac{1}{\sin\theta}\, h_0(r)\,\partial_\varphi & -\sin\theta\, h_0(r)\,\partial_\theta \\ & & \frac{1}{\sin\theta}\, h_1(r)\,\partial_\varphi & -\sin\theta\, h_1(r)\,\partial_\theta \\ \text{sym} & \text{sym} & \frac{1}{\sin\theta}\, h_2(r)(\partial_\theta\partial_\varphi - \operatorname{cotan}\theta\,\partial_\varphi) & -\sin\theta\, h_2(r)(\frac{\ell(\ell+1)}{2} - \partial_\theta^2) \\ \text{sym} & \text{sym} & \text{sym} & -\sin\theta\, h_2(r)(\partial_\theta\partial_\varphi - \operatorname{cotan}\theta\,\partial_\varphi) \end{pmatrix} Y_{\ell m}\, e^{-i\omega t}$$

Polar modes: even-parity perturbations

$$h_{\mu\nu} = \begin{pmatrix} A\, H_0(r) & H_1(r) & \beta(r)\partial_\theta & \beta(r)\partial_\varphi \\ \text{sym} & B^{-1}\, H_2(r) & \alpha(r)\partial_\theta & \alpha(r)\partial_\varphi \\ \text{sym} & \text{sym} & K(r) + G(r)\partial_\theta^2 & -G(r)\operatorname{cotan}\theta\,\partial_\varphi \\ \text{sym} & \text{sym} & \text{sym} & \sin^2(\theta)\, K(r) + G(r)(\partial_\varphi^2 + \sin\theta\cos\theta\,\partial_\theta) \end{pmatrix} Y_{\ell m}\, e^{-i\omega t}$$

Separating the degrees of freedom

1. Start with the Einstein-Hilbert action

$$S[g_{\mu\nu}] = \int \mathrm{d}^4x \sqrt{-g}\, R$$

2. Impose static spherically symmetric background
3. Perturb the metric: $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$, inject RW gauge and linearise Einstein's equations

$$\Rightarrow \text{obtain 10 equations for 10 functions}$$

4. The system separates by parity: polar (even) and axial (odd) modes
5. Gauge fixing via $h_{\mu\nu} \longrightarrow h_{\mu\nu} + \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu$: can set h_2, α, β and G to zero
    - Polar modes: 7 equations for K, H_0, H_1, H_2
    - Axial modes: 3 equations for h_0, h_1

Reducing the number of equations

Two systems with more equations than variables $\rightarrow$ overconstrained?

Axial modes

- 2 first-order equations
- 1 second-order equation

Polar modes

- 4 first-order equations
- 2 second-order equations
- 1 algebraic equation

Interestingly, each system is equivalent to a 2-dimensional system [Regge+57; Zerilli+70]

$$\frac{\mathrm{d}X_{\text{axial}}}{\mathrm{d}r} = M_{\text{axial}}(r)X_{\text{axial}}\,, \qquad \frac{\mathrm{d}X_{\text{polar}}}{\mathrm{d}r} = M_{\text{polar}}(r)X_{\text{polar}}$$

Final system of equations

Axial modes

$$X_{\text{axial}} = {}^t\begin{pmatrix} h_0 & h_1/\omega \end{pmatrix}$$

$$M_{\text{axial}} = \begin{pmatrix} \frac{2}{r} & 2i\lambda\frac{r-\mu}{r^3} - i\omega^2 \\ -\frac{r^2}{(r-\mu)^2} & -\frac{\mu}{r(r-\mu)} \end{pmatrix}$$

(set $2(\lambda+1) = \ell(\ell+1)$)

Polar modes

$$X_{\text{polar}} = {}^t\begin{pmatrix} K & H_1/\omega \end{pmatrix}$$

$$M_{\text{polar}} = \frac{1}{3\mu + 2\lambda r}\begin{pmatrix} \frac{a_{11}(r)+b_{11}(r)\omega^2}{r(r-\mu)} & \frac{a_{12}(r)+b_{12}(r)\omega^2}{r^2} \\ \frac{a_{21}(r)+b_{21}(r)\omega^2}{2(r-\mu)^2} & \frac{a_{22}(r)+b_{22}(r)\omega^2}{r(r-\mu)} \end{pmatrix}$$

$\Rightarrow$ goal to achieve: simplify these complicated differential systems

Effect of a change of variables

Changing the functions in X is not a change of basis for M!

Change of variables

$$\frac{\mathrm{d}X}{\mathrm{d}r} = M(r)X\,, \quad X = P(r)\tilde{X}$$

$$\frac{\mathrm{d}\tilde{X}}{\mathrm{d}r} = \tilde{M}(r)\tilde{X}\,, \quad \tilde{M} = P^{-1}MP - P^{-1}\frac{\mathrm{d}P}{\mathrm{d}r}$$

Main idea: find a change of variables that will put the equation into a better form

Usual change of variables: propagation equation

Canonical form for $\tilde{M}$:

$$\tilde{M} = n(r) \begin{pmatrix} 0 & 1 \\ V(r) - \frac{\omega^2}{c^2} & 0 \end{pmatrix}$$

Physical interpretation

$$\begin{cases} \dfrac{\mathrm{d}\tilde{X}_0}{\mathrm{d}r_*} = \tilde{X}_1 \,, \\ \dfrac{\mathrm{d}\tilde{X}_1}{\mathrm{d}r_*} = \left(V(r) - \omega^2/c^2\right)\tilde{X}_0 \end{cases} \Rightarrow \quad \frac{\mathrm{d}^2\tilde{X}_0}{\mathrm{d}r_*^2} + \left(\frac{\omega^2}{c^2} - V(r)\right)\tilde{X}_0 = 0 \,, \quad \frac{\mathrm{d}r}{\mathrm{d}r_*} = n(r)$$

Schrödinger equation with potential V

r_*: "tortoise coordinate", $r = \mu \longrightarrow r_* = -\infty$ and $r = +\infty \longrightarrow r_* = +\infty$

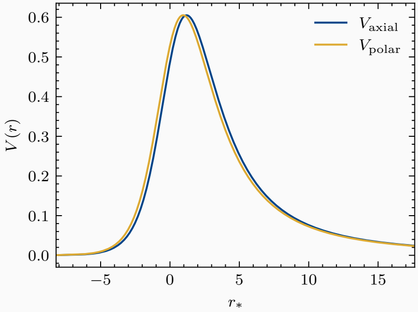
Interpretation of the equations

Axial case:

$$P_{\text{axial}} = \begin{pmatrix} 1-\mu/r & r \\ ir^2/(r-\mu) & 0 \end{pmatrix}, \quad c=1$$

Polar case:

$$P_{\text{polar}} = \begin{pmatrix} \frac{3\mu^2+3\lambda\mu r+2r^2\lambda(\lambda+1)}{2r^2(3\mu+2\lambda r)} & 1 \\ -i+\frac{i\mu}{2(r-\mu)}+\frac{3i\mu}{2\mu+2\lambda r} & -\frac{ir^2}{r-\mu} \end{pmatrix}, \quad c=1$$



Physical interpretation

At the horizon and infinity: $X_0(t,r) \propto e^{-i\omega(t\pm r_*)}$

$\Rightarrow$ Propagation of waves

Going back to the original variables

Summing up the change of variables:

$$
\begin{aligned}
h_0 &= \left(1 - \frac{\mu}{r}\right)\psi_{\text{axial}} + r\frac{\mathrm{d}\psi_{\text{axial}}}{\mathrm{d}r_*} \\
\frac{h_1}{\omega} &= \frac{ir^2}{r-\mu}\psi_{\text{axial}} \\
K &= \frac{3\mu^2 + 3\lambda\mu r + 2r^2\lambda(\lambda+1)}{2r^2(3\mu + 2\lambda r)}\psi_{\text{polar}} + \frac{\mathrm{d}\psi_{\text{polar}}}{\mathrm{d}r_*} \\
\frac{H_1}{\omega} &= \left[-i + \frac{i\mu}{2(r-\mu)} + \frac{3i\mu}{2\mu + 2\lambda r}\right]\psi_{\text{polar}} - \frac{ir^2}{r-\mu}\frac{\mathrm{d}\psi_{\text{polar}}}{\mathrm{d}r_*}
\end{aligned}
\quad\Rightarrow\quad
\begin{aligned}
&\frac{\mathrm{d}^2\psi_{\text{axial}}}{\mathrm{d}r_*^2} + \left(\omega^2 - V_{\text{axial}}(r)\right)\psi_{\text{axial}} = 0 \\
&\frac{\mathrm{d}^2\psi_{\text{polar}}}{\mathrm{d}r_*^2} + \left(\omega^2 - V_{\text{polar}}(r)\right)\psi_{\text{polar}} = 0
\end{aligned}
$$

Summary: computation of modes in GR

Linearized Einstein's equations

Fix gauge

Separate by parity

Odd parity:

$$X' = \begin{pmatrix} * & * \\ * & * \end{pmatrix} X \longrightarrow$$

Master equation:

$$Y'' + (\omega^2 - V_{\mathrm{a}})\,Y = 0$$

Even parity:

$$X' = \begin{pmatrix} * & * \\ * & * \end{pmatrix} X \longrightarrow$$

Master equation:

$$Y'' + (\omega^2 - V_{\mathrm{p}})\,Y = 0$$

Fully decoupled system

Numerics

Newman-Penrose formalism

No equivalent of RW gauge for rotating BH perturbation

Separation of Kerr perturbations

- Use of Newman-Penrose formalism
- Perturbation of 5 NP scalars, 12 spin coefficients, 4 tetrad components
- Computational *tour de force*: complete reduction of system to separated radial and angular equations for both polar and axial perturbations (Teukolsky equation) [Teukolsky+72] [Chandrasekhar+85]
- Polar perturbations: $\delta\Psi_0$, axial perturbations: $\delta\Psi_4$

From GR to modified gravity

DHOST: principle of construction

- DHOST: Degenerate Higher-Order Scalar-Tensor
- Add scalar field ϕ + higher-derivatives to break Lovelock
- Degeneracy conditions to ensure only one additional degree of freedom
- Action contains first and second derivatives of ϕ
- Obtain all possible terms and classify by powers of derivatives

$$\text{DHOST} = \boxed{\text{GR}} \times \boxed{\text{Coupling}} + \boxed{\begin{array}{c}\text{Orders 0 and}\\ \text{1 in } \nabla\nabla\phi\end{array}} + \boxed{(\nabla\nabla\phi)^2} + \boxed{(\nabla\nabla\phi)^3}$$

Lagrangian building blocks

$$S[g_{\mu\nu},\phi] = \int \mathrm{d}^4x\,\sqrt{-g}\left(F_2 R + P + Q\Box\phi + \sum_{i=1}^{5} A_i L_i^{(2)} + F_3\, G^{\mu\nu}\phi_{\mu\nu} + \sum_{i=1}^{10} B_i L_i^{(3)}\right),$$

$$\phi_\mu = \nabla_\mu\phi\,, \quad \phi_{\mu\nu} = \nabla_\mu\nabla_\nu\phi\,, \quad X = \phi_\mu\phi^\mu$$

Quadratic terms

$$L_1^{(2)} = \phi_{\mu\nu}\phi^{\mu\nu}\,, \quad L_2^{(2)} = (\Box\phi)^2$$
$$L_3^{(2)} = \phi^\mu\phi_{\mu\nu}\phi^\nu\Box\phi\,, \quad \ldots$$

Cubic terms

$$L_1^{(3)} = (\Box\phi)^3\,, \quad L_2^{(3)} = \phi_{\mu\nu}\phi^{\mu\nu}\Box\phi$$
$$L_3^{(3)} = \phi_{\mu\nu}\phi^{\nu\rho}\phi_\rho^\nu\,, \quad \ldots$$

All functions depend on ϕ and X (only X if shift-symmetric)

General form of black hole solutions

Metric sector

$$\mathrm{d}s^2 = -A(r)\,\mathrm{d}t^2 + \frac{1}{B(r)}\,\mathrm{d}r^2 + C(r)\,\mathrm{d}\Omega^2$$

Scalar sector

$$\phi(t, r) = qt + \psi(r)$$

- Choose specific form for DHOST functions A_i, B_i, etc.
- Escape no-hair theorems

Illustrative solutions

	Only quadratic	$q=0$	$A(r)(=B(r))$	Remarks
Stealth	✓	✗	$1-\dfrac{\mu}{r}$	X is constant
BCL [Babichev+17]	✓	✓	$1-\dfrac{\mu}{r}-\dfrac{\xi\mu^2}{2r^2}$	
4dEGB [Lu+20]	✗	✓	$1-\dfrac{2\mu/r}{1+\sqrt{1+4\alpha\mu/r^3}}$	Motivated from higher dimensions
EsGB [Julié+19]	✗	✓	$1-\dfrac{\mu}{r}+a_2(r)\varepsilon^2+\ldots$	Known only as expansion

General setup

Perturbations of the metric

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}\,, \quad \phi = \bar{\phi} + \delta\phi$$

$$\bar{g}_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = -A(r)\,\mathrm{d}t^2 + \mathrm{d}r^2\,/B(r) + C(r)\,\mathrm{d}\Omega^2\,, \quad \bar{\phi} = qt + \psi(r)$$

Difficulties arising in modified gravity

- Coupling between scalar mode and gravitational mode
- More free functions in the action
- New timelike direction $\nabla_\mu\phi$ in some cases

Axial modes

Axial modes: odd-parity perturbations

$$h_{\mu\nu} = \begin{pmatrix} & & \frac{1}{\sin\theta}\, h_0(r)\,\partial_\varphi & -\sin\theta\, h_0(r)\,\partial_\theta \\ & & \frac{1}{\sin\theta}\, h_1(r)\,\partial_\varphi & -\sin\theta\, h_1(r)\,\partial_\theta \\ \text{sym} & \text{sym} & \frac{1}{\sin\theta}\, h_2(r)(\partial_\theta\partial_\varphi - \operatorname{cotan}\theta\partial_\varphi) & -\sin\theta\, h_2(r)(\frac{\ell(\ell+1)}{2} - \partial_\theta^2) \\ \text{sym} & \text{sym} & \text{sym} & -\sin\theta\, h_2(r)(\partial_\theta\partial_\varphi - \operatorname{cotan}\theta\partial_\varphi) \end{pmatrix} Y_{\ell m}\, e^{-i\omega t}, \quad \delta\phi = 0$$

Polar modes: even-parity perturbations

$$h_{\mu\nu} = \begin{pmatrix} A\, H_0(r) & H_1(r) & \beta(r)\partial_\theta & \beta(r)\partial_\varphi \\ \text{sym} & B^{-1}\, H_2(r) & \alpha(r)\partial_\theta & \alpha(r)\partial_\varphi \\ \text{sym} & \text{sym} & K(r) + G(r)\partial_\theta^2 & -G(r)\operatorname{cotan}\theta\partial_\varphi \\ \text{sym} & \text{sym} & \text{sym} & \sin^2(\theta)\, K(r) + G(r)(\partial_\varphi^2 + \sin\theta\cos\theta\partial_\theta) \end{pmatrix} Y_{\ell m}\, e^{-i\omega t},$$

$$\delta\phi = \delta\varphi(r)\, Y_{\ell m}\, e^{-i\omega t}$$

What changes in modified gravity

Linearized modified gravity equations ⟶ Fix gauge ⟶ Separate by parity

Odd parity:

$$X' = \begin{pmatrix} * & * \\ * & * \end{pmatrix} X \longrightarrow$$

Master equation:

$$Y'' + (\omega^2 - V_{\mathrm{o}})\, Y = 0 \longrightarrow \text{Numerics}$$

Even parity:

$$X' = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix} X \longrightarrow$$

In general: no decoupling, usual numerics not possible

New even scalar mode coupled to gravitation

Modified Schrödinger equation

- Obtain form of first-order system for axial perturbations for any DHOST [Langlois+22]:

$$M_{\mathrm{axial}}(r) = \begin{pmatrix} C'/C + i\omega\Psi & -i\omega^2 + 2i\lambda\Phi/C \\ -i\Gamma & \Delta + i\omega\Psi \end{pmatrix}$$

- Modified propagation speed and effective potential from Ψ, Φ, Γ and Δ
- Many divergences at black hole horizon
- Not coordinate invariant quantities

Expression of building functions

$$\mathcal{F} = AF_2 - (q^2 + AX)A_1 - \frac{1}{2}AB\psi' X' F_{3X} - \frac{1}{2}B\psi'(AX)'B_2 - \frac{A}{2B}(B\psi')^3 X' B_6\,,$$

$$\frac{\mathcal{F}}{\Phi} = F_2 - XA_1 - \frac{1}{2}B\psi' X' F_{3X} - \frac{1}{2}B\psi' \frac{(CX)'}{C}B_2 - \frac{1}{2}B\psi' XX' B_6\,,$$

$$\mathcal{F}\Psi = q\psi' A_1 + \frac{q}{2}\left(B\psi'^2\right)' F_{3X} + \frac{q}{2}\frac{(AX)'}{A}B_2 + \frac{q}{4}\left(B^2\psi'^4\right)' B_6\,,$$

$$\Gamma = \Psi^2 + \frac{1}{2AB\mathcal{F}}\left(2q^2 A_1 + 2AF_2 + AB\psi' X' F_{3X} + q^2\frac{(AX)'}{A\psi'}B_2 + q^2 B\psi' X' B_6\right),$$

$$\Delta = -\frac{\mathcal{F}'}{\mathcal{F}} - \frac{B'}{2B} + \frac{A'}{2A} = -\frac{\mathrm{d}}{\mathrm{d}r}\left(\ln\left(\sqrt{\frac{B}{A}}\mathcal{F}\right)\right).$$

Modified potential

Generalized change of variables $\rightarrow$ generalized Schrödinger equation

$$\frac{\mathrm{d}^2\psi_{\text{axial}}}{\mathrm{d}r_*^2} + \left(\frac{\omega^2}{c^2} - V_{\text{axial}}(r)\right)\psi_{\text{axial}} = 0\,, \qquad \frac{\mathrm{d}r}{\mathrm{d}r_*} = n$$

$$V_{\text{axial}}(r) = 2n^2\lambda\frac{\Gamma\Phi}{C} + n^2\,V_0[n, C, \Gamma, \Delta]$$

Physical interest

Possibility to study perturbations in a manner very similar to GR!

Main subtility

Expression of V_{axial} is coordinate-dependant as it depends on n.

$\Rightarrow$ find a way to obtain coordinate-independant statements?

Equivalence with spin 2 in GR

Propagation of axial perturbations in cubic DHOST with metric $g_{\mu\nu}$	$\longleftrightarrow$	Propagation of axial massless spin 2 in GR with metric $\tilde{g}_{\mu\nu}$

Effective metric

$$\mathrm{d}\tilde{s}^2 = \tilde{g}_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = \tilde{A}\left(-\,\mathrm{d}t_*{}^2 + \mathrm{d}r_*{}^2 + \frac{\tilde{C}}{\tilde{A}}\,\mathrm{d}\Omega^2\right)$$

$$\mathrm{d}t_* = \mathrm{d}t - \Psi\,\mathrm{d}r\ , \quad \sqrt{\tilde{A}\tilde{B}}\,\mathrm{d}r_* = \mathrm{d}r$$

($\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ too complex to write here)

Properties of the effective metric

Stealth black hole

- Effective metric is still a stealth BH metric
- However, the horizon of this BH is displaced [Tomikawa+21]

4dEGB black hole

- The effective metric is not a black hole metric
- Naked singularity instead of horizon

EsGB and BCL black holes

No issue for the effective metric

Love numbers and QNMs

Tidal deformations

Love number: definition

Ratio between the linear response to an external static field and the field itself

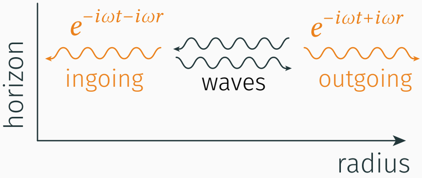
Computation

- Set $\omega = 0$ in master equation
- BC at horizon: regularity
- BC at infinity: normalization
- Identify linear response with subleading order at infinity

Properties

- All numbers are zero for $D = 4$ Schwarzschild [Hui+21], linked to symmetry [Ben Achour+22]
- No longer true in modified gravity theories [Cardoso+17] and for $D = 4$ Kerr [Tiec+21]
- Detectable in late inspiral phase

As an eigenvalue problem



- 2 boundary conditions: eigenvalue problem (similar to plucked string)
- **Complex spectrum** due to energy loss
- Depend on background and theory: very interesting test!

Possible issues

- Differential operator is not self-adjoint due to complex boundary conditions
- Choosing physical boundary conditions not always possible [Noui+23]

Definition from Green function

Setup

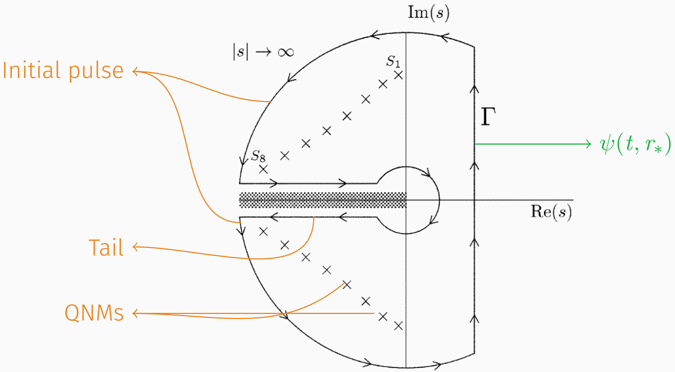
- Differential equation on $\psi(t, r_*)$: $\dfrac{\partial^2 \psi}{\partial r_*^2} - \dfrac{\partial^2 \psi}{\partial t^2} - V\psi = 0$
- Initial data $\psi(0, r)$ localized in r_*

$\Rightarrow$ compute $\psi(t, r_*)$ using Laplace transform

$$\psi(t, r_*) = \frac{1}{2\pi i} \int_{\varepsilon - i\infty}^{\varepsilon + i\infty} \mathrm{d}s\, e^{st} \int_{-\infty}^{+\infty} \mathrm{d}r' \, \underbrace{G(s, r_*, r')}_{\text{Green's function}} \overbrace{I(s, r')}^{\text{Initial data}}$$

Green function: complex integration

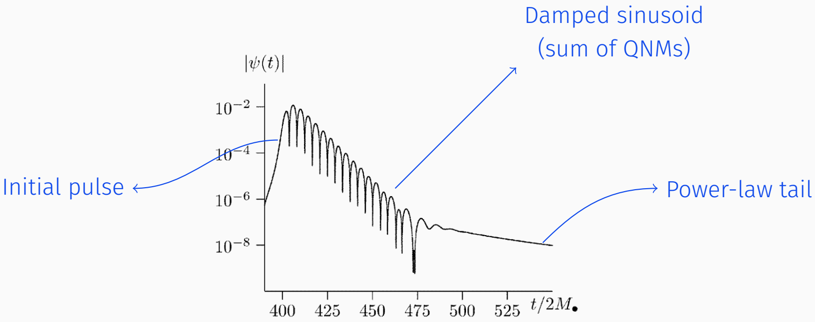
Various contributions in the complex plane [Nollert+99]



QNMs are the same as defined before!

Example of signal

Integration of the Schrödinger equation [Nollert+99]



However: this is not exactly the ringdown signal (non-linearities, spectral instabilities...)

Direct integration

From the horizon

Integrate equation with

$$\psi(r_* \to -\infty) \propto e^{-i\omega r_*}$$

$\to$ solution ψ_-

$\leftrightarrow$

MATCHING

$$W(\psi_-, \psi_+) = 0$$

$\leftrightarrow$

From infinity

Integrate equation with

$$\psi(r_* \to +\infty) \propto e^{i\omega r_*}$$

$\to$ solution ψ_+

- Easy to implement
- High numerical instability: rounding errors feed parasite solutions $e^{+i\omega r_*}$ at the horizon and $e^{-i\omega r_*}$ at infinity [Chandrasekhar+75]

Continued fraction

Main idea

- Ansatz $\psi(r) = \psi_\infty(r) \times \psi_{\text{horiz}}(r) \times f(r)$
- Decompose f as power series: $f(r) = \sum_{n=0}^{+\infty} f_n \left(\frac{r-r_h}{r}\right)^n$
- Look for ω such that f is bounded

- Get recurrence relation for f_n: $\alpha_n f_{n+1} + \beta_n f_n + \gamma_n f_{n-1} = 0$
- f is bounded when one has: [Gautschi+67; Leaver+97]

$$\frac{\beta_0}{\alpha_0} = \frac{\gamma_1}{\beta_1 -} \frac{\alpha_1\gamma_2}{\beta_2 -} \frac{\alpha_2\gamma_3}{\beta_3 -} \cdots$$

$\rightarrow$ continued fraction equation

- Can compute QNMs precisely at nearly any overtone

WKB

Qualitative interpretation

Understand QNMs as waves trapped in the light ring (corresponding to the max of V) and slowly leaking out

Quantitative realisation

Decompose $V-\omega^2$ around r_*^{max} :

$$V-\omega^2 = Q_0 + \frac{1}{2} Q_0^{(2)} (r_* - r_*^{\text{max}})^2 + \dots$$

- Proposed in [Goebel+72], improved in [Iyer+87; Iyer+87] and [Konoplya+03]
- Main advantage: QNMs as roots of algebraic equation
- Works better at high ℓ

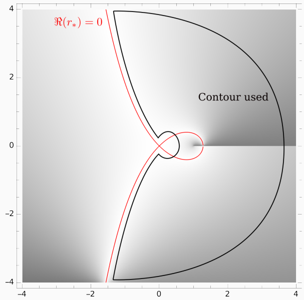
Monodromy

Main idea

- Extend the master equation to a complex r_*
- Solve in the regime $\mathrm{Re}(\omega) \ll \mathrm{Im}(\omega)$: $\omega \in i\mathbb{R}$
- Make use of the analyticity of the solution ψ

- BC at $r = \mu$: monodromy of function around singularity
- BC at $r = \infty$: imposed on line $\mathrm{Re}(r_*) = 0$ ($e^{\pm i\omega r_*}$ bounded)
- Recover asymptotic regime of QNMs: [Motl+03]

$$2\pi M\omega_n = \ln(3) + (2n+1)i\pi$$



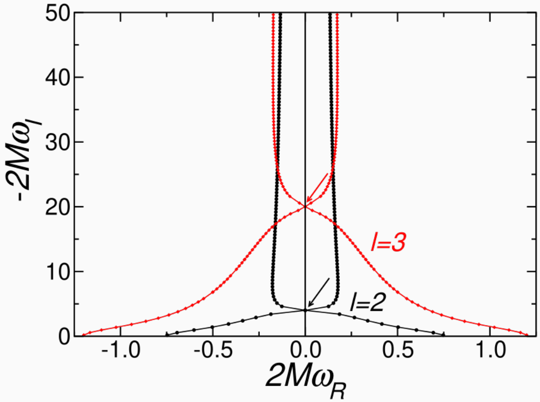
Comparison of methods

Method	Analytical	Difficulty	Modes computable	Validity
Direct integration	✗	+	~ 10	Low n
Continued fraction	✗	+++	$\sim 10^3$	Anywhere
WKB	✓	-	~ 10	Low n, high ℓ
Monodromy	✓	+	Asymptote	High n

Properties of the QNM spectrum

Positions of the modes

Schwarzschild spectrum obtained with continued fraction method [Berti+09]



Main properties of the spectrum

Stability

All modes have $\mathrm{Im}(\omega) < 0$: perturbations exponentially decreasing in time

Asymptote

Vertical asymptote independant of ℓ, coherent with the monodromy method

Algebraically special

For each ℓ one mode has $\mathrm{Re}(\omega) = 0$: *algebraically special* mode, linked to exact Robinson-Trautman solution [Qi+93]

Isospectrality

Values of ω for axial and polar perturbations are identical: linked to specific symmetry between V_{axial} and V_{polar} [Chandrasekhar+85]

Main challenges

Coupling of even modes

Polar perturbations couple with scalar: can only get *coupled Schrödinger equations*

In general: might not get Schrödinger formulation even for odd perturbations (ex: MTMG)

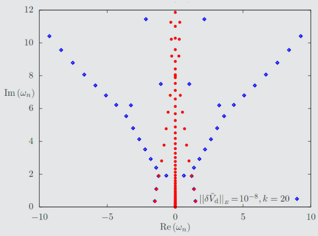
Boundary conditions

Not all potentials have ingoing and outgoing wave solutions at horizon and infinity

Spectral instability

Positions of the modes vary in an arbitrarily large manner when deviating from GR

[Jaramillo+21]



Existing results

- Computation of QNMs: done for the axial sector in various setups, but no study of asymptotes via monodromy technique
- Polar QNMs: obtained for EsGB [Blázquez-Salcedo+17]
- QNMs of slowly rotating EsGB [Pierini+22]
- Proposed methods for dealing with coupled systems [Hui+22; Langlois+21]
- EFT formulation of scalar-tensor [Mukohyama+22; Mukohyama+22; Mukohyama+23]
- Parametrized QNM spectrum [Cardoso+19; McManus+19]
- Inverse problem: recovering metric from QNMs [Völkel+20]

Topics for discussion

- Relative importance of QNM overtones vs quadratic perturbations in the ringdown
- Love numbers in modified gravity
- Application of the monodromy method to rotating modified BH
- Possibility to generalize Teukolsky equation to modified BH
- Are there modified gravity theories in which the symmetries of the QNM spectrum are maintained?

Thank you for your attention!

New black holes in DHOST: stealth solution

Metric sector: mimic GR

$$\mathrm{d}s^2 = -(1-\mu/r)\,\mathrm{d}t^2 + (1-\mu/r)^{-1}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2$$

Scalar sector

$$\phi = qt + \psi(r)$$

$$X = -q^2 \;\Rightarrow\; \psi'(r) = q\frac{\sqrt{r\mu}}{r-\mu}$$

Properties

- Metric sector: similar to Schwarzschild, time-dependant scalar field
- $X = \mathrm{cst} \Rightarrow$ functions of X reduced to constants

Effective metric for stealth Schwarzschild

$$\mathrm{d}s^2 = -A(r)\,\mathrm{d}t^2 + \frac{1}{A(r)}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2 \qquad A(r) = 1 - \frac{\mu}{r}$$

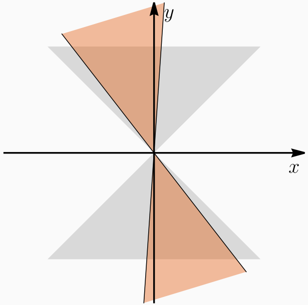
$$\mathrm{d}\tilde{s}^2 = \sqrt{1+\zeta}\left(-\frac{1}{1+\zeta}\left(1-\frac{r_g}{r}\right)\mathrm{d}{t_*}^2 + \left(1-\frac{r_g}{r}\right)^{-1}\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2\right),$$

$$\zeta = q^2 A_1 = \mathrm{cst}\,, \qquad r_g = (1+\zeta)\mu$$

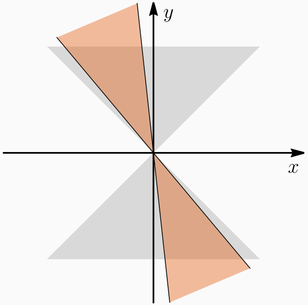
Properties

- Corresponds to Schwarzschild BH with $R = (1+\zeta)^{1/4} r$ and $T = (1+\zeta)^{-1/4} t_*$
- Horizon at $R = (1+\zeta)^{5/4}\mu$, corresponding to $r = r_g \neq \mu$
- The horizon seen by axial perturbations is displaced [Tomikawa+21]

Lightcones for stealth Schwarzschild



Lightcone for $r > r_g$



Lightcone for $\mu < r < r_g$

$\Rightarrow$ metrics are compatible despite the shift of horizon!

New black holes in Horndeski: EGB theory

Einstein-Gauss-Bonnet Lagrangian:

$$S = \int \mathrm{d}^D x \sqrt{-g}(R + \alpha'(\underbrace{R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2}_{\text{Gauss-Bonnet term } \mathcal{G}}))$$

Compactification procedure [Lu+20]

$$\mathrm{d}s_D = \mathrm{d}s + e^{2\phi}\,\mathrm{d}\Sigma^2 \quad \text{and} \quad \alpha' = \frac{\alpha}{D-4}$$

Take $D \longrightarrow 4$: get motivated choice of parameters of Horndeski given by

$$F(X) = 1 - 2\alpha X \quad P(X) = 2\alpha X^2\,, \quad Q(X) = -4\alpha X\,, \quad G(X) = -4\alpha \ln(X)$$

New black holes in Horndeski: EGB solution

Metric sector

$$\mathrm{d}s^2 = -A(r)\,\mathrm{d}t^2 + \frac{1}{A(r)}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2$$

$$A(r) = 1 - \frac{M(r)}{r}, \qquad M(r) = \frac{2\mu}{1+\sqrt{1+4\alpha\mu/r^3}}$$

Scalar sector

$$\phi = \psi(r)$$

$$\psi'(r) = \frac{-1+\sqrt{A}}{r\sqrt{A}}$$

Properties

- One horizon at $r = r_h = 1/2(\mu + \sqrt{\mu^2 - 4\alpha})$
- Constant α verifies $0 \leq \alpha \leq r_h^2$

Effective metric for EGB

$$\mathrm{d}s^2 = -A(r)\,\mathrm{d}t^2 + \frac{1}{A(r)}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2 \qquad A(r) = 1 - \frac{2\mu/r}{1+\sqrt{1+4\alpha\mu/r^3}}$$

$$\mathrm{d}\tilde{s}^2 = -\frac{1}{r^2}\sqrt{\frac{A^{1/2}\gamma_1^3\gamma_2}{\gamma_3^3}}\,\mathrm{d}{t_*}^2 + \frac{1}{r^2}\sqrt{\frac{\gamma_1\gamma_2^3}{A^{5/2}\gamma_3^5}}\,\mathrm{d}r^2 + \sqrt{\frac{\gamma_1\gamma_2}{A^{1/2}\gamma_3}}\,\mathrm{d}\Omega^2$$

- γ_1 and γ_3 are nonzero functions
- γ_2 has a zero at $r_2 = \sqrt[3]{2\alpha\mu}$
- A is zero at r_h only

Behaviour at the coordinate singularities

At $r = r_h$

$$d\tilde{s}^2 \sim -c_1(r - r_h)^{1/4}\, dt_*{}^2 + \frac{c_2}{(r - r_h)^{5/4}}\, dr^2 + \frac{c_3}{(r - r_h)^{1/4}}\, d\Omega^2$$

$\Rightarrow$ the Ricci scalar is singular at $r = r_h$: **curvature singularity** at the horizon

At $r = r_2$

$$d\tilde{s}^2 \sim -c_4(r - r_2)^{1/2}\, dt_*{}^2 + c_5(r - r_2)^{3/2}\, dr^2 + c_6(r - r_2)^{1/2}\, d\Omega^2$$

$\Rightarrow$ the Ricci scalar is singular at $r = r_2$: another **curvature singularity**

Property

The axial modes propagate in a metric with naked singularities